Chapter 1
What Are Partial Differential Equations?

Let us begin by delineating our field of study. A *differential equation* is an equation that relates the derivatives of a (scalar) function depending on one or more variables. For example,

$$\frac{d^4u}{dx^4} + \frac{d^2u}{dx^2} + u^2 = \cos x \tag{1.1}$$

is a differential equation for the function $u(x)$ depending on a single variable x, while

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u \tag{1.2}$$

is a differential equation involving a function $u(t, x, y)$ of three variables.

A differential equation is called *ordinary* if the function u depends on only a single variable, and *partial* if it depends on more than one variable. Usually (but not quite always) the dependence of u can be inferred from the derivatives that appear in the differential equation. The *order* of a differential equation is that of the highest-order derivative that appears in the equation. Thus, (1.1) is a fourth-order ordinary differential equation, while (1.2) is a second-order partial differential equation.

Remark: A differential equation has order 0 if it contains no derivatives of the function u. These are more properly treated as *algebraic equations*,† which, while of great interest in their own right, are not the subject of this text. To be a bona fide *differential equation*, it must contain at least one derivative of u, and hence have order ≥ 1.

There are two common notations for partial derivatives, and we shall employ them interchangeably. The first, used in (1.1) and (1.2), is the familiar Leibniz notation that employs a d to denote ordinary derivatives of functions of a single variable, and the ∂ symbol (usually also pronounced "dee") for partial derivatives of functions of more than one variable. An alternative, more compact notation employs subscripts to indicate partial derivatives. For example, u_t represents $\partial u/\partial t$, while u_{xx} is used for $\partial^2 u/\partial x^2$, and $\partial^3 u/\partial x^2 \partial y$ for u_{xxy}. Thus, in subscript notation, the partial differential equation (1.2) is written

$$u_t = u_{xx} + u_{yy} - u. \tag{1.3}$$

† Here, the term "algebraic equation" is used only to distinguish such equations from true "differential equations". It does not mean that the defining functions are necessarily algebraic, e.g., polynomials. For example, the transcendental equation $\tan u = u$, which appears later in (4.50), is still regarded as an algebraic equation in this book.

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We will similarly abbreviate partial differential operators, sometimes writing $\partial/\partial x$ as ∂_x, while $\partial^2/\partial x^2$ can be written as either ∂_x^2 or ∂_{xx}, and $\partial^3/\partial x^2 \partial y$ becomes $\partial_{xxy} = \partial_x^2 \partial_y$.

It is worth pointing out that the preponderance of differential equations arising in applications, in science, in engineering, and within mathematics itself are of either first or second order, with the latter being by far the most prevalent. Third-order equations arise when modeling waves in dispersive media, e.g., water waves or plasma waves. Fourth-order equations show up in elasticity, particularly plate and beam mechanics, and in image processing. Equations of order ≥ 5 are very rare.

A basic prerequisite for studying this text is the ability to solve simple ordinary differential equations: first-order equations; linear constant-coefficient equations, both homogeneous and inhomogeneous; and linear systems. In addition, we shall assume some familiarity with the basic theorems concerning the existence and uniqueness of solutions to initial value problems. There are many good introductory texts, including [**18**, **20**, **23**]. More advanced treatises include [**31**, **52**, **54**, **59**]. Partial differential equations are considerably more demanding, and can challenge the analytical skills of even the most accomplished mathematician. Many of the most effective solution strategies rely on reducing the partial differential equation to one or more ordinary differential equations. Thus, in the course of our study of partial differential equations, we will need to develop, ab initio, some of the more advanced aspects of the theory of ordinary differential equations, including boundary value problems, eigenvalue problems, series solutions, singular points, and special functions.

Following the introductory remarks in the present chapter, the exposition begins in earnest with simple first-order equations, concentrating on those that arise as models of wave phenomena. Most of the remainder of the text will be devoted to understanding and solving the three essential linear second-order partial differential equations in one, two, and three space dimensions:† the *heat equation*, modeling thermodynamics in a continuous medium, as well as diffusion of animal populations and chemical pollutants; the *wave equation*, modeling vibrations of bars, strings, plates, and solid bodies, as well as acoustic, fluid, and electromagnetic vibrations; and the *Laplace equation* and its inhomogeneous counterpart, the *Poisson equation*, governing the mechanical and thermal equilibria of bodies, as well as fluid-mechanical and electromagnetic potentials.

Each increase in dimension requires an increase in mathematical sophistication, as well as the development of additional analytic tools — although the key ideas will have all appeared once we reach our physical, three-dimensional universe. The three starring examples — heat, wave, and Laplace/Poisson — are not only essential to a wide range of applications, but also serve as instructive paradigms for the three principal classes of linear partial differential equations — parabolic, hyperbolic, and elliptic. Some interesting nonlinear partial differential equations, including first-order transport equations modeling shock waves, the second-order Burgers' equation governing simple nonlinear diffusion processes, and the third-order Korteweg–de Vries equation governing dispersive waves, will also be discussed. But, in such an introductory text, the further reaches of the vast realm of nonlinear partial differential equations must remain unexplored, awaiting the reader's more advanced mathematical excursions.

More generally, a *system of differential equations* is a collection of one or more equations relating the derivatives of one or more functions. It is essential that all the functions

† For us, *dimension* always refers to the number of space dimensions. Time, although theoretically also a dimension, plays a very different physical role, and therefore (at least in nonrelativistic systems) is to be treated on a separate footing.

occurring in the system *depend on the same set of variables*. The symbols representing these functions are known as the *dependent variables*, while the variables that they depend on are called the *independent variables*. Systems of differential equations are called *ordinary* or *partial* according to whether there are one or more independent variables. The *order* of the system is the highest-order derivative occurring in any of its equations.

For example, the three-dimensional *Navier–Stokes equations*

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} &= -\,\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right),\\
\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z} &= -\,\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right),\\
\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z} &= -\,\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right),\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0,
\end{aligned} \tag{1.4}$$

is a second-order system of differential equations that involves four functions, $u(t,x,y,z)$, $v(t,x,y,z)$, $w(t,x,y,z)$, $p(t,x,y,z)$, each depending on four variables, while $\nu \geq 0$ is a fixed constant. (The function p necessarily depends on t, even though no t derivative of it appears in the system.) The independent variables are t, representing time, and x,y,z, representing space coordinates. The dependent variables are u,v,w,p, with $\mathbf{v} = (u,v,w)$ representing the velocity vector field of an incompressible fluid flow, e.g., water, and p the accompanying pressure. The parameter ν measures the viscosity of the fluid. The Navier–Stokes equations are fundamental in fluid mechanics, [**12**], and are notoriously difficult to solve, either analytically or numerically. Indeed, establishing the existence or nonexistence of solutions for all future times remains a major unsolved problem in mathematics, whose resolution will earn you a $1,000,000 prize; see http://www.claymath.org for details. The Navier–Stokes equations first appeared in the early 1800s in works of the French applied mathematician/engineer Claude-Louis Navier and, later, the British applied mathematician George Stokes, whom you already know from his eponymous multivariable calculus theorem.† The inviscid case, $\nu = 0$, is known as the *Euler equations* in honor of their discoverer, the incomparably influential eighteenth-century Swiss mathematician Leonhard Euler.

We shall be employing a few basic notational conventions regarding the variables that appear in our differential equations. We always use t to denote time, while x,y,z will represent (Cartesian) space coordinates. Polar coordinates r,θ, cylindrical coordinates r,θ,z, and spherical coordinates‡ r,θ,φ, will also be used when needed. An *equilibrium equation* models an unchanging physical system, and so involves only the space variable(s). The time variable appears when modeling *dynamical*, meaning time-varying, processes. Both time and space coordinates are (usually) independent variables. The dependent variables will mostly be denoted by u,v,w, although occasionally — particularly in representing

† Interestingly, Stokes' Theorem was taken from an 1850 letter that Lord Kelvin wrote to Stokes, who turned it into an undergraduate exam question for the Smith Prize at Cambridge University in England. However, unbeknownst to either, the result had, in fact, been discovered earlier by George Green, the father of Green's Theorem and also the Green's function, which will be the subject of Chapter 6.

‡ See Section 12.2 for our notational convention.

particular physical quantities — other letters may be employed, e.g., the pressure p in (1.4). On the other hand, the letters f, g, h typically represent specified functions of the independent variables, e.g., forcing or boundary or initial conditions.

In this introductory text, we must confine our attention to the most basic analytic and numerical solution techniques for a select few of the most important partial differential equations. More advanced topics, including all systems of partial differential equations, must be deferred to graduate and research-level texts, e.g., [**35**, **38**, **44**, **61**, **99**]. In fact, many important issues remain incompletely resolved and/or poorly understood, making partial differential equations one of the most active and exciting fields of contemporary mathematical research. One of my goals is that, by reading this book, you will be both inspired and equipped to venture much further into this fascinating and essential area of mathematics and/or its remarkable range of applications throughout science, engineering, economics, biology, and beyond.

Exercises

1.1. Classify each of the following differential equations as ordinary or partial, and equilibrium or dynamic; then write down its order. (a) $\dfrac{du}{dx} + x\,u = 1$, (b) $\dfrac{\partial u}{\partial t} + u\,\dfrac{\partial u}{\partial x} = x$, (c) $u_{tt} = 9\,u_{xx}$, (d) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x}$, (e) $-\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = x^2 + y^2$, (f) $\dfrac{d^2 u}{dt^2} + 3\,u = \sin t$, (g) $u_{xx} + u_{yy} + u_{zz} + (x^2 + y^2 + z^2)u = 0$, (h) $u_{xx} = x + u^2$, (i) $\dfrac{\partial u}{\partial t} + \dfrac{\partial^3 u}{\partial x^3} + u\,\dfrac{\partial u}{\partial x} = 0$, (j) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y\,\partial z} = u$, (k) $u_{tt} = u_{xxxx} + 2\,u_{xxyy} + u_{yyyy}$.

1.2. In two space dimensions, the *Laplacian* is defined as the second-order partial differential operator $\Delta = \partial_x^2 + \partial_y^2$. Write out the following partial differential equations in (i) Leibniz notation; (ii) subscript notation: (a) the Laplace equation $\Delta u = 0$; (b) the Poisson equation $-\Delta u = f$; (c) the two-dimensional heat equation $\partial_t u = \Delta u$; (d) the von Karman plate equation $\Delta^2 u = 0$.

1.3. Answer Exercise 1.2 for the three-dimensional Laplacian $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$.

1.4. Identify the independent variables, the dependent variables, and the order of the following systems of partial differential equations: (a) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$; (b) $u_{xx} + v_{yy} = \cos(x+y)$, $u_x v_y - u_y v_x = 1$; (c) $\dfrac{\partial u}{\partial t} = \dfrac{\partial v}{\partial x}$, $\dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$; (d) $u_t + u\,u_x + v\,u_y = p_x$, $v_t + u\,v_x + v\,v_y = p_y$, $u_x + v_y = 0$; (e) $u_t = v_{xxx} + v(1-v)$, $v_t = u_{xxy} + v\,w$, $w_t = u_x + v_y$.

Classical Solutions

Let us now focus our attention on a single differential equation involving a single, scalar-valued function u that depends on one or more independent variables. The function u

is usually real-valued, although complex-valued functions can, and do, play a role in the analysis. Everything that we say in this section will, when suitably adapted, apply to systems of differential equations.

By a *solution* we mean a sufficiently smooth function u of the independent variables that satisfies the differential equation at every point of its domain of definition. We do not necessarily require that the solution be defined for all possible values of the independent variables. Indeed, usually the differential equation is imposed on some domain D contained in the space of independent variables, and we seek a solution defined only on D. In general, the *domain* D will be an open subset, usually connected and, particularly in equilibrium equations, often bounded, with a reasonably nice boundary, denoted by ∂D.

We will call a function *smooth* if it can be differentiated sufficiently often, at least so that all of the derivatives appearing in the equation are well defined on the domain of interest D. More specifically, if the differential equation has order n, then we require that the solution u be of *class* C^n, which means that it and all its derivatives of order $\leq n$ are continuous functions in D, and such that the differential equation that relates the derivatives of u holds throughout D. However, on occasion, e.g., when dealing with shock waves, we will consider more general types of solutions. The most important such class consists of the so-called "weak solutions" to be introduced in Section 10.4. To emphasize the distinction, the smooth solutions described above are often referred to as *classical solutions*. In this book, the term "solution" without extra qualification will usually mean "classical solution".

Example 1.1. A classical solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1.5}$$

is a function $u(t,x)$, defined on a domain $D \subset \mathbb{R}^2$, such that all of the functions

$$u(t,x), \quad \frac{\partial u}{\partial t}(t,x), \quad \frac{\partial u}{\partial x}(t,x), \quad \frac{\partial^2 u}{\partial t^2}(t,x), \quad \frac{\partial^2 u}{\partial t\,\partial x}(t,x) = \frac{\partial^2 u}{\partial x\,\partial t}(t,x), \quad \frac{\partial^2 u}{\partial x^2}(t,x),$$

are well defined and continuous[†] at every point $(t,x) \in D$, so that $u \in \mathrm{C}^2(D)$, and, moreover, (1.5) holds at every $(t,x) \in D$. Observe that, even though only u_t and u_{xx} explicitly appear in the heat equation, we require continuity of *all* the partial derivatives of order ≤ 2 in order that u qualify as a classical solution. For example,

$$u(t,x) = t + \tfrac{1}{2}x^2 \tag{1.6}$$

is a solution to the heat equation that is defined on the full domain $D = \mathbb{R}^2$ because it is[‡] C^2, and, moreover,

$$\frac{\partial u}{\partial t} = 1 = \frac{\partial^2 u}{\partial x^2}\,.$$

Another, more complicated but extremely important, solution is

$$u(t,x) = \frac{e^{-x^2/(4t)}}{2\sqrt{\pi t}}\,. \tag{1.7}$$

[†] The equality of the mixed partial derivatives follows from a general theorem in multivariable calculus, [**8, 97, 108**]. Classical solutions automatically enjoy equality of all their relevant mixed partial derivatives.

[‡] In fact, the function (1.6) is C^∞, meaning infinitely differentiable, on all of $\mathbb{R}^2$.

One easily verifies that $u \in \mathrm{C}^2$ and, moreover, solves the heat equation on the domain $D = \{\,(t,x) \mid t > 0\,\} \subset \mathbb{R}^2$. The reader is invited to verify this by computing $\partial u/\partial t$ and $\partial^2 u/\partial x^2$, and then checking that they are equal. Finally, with $\mathrm{i} = \sqrt{-1}$ denoting the imaginary unit, we note that

$$u(t,x) = e^{-t+\mathrm{i}\,x} = e^{-t}\cos x + \mathrm{i}\,e^{-t}\sin x, \tag{1.8}$$

the second expression following from Euler's formula (A.11), defines a complex-valued solution to the heat equation. This can be verified directly, since the rules for differentiating complex exponentials are identical to those for their real counterparts:

$$\frac{\partial u}{\partial t} = -\,e^{-t+\mathrm{i}\,x}, \qquad \frac{\partial u}{\partial x} = \mathrm{i}\,e^{-t+\mathrm{i}\,x}, \qquad \text{and so} \qquad \frac{\partial^2 u}{\partial x^2} = -\,e^{-t+\mathrm{i}\,x} = \frac{\partial u}{\partial t}\,.$$

It is worth pointing out that both the real part, $e^{-t}\cos x$, and the imaginary part, $e^{-t}\sin x$, of the complex solution (1.8) are individual real solutions, which is indicative of a fairly general property.

Incidentally, most partial differential equations arising in physical applications are real, and, although complex solutions often facilitate their analysis, at the end of the day we require real, physically meaningful solutions. A notable exception is quantum mechanics, which is an inherently complex-valued physical theory. For example, the one-dimensional *Schrödinger equation*

$$\mathrm{i}\,\hbar\,\frac{\partial u}{\partial t} = -\,\frac{\hbar^2}{2\,m}\,\frac{\partial^2 u}{\partial x^2} + V(x)\,u, \tag{1.9}$$

with $\hbar$ denoting *Planck's constant*, which is real, governs the dynamical evolution of the complex-valued wave function $u(t,x)$ describing the probabilistic distribution of a quantum particle of mass m, e.g., an electron, moving in the force field prescribed by the (real) potential function $V(x)$. While the solution u is complex-valued, the independent variables t, x, representing time and space, remain real.

Initial Conditions and Boundary Conditions

How many solutions does a partial differential equation have? In general, lots. Even ordinary differential equations have infinitely many solutions. Indeed, the general solution to a single n^{th} order ordinary differential equation depends on n arbitrary constants. The solutions to partial differential equations are yet more numerous, in that they depend on *arbitrary functions*. Very roughly, we can expect the solution to an n^{th} order partial differential equation involving m independent variables to depend on n arbitrary functions of $m-1$ variables. But this must be taken with a large grain of salt — only in a few special instances will we actually be able to express the solution in terms of arbitrary functions.

The solutions to dynamical ordinary differential equations are singled out by the imposition of initial conditions, resulting in an *initial value problem*. On the other hand, equations modeling equilibrium phenomena require boundary conditions to specify their solutions uniquely, resulting in a *boundary value problem*. We assume that the reader is already familiar with the basics of initial value problems for ordinary differential equations. But we will take time to develop the perhaps less familiar case of boundary value problems for ordinary differential equations in Chapter 6.

A similar specification of auxiliary conditions applies to partial differential equations. Equations modeling equilibrium phenomena are supplemented by boundary conditions imposed on the boundary of the domain of interest. In favorable circumstances, the boundary conditions serve to single out a unique solution. For example, the equilibrium temperature of a body is uniquely specified by its boundary behavior. If the domain is unbounded, one must also restrict the nature of the solution at large distances, e.g., by asking that it remain bounded. The combination of a partial differential equation along with suitable boundary conditions is referred to as a *boundary value problem*.

There are three principal types of boundary value problems that arise in most applications. Specifying the value of the solution along the boundary of the domain is called a *Dirichlet boundary condition*, to honor the nineteenth-century analyst Johann Peter Gustav Lejeune Dirichlet. Specifying the normal derivative of the solution along the boundary results in a *Neumann boundary condition*, named after his contemporary Carl Gottfried Neumann. Prescribing the function along part of the boundary and the normal derivative along the remainder results in a *mixed boundary value problem*. For example, in thermal equilibrium, the Dirichlet boundary value problem specifies the temperature of a body along its boundary, and our task is to find the interior temperature distribution by solving an appropriate partial differential equation. Similarly, the Neumann boundary value problem prescribes the heat flux through the boundary. In particular, an insulated boundary has no heat flux, and hence the normal derivative of the temperature is zero on the boundary. The mixed boundary value problem prescribes the temperature along part of the boundary and the heat flux along the remainder. Again, our task is to determine the interior temperature of the body.

For partial differential equations modeling dynamical processes, in which time is one of the independent variables, the solution is to be specified by one or more initial conditions. The number of initial conditions required depends on the highest-order time derivative that appears in the equation. For example, in thermodynamics, which involves only the first-order time derivative of the temperature, the initial condition requires specifying the temperature of the body at the initial time. Newtonian mechanics describes the acceleration or second-order time derivative of the motion, and so requires two initial conditions: the initial position and initial velocity of the system. On bounded domains, one must also impose suitable boundary conditions in order to uniquely characterize the solution and hence the subsequent dynamical behavior of the physical system. The combination of the partial differential equation, the initial conditions, and the boundary conditions leads to an *initial-boundary value problem*. We will encounter, and solve, many important examples of such problems during the course of this text.

Remark: An additional consideration is that, besides any smoothness required by the partial differential equation within the domain, the solution and any of its derivatives specified in any initial or boundary condition should also be continuous at the initial or boundary point where the condition is imposed. For example, if the initial condition specifies the function value $u(0,x)$ for $a < x < b$, while the boundary conditions specify the derivatives $\dfrac{\partial u}{\partial x}(t,a)$ and $\dfrac{\partial u}{\partial x}(t,b)$ for $t > 0$, then, in addition to any smoothness required inside the domain $\{a < x < b,\ t > 0\}$, we also require that u be continuous at all initial points $(0,x)$, and that its derivative $\dfrac{\partial u}{\partial x}$ be continuous at all boundary points (t,a) and (t,b), in order that $u(t,x)$ qualify as a *classical solution* to the initial-boundary value problem.

Exercises

1.5. Show that the following functions $u(x,y)$ define classical solutions to the two-dimensional Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Be careful to specify an appropriate domain.
(a) $e^x \cos y$, (b) $1+x^2-y^2$, (c) x^3-3xy^2, (d) $\log(x^2+y^2)$, (e) $\tan^{-1}(y/x)$, (f) $\dfrac{x}{x^2+y^2}$.

1.6. Find all solutions $u = f(r)$ of the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ that depend only on the radial coordinate $r = \sqrt{x^2+y^2}$.

1.7. Find all (real) solutions to the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ of the form $u = \log p(x,y)$, where $p(x,y)$ is a quadratic polynomial.

1.8. (a) Find all quadratic polynomial solutions of the three-dimensional Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$. (b) Find all the homogeneous cubic polynomial solutions.

1.9. Find all polynomial solutions $p(t,x)$ of the heat equation $u_t = u_{xx}$ with $\deg p \le 3$.

1.10. Show that each of the following functions $u(t,x)$ is a solution to the wave equation $u_{tt} = 4u_{xx}$: (a) $4t^2 - x^2$; (b) $\cos(x+2t)$; (c) $\sin 2t \cos x$; (d) $e^{-(x-2t)^2}$.

1.11. Find all polynomial solutions $p(t,x)$ of the wave equation $u_{tt} = u_{xx}$ with
(a) $\deg p \le 2$, (b) $\deg p = 3$.

1.12. Suppose $u(t,x)$ and $v(t,x)$ are C^2 functions defined on $\mathbb{R}^2$ that satisfy the first-order system of partial differential equations $u_t = v_x$, $v_t = u_x$.
(a) Show that both u and v are classical solutions to the wave equation $u_{tt} = u_{xx}$. Which result from multivariable calculus do you need to justify the conclusion?
(b) Conversely, given a classical solution $u(t,x)$ to the wave equation, can you construct a function $v(t,x)$ such that $u(t,x), v(t,x)$ form a solution to the first-order system?

1.13. Find all solutions $u = f(r)$ of the three-dimensional Laplace equation
$u_{xx} + u_{yy} + u_{zz} = 0$ that depend only on the radial coordinate $r = \sqrt{x^2+y^2+z^2}$.

1.14. Let $u(x,y)$ be defined on a domain $D \subset \mathbb{R}^2$. Suppose you know that all its second-order partial derivatives, $u_{xx}, u_{xy}, u_{yx}, u_{yy}$, are defined and continuous on all of D. Can you conclude that $u \in C^2(D)$?

1.15. Write down a partial differential equation that has
(a) no real solutions; (b) exactly one real solution; (c) exactly two real solutions.

1.16. Let $u(x,y) = xy\,\dfrac{x^2-y^2}{x^2+y^2}$ for $(x,y) \ne (0,0)$, while $u(0,0) = 0$. Prove that

$$\frac{\partial^2 u}{\partial x\,\partial y}(0,0) = 1 \ne -1 = \frac{\partial^2 u}{\partial y\,\partial x}(0,0).$$

Explain why this example does not contradict the theorem on the equality of mixed partials.

Linear and Nonlinear Equations

As with algebraic equations and ordinary differential equations, there is a crucial distinction between linear and nonlinear partial differential equations, and one must have a firm grasp of the linear theory before venturing into the nonlinear wilderness. While linear algebraic equations are (modulo numerical difficulties) eminently solvable by a variety of techniques, linear ordinary differential equations, of order ≥ 2, already present a challenge, as most cannot be solved in terms of elementary functions. Indeed, as we will learn in Chapter 11, solving many of those equations that arise in applications requires introducing new types of "special functions" that are typically not encountered in a basic calculus course. Linear partial differential equations are of a yet higher level of difficulty, and only a small handful of specific equations can be completely solved. Moreover, explicit solutions tend to be expressible only in the form of infinite series, requiring subtle analytic tools to understand their convergence and properties. For the vast majority of partial differential equations, the only feasible means of producing general solutions is through numerical approximation. In this book, we will study the two most basic numerical schemes: finite differences and finite elements. Keep in mind that, in order to develop and understand numerics for partial differential equations, one must already have a good understanding of their analytical properties.

The distinguishing feature of linearity is that it enables one to straightforwardly combine solutions to form new solutions, through a general Superposition Principle. Linear superposition is universally applicable to all linear equations and systems, including linear algebraic systems, linear ordinary differential equations, linear partial differential equations, linear initial and boundary value problems, as well as linear integral equations, linear control systems, and so on. Let us introduce the basic idea in the context of a single differential equation.

A differential equation is called *homogeneous linear* if both sides are sums of terms, each of which involves the dependent variable u *or* one of its derivatives to the first power; on the other hand, there is no restriction on how the terms involve the independent variables. Thus,

$$\frac{d^2u}{dx^2} + \frac{u}{1+x^2} = 0$$

is a homogeneous linear second-order ordinary differential equation. Examples of homogeneous linear partial differential equations include the heat equation (1.5), the partial differential equation (1.2), and the equation

$$\frac{\partial u}{\partial t} = e^x \, \frac{\partial^2 u}{\partial x^2} + \cos(x-t)\, u.$$

On the other hand, Burgers' equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \tag{1.10}$$

is not linear, since the second term involves the product of u and its derivative u_x. A similar terminology is applied to systems of partial differential equations. For example, the Navier–Stokes system (1.4) is not linear because of the terms $u\,u_x, v\,u_y$, etc. — although its final constituent equation is linear.

A more precise definition of a homogeneous linear differential equation begins with the concept of a *linear differential operator* L. Such operators are assembled by summing the basic partial derivative operators, with either constant coefficients or, more generally, coefficients depending on the independent variables. The operator acts on sufficiently smooth functions depending on the relevant independent variables. According to Definition B.32, *linearity* imposes two key requirements:

$$L[u+v] = L[u] + L[v], \qquad L[cu] = c\,L[u], \tag{1.11}$$

for any two (sufficiently smooth) functions u, v, and any constant c.

Definition 1.2. A *homogeneous linear differential equation* has the form

$$L[u] = 0, \tag{1.12}$$

where L is a linear differential operator.

As a simple example, consider the second-order differential operator

$$L = \frac{\partial^2}{\partial x^2}, \qquad \text{whereby} \qquad L[u] = \frac{\partial^2 u}{\partial x^2}$$

for any C^2 function $u(x,y)$. The linearity requirements (1.11) follow immediately from basic properties of differentiation:

$$\begin{aligned} L[u+v] &= \frac{\partial^2}{\partial x^2}(u+v) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} = L[u] + L[v], \\ L[cu] &= \frac{\partial^2}{\partial x^2}(cu) = c\,\frac{\partial^2 u}{\partial x^2} = c\,L[u], \end{aligned}$$

which are valid for any C^2 functions u, v and any constant c. The corresponding homogeneous linear differential equation $L[u] = 0$ is

$$\frac{\partial^2 u}{\partial x^2} = 0.$$

The heat equation (1.5) is based on the linear partial differential operator

$$L = \partial_t - \partial_x^2, \qquad \text{with} \qquad L[u] = \partial_t u - \partial_x^2 u = u_t - u_{xx} = 0. \tag{1.13}$$

Linearity follows as above:

$$\begin{aligned} L[u+v] &= \partial_t(u+v) - \partial_x^2(u+v) = (\partial_t u - \partial_x^2 u) + (\partial_t v - \partial_x^2 v) = L[u] + L[v], \\ L[cu] &= \partial_t(cu) - \partial_x^2(cu) = c\,(\partial_t u - \partial_x^2 u) = c\,L[u]. \end{aligned}$$

Similarly, the linear differential operator

$$L = \partial_t^2 - \partial_x\,\kappa(x)\,\partial_x = \partial_t^2 - \kappa(x)\,\partial_x^2 - \kappa'(x)\,\partial_x,$$

where $\kappa(x)$ is a prescribed C^1 function of x alone, defines the homogeneous linear partial differential equation

$$L[u] = \partial_t^2 u - \partial_x(\kappa(x)\,\partial_x u) = u_{tt} - \partial_x(\kappa(x)\,u_x) = u_{tt} - \kappa(x)\,u_{xx} - \kappa'(x)\,u_x = 0,$$

which is used to model vibrations in a nonuniform one-dimensional medium.

The defining attributes of linear operators (1.11) imply the key properties shared by all homogeneous linear (differential) equations.

Proposition 1.3. *The sum of two solutions to a homogeneous linear differential equation is again a solution, as is the product of a solution with any constant.*

Proof: Let u_1, u_2 be solutions, meaning that $L[u_1] = 0$ and $L[u_2] = 0$. Then, thanks to linearity,

$$L[u_1 + u_2] = L[u_1] + L[u_2] = 0,$$

and hence their sum $u_1 + u_2$ is a solution. Similarly, if c is any constant and u any solution, then

$$L[c\,u] = c\,L[u] = c\,0 = 0,$$

and so the constant multiple $c\,u$ is also a solution. *Q.E.D.*

As a result, starting with a handful of solutions to a homogeneous linear differential equation, by repeating these operations of adding solutions and multiplying by constants, we are able to build up large families of solutions. In the case of the heat equation (1.5), we are already in possession of two solutions, namely (1.6) and (1.7). Multiplying each by a constant produces two infinite families of solutions:

$$u(t,x) = c_1(t + \tfrac{1}{2}x^2) \qquad \text{and} \qquad u(t,x) = \frac{c_2\, e^{-x^2/(4t)}}{2\sqrt{\pi\, t}}\,,$$

where c_1, c_2 are arbitrary constants. Moreover, one can add the latter solutions together, producing a two-parameter family of solutions

$$u(t,x) = c_1(t + \tfrac{1}{2}x^2) + \frac{c_2\, e^{-x^2/(4t)}}{2\sqrt{\pi\, t}}\,,$$

valid for any choice of the constants c_1, c_2.

The preceding construction is a special case of the general *Superposition Principle* for homogeneous linear equations:

Theorem 1.4. *If $u_1, \ldots, u_k$ are solutions to a common homogeneous linear equation $L[u] = 0$, then the* linear combination, *or* superposition, $u = c_1 u_1 + \cdots + c_k u_k$ *is a solution for any choice of constants $c_1, \ldots, c_k$.*

Proof: Repeatedly applying the linearity requirements (1.11), we find

$$\begin{aligned} L[u] &= L[c_1 u_1 + \cdots + c_k u_k] = L[c_1 u_1 + \cdots + c_{k-1} u_{k-1}] + L[c_k u_k] \\ &= \;\cdots\; = L[c_1 u_1] + \cdots + L[c_k u_k] = c_1 L[u_1] + \cdots + c_k L[u_k]. \end{aligned} \tag{1.14}$$

In particular, if the functions are solutions, so $L[u_1] = 0, \ \ldots\ , L[u_k] = 0$, then the right-hand side of (1.14) vanishes, proving that u also solves the equation $L[u] = 0$. *Q.E.D.*

In the linear algebraic language of Appendix B, Theorem 1.4 tells us that the solutions to a homogeneous linear partial differential equation form a vector space. The same holds true for linear algebraic equations, [**89**], and linear ordinary differential equations, [**18, 20, 23, 52**]. In the latter two situations, once one finds a sufficient number of independent solutions, the general solution is obtained as a linear combination thereof. In the language of linear algebra, the solution space is finite-dimensional. In contrast, most linear systems of partial differential equations admit an infinite number of independent solutions, meaning that the solution space is infinite-dimensional, and, as a consequence, one cannot hope to build the general solution by taking *finite* linear combinations. Instead, one requires the far more delicate operation of forming infinite series involving the basic solutions. Such considerations will soon lead us into the heart of Fourier analysis, and require spending an entire chapter developing the required analytic tools.

Definition 1.5. An *inhomogeneous linear differential equation* has the form

$$L[v] = f, \tag{1.15}$$

where L is a linear differential operator, v is the unknown function, and f is a prescribed nonzero function of the independent variables alone.

For example, the inhomogeneous form of the heat equation (1.13) is

$$L[v] = \partial_t v - \partial_x^2 v = v_t - v_{xx} = f(t,x), \tag{1.16}$$

where $f(t,x)$ is a specified function. This equation models the thermodynamics of a one-dimensional medium subject to an external heat source.

You already learned the basic technique for solving inhomogeneous linear equations in your study of elementary ordinary differential equations. Step one is to determine the general solution to the homogeneous equation. Step two is to find a particular solution to the inhomogeneous version. The general solution to the inhomogeneous equation is then obtained by adding the two together. Here is the general version of this procedure:

Theorem 1.6. *Let $v_\star$ be a particular solution to the inhomogeneous linear equation $L[v_\star] = f$. Then the general solution to $L[v] = f$ is given by $v = v_\star + u$, where u is the general solution to the corresponding homogeneous equation $L[u] = 0$.*

Proof: Let us first show that $v = v_\star + u$ is also a solution whenever $L[u] = 0$. By linearity,

$$L[v] = L[v_\star + u] = L[v_\star] + L[u] = f + 0 = f.$$

To show that every solution to the inhomogeneous equation can be expressed in this manner, suppose v satisfies $L[v] = f$. Set $u = v - v_\star$. Then, by linearity,

$$L[u] = L[v - v_\star] = L[v] - L[v_\star] = 0,$$

and hence u is a solution to the homogeneous differential equation. Thus, $v = v_\star + u$ has the required form. *Q.E.D.*

In physical applications, one can interpret the particular solution $v_\star$ as a response of the system to the external forcing function. The solution u to the homogeneous equation represents the system's internal, unforced behavior. The general solution to the inhomogeneous linear equation is thus a combination, $v = v_\star + u$, of the external and internal responses.

Finally, the *Superposition Principle* for *inhomogeneous* linear equations allows one to combine the responses of the system to different external forcing functions. The proof of this result is left to the reader as Exercise 1.26.

Theorem 1.7. *Let $v_1, \ldots, v_k$ be solutions to the inhomogeneous linear systems $L[v_1] = f_1, \ \ldots \ , L[v_k] = f_k$, involving the* same *linear operator L. Then, given any constants $c_1, \ldots, c_k$, the* linear combination *$v = c_1 v_1 + \cdots + c_k v_k$ solves the inhomogeneous system $L[v] = f$ for the combined forcing function $f = c_1 f_1 + \cdots + c_k f_k$.*

The two general Superposition Principles furnish us with powerful tools for solving linear partial differential equations, which we shall repeatedly exploit throughout this text. In contrast, nonlinear partial differential equations are much tougher, and, typically, knowledge of several solutions is of scant help in constructing others. Indeed, finding even one solution to a nonlinear partial differential equation can be quite a challenge. While this text will primarily concentrate on analyzing the solutions and their properties to some of the most basic and most important linear partial differential equations, we will have occasion to briefly venture into the nonlinear realm, introducing some striking recent developments in this fascinating arena of contemporary research.

Exercises

1.17. Classify the following differential equations as either
(i) homogeneous linear; (ii) inhomogeneous linear; or (iii) nonlinear:
(a) $u_t = x^2 u_{xx} + 2x u_x$, (b) $-u_{xx} - u_{yy} = \sin u$; (c) $u_{xx} + 2y u_{yy} = 3$;
(d) $u_t + u u_x = 3u$; (e) $e^y u_x = e^x u_y$; (f) $u_t = 5u_{xxx} + x^2 u + x$.

1.18. Write down all possible solutions to the Laplace equation you can construct from the various solutions provided in Exercise 1.5 using linear superposition.

1.19. (a) Show that the following functions are solutions to the wave equation $u_{tt} = 4u_{xx}$:
(i) $\cos(x - 2t)$, (ii) e^{x+2t}; (iii) $x^2 + 2xt + 4t^2$.
(b) Write down at least four other solutions to the wave equation.

1.20. The displacement $u(t,x)$ of a forced violin string is modeled by the partial differential equation $u_{tt} = 4u_{xx} + F(t,x)$. When the string is subjected to the external forcing $F(t,x) = \cos x$, the solution is $u(t,x) = \cos(x - 2t) + \frac{1}{4}\cos x$, while when $F(t,x) = \sin x$, the solution is $u(t,x) = \sin(x - 2t) + \frac{1}{4}\sin x$. Find a solution when the forcing function $F(t,x)$ is
(a) $\cos x - 5\sin x$, (b) $\sin(x - 3)$.

1.21. (a) Show that the partial derivatives $\partial_x[f] = \dfrac{\partial f}{\partial x}$ and $\partial_y[f] = \dfrac{\partial f}{\partial y}$ both define linear operators on the space of continuously differentiable functions $f(x,y)$. (b) For which values of a,b,c,d is the differential operator $L[f] = a\,\dfrac{\partial f}{\partial x} + b\,\dfrac{\partial f}{\partial y} + c f + d$ linear?

1.22. (a) Prove that the Laplacian $\Delta = \partial_x^2 + \partial_y^2$ defines a linear differential operator.
(b) Write out the Laplace equation $\Delta[u] = 0$ and the Poisson equation $-\Delta[u] = f$.

1.23. Prove that, on $\mathbb{R}^3$, the gradient, curl, and divergence all define linear operators.

1.24. Let L and M be linear partial differential operators. Prove that the following are also linear partial differential operators: (a) $L - M$, (b) $3L$, (c) $f L$, where f is an arbitrary function of the independent variables; (d) $L \circ M$.

1.25. Suppose L and M are linear differential operators and let $N = L + M$.
(a) Prove that N is a linear operator. (b) *True or false*: If u solves $L[u] = f$ and v solves $M[v] = g$, then $w = u + v$ solves $N[w] = f + g$.

♢ 1.26. Prove Theorem 1.7.

1.27. Solve the following inhomogeneous linear ordinary differential equations:
(a) $u' - 4u = x - 3$, (b) $5u'' - 4u' + 4u = e^x \cos x$, (c) $u'' - 3u' = e^{3x}$.

1.28. Use superposition to solve the following inhomogeneous ordinary differential equations:
(a) $u' + 2u = 1 + \cos x$, (b) $u'' - 9u = x + \sin x$, (c) $9u'' - 18u' + 10u = 1 + e^x \cos x$,
(d) $u'' + u' - 2u = \sinh x$, where $\sinh x = \frac{1}{2}(e^x - e^{-x})$, (e) $u''' + 9u' = 1 + e^{3x}$.