# Chapter 24
Decoherence

> The theory of decoherence solves some significant problems associated with the measurement process in quantum mechanics. The basic idea is to take into account the effect of the environment on a quantum system.

As we have seen in the preceding chapters, quantum mechanics is a theory that can predict the outcome of measurements with great accuracy—fine for all practical purposes, *fapp*. However, fundamental questions of interpretation remain open, and this is particularly the case when we are dealing with the concept of 'measurement', as we have repeatedly seen. In our formulation so far, we have on the one hand the deterministic SEq, on the other hand the measurement process which introduces probabilities into the theory—how do these two aspects fit together? Another ambiguity: Just previous to the time of a measurement, a quantum-mechanical system is in general a superposition of different states—but due to the measurement, *one* of the states is selected out of this superposition; we do not obtain any sort of 'superimposed' states as the result of a measurement. How does this 'collapse' of the wavefunction take place, and on which time scale does it occur?

This is essentially the already often-mentioned *quantum mechanical measurement problem*. At least a part of the unclear issues is answered by the theory of *decoherence*, which we now wish to discuss briefly. The basic idea is that one takes into account the interactions of a quantum system with its environment. Indeed, the term 'isolated system' is *per se* an idealization that never can be realized, strictly speaking, except possibly for the entire universe. In experimental practice, it may require a very elaborate procedure to isolate quantum systems even approximately or to separate them sufficiently from their surroundings.[1]

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[1] Indeed, the experimental challenges are enormous if one tries to isolate certain quantum objects from their environment. Serge Haroche and David Wineland developed new ground-breaking experimental methods, making it possible to measure non-destructively those quantum objects and to control them, which was previously thought to be impossible. In 2012, they were awarded the Nobel Prize in physics. As the BBC wrote (http://www.bbc.co.uk/news/science-environment-19879890 (November 2012)), "But for physicists, the import of the pair's techniques is outlined in a layman's summary on the Nobel site: they preserve the delicate quantum-mechanical states

Fig. 24.1 System $\mathcal{S}$, measuring device $\mathcal{M}$ and environment $\mathcal{U}$
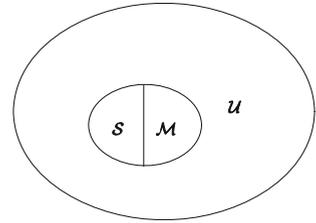
We must therefore consider not only the measured quantum object $\mathcal{S}$ and the measuring apparatus[2] $\mathcal{M}$, but in addition the two must be understood as an *open system* interacting with its environment $\mathcal{U}$; see Fig. 24.1.[3] It turns out that the influence of the environment actually destroys superpositions and makes (relative) phases unobservable. In this way, decoherence explains the non-occurrence of macroscopic superpositions, discussed e.g. in the example of Schrödinger's cat.

In the following, we want to illustrate the basic idea by using a simple example, before we present a slightly more formal approach.

24.1   A Simple Example

To conceive an intuitive idea of decoherence, we discuss a very simplified example. We assume a material object, whose dimensions may be microscopic (e.g. an electron) or macroscopic (e.g. a grain of sand), and which can be in two states $|z_1\rangle$ and $|z_2\rangle$, with sharp energies E_1 and E_2. The time evolution of the states is then given by $|z_n\rangle \rightarrow e^{-iE_n t/\hbar} |z_n\rangle$. The initial state $|z\rangle$ is supposed to be a superposition of the two states, i.e. $|z\rangle = c_1 |z_1\rangle + c_2 |z_2\rangle$. The unitary time evolution of this state is then given by[4]:

$$|z\rangle \rightarrow e^{-iE_1 t/\hbar} c_1 |z_1\rangle + e^{-iE_2 t/\hbar} c_2 |z_2\rangle. \tag{24.1}$$

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of the photons and ions—states that theorists had for decades hoped to measure in the laboratory, putting the ideas of quantum mechanics on a solid experimental footing. Those include the slippery quantum-mechanical ideas of entanglement—the seemingly ethereal connection between two distant particles . . . and of decoherence, in which the quantum nature of a particle slowly slips away through its interactions with other matter."

[2]Of course, $\mathcal{M}$ is not confined to the usual equipment of the physics laboratory. In the case of e.g. Schrödinger's cat, the cat is the measuring apparatus with the pointer states 'dead' and 'alive', which measure the state of the radioactive atom.

[3]Some remarks on the concepts of 'open', 'isolated', etc. can be found in Appendix S, Vol. 1.

[4]Note: For a grain of sand this is a *cat-state*. Today, this term is understood as the superposition of two quantum states that are macroscopically distinguishable. Originally, the term referred to an entangled state between a macrostate (cat) and a microstate (radioactive nucleus). Schrödinger cat states are 'normal' quantum mechanical states which for instance can be entangled, see e.g. C. Wang et al. 'A Schrödinger cat living in two boxes, *Science* 352, 1087 (2016), https://doi.org/10. 1126/Science.aaf2941.

Table 24.1 Frequencies ω for different distances and masses

ω	$\Delta z = 1$ nm	$\Delta z = 1$ m
$m = 10^{-30}$ kg	10^{-4} s^{-1}	10^{5} s^{-1}
$m = 10^{-3}$ kg	10^{23} s^{-1}	10^{32} s^{-1}

We can factor out one of the two exponential terms (which results in a physically uninteresting global phase as overall factor), for instance:

$$|z\rangle \to e^{-iE_1t/\hbar}\left[c_1\,|z_1\rangle + e^{-i\omega t}c_2\,|z_2\rangle\right], \tag{24.2}$$

with $\omega = (E_2 - E_1)/\hbar$. Of course, this also works with E_2 instead of E_1. To get an idea of the order of magnitude of ω and the period of oscillation, we assume that the object is in the earth's gravitational field. Then we have $\omega = \frac{mg\Delta z}{\hbar}$. For an electron and a separation of the wave packets of $\Delta z = 1$ nm or $\Delta z = 1$ m, we obtain $\omega = 10^{-4}$ or 10^{5} s^{-1}; for a mass of 1 g, these values are $\omega = 10^{23}$ or 10^{32} s^{-1}, see Table 24.1. To compare: 10^{-22} s is the time which light requires to 'pass through' an atomic nucleus.

This means that for macroscopic masses and distances, the phase ωt in (24.2) rotates so fast that during the measurement one records only its average. In other words, when phases are changing so rapidly, only *one* of two states $|z_1\rangle$ and $|z_2\rangle$ can be detected. Which one of the two states this actually applies to remains an open question—not only with this simple heuristic reasoning, but also within the scope of the approaches to decoherence which we discuss in the following.

We can also treat the problem using the density matrix. We have from (24.1):

$$\rho = \begin{pmatrix} |c_1|^2 & c_1c_2^*e^{i\omega t} \\ c_1^*c_2e^{-i\omega t} & |c_2|^2 \end{pmatrix}. \tag{24.3}$$

Again, we see a rapid 'flickering' in the superpositions or coherences. If we want to carry out a measurement, we must remember that *every* measurement requires a finite amount of time T, although this may seem to our everyday understanding vanishingly small (e.g. 10^{-12} s). Thus, we have to average the density matrix over the measuring time T (of course, this is again a heuristic argument) and obtain (see exercises):

$$\frac{1}{T}\int_0^T \rho \mathrm{d}t = \begin{pmatrix} |c_1|^2 & c_1c_2^* \cdot s(T) \\ c_1^*c_2 \cdot s^*(T) & |c_2|^2 \end{pmatrix} \tag{24.4}$$

with

$$s(T) = e^{i\omega T/2}\frac{\sin \omega T/2}{\omega T/2} \underset{\omega T\to\infty}{\to} 0. \tag{24.5}$$

Hence, for sufficiently long averaging times, we can write

$$\rho \approx \begin{pmatrix} |c_1|^2 & 0 \\ 0 & |c_2|^2 \end{pmatrix}. \qquad (24.6)$$

For e.g. $\omega = 10^{32}\,\text{s}^{-1}$ and a measurement lasting $T = 10^{-16}\,\text{s}$, we have $|s(T)|$ $\sim 10^{-16}$; under these circumstances, one effectively cannot see the coherences or superpositions. This means that the off-diagonal elements of the density matrix, i.e. the superpositions, disappear very quickly—in this way, a statistical mixture emerges out of a pure state. Which of the two c_i is finally selected by the measuring process cannot be said at this point.

As we pointed out in the introduction, we have to work with open systems, i.e. to take into account the influence of the environment. Here, we have modelled this effect by considering the finite time resolution of the measuring apparatus. We did not try to describe the effects of the environment as realistically as possible. The fact that we still obtain a result such as (24.6) indicates that decoherence in reality does not depend on specific interactions or conditions, but rather that it is robust with regard to them and constitutes a universal phenomenon.

24.2   Decoherence

In this section, we describe the process of measurement on a very schematic level. One reason is that we have considered only Hamiltonians that do not depend on time. But here, we consider *open* systems interacting with their environment; they of course experience an evolution in time (e.g. due to the fact that the measuring device adjusts itself to a new value), and we have no conceptual tools[5] at hand in this regard. A further justification for the following simple approach (which is representative of the standard scenario) is due to the aforementioned universality of decoherence— the details do not matter here if one wants to explain the phenomenon in a more qualitative than quantitative manner. And finally, the basic idea of decoherence may be worked out better by means of systems which are as simple as possible (i.e. toy models). The following considerations are therefore characterized in some places by plausibility arguments rather than strict mathematics, but this does not diminish their general validity.

First, we repeat an example treated in Chap. 20, as shown in Fig. 24.2.

The photon can occupy the state $|H\rangle$ or $|V\rangle$ (horizontal or vertical). We describe the detectors (i.e. the measuring apparatus) by the ket $|10\rangle$ or $|01\rangle$, if DH or DV clicks, respectively. If both detectors are inactive, we have $|00\rangle$. With this notation we can write the process in short form as:

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[5]Such as time-ordering operators etc.

Fig. 24.2 A photon is incident on a beam splitter BS and is detected in one of the detectors DH or DV
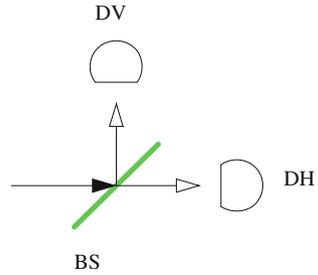
$$|H\rangle |00\rangle \rightarrow \frac{1+i}{2} [|H\rangle + i |V\rangle] |00\rangle \rightarrow \frac{1+i}{2} [|H\rangle |10\rangle + i |V\rangle |01\rangle]. \quad (24.7)$$

In words: The photon in the state $|H\rangle$ propagates towards the beam splitter, no detector is activated and the total state is factorizable. This is true even after it passes the first beam splitter. Eventually, the photon will hit and activate one of the two detectors with 50 % probability, e.g. if it is in the state $|H\rangle$, it will activate detector DH. Obviously, the final state in (24.7) is entangled. With regard to the following considerations, we denote $[|H\rangle + i |V\rangle] |00\rangle$ as the initial state and $[|H\rangle |10\rangle + i |V\rangle |01\rangle]$ as the final state, which means that we confine ourselves to the process $[|H\rangle + i |V\rangle] |00\rangle \rightarrow [|H\rangle |10\rangle + i |V\rangle |01\rangle]$.

We now generalize this example. To this end, we assume a quantum system $\mathcal{S}$ which has the states $|S_m\rangle$. $\mathcal{S}$ is to be measured by a measuring apparatus $\mathcal{M}$ with the states $|M_k\rangle$. For simplicity and without loss of generality, we assume that $\mathcal{S}$ and $\mathcal{M}$ have the same number M of states; typically, M has values of less than 10^4.

At the beginning of our observations, the system is in the state $|S_m\rangle$ and the measuring apparatus is in its initial state $|M_0\rangle$. An ideal (recoilless) measurement then eventually causes the measuring apparatus to indicate the state of $\mathcal{S}$:

$$|S_m\rangle |M_0\rangle \rightarrow |S_m\rangle |M_m\rangle. \quad (24.8)$$

If the initial state of $\mathcal{S}$ is a superposition, i.e. $\sum s_m |S_m\rangle$, then it follows due to the linearity of the dynamics[6] that:

$$\sum_m s_m |S_m\rangle |M_0\rangle \rightarrow \sum_m s_m |S_m\rangle |M_m\rangle. \quad (24.9)$$

We see that we have on the right-hand side an entangled state whose density operator is given by

$$\rho = \sum_{mm'} s_m s_{m'}^* |S_m\rangle |M_m\rangle \langle M_{m'}| \langle S_{m'}|. \quad (24.10)$$

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[6] Just as in the example of the beam splitter considered above.

Evidently, coherences occur, i.e. terms with $m \neq m'$. On the other hand, superpositions are never observed in the macroscopic domain, and the density operator for a mixture of non-interfering states would be

$$\rho = \sum_m |s_m|^2 |S_m\rangle |M_m\rangle \langle M_m| \langle S_m|. \tag{24.11}$$

How does it happen that the coherences disappear? If one considers only $\mathcal{S}$ and $\mathcal{M}$, this question cannot be answered.

24.2.1 The Effect of the Environment I

The dilemma can be resolved only if—and this is the essential idea of decoherence— the environment $\mathcal{U}$ is included in the considerations. The states of $\mathcal{U}$ are $|U_n\rangle$. Their number is N, where N is a very large number, of order 10^{20}, 10^{30} or larger and in any case substantially greater than M.

Initially, we assume a factorized state of the form:

$$|\psi\rangle_0 = \sum_{m=1}^{M} s_m |S_m\rangle |M_0\rangle \sum_{n=1}^{N} u_n |U_n\rangle. \tag{24.12}$$

In the course of time, the system, measuring apparatus and environment interact and the state changes to:

$$|\psi\rangle = \sum_{m,n=1}^{M,N} c_{mn} |S_m\rangle |M_m\rangle |U_n\rangle. \tag{24.13}$$

The coupling is in the factors c_{mn} which do not generally factorize in the form $c_{mn} = a_m b_n$. In other words, the state (24.13) is entangled, and this with respect to the $(\mathcal{S}, \mathcal{M})$ states on the one hand and the environmental states on the other hand.

Thus, the essential mechanism here is not the direct effect of the environment on $(\mathcal{S}, \mathcal{M})$, which would possibly change the states of $\mathcal{S}$ or $\mathcal{M}$ (noise). Rather, we assume that these states remain unchanged. Instead, the fact that the evolution of $(\mathcal{S}, \mathcal{M})$ leaves its mark on the environment due to the entanglement—in other words, that information about $(\mathcal{S}, \mathcal{M})$ 'seeps out' to the environment, is decisive.

The density operator for the state (24.13) as an element of $\mathcal{H}_S \otimes \mathcal{H}_M \otimes \mathcal{H}_U$ is

$$\rho = \sum_{m,n,m',n'} c_{mn} c^*_{m'n'} |S_m\rangle |M_m\rangle |U_n\rangle \langle S_{m'}| \langle M_{m'}| \langle U_{n'}|. \tag{24.14}$$

We note that this density operator describes a pure state and that, at this point, we have the *total* information about the triplet $(\mathcal{S}, \mathcal{M}, \text{and } \mathcal{U})$ at our disposal.

Crucial for the following argument is the *entanglement* between $(\mathcal{S}, \mathcal{M})$ and the environment $\mathcal{U}$; thus, the state (24.13) must not factorize (e.g. as denoted by $c_{mn} = a_m b_n$). Without this entanglement, there is no decoherence; the environment *must* be included in the considerations. This naturally raises quite difficult questions (how far does the environment extend?), but on a *fapp* level, we need only the argument that the environment has very many degrees of freedom; whether there are 10^{20} or 10^{30} is irrelevant.

The coherences (i.e. superpositions) occurring in the density operator (24.14) can be eliminated by averaging over the environment variables. Since there is an enormous number of environmental variables or states interacting with $(\mathcal{S}, \mathcal{M})$, we see—as in the ideal gas[7]—not every single contribution, but a mean value; this is analogous to the averaging over the phase that we performed in the above 'grain of sand' example. The averaging over environmental states (trace over $\mathcal{U}$, here also called *trace over unobservables*) leads in a first step to the reduced density operator

$$
\rho_{S,red} = tr_U\,(\rho) = \sum_{k=1}^{N} \langle U_k|\,\rho\,|U_k\rangle
$$

$$
= \sum_{k=1}^{N} \sum_{m,n,m',n'} c_{mn}c^*_{m'n'}\,|S_m\rangle\,|M_m\rangle\,\delta_{nk}\,\langle S_{m'}|\,\langle M_{m'}|\,\delta_{n'k}
$$

$$
= \sum_{m,m'} \sum_{k=1}^{N} c_{mk}c^*_{m'k}\,|S_m\rangle\,|M_m\rangle\,\langle S_{m'}|\,\langle M_{m'}|. \tag{24.15}
$$

In the representation as density matrix (of dimension $M \times M$), we can write this as in Chap. 22 as the product of an $M \times N$ matrix C with its adjoint:

$$
\rho_{S,red} = CC^\dagger \ \text{ with } \ C = (c_{mn}), \tag{24.16}
$$

and the matrix elements of $\rho_{S,red}$ are then the dot product of the N-dimensional mth row of C with the m'th column of $C^\dagger$ (this is shown explicitly in the exercises).

In a second step, we want to estimate the order of magnitude of the matrix elements of $\rho_{S,red}$. We start with the diagonal elements that must be positive, being absolute squared values. Because of the normalization (or $tr\rho = 1$), they are $1/M$ on average.

On this basis, also the off-diagonal terms are on average $1/M$, so that we can write this common factor in front of the matrix. The diagonal terms within the matrix then are of order 1. Concerning the off-diagonal elements, we have to sum over generally both positive and negative contributions (from real and imaginary terms). For very large N (as said, we have for macroscopic environments values of $N \approx 10^{20}$ or more), these contributions will cancel out on average; if we can assume a normal distribution, the deviation from the mean value zero is on the order of in the magnitude of the

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[7]There, we do not know the phase-space coordinates of 10^{23} particles (more precisely, we neither can know them nor want to know them).

relative statistical error, i.e. $\sim \frac{1}{\sqrt{N}}$. Overall, the estimate of the order of magnitude of the matrix elements reads:

$$\left(\rho_{S,\text{red}}\right)_{ij} = O\left(\frac{1}{M}\right)\left(\delta_{ij} + O\left(\frac{1}{\sqrt{N}}\right)\right). \tag{24.17}$$

A more detailed analysis is given in the exercises.

For coupling to a macroscopic environment, we have the result that the off-diagonal elements of the reduced density matrix disappear, so to speak for statistical reasons alone[8]; to a good approximation, it holds that:

$$\rho_{S,\text{red}} = \sum_m p_m \, |S_m\rangle \, |M_m\rangle \, \langle S_m| \, \langle M_m| \, ; \;\; p_m = \sum_{k=1}^{N} |c_{mk}|^2 \, . \tag{24.18}$$

In other words, due to the influence of the environment, the coherences seem to have disappeared from the system; the p_m are the measurement probabilities of the *individual* configurations.

24.2.2   Simplified Description

We can also simplify by assuming that after the measurement, we consider *one* state of the environment for each setting of $(\mathcal{S}, \mathcal{M})$, and write the total state correspondingly instead of (24.13) as:

$$|\psi\rangle = \sum_{m=1}^{M} d_m \, |S_m\rangle \, |M_m\rangle \, |U_m\rangle \, . \tag{24.19}$$

Then the density operator reads

$$\rho = \sum_{m,m'} d_m d_{m'}^* \, |S_m\rangle \, |M_m\rangle \, |U_m\rangle \, \langle S_{m'}| \, \langle M_{m'}| \, \langle U_{m'}| \tag{24.20}$$

and for the reduced density operator, it follows for the moment:

$$\rho_{S,\text{red}} = tr_U \, (\rho) = \sum_{k=1}^{M} \sum_{m,m'} d_m d_{m'}^* \, |S_m\rangle \, |M_m\rangle \, \langle U_k| \, U_m\rangle \, \langle U_{m'}| \, U_k\rangle \, \langle S_{m'}| \, \langle M_{m'}| \, . \tag{24.21}$$

Since the states $|U_m\rangle$ come from a high-dimensional space (as we said above, dimension 10^{20} or more), two arbitrarily chosen states are orthogonal to each other with a high probability (this is a similar argument to the one leading to the estimate (24.17));

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[8]One can imagine that the system itself has lost the information about certain interference terms because it has migrated into the environment. Therefore, the coupling to a *great many* degrees of freedom is substantial—then the process is virtually irreversible.

actually we have $\langle U_k | U_m \rangle \approx \delta_{mk}$. This means that we arrive at the same result as above:

$$\rho_{S,\text{red}} = \sum_m p_m |S_m\rangle |M_m\rangle \langle S_m| \langle M_m|; \quad p_m = |d_m|^2 . \tag{24.22}$$

Thus, the system looks *fapp* like a mixture.

24.2.3 *The Effect of the Environment II*

The influence of the environment is not limited to receiving information about the system. It also structures the possible results and resolves ambiguities. To consider an example, we assume a spin-1/2 system S in the state

$$|\alpha\rangle = \frac{|z+\rangle + |z-\rangle}{\sqrt{2}}, \tag{24.23}$$

where $|z+\rangle$ and $|z-\rangle$ are the eigenstates of σ_z with the eigenvalues $+1$ and -1.[9] S interacts with $\mathcal{M}$, which in our example is a two-dimensional system, also. After some time, the state evolves to:

$$|\psi\rangle = \frac{|z+\rangle |m_{z+}\rangle + |z-\rangle |m_{z-}\rangle}{\sqrt{2}}. \tag{24.24}$$

We can think of $|m_{z+}\rangle$ and $|m_{z-}\rangle$ as the two eigenstates of a pointer variable (pointer observable) M_z of $\mathcal{M}$.

The state $|\psi\rangle$ has the form of a *biorthonormal decomposition*.[10] Such decompositions are not unique if the squared values of the coefficients are all equal. This is the case here, and indeed we can represent $|\psi\rangle$ e.g. by the following biorthonormal decomposition with respect to x (the example is considered in the exercises):

$$|\psi\rangle = \frac{|x+\rangle |m_{x+}\rangle + |x-\rangle |m_{x-}\rangle}{\sqrt{2}} \tag{24.25}$$

where the $|m_{x\pm}\rangle$ are eigenstates of a pointer variable M_x which are related to the $|m_{z\pm}\rangle$ by

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[9] In vector notation, we have $|z+\rangle \cong \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|z-\rangle \cong \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Correspondingly, $|\alpha\rangle$ is the 'up' state of σ_x.

[10] Each state $|\psi\rangle$ of the total system $S + \mathcal{M}$ can be represented in the form

$$|\psi\rangle = \sum c_i |u_i\rangle |v_i\rangle$$

where $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ are CONS in $\mathcal{H}_S$ and $\mathcal{H}_{\mathcal{M}}$ (Schmidt decomposition, biorthogonal decomposition). The decomposition is unique iff the coefficients $|c_i|^2$ are all different.

$$|m_{x+}\rangle = \frac{|m_{z+}\rangle + |m_{z-}\rangle}{\sqrt{2}}; \ |m_{x-}\rangle = \frac{-|m_{z+}\rangle + |m_{z-}\rangle}{\sqrt{2}}. \tag{24.26}$$

We note that (24.24) and (24.25) are two different biorthonormal decompositions of the same state $|\psi\rangle$. Consequently, certain entangled states of $\mathcal{S} + \mathcal{M}$ can represent both a correlation between σ_z values and the values of an observable M_z of $\mathcal{M}$, and simultaneously a corresponding correlation between σ_x values and the values of an observable M_x von $\mathcal{M}$. Of course, this is unsatisfactory, if (as in our example) M_x does not commute with M_z. One way out of this dilemma is the inclusion of the environment.

For this, we consider in our model in addition a third, likewise two-dimensional system $\mathcal{U}$ (our model environment), which interacts with $\mathcal{M}$. In the course of the measurement, the state changes over time and reads:

$$|\Psi\rangle = \frac{|z+\rangle\,|m_{z+}\rangle\,|u_{z+}\rangle + |z-\rangle\,|m_{z-}\rangle\,|u_{z-}\rangle}{\sqrt{2}}, \tag{24.27}$$

where $|u_{z+}\rangle$ and $|u_{z-}\rangle$ are two orthogonal states which span $\mathcal{H}_{\mathcal{U}}$. We have three components, $\mathcal{S} + \mathcal{M} + \mathcal{U}$, and accordingly a triorthonormal decomposition. For this we can use another decomposition theorem (theorem of triorthogonal decompositions[11]) which ensures the uniqueness of the state (24.27) when the three pairs of states $\{|z+\rangle, |z-\rangle\}$, $\{|m_{z-}\rangle, |m_{z+}\rangle\}$ and $\{|u_{z+}\rangle, |u_{z-}\rangle\}$ are orthogonal in their respective Hilbert spaces, independently of the expansion coefficients. We have therefore, in contrast to the biorthonormal decomposition, the result that M_x and M_z cannot be measured simultaneously, or

$$|\Psi\rangle \neq \frac{|x+\rangle\,|m_{x+}\rangle\,|u_{x+}\rangle + |x-\rangle\,|m_{x-}\rangle\,|u_{x-}\rangle}{\sqrt{2}}. \tag{24.28}$$

The disturbing ambiguity appearing in (24.24) and (24.25) is therefore removed by the inclusion of the model environment (technically: tri- instead of biorthogonal decompositions).

We can, on the basis of our model system, summarize our considerations as follows: We include the environment by extending the system $(\mathcal{S}, \mathcal{M})$ to the system $(\mathcal{S}, \mathcal{M}, \mathcal{U})$, where we assume that there is an interaction $\mathcal{W}$ between $\mathcal{M}$ and $\mathcal{U}$, although at this point it is unknown.[12] $\mathcal{W}$ commutes with a $\mathcal{M}$ observable (the pointer variable[13]); we call it M_z. Under these conditions, the system $(\mathcal{S}, \mathcal{M})$ will be perfectly correlated in *one* product basis (in the example $\{|z+\rangle\,|m_{z+}\rangle, |z-\rangle\,|m_{z-}\rangle\}$). Hence, $\{|m_{z+}\rangle, |m_{z-}\rangle\}$ is the pointer basis of $\mathcal{M}$, which appears in the course of time in the diagonal of the reduced density matrix that is obtained by tracing out over the

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[11]The theorem of triorthogonal decompositions can be generalized to vectors that are linearly independent (and not necessarily orthogonal), as well as to n systems (instead of 3).

[12]Of course, one knows these interactions in model computations.

[13]This pointer variable contains the possible states of the measuring apparatus.

environmental modes. Measurements of other spin directions are impossible. The pointer observable is thus actually determined by the interaction with $\mathcal{U}$, because this 'monitoring' of $\mathcal{M}$ by $\mathcal{U}$ leads to an almost immediate decoherence between different pointer states. So we have an effective 'collapse' of the total state into a tensor product of a pointer state and a correlated eigenstate of $\mathcal{S}$.

24.2.4 Interim Review

Following the model concepts which we have just outlined, the major mechanism is that information about the system $\mathcal{S}$ (and the measuring apparatus $\mathcal{M}$) finds its way into the environment $\mathcal{U}$. On the one hand, the enormous number of degrees of freedom of the environment ensures thereby that this process is virtually irreversible. On the other hand, the environment specifies the pointer basis via its interaction with $\mathcal{M}$ and in this way prevents ambiguities and contradictions. For this reason, we speak of *measurement by the environment* (environmental monitoring)[14]—in accordance with these ideas, macroscopic properties are created by the environment.

Of course, the dividing lines between $\mathcal{S}$, $\mathcal{M}$ and $\mathcal{U}$ are not sharply defined and are to a certain extent arbitrary. At first glance, this is similar to the problem of demarcation between quantum mechanics and classical mechanics (the Heisenberg cut), discussed e.g. in Chap. 14, Vol. 1, which arises when one postulates that the measuring apparatus must obey the rules of classical mechanics. But there is the fundamental difference that in the decoherence theory, $\mathcal{S}$, $\mathcal{M}$ and $\mathcal{U}$, i.e. *all* systems, obey the rules of quantum mechanics—we no longer have to make use of classical mechanics, but have a coherent and consistent representation within the framework of quantum mechanics alone. In other words, the problems discussed in Chap. 14, Vol. 1, regarding the boundary between quantum mechanics and classical mechanics have been resolved by the introduction of decoherence.

Finally, a note with regard to the measuring apparatus. It plays only the role of an agent which shows us macroscopically the microscopic result. But even without measuring apparatus, entanglement of the system states with those of the environment occurs, and hence decoherence. Since measurement by the environment is always present (unless the system were indeed isolated), we can for the sake of simplicity dispense with the explicit consideration of the measuring apparatus in the frame of certain considerations (or ascribe it either to the system or to the environment), and restrict ourselves to the pair $(\mathcal{S}, \mathcal{U})$.[15] In this way, the state $|\psi\rangle_0 = \sum_{m=1}^{M} s_m |S_m\rangle \sum_{n=1}^{N} u_n |U_n\rangle$ becomes $|\psi\rangle = \sum_{m,n=1}^{M,N} c_{mn} |S_m\rangle |U_n\rangle$, and by decoherence effects this leads to the reduced density operator $\rho_{S,red} = \sum_m p_m |S_m\rangle \langle S_m|$ with $p_m = \sum_{k=1}^{N} |c_{mk}|^2$.

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[14]Other designations are *environmentally induced decoherence* or *einselection*, an abbreviation of *environment-induced selection*.

[15]One can guarantee the uniqueness (or tri- instead of biorthogonal decomposition) by (formally) splitting the environment into two or more systems.

24.2.5   Formal Treatment

To give a rough outline of the formal treatment, we start with the total Hamiltonian
(for clarity, we dispense here with the measuring apparatus):

$$H = H_S + H_U + H_{US}, \tag{24.29}$$

where H_{US} describes the interaction between U and S. In this context, the environ-
ment is often referred to as a '(thermal) bath' or 'reservoir' (based on the concepts
of thermodynamics).

The time evolution of the total density operator reads

$$\rho(t) = \hat{U}(t)\,\rho(0)\,\hat{U}^\dagger(t);\ \ \hat{U}(t) = e^{-i\frac{Ht}{\hbar}}, \tag{24.30}$$

and the reduced density operator is given by the trace over the degrees of freedom
of the environment:
$$\rho_S(t) = tr_U\left[\hat{U}(t)\,\rho(0)\,\hat{U}^\dagger(t)\right]. \tag{24.31}$$

We assume that initially, system and environment are not entangled, i.e.

$$\rho(0) = \rho_S(0) \otimes \rho_U(0). \tag{24.32}$$

Furthermore, we assume that we know the orthogonal basis states $|n\rangle$ of the envi-
ronment (which of course, strictly speaking, is the case only if we define a model
environment). Since we do not know in which exact state of superposition $\mathcal{U}$ is
initially, we assume a statistical mixture:

$$\rho_U(0) = \sum_n p_n |n\rangle\langle n|\,;\ \ \sum_n p_n = 1. \tag{24.33}$$

Then we have in S the reduced density operator

$$\begin{aligned}
\rho_S(t) &= \sum_m \langle m|\left[\hat{U}(t)\,\rho_S(0) \otimes \sum_n p_n |n\rangle\langle n|\,\hat{U}^\dagger(t)\right]|m\rangle \\
&= \sum_{m,n} \sqrt{p_n}\,\langle m|\,\hat{U}(t)\,|n\rangle\,\rho_S(0)\,\sqrt{p_n}\,\langle n|\,\hat{U}^\dagger(t)\,|m\rangle \\
&= \sum_{i=(m,n)} A_i(t)\,\rho_S(0)\,A_i^\dagger(t);\ \ A_{i=(m,n)}(t) = \sqrt{p_n}\,\langle m|\,\hat{U}(t)\,|n\rangle. \tag{24.34}
\end{aligned}$$

The operators $A_i(t)$ act in the Hilbert space of S. Due to the unitarity of the propa-
gator, they satisfy (see the exercises):

$$\sum_{i=(m,n)} A_i^\dagger(t)\,A_i(t) = 1. \tag{24.35}$$

With the equation $\rho_S(t) = \sum_i A_i(t)\,\rho_S(0)\,A_i^{\dagger}(t)$, the problem is formally solved. An explicit calculation of $A_i(t)$ and therefore of $\rho_S(t)$, of course, demands specific assumptions about the system and the environment and requires considerable calculation. Therefore, we leave the discussion of the formal treatment with these remarks.

24.3 Time Scales, Universality

Decoherence remained unnoticed for quite a while (see also the 'Historical side note' below). Partly responsible for this is the extremely high speed with which it proceeds; cf. the above example of the 'grain of sand'. Theoretical and experimental work suggests that for macroscopic conditions, the time scales are of order 10^{-20}, 10^{-30}, 10^{-40} s or even higher orders of magnitude.

As pointed out above, these values are obtained from various model calculations. In these, micro- and macroscopic objects are placed in different model environments. Let us for example assume that the initial state of the object is a superposition of two states (given e.g. by two bell curves) at the positions x and x' (the locations of the peaks of the bell curves). Then, under suitable assumptions, one can describe [16] the evolution by a density matrix of the form $\rho(x, x', t) = \rho(x, x', 0)\,e^{-\Lambda t(x-x')^2}$. Clearly, this expression eventually becomes diagonal, $\rho(x, x', t) \to \rho(x, x, t)\,\delta_{xx'}$. The localization rate $\Lambda > 0$ is a measure of the speed at which this process carries through. Numerical values of the model calculations are given in Table 24.2. In this case, the dust particle has a diameter of 10^{-5} m, and the (large) molecule is 10^{-8} m in diameter.

We see that for dust particles in the air, even an overlap of $x - x' = 10^{-10}$ m has decayed in a time of about 10^{-20} s.[17] Therefore, if Schrödinger's cat were in a superposition state between death and life, it would last at most a period of perhaps the order of the Planck time. Even molecules are already on the border between quantum mechanics and classical mechanics, and also in microscopic bodies, coherences may fade away in 10^{-12} s or similarly short times. These short decoherence times give the impression of a jump or collapse. In fact, the momentum is continuous, so it is only an 'apparent' collapse.

All these considerations are quite general; at least in the macroscopic world, decoherence has a certain universality. Studies show that decoherence is the only relevant dynamics on extremely short time scales. Details of the system or environment do not play a role; the result is insensitive to them. Hence, decoherence appears as a universal phenomenon of macroscopic superpositions.

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[16]E. Joos and H.D. Zeh, 'The Emergence of Classical Properties Through Interaction with the Environment', *Z. Phys.* B59 223-243 (1985); M. Tegmark, 'Apparent Wave Function Collapse Caused by Scattering', *Found. Phys. Lett.* 6, 571–590 (1993).

[17]The specific values—whether 10^{17} or 10^{19}—do not matter; just the orders of magnitude are relevant.

Table 24.2 Localization rate Λ (unit $m^{-2}s^{-1}$) for different objects and environments

Model environment\object	Electron	Molecule	Dust particle	Bowling ball
Cosmic background radiation	10^{-6}	10^{-8}	10^{10}	10^{21}
Sunlight on earth	10^{5}	10^{17}	10^{24}	10^{32}
Vacuum (10^{3}particles/cm^{3})	10^{22}	10^{21}	10^{27}	10^{35}
Air molecules (s.t.p.)	10^{35}	10^{34}	10^{40}	10^{49}

24.4 Decoherence-Free Subspaces, Basis

It is clear that the very short decoherence times (together with the universal validity of decoherence) may be 'deadly' for quantum computers and other applications that operate on the basis of the superposition principle of quantum mechanics. The decoherence time itself is not crucial, but rather the ratio of 'switching time' of quantum gates (see Chap. 26) to the decoherence time, i.e. the number N_{op} of possible operations during the lifetime of the system. For N_{op} of e.g. the systems quantum dots/MRI/ion traps, currently (2011), numbers of the order of $10^{3}/10^{7}/10^{10}$ have been reported or considered possible.

There are several strategies as to how to escape the problem of short decoherence times; a keyword in this context is 'decoherence-free subspaces'.[18] Intuitively explained, these are subspaces of the state space to whose individual states the environment reacts identically. Accordingly, the phase relationships in superpositions will remain unchanged under the influence of the environment, and the coherences can survive.

We consider a simple model: We have two states $|0\rangle$ and $|1\rangle$; the influence of the environment lies in the fact that it adds to each state a random phase: $|0\rangle \rightarrow e^{i\varphi_0} |0\rangle$ and $|1\rangle \rightarrow e^{i\varphi_1} |1\rangle$ (for the sake of simplicity, we omit the measuring apparatus). If we first consider a simple superposition, we obtain

$$|0\rangle + |1\rangle \rightarrow e^{i\varphi_0} |0\rangle + e^{i\varphi_1} |1\rangle = e^{i\varphi_0} \left[|0\rangle + e^{i(\varphi_1 - \varphi_0)} |1\rangle \right], \qquad (24.36)$$

and (averaged over the difference between the uncorrelated random phases) the coherences disappear, similar to the example of the 'grain of sand' discussed above. But things look different if we consider two quantum objects in the (entangled) states

$$\left| \psi^{\pm} \right\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}. \qquad (24.37)$$

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[18]Other methods to protect decoherence are to rely on the quantum Zeno effect (cf. Appendix L, Vol. 1), or on a special quantum measuring technique (called *weak measurement*); see Sabrina Maniscalco et al. 'Protecting entanglement via the quantum Zeno effect', *Phys. Rev. Lett.* 100, 090503 (2008), or Yong-Su Kim et al. 'Protecting entanglement from decoherence using weak measurement and quantum measurement reversal', Nature Physics (2011), https://doi.org/10.1038/nphys2178. See also H. Le Jeannic et al. 'Slowing Quantum Decoherence by Squeezing in Phase Space', *Phys. Rev. Lett.* 120, 073603 (Feb 2018).

The coupling to our model environment then leads to

$$\left|\psi^{\pm}\right\rangle \rightarrow e^{i(\varphi_0+\varphi_1)}\frac{|01\rangle \pm |10\rangle}{\sqrt{2}}. \tag{24.38}$$

Since the global phase is unobservable, the initial state will in fact be conserved—even if we take the environment into account.

Hence, if there is no entanglement in the overall state (24.13) and, consequently, no coherences in the full density matrix (24.14), then the environment cannot distinguish between the individual $\mathcal{S}$ states, and we have no decoherence—in other words, $\mathcal{S}$ behaves as an isolated system. In accordance with the simple examples just treated, we can therefore conclude that a subspace is decoherence-free if the environment cannot distinguish between its components.

However, this conclusion holds in the above example only if we restrict the discussion to the basis $\{|0\rangle, |1\rangle\}$. In another basis, such as

$$|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}, \tag{24.39}$$

we have e.g.

$$\left|\psi^+\right\rangle = \frac{|++\rangle - |--\rangle}{\sqrt{2}}. \tag{24.40}$$

When now the environment adds random phases to each state, that is $|+\rangle \rightarrow e^{i\varphi_+} |+\rangle$ and $|-\rangle \rightarrow e^{i\varphi_-} |-\rangle$, it follows (see exercises) that:

$$\left|\psi^+\right\rangle \rightarrow e^{2i\varphi_+}\frac{|++\rangle - e^{2i(\varphi_- -\varphi_+)} |--\rangle}{\sqrt{2}}, \tag{24.41}$$

and clearly, this state is not decoherence-free for $\varphi_- - \varphi_+ \neq 2m\pi$.

In order to obtain a unique situation, we can consider e.g. the measuring apparatus or formally split the environment into two or more parts, as we have outlined above. With three systems such as $\mathcal{S}$, $\mathcal{M}$ and $\mathcal{U}$, we can then perform a triorthogonal decomposition, whereby a unique decomposition (or pointer basis) can be achieved due to the interaction of the environment with the measuring apparatus.

24.5 Historical Side Note

For decades, the attitude of the 'old' Copenhagen school was authoritative; it claimed that the physical analysis of the measurement process in quantum-mechanical terms would be a pointless undertaking (see also Chap. 28). However, many people found it quite unsatisfactory to 'split' the world into a quantum realm dominated by the SEq and a separate realm of classical instruments. Where and by which criteria should

one draw the line? A quote[19]: "The principle of superposition was suspended by 'decree' in the classical domain. This point of view—known as the Copenhagen Interpretation (CI)—has kept many a physicist out of despair. On the other hand, as long as a compelling reason for the quantum-classical border could not be found, the CI universe would be governed by two sets of laws, with poorly defined domains of jurisdiction. This fact has kept many a student, not to mention their teachers, in despair."

That the 'thought control' of the Copenhagen interpretation actually dominated quantum physics in the past to a great extent was experienced by H. Dieter Zeh, one of the first protagonists of the idea of decoherence, among others. He published his ideas in 1970 in a paper 'On the interpretation of measurement in quantum theory'. Before that, he had submitted an earlier version of the paper to the renowned physics journal *Nuovo Cimento*. That version was rejected because of the devastating judgment of the referee: "The paper is completely senseless. It is clear that the author has not fully understood the problem and the previous contributions in the field".[20] We see that even in the natural sciences, the 'right thing' can prevail more readily if it is generally accepted (and understood).

In the meantime, the theory of decoherence is considered an important element that can contribute to the explanation of the measurement problem.

24.6   Conclusions

Decoherence is a purely quantum-mechanical phenomenon that classical mechanics cannot even begin to explain. It is caused by the interaction of an open system with the environment, which thereby absorbs information about the system—the quantum nature of the system 'leaks out', so to speak, into the environment.[21] Since the environment in general has very many degrees of freedom among which the information about the system is distributed, the process is virtually irreversible; we cannot know the state of all environmental degrees of freedom (if we could, we would see superpositions in the macroscopic domain, also).

This information transfer becomes apparent in *entangled* states, which are a purely quantum-mechanical phenomenon. As soon as enough information about the entanglement has found its way into the environment, so that it can distinguish between

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[19]W.H. Zurek, 'Decoherence, einselection and the quantum origins of the classical', quant-ph/0105127 v2, 11.7.2002

[20]Quoted in E. Joos, 'Elements of environmental decoherence', quant-ph/9908008 v1, 2.8.1999. In retrospect, it is clear that it was the referee who did not understand the problem fully.

[21]In J. Samuel, 'Gravity and decoherence: the double slit experiment revisited', *Classical and Quantum Gravity* 35 045004, https://doi.org/10.1088/1361-6382/aaa313 (Jan 2018) the view is represented that gravity is responsible for decoherence.

system states, the states can no longer interfere. Thus, entanglement as a *nonlocal* phenomenon of quantum mechanics leads to the *local* classical properties.[22]

It is only in isolated systems that superpositions and entanglement can be maintained over extended periods. For open systems, with their inevitable and uncontrollable interactions with their environment, decoherence acts on extremely short time scales. Thus it is understandable that one does not see superpositions, although they are allowed by quantum mechanics, under macroscopic conditions (not even for very short times in everyday terms), and why classical objects always maintain their familiar properties or, strictly speaking, *seem to have* them.[23]

Following the ideas outlined above, we can assume that the pointer variables (i.e. the display states) of the measuring apparatus $\mathcal{M}$ are determined by the interaction of $\mathcal{M}$ with $\mathcal{U}$. At least this assumption applies in simple model systems. In any case, it is certain that we do not need classical mechanics to describe the measurement process; by virtue of this point alone, decoherence contributes significantly to the internal consistency of quantum mechanics.

In principle, decoherence has the same effect as the postulate of the reduction of the wavefunction (wavefunction collapse), but in contrast, it describes a physically (and mathematically) plausible process. So we can dispense with the collapse postulate (see Chap. 14, Vol. 1, postulate 2.3) and hence the ominous distinction between the two dynamics (SEq and measurement); also the discussion about the boundary between quantum mechanics and classical mechanics may be placed on a different basis. For practical purposes, so to say as a working tool, we can of course continue to use the state reduction concept—we know now that it is simply an abbreviation for the process described above.

Given the dependence on the environment, one speaks of environment-induced decoherence; more vivid terms are *measurement by the environment* or *environment as a witness*.

Although decoherence provides a catchy explanation of why the world around us appears so classical, it cannot solve the measurement problem in its entirety. We note that decoherence does not provide a mechanism for the actual collapse, but rather a mechanism for the *appearance* of the collapse. In addition, decoherence does not explain why in a particular experiment precisely one out of many possibilities for the measured result is realized (i.e. that one which is actually measured). In addition, one would also like to know e.g. why certain superpositions are not observed (superselection rules[24]); is that due to the mechanism of decoherence? A further problem is provided by quantum cosmology, which attempts to describe quantum states of the entire universe; of course, there is after all no environment in this situation which could produce decoherence. Finally, one can argue that decoherence

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[22]See e.g. J. Richens et al. 'Entanglement is Necessary for Emergent Classicality in All Physical Theories', *Phys. Rev. Lett.* 119, 080503 (Aug 2017).

[23]"The unresolved problem today is rather the classical physics itself. How is it possible that there is after all something like the familiar ideal world of classical physics?" Peter Mittelstaedt, 'Quantum Mechanics at the End of the 20th Century', *Physikalische Blätter* 56 (2000), No. 12, p. 65.

[24] For example, there are no interferences between fermions and bosons.

only claims irreversibility, when in fact all time evolutions are reversible in theory. That would of course be different if we could show that the correlations lost between the environment and the system can, even in principle, never be recovered. In short, decoherence answers some questions regarding the measurement process, but leaves others open. In essence, we still do not know how the transition from 'possible' to 'factual' really happens.

Be that as it may be—decoherence is at least a very common approach these days (if not one of the most common approaches) in quantum mechanics, which is also due to its physically and mathematically compelling content. Obviously, the opinion has found more and more followers in recent years that with decoherence, a step has been made in the right direction concerning the treatment of the measurement problem.

Among other things, decoherence has led to the very significant finding that quantum mechanics must be considered not only in isolated, but also in open systems. This entails the understanding that superpositions of states, such as those used for example in quantum computers, are very fragile constructs under normal conditions, and can be kept 'alive' only when the system is sufficiently isolated from its environment.[25] Thus, decoherence is a great, if not *the* great obstacle to the construction of a quantum computer. Such systems have to be very carefully shielded from their environments.

24.7   Exercises

1. Given the density matrix

$$\rho = \begin{pmatrix} |c_1|^2 & c_1 c_2^* e^{i\omega t} \\ c_1^* c_2 e^{-i\omega t} & |c_2|^2 \end{pmatrix}; \tag{24.42}$$

   calculate $\frac{1}{T} \int_0^T \rho \, dt$.

2. Consider the reduced density matrix $\rho_{S,red} = CC^\dagger$ of (24.16), where C is given as an $M \times N$-matrix:

$$C = (c_{mn}) = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1N} \\ c_{21} & c_{22} & \dots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{M1} & c_{M2} & \dots & c_{MN} \end{pmatrix}. \tag{24.43}$$

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[25] "The fundamental limitation to an observer's ability is of a different nature: he must restrict his observations to a finite part of the Universe. Insurmountable difficulties do not arise from having to handle large, complicated systems; rather, they appear with limited and not perfectly isolated systems." A. Peres, *Quantum Theory*, p. 346.

Hence, the system has M states, and the environment has N. Estimate the order of magnitude of the elements of $\rho_{S,\text{red}}$.

3. Calculate explicitly the eigenvalues of the density matrix

$$\rho = \begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{pmatrix} \tag{24.44}$$

with $|c_1| + |c_2|^2 = 1$.

4. We consider two quantum objects with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_1$. The CONS $\{|0\rangle, |1\rangle\}$ is a basis of $\mathcal{H}_1$.

   (a) Show that the states

$$|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}} \tag{24.45}$$

   are also a CONS in $\mathcal{H}_1$.

   (b) Write down the states

$$\left|\psi^\pm\right\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \tag{24.46}$$

   in the basis $\{|+\rangle, |-\rangle\}$.

   (c) As assumed in the text, the effect of the environment is to add to each basis state a corresponding random phase. How are the new states $\left|\psi^\pm\right\rangle$ formulated?

5. Show that

$$\sum_{i=(m,n)} A_i^\dagger(t)\, A_i(t) = 1; \quad A_{i=(m,n)}(t) = \sqrt{p_n}\,\langle m|\,\hat{U}(t)\,|n\rangle. \tag{24.47}$$

see (24.34).

6. Two quantum objects each have a two-dimensional Hilbert space with the orthonormal basis vectors $|0\rangle$ and $|1\rangle$. They are in the ground state:

$$|\psi\rangle = c_0 |0\rangle |0\rangle + c_1 |1\rangle |1\rangle. \tag{24.48}$$

We now perform a change of basis via

$$|0\rangle = a_{11} |+\rangle + a_{12} |-\rangle; \ |1\rangle = a_{21} |+\rangle + a_{22} |-\rangle, \tag{24.49}$$

where $|+\rangle$ and $|-\rangle$ are also an orthonormal basis. Under which conditions does $|\psi\rangle = d_+ |+\rangle |+\rangle + d_- |-\rangle |-\rangle$ hold?