Chapter 11
Term Structure of Loss Cascades in Portfolio Securitisation

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Abstract We report on the term structure of loss cascades generated through portfolio tranching. The results are based on the analytical form of the loss distribution for uniform loan portfolios and show that the expected loss of the first loss position increases roughly linear whereas the expected losses of the more senior tranches increase exponentially over time depending on the relation between mean default probability and tranching limits.

11.1 Introduction

Asset Backed Securities (ABS) and related portfolio dependent financial products like collateralised loan obligations (CLO) are used for several purposes, namely to transfer and manage credit risk, as a balance sheet management tool in order to obtain capital relief, and gain liquidity. From the methodological point of view these structures boil down to a repartition of interest earnings in exchange to loss burdens among possible investors whereas both are alloted according to the investors seniority. In the present note we treat only the second point, i.e. the allocation of

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losses through the various tranches and their evolution in time. These structures have obtained a lot of attraction before and during the credit crisis 2007/08. Many banks built large trading desks for these structures, called correlation desks. In analogy to the volatility trading desks based on Black-Scholes model and its extension, the correlation desks were trading "implied correlation" which were based on the "base correlation approach" (cf. Li (2000), McGinty and Ahluwalia (2004) or Bluhm and Overbeck (2006)). This is a simplified default time model with a uniform Gaussian copula. For our focus, the timing of aggregate losses, we do not model the default time of the single entities in the portfolio, but model for each time step the aggregate loss in the portfolio. This is in some aspects a simple top-down approach, which is a more recent stream of modelling for structured products (cf. e.g. Sidenius et al. (2008), Bennani (2005), Schönbucher (2005) and Filipovic et al. (2011)). In the present chapter we assume a uniform portfolio and use loss distributions which are available in analytic form. The main result of the paper is that even is this simplified approach, the fact that losses are back loaded in senior tranche. This made it plausible that the down-rating in the credit crisis was especially severe on senior tranches. Also compared to migration behaviour of well rated counterparties, well rated tranches will migrate in a more non-linear way. Most of the downward migration will come and the end of the lifte-time of the transaction.

11.2 Loss Distribution of Uniform Portfolio

It is well known (cf. Vasicek (1987) or Bluhm et al. (2010)) that for a uniform portfolio of m loans, i.e. equal exposure $1/m$, equal default probability p and equal pairwise asset correlation ρ, the limiting distribution for $m \longrightarrow \infty$ is the so called normal inverse distribution $NID(p, \rho)$ (The underlying asset returns in this model are assumed to be normal distributed.). The distribution of the portfolio losses $0 \leq x \leq 1$ is given by the cumulative distribution function

$$NID(x, p, \rho) = N\left\{\frac{1}{\sqrt{\rho}}\left[\sqrt{1-\rho}N^{-1}(x) - N^{-1}(p)\right]\right\} \tag{11.1}$$

and its density

$$\phi(x, p, \rho) = \sqrt{\frac{1-\rho}{\rho}} \exp\left\{-\frac{1}{2\rho}\left\{(1-2\rho)\left[N^{-1}(x)\right]^2 - 2\sqrt{1-\rho}N^{-1}(x)N^{-1}(p) + \left[N^{-1}(p)\right]^2\right\}\right\} \tag{11.2}$$

with $0 < p, \rho < 1$, with mean p and variance $\sigma^2 = N_2\left(N^{-1}(p), N^{-1}(p); \rho\right) - p^2$, where N denotes the standard normal distribution function and $N_2(x, y; \rho)$ denotes the bivariate normal distribution function with zero expectation vector and covariance matrix showing units on the diagonal and ρ off the diagonal.

11.3 Time Slicing

Now, let us observe a portfolio and its losses on a discrete time grid $0 = t_0 < t_1 < t_2 \cdots < t_{n-1} < t_n$. Denote X_i the relative portfolio loss (relative to the remaining exposure) during time step i, then the absolute loss at i, assuming the balance at time 0 to be 1, is

$$Y_i = \prod_{j=1}^{i-1}(1 - X_j)\, X_i \qquad \text{for} \quad i = 1, \ldots n\,, \tag{11.3}$$

and accumulates over time to

$$\tilde{Y}_i = \sum_{j=1}^{i} Y_i\,. \tag{11.4}$$

Suppose further that our portfolio and the residues after losses can be considered as being uniform with possible changes only being reflected by time dependent portfolio parameters, p_i, ρ_i, $i = 1 \ldots n$. We can then draw the random variable X_i in step i from the normal inverse distribution

$$X_i \sim NID(x, p_i, \rho_i) \tag{11.5}$$

to obtain the absolut loss Y_i. The respective density function can in principle be calculated by product folding, but it does not seem to be possible to state the results in a closed form.

We therefore resort to Monte Carlo simulations of the loss distribution, whereby the random variables are generated according to Eq. (11.5). For this, we first take uniformly distributed random variables $Z \sim U(0, 1)$ and transform with

$$x = NID^{-1}(z, p_i, \rho_i) = N\left(\frac{1}{\sqrt{1-\rho}}\left(N^{-1}(p) - \sqrt{\rho}\, z\right)\right).$$

11.4 Loss Cascades

As already mentioned in the introduction during securitization transactions the portfolio losses L are allocated subsequently to various tranches according to their seniority, i.e. investor 1 holds for losses up to α_1%, investor 2 for remaining losses but smaller than α_2%, and so on. In mathematical notation this reads

$$L_i = (L - \alpha_{i-1})^+ \wedge (\alpha_i - \alpha_{i-1})$$

where $0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_k$ are the boundaries of the tranches and L_i denotes the loss to be borne by tranche i. Thus, tranches are 'served' in cascades, if one tranche has overflowed further losses are allocated to the next senior tranche. The first tranche is usually kept by the issuer and is called first loss position (FLP). The mezzanine tranches are usually brought to the market as notes and the senior tranche is securitized by a credit default swap. For the rating and spreads of the various tranches an interesting quantity is the expected loss per tranche

$$EL_i = \int \frac{(x - \alpha_{i-1})^+ \wedge (\alpha_i - \alpha_{i-1})}{\alpha_i - \alpha_{i-1}} \, df(x) \,, \tag{11.6}$$

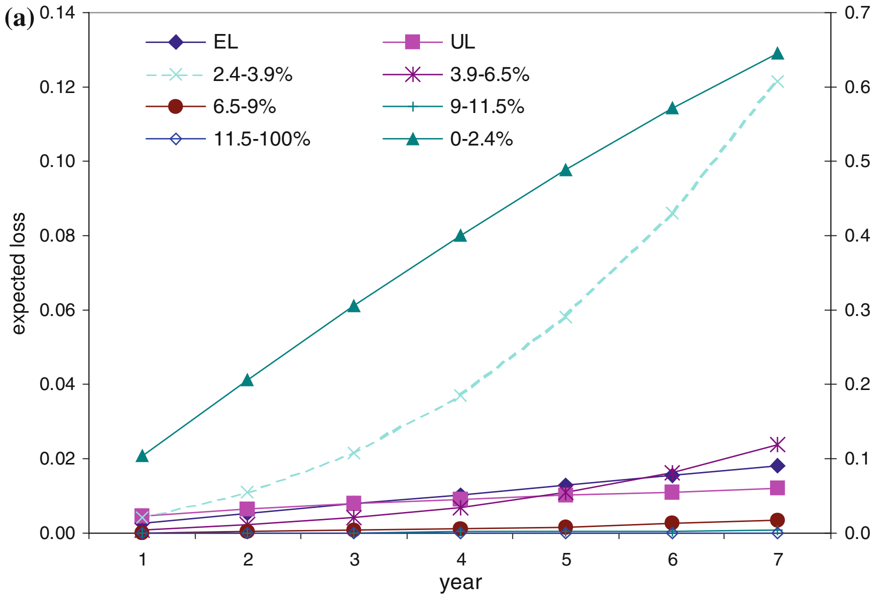
with $f(x)$ being the probability measure of some loss distribution.

11.5 Results

Table 11.1 shows the results of a Monte Carlo simulation with 10^6 simulations of a sequence of normal inverse distributed portfolio losses $X_i, i = 1, \ldots, 7$, Eq. (11.5), with constant portfolio parameters $p = 0.0026$ and $\rho = 0.17$. The first column denotes the year, the second the respective (forward) default rate p and the third and fourth column give the mean and the standard deviation of the accumulated loss, EL and UL. The remaining columns report on the accumulated expected loss per tranche, where some typical boundaries have been chosen. All quantities increase monotoneously, but the more interesting result can be seen in Fig. 11.1 (linear and logarithmic plot). Whereas the expected loss of the first tranche increases linearly (scaled on the right axis in the linear plot) the ELs of the other tranches increase exponentially over the years. In Fig. 11.2 we attempt a direct comparison of the default-probability term structure for the tranches $[2.4 - 3.9\%]$, $[3.9 - 6.5\%]$ and $[6.5 - 9\%]$ with respective corporate zero bonds (calibration based on rating reports of Standard & Poor's and Moody's Investors Services, Moody (2001). For this, we

Table 11.1 Vasicek (normal inverse) distribution

EL per tranche									
Year	p	EL	UL	0–2.4%	2.4–3.9%	3.9–6.5%	6.5–9%	9–11.5%	11.5–100%
1	0.0026	0.002593	0.004588	0.104352	0.004135	0.000830	0.000157	0.000041	0.000001
2	0.0026	0.005186	0.006491	0.206350	0.010987	0.002129	0.000389	0.000110	0.000001
3	0.0026	0.007771	0.007921	0.304995	0.021439	0.004068	0.000683	0.000177	0.000002
4	0.0026	0.010356	0.009126	0.399452	0.036819	0.006922	0.001108	0.000261	0.000003
5	0.0026	0.012934	0.010181	0.488493	0.058068	0.010884	0.001691	0.000392	0.000004
6	0.0026	0.015500	0.011127	0.570866	0.086047	0.016401	0.002502	0.000559	0.000006
7	0.0026	0.018059	0.011991	0.645940	0.121363	0.023847	0.003575	0.000777	0.000008



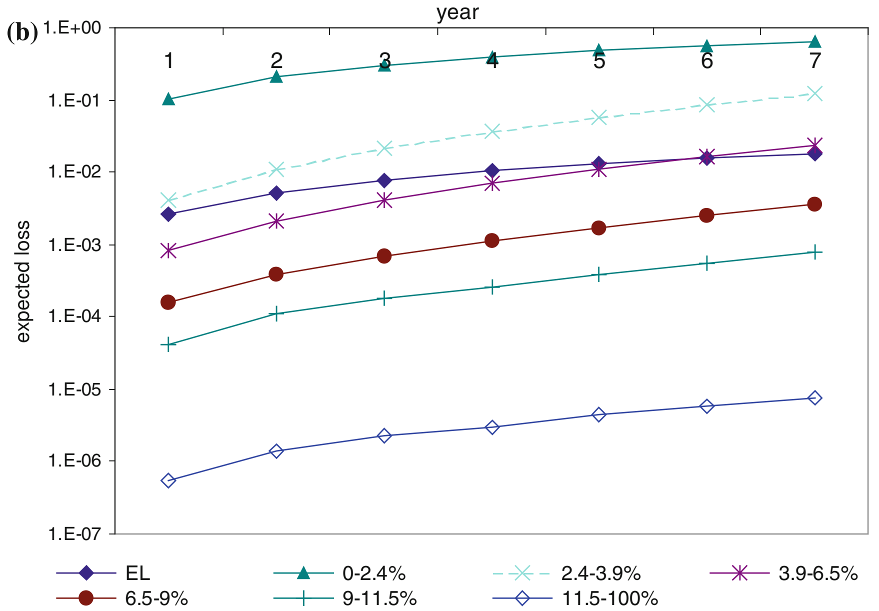


Fig. 11.1 Term structure of expected losses in tranches with $p = 0.0026$, $\rho = 0.17$, linear and logarithmic scale. Note that in the upper plot EL$[0 - 2.4\%]$ scales with the *right* axis

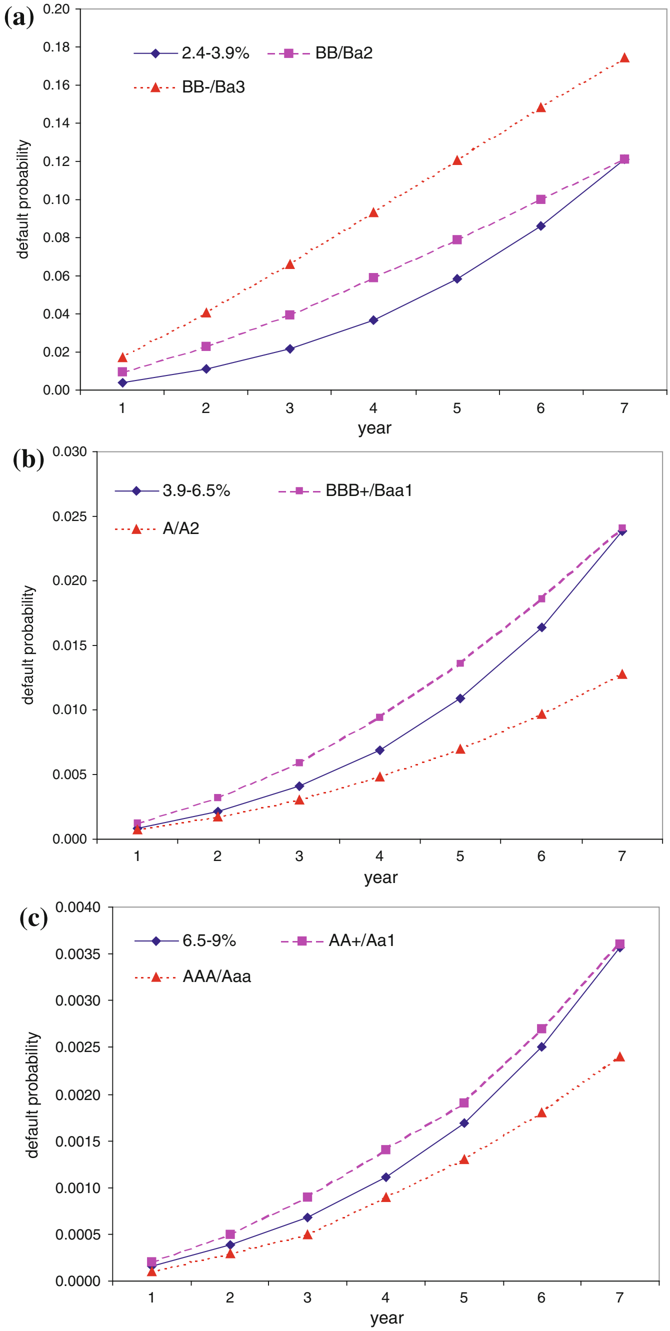


Fig. 11.2 Term structure of expected losses in tranches [2.4 − 3.9%], [3.9 − 6.5%], [6.5 − 9%], with $p = 0.0026$, $\rho = 0.17$, and corporate zero bonds for comparison

either try to match 'one-year' expected loss or the accumulated '7-years' expected loss per tranche to the respective default probabilities assigned by Moody's to a suitable corporate bond.

Since the first loss position is usually kept by the issuer he can expect linear increasing loss burdens over time whereas the investors buying the notes have to anticipate exponentially increasing loss burdens. This is different to the term structure of the expected loss for similar rated corporate bonds, these show a less convex increase in expected loss during time. The term structure of securitized tranches might therefore serve a non-linear risk appetite on the investors side. We can now estimate the respective rates r_i in tranche i given to the investors by calculating the net present value of the expected cash flows according to

$$\sum_{j=1}^{n-1} \frac{(1 - EL_i^j) r_i}{\prod_{l=1}^{j}(1 + z_l)} + \frac{(1 - EL_i^n)(1 + r_i)}{\prod_{l=1}^{n}(1 + z_l)} = 1\,, \qquad (11.7)$$

where EL_i^j denotes the accumulated expected loss in tranche i up to year j and z_l represents the risk free zero forward rate.

Using Eq. (11.7) and a constant risk free rate $z = z_l = 5.0\%$ we arrive at:

tranche	0–2.4%	2.4–3.9%	3.9–6.5%	6.5–9%	9–11.5%	11.5–100%
rate	20.560%	6.794%	5.339%	5.051%	5.011%	5.000%

In reality, the spreads given to investors are considerably higher. Clearly, this is again the discussion of real-world versus risk-neutral probabilities. But one justification for higher risk neutral spreads, besides liquidity or other additional risks, can be found in Fig. 11.3 where the ratios of unexpected to expected loss, UL/EL, for the whole portfolio (total) and all tranches are shown. All ratios decrease in time, but the more interesting result is that they differ considerably in orders of magnitude. Whereas the whole portfolio and the first loss piece $[0 - 2.4\%]$ yield a ratio of order one already the second tranche $[2.4 - 3.9\%]$ exhibits a ratio of order 10 and all more senior ratios increase roughly by a factor of two. This means that the variation of the losses around the expected value is much higher for the investors tranches than for the first loss position and requires an additional risk premium.

11.5.1 Other Loss Distributions

Since the tail behavior of loan loss distributions is a rather critical part in risk considerations we also experiment with other possible distributions. The following comparison is based on an EL/UL match, i.e. we choose the parameters such that the first two moments match to the ones obtained from the normal inverse distribution.

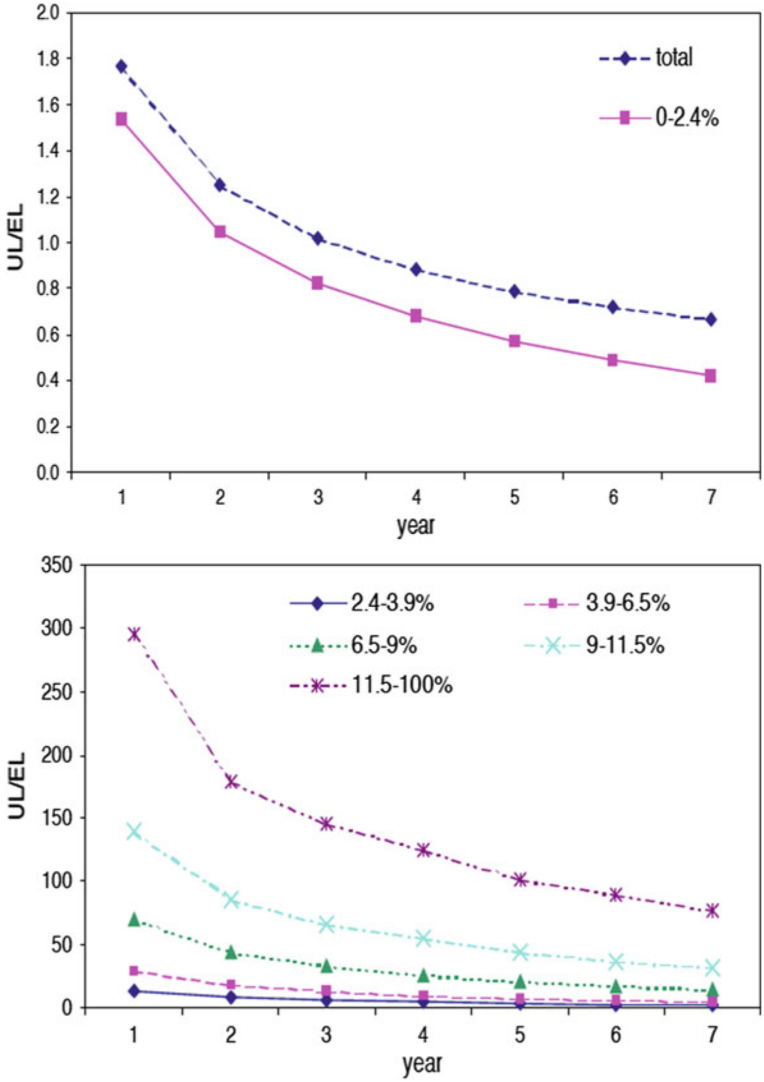


Fig. 11.3 Term structure of ratio UL/EL for the whole portfolio (total) and the tranches with $p = 0.0026$, $\rho = 0.17$

Beta Distribution

Choosing the parameters for the Beta-distribution with density

$$f_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 \le x \le 1, \tag{11.8}$$

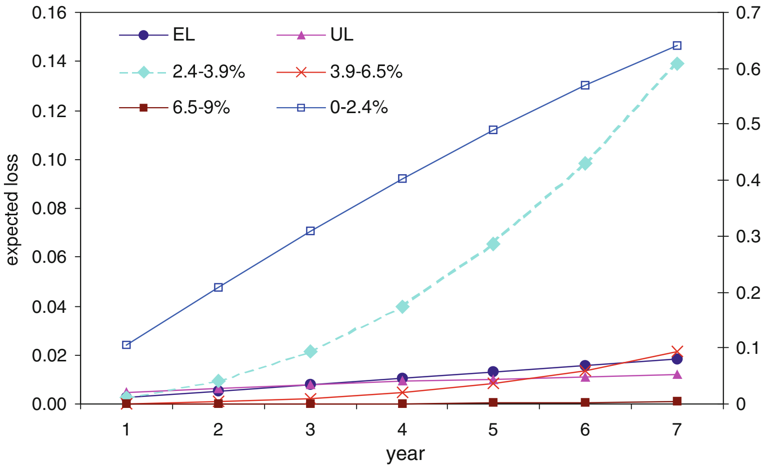


Fig. 11.4 Term structure of expected losses in tranches where the parameters of Beta-distribution ($\alpha = 0.315878$, $\beta = 121.176$) are chosen such that EL and UL match to the previous example (Vasicek-distribution with $p = 0.0026$, $\rho = 0.17$). Note that in the upper plot EL[$0 - 2.4\%$] scales with the *right* axis

Table 11.2 Beta distribution

EL per tranche								
Year	EL	UL	0–2.4%	2.4–3.9%	3.9–6.5%	6.5–9%	9–11.5%	11.5–100%
1	0.002569	0.004512	0.105027	0.002851	0.000209	4.48E-06	0	0
2	0.005176	0.006450	0.208788	0.009512	0.00085	2.14E-05	0	0
3	0.007789	0.007294	0.308685	0.021589	0.002135	6.06E-05	0	0
4	0.010391	0.009154	0.402956	0.039937	0.004462	0.000185	1.62E-06	0
5	0.012955	0.010195	0.489843	0.065128	0.008176	0.000377	2.56E-06	0
6	0.015510	0.011122	0.569401	0.098164	0.013608	0.000714	8.69E-06	0
7	0.018078	0.011989	0.641583	0.138913	0.021659	0.001298	2.98E-05	0

mean $\mu_B = \frac{\alpha}{\alpha+\beta}$ and variance $\sigma_B = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$ as $\alpha = 0.315878$, $\beta = 121.176$ leads to a good EL/UL match with the Vasicek distribution under $p = 0.0026$, $\rho = 0.17$. Table 11.2 and Fig. 11.4 show the result where the yearly portfolio loss X_i is now drawn according to Eq. (11.8).

Negative Binomial Distribution

Another prominent loss distribution in extreme event statistics is the Negative-Binomial distribution with frequency function

$$P[Loss = n] = f_{\alpha,\beta}(n) = \frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)}\left(1-\frac{\beta}{1+\beta}\right)^{\alpha}\left(\frac{\beta}{1+\beta}\right)^{n} \tag{11.9}$$

with mean $\mu_{NB} = \alpha\beta$ and variance $\sigma^2_{NB} = \alpha\beta(1+\beta)$. Note that the Negative Binomial distribution can be constructed as a composition of a Poisson distribution conditional on Gamma distributed intensities Λ see e.g. Rice (1995), for a motivation in credit risk management see CreditRisk$^+$ (1997). The respective parameters α, β can then be found by matching first and second moment with the normal inverse distribution (Sect. 11.2) under the constraint

$$\frac{\mu_{NB}}{m} = p \quad \text{and} \quad \frac{\sigma^2_{NB}}{m^2} = \sigma^2 \qquad (m \text{ sufficiently large}).$$

Choosing $m = 10^6$ yields a fairly good approximation of the corresponding percentage loss, i.e. $Loss/m$, since the probabilities $P\left[Loss = k\right]$ are negligible for $k \geq 10^6$ and results in $\alpha = 0.3193$ and $\beta = 8.1416 \times 10^3$. Table 11.2 shows the results for the different tranches (Table 11.3). The expected (EL) loss, the unexpected loss (UL) and the expected loss in the first tranche $[0 - 2.4\%]$ match pretty well for all three distributions. As we move further into the tails to higher tranches, see also Fig. 11.5, we observe an increasing difference in the term structure between normal inverse respectively Beta-/Negative-Binomial distribution reflecting the different 'fatness' of tails, see especially tranche $[6 - 9\%]$. Due to the very asymmetric and 'extreme event'-like behavior of credit loss, we think that the normal inverse distribution more truthfully reflects the 'loss reality' (Bluhm et al. (2010)). Surprisingly, the term structure of beta-distribution and negative binomial distribution are fairly equal. For further investigation we generated a q-q-plot (Fig. 11.7) for both distributions with matched first two moments. Remember that in case of the Negativ-Binomial Distribution the discrete losses $n \in \mathbb{N}_0$ have to be divided by some large number m (For

Table 11.3 Negative Binomial distribution

EL per tranche								
Year	EL	UL	0–2.4%	2.4–3.9%	3.9–6.5%	6.5–9%	9–11.5%	11.5–100%
1	0.002587	0.004581	0.105659	0.003039	0.000205	0.000010	0.00E+00	0.00E+00
2	0.005197	0.006476	0.209482	0.009744	0.000864	0.000017	0.00E+00	0.00E+00
3	0.007777	0.007932	0.308187	0.021358	0.002205	0.000091	3.46E-06	0.00E+00
4	0.010356	0.009115	0.402161	0.038983	0.004387	0.000225	4.36E-06	0.00E+00
5	0.012926	0.010175	0.489310	0.063951	0.008158	0.000425	2.76E-05	0.00E+00
6	0.015511	0.011120	0.569918	0.096979	0.013770	0.000770	4.97E-05	0.00E+00
7	0.018083	0.011988	0.642122	0.138027	0.021825	0.001309	7.55E-05	0.00E+00

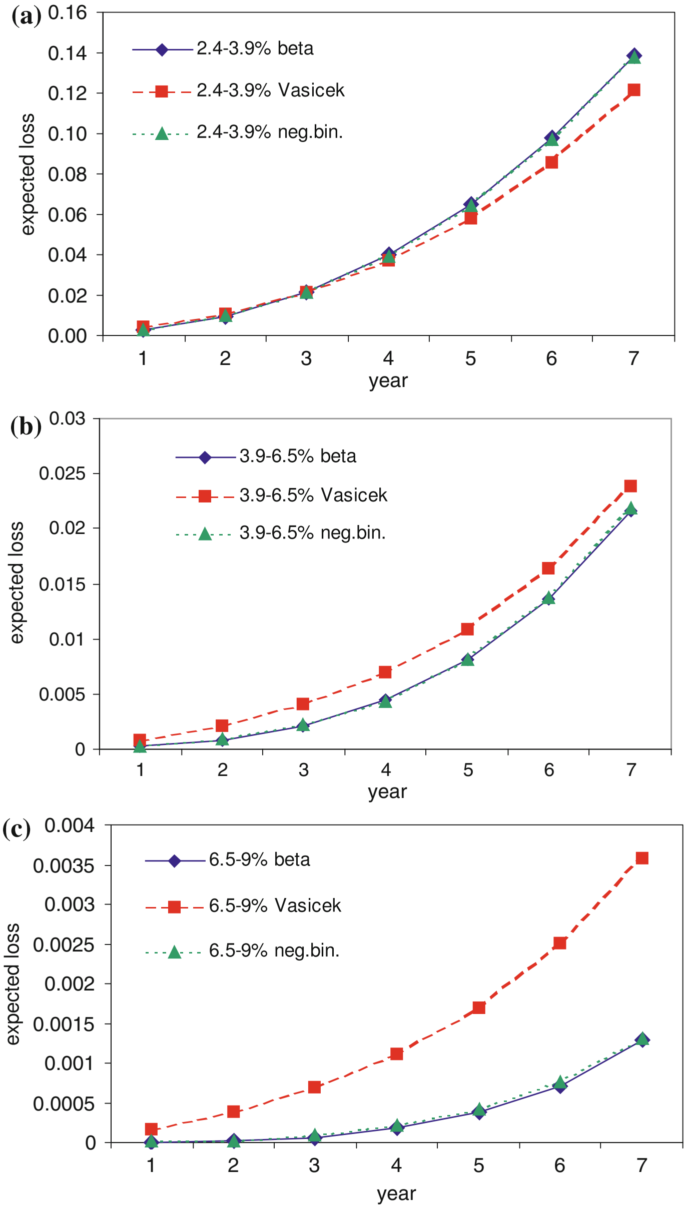


Fig. 11.5 Comparison of term structures of expected losses in tranches two, three and four with different underlying loss distributions under the constraint of matching the first two moments

the plot we chose $m = 1000$ whichresults in $\alpha = 0.323278$ and $\beta = 80.4258$.). The q-q-plot shows cummulative probabilities up to 99.995% and is well on the diagonal. Only in the right upper corner the points begin to fall below the diagonal, but this clearly depends on the cut-off m. These coinciding probability masses far out into the tails thus explain the identical tranching results.

11.5.2 Variable Portfolio Quality

Due to credit migrations we are often confronted with variable portfolio quality during the term of the transaction. In the following we investigate the consequences of two extreme cases, i.e. strictly deteriorating (back loaded) and strictly improving (front loaded) quality on our tranching structure.

Deteriorating Portfolio

A variable portfolio quality can represented through different one-year default probabilities, thus we simply choose a sequence of increasing ‘forward’ default probabilities p as in Table 11.4. The corresponding plots are shown in Fig. 11.6a. A comparison with the Table 11.1 resp. Figure 11.1 shows that the slopes of the EL-per-tranche curves over the years increases as expected. Only the first loss position flattens with year five since the first tranche begins to fill up (Fig. 11.7).

Improving Portfolio

Conversely, we can represent an improving portfolio quality by decreasing ‘forward’ yearly default probabilities, Table 11.5. The plot in Fig. 11.6b depicts again that the cumulative expected loss of the first loss position increases less than linear. The reason is that now the upper limit of the first tranche is small compared to the high default probabilities in the first years, i.e. again the first tranche fills up rather quickly and losses are passed to the next higher tranche.

Table 11.4 Increasing default probability

EL per tranche									
Year	p (forward)	EL	UL	0–2.4%	2.4–3.9%	3.9–6.5%	6.5–9%	9–11.5%	11.5–100%
1	0.0026	0.002593	0.004579	0.104369	0.004101	0.000831	0.000147	0.000039	0.000001
2	0.0036	0.006178	0.007552	0.241895	0.016947	0.003589	0.000685	0.000200	0.000004
3	0.0043	0.010451	0.010198	0.393470	0.045190	0.010055	0.001928	0.000516	0.000008
4	0.0048	0.015207	0.012650	0.542116	0.096050	0.023047	0.004522	0.001182	0.000016
5	0.0051	0.020238	0.014826	0.673930	0.172279	0.044946	0.009036	0.002325	0.000030
6	0.0053	0.025426	0.016816	0.780726	0.271508	0.078663	0.016656	0.004250	0.000054
7	0.0054	0.030681	0.018618	0.860469	0.386342	0.125166	0.028539	0.007366	0.000094

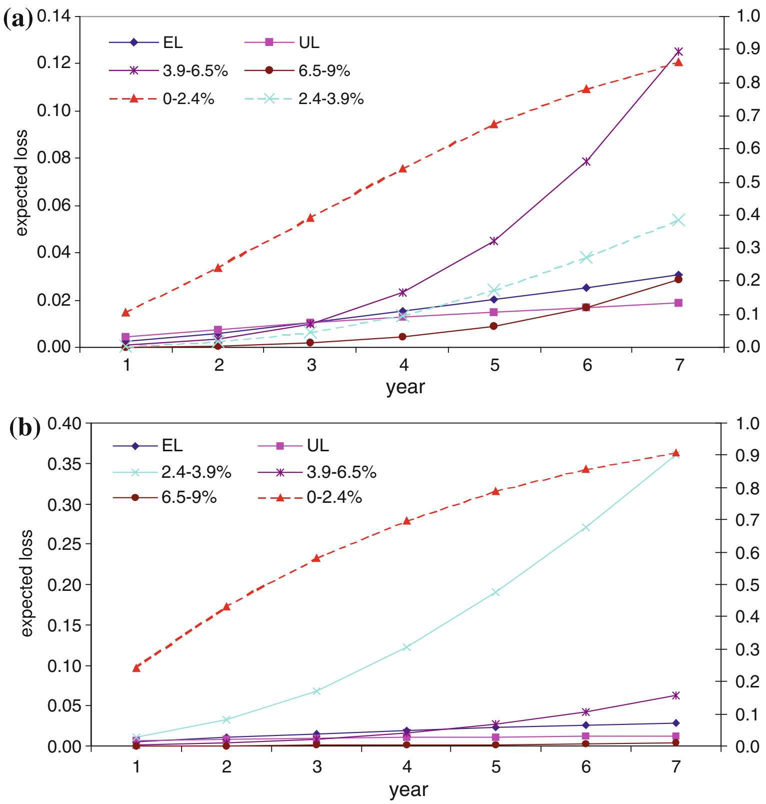


Fig. 11.6 Expected loss in tranches for a portfolio with deteriorating quality **a** and improving quality **b**. The *dashed lines* scale with the *right* axis, the *solid lines* scale with the *left* axis

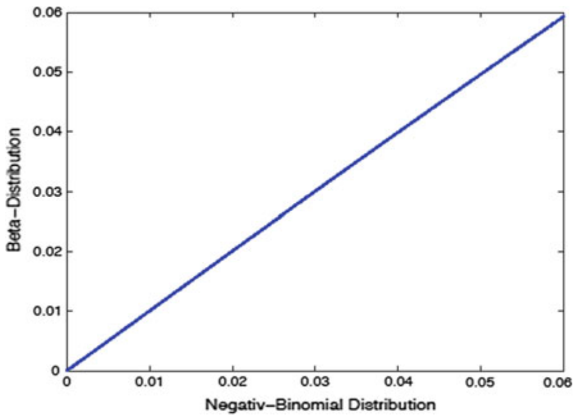


Fig. 11.7 q-q-plot for Beta- and Negativ-Binomial Distribution under the constraint of matched first two moments. The Negativ-Binomial Distribution is rescaled to [0, 1] via a division by $m = 1000$. The cumulative probabilities are shown up to 99.995%

Table 11.5 Decreasing default probability

EL per tranche									
Year	p (forward)	EL	UL	0–2.4%	2.4–3.9%	3.9–6.5%	6.5–9%	9–11.5%	11.5–100%
1	0.0060	0.005992	0.006269	0.241622	0.010208	0.001379	0.000144	2.31E-05	1.27E-07
2	0.0050	0.010961	0.008220	0.431596	0.032250	0.004144	0.000389	6.05E-05	3.00E-07
3	0.0043	0.015214	0.009431	0.580569	0.068669	0.008830	0.000719	9.70E-05	5.02E-07
4	0.0039	0.019060	0.010333	0.698890	0.121829	0.016286	0.001245	1.47E-04	6.63E-07
5	0.0036	0.022595	0.011046	0.790304	0.190679	0.027340	0.002020	2.25E-04	9.83E-07
6	0.0033	0.025819	0.011616	0.857477	0.270135	0.042336	0.003105	3.23E-04	1.39E-06
7	0.0032	0.028936	0.012125	0.907148	0.359962	0.062859	0.004687	4.54E-04	1.86E-06

11.6 Conclusion

We have investigated the term structure of loss cascades in a tranched portfolio structure, commonly found in loan portfolio securitisation. The yearly loss is first simulated via a normal inverse distribution. The resulting expected losses per tranche increase roughly linear for the first loss position and exponential for the higher tranches. We show how to derive a corresponding rating for each tranche, first by comparison with corporate zero bonds, and second by calculating the spreads representing the expected default risk. The spreads of tranches implied by our analysis of a securitized portfolio show a more convex term structure than for comparable corporate bonds. Next, using other possible loss distribution (Beta-/Negative-Binomial distribution) we find that the expected losses for tranches higher than the first depend heavily on the chosen distribution. The respective parameters have been calibrated by matching the first two moments and reveal again the tail-'fatness' of the normal inverse distribution compared to the other two. It is well known (eg. Bluhm et al. (2010)) that the tail behavior of the normal inverse distribution captures the extreme type behavior of credit losses better than the other two distributions. Interestingly, Beta- and Negativ-Binomial distribution yield coinciding results. A brief look at the q-q-plot reveals that both distributions seem to coincide as used above. Eventually, we have a brief look on how variable portfolio quality can be treated in our context and how tranching limits and yearly default probabilities interact in the term structure of loss cascades.

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