

Chapter 20

Entanglement, EPR, Bell



For two or more quantum objects, there exist states that are typically quantum-mechanical and have no classical analogues. These entangled states are of central importance for the understanding of quantum mechanics and especially for modern developments such as quantum computers.

Up to now, our discussion of practical problems was confined to *one* quantum object.¹ In this chapter, we consider systems of two quantum objects that each can assume different states independently of one another (e.g. spin, polarization...). The ideas discussed can easily be generalized to several quantum objects.

20.1 Product Space

First, some words about the state space of a system of two quantum objects, such as two photons or an electron and a positron. The two Hilbert spaces are \mathcal{H}_1 and \mathcal{H}_2 , with the dimensions N and M . The state of the overall system is then determined by the simultaneous specification of the vectors $|\varphi\rangle \in \mathcal{H}_1$ and $|\chi\rangle \in \mathcal{H}_2$; the pair $\{|\varphi\rangle, |\chi\rangle\}$ can be regarded as a vector of a vector space of dimension $N \cdot M$. This vector space is called the *product space* of the spaces \mathcal{H}_1 and \mathcal{H}_2 (or *tensor product* of the two spaces) and is denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$. If the spaces \mathcal{H}_1 and \mathcal{H}_2 have the bases $\{|n\rangle\}$ and $\{|m\rangle\}$, then the basis system of the product space is the set of all pairs $\{|n\rangle |m\rangle\}$ ²; these vectors are written as $|n \otimes m\rangle$ or $|n\rangle \otimes |m\rangle$. The vectors of the

¹In Chap. 17, we treated the hydrogen atom as a system of two quantum objects (nucleus and electron), but we reduced it to *one* equivalent quantum object.

²In detail, $|1\rangle |1\rangle, |1\rangle |2\rangle, \dots, |2\rangle |1\rangle, |2\rangle |2\rangle, \dots, |3\rangle |1\rangle, \dots$

individual spaces are given by $|\varphi\rangle = \sum_{n=1}^N c_n |n\rangle$ and $|\chi\rangle = \sum_{m=1}^M d_m |m\rangle$; the total state is then $|\varphi \otimes \chi\rangle = \sum_{n,m} c_n d_m |n \otimes m\rangle$.³ A general state vector therefore takes the form

$$|\psi\rangle = \sum_{n,m} a_{nm} |n \otimes m\rangle. \quad (20.1)$$

By the way, if there is no danger of misunderstanding, one often writes simply $|nm\rangle$ or a similar formulation, instead of $|n \otimes m\rangle$.

Some remarks:

1. Up to now, if we wrote e.g. $|j, m\rangle$ for angular momentum states, we meant *two* quantum numbers for *one* quantum object. In this chapter, the notation $|nm\rangle$ always means *two* quantum objects, the first in the state $|n\rangle$, the second in the state $|m\rangle$.
2. The order of the product states is not changed on taking the adjoint: $|nm\rangle^\dagger = \langle nm|$.
3. Regarding the notation, one uses indices where appropriate. There are various equivalent formulations such as

$$|nm\rangle = |n\rangle |m\rangle = |n_1 m_2\rangle = |n_1\rangle |m_2\rangle = |n\rangle_1 |m\rangle_2. \quad (20.2)$$

In the following, we will use that notation which is best suited for the corresponding topic at hand.

4. In addition, one can use the explicit representation as a column vector in certain cases; we have e.g.⁴

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}. \quad (20.3)$$

5. More on the basics of tensor products including some examples can be found in Appendix C, Vol. 2.

20.2 Entangled States

We now consider *entangled states*.⁵ As we shall see, the individual quantum objects have no well-defined status in this case. It is only through measurement that they obtain definite properties.

³Strictly speaking, this applies initially only if the two systems are independent of each other. But even in the presence of interactions, we make the plausible assumption that the state space is the product space $\mathcal{H}_1 \otimes \mathcal{H}_2$.

⁴Mnemonic: ‘The right index changes the fastest.’

⁵The term was coined 1935 by E. Schrödinger; it is possibly related to ‘clasped hands’. In that situation, one cannot change one hand without changing the other one.

20.2.1 Definition

To work out the essentials, we restrict ourselves to *two* quantum objects that may exist in two states. As a concrete example, we consider the two linear polarization states $|h\rangle$ and $|v\rangle$ of two photons.⁶ The basis states of the four-dimensional product space are then $|hh\rangle$, $|hv\rangle$, $|vh\rangle$, $|vv\rangle$. Accordingly, a general state of the product space reads

$$|\Psi\rangle = a_{hh}|hh\rangle + a_{hv}|hv\rangle + a_{vh}|vh\rangle + a_{vv}|vv\rangle. \quad (20.4)$$

In such states, the question arises as to whether the single quantum object 1 or 2 is in a well-defined state, i.e. if one can say that photon 1 has a specific linear polarization state. In any case, this is not directly visible from (20.4). If each of the quantum objects is in a well-defined linear polarization state, we can write

$$|\varphi_1\rangle = a_{1h}|h\rangle + a_{1v}|v\rangle; \quad |\varphi_2\rangle = a_{2h}|h\rangle + a_{2v}|v\rangle. \quad (20.5)$$

It follows that

$$|\varphi_1\varphi_2\rangle = a_{1h}a_{2h}|hh\rangle + a_{1h}a_{2v}|hv\rangle + a_{1v}a_{2h}|vh\rangle + a_{1v}a_{2v}|vv\rangle. \quad (20.6)$$

One says that this state *factorizes*, so it may be written as a product of two individual states (20.5) and is therefore also called a *product state*. However, the state (20.4) has this form only if

$$a_{hh} = a_{1h}a_{2h}; \quad a_{hv} = a_{1h}a_{2v}; \quad a_{vh} = a_{1v}a_{2h}; \quad a_{vv} = a_{1v}a_{2v}. \quad (20.7)$$

From this, we obtain immediately the condition

$$a_{hh} \cdot a_{vv} = a_{hv} \cdot a_{vh} \hat{=} \text{state is product state.} \quad (20.8)$$

If $|\Psi\rangle$ in (20.4) is not of the form (20.6), i.e. it does not factorize, one speaks of an *entangled* state:

$$a_{hh} \cdot a_{vv} \neq a_{hv} \cdot a_{vh} \hat{=} \text{state is entangled.} \quad (20.9)$$

An example is the vector

$$|\Psi\rangle = \frac{|hv\rangle - |vh\rangle}{\sqrt{2}}. \quad (20.10)$$

Evidently, it holds that $a_{hh} \cdot a_{vv} = 0$ and $a_{hv} \cdot a_{vh} = -1$; condition (20.8) is not fulfilled and the state is not factorized, but rather entangled.⁷

⁶We recall that $\{|h\rangle, |v\rangle\}$ is a CONS.

⁷One cannot demonstrate this situation in an intuitive manner because it is set in a four-dimensional space.

For systems of more than two quantum objects, one can, if necessary, set up equations similar to (20.8). In general, a system of several quantum objects is called ‘entangled’ if the total state cannot be represented as a product of the individual states, i.e. if it is not factorized.⁸

We emphasize once again that the equations imply that the individual components of an entangled state have no well-defined properties. In today’s view, this fact does not mean that our knowledge or quantum mechanics as a theory is (still) not sophisticated enough, but rather it is due to the structure of quantum mechanics itself (in the end, to the fact that the SEq is linear). Entangled states are typical of quantum mechanics—they do not exist in classical mechanics, and they are not even imaginable there.⁹ In quantum mechanics, however, they not only exist, but are fundamental to some modern applications such as quantum computers and quantum teleportation.¹⁰

Entangled states can nowadays be produced routinely (see Appendix I, Vol. 2).¹¹ To this end, one generates pairs of quantum objects which move apart from each other and are entangled with respect to a certain property; an example is polarization-entangled photons.¹² There are of course other methods; for example, one can

⁸We take up again at this point the remark made in the Introduction that the popular-scientific presentation of quantum mechanics is possible only to a very limited extent, and this applies especially to purely quantum-mechanical phenomena such as entanglement. As an example, we illustrate this fact by a quotation from the French scientific journal ‘*La Recherche*’: “The term (i.e. entanglement) was introduced to refer to a pair of particles which are separated from each other, but have interacted at a previous time with each other, and whose state is described more completely by their common properties (called ‘entangled’ or ‘correlated’) than by their individual properties.” (Anton Zeilinger and Markus Aspelmeyer, *The incredible illusion of reality*, in *La Recherche*, dossier 38, Feb. 2010, p. 19, translated). The authors are physicists and internationally accepted experts in the field of basic quantum-mechanics research; they also know very well how to present their field of research in a popular-scientific way. And yet their explanation is fuzzy (one might say, of course, that it has to be), and in the end not very helpful because it fits everything possible and does not grasp the essential point of entanglement. By contrast, ‘factorization’ is a precise definition, which of course presupposes that the reader knows at least some basic aspects of the mathematical apparatus (in this case, as the example shows, one can describe the situation without formulas). For both ‘explanations’, the consequences of entanglement are not clear at this point. But it is only the access via ‘factorization’ which leads on to further conclusions.

⁹They can, however, excite the fantasies of authors. Philippe Djian writes for example in *Vers chez les blancs*: “Edith’s presence made me more human, not so petty; I had often noticed that. By the way, in physics the phenomenon of interlacing is well established. In contrast to Bohr, Einstein never believed in it. But the action-at-a-distance which connects two widely-separated particles has in the meantime been observed using photons. Edith and I were »entangled« with each other; that was the truth in its most extreme and pitiless certainty.”

¹⁰See Chap. 26 (quantum information). Entangled photons may also be used in quantum cryptography.

¹¹We note by the way that entanglement may also be generated by means of the quantum Zeno effect (cf. Appendix L, Vol. 1); see Nathan S. Williams & Andrew N. Jordan, ‘Entanglement genesis under continuous parity measurement’, *Phys. Rev. A* 78, 062322 (2008).

¹²Currently (2017), ten photons can be entangled; see for instance Xi-Ling Wang et al., ‘Experimental Ten-Photon Entanglement’, *Phys. Rev. Lett.* 117, 219502 (2016). ((The paper claims to report the first experimental demonstration of entanglement among ten spatially separated single photons.))

entangle a positron–electron pair using a double Mach–Zehnder interferometer (Hardy’s experiment, see Appendix J, Vol. 2).

20.2.2 *Single Measurements on Entangled States*

Only an appropriate measurement forces the two quantum objects to assume well-defined properties. As an example, we illustrate this fact by means of the state (20.10). As usual, we describe the measurement by a projection onto the final state; each quantum object can be measured as $|h\rangle$ or $|v\rangle$. If we detect both photons, we obtain the possible amplitudes

$$\begin{aligned} \langle hh | \Psi \rangle &= 0; & \langle hv | \Psi \rangle &= \frac{1}{\sqrt{2}} \\ \langle vh | \Psi \rangle &= -\frac{1}{\sqrt{2}}; & \langle vv | \Psi \rangle &= 0. \end{aligned} \quad (20.11)$$

In other words, we find two possible readings, each with probability $1/2$. Either the quantum object 1 is in the state $|h\rangle$ and quantum object 2 in $|v\rangle$, or *vice versa*. The other two options, $|hh\rangle$ or $|vv\rangle$, are eliminated.

The really interesting question however is: What happens if we measure the state of only *one* of the quantum objects? In order to treat this clearly, we denote for the moment e.g. the state in which quantum object 1 is horizontally and quantum object 2 is vertically polarized by $|h_1\rangle |v_2\rangle$. The state (20.10) is then written as

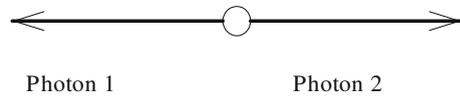
$$|\Psi\rangle = \frac{|h_1\rangle |v_2\rangle - |v_1\rangle |h_2\rangle}{\sqrt{2}}. \quad (20.12)$$

We now measure the state of only the first quantum object, and ask if it is horizontally polarized. As usual, we represent the measurement by the corresponding projection operator, i.e. by $|h_1\rangle \langle h_1| \otimes I_2$, where I_2 is the identity (the 1-operator) in the space 2. We find

With a completely different technique, namely detecting whether photons share polarizations under certain conditions, the entanglement of an even larger number of photons may be demonstrated; see for instance T. Sh. Iskhakov et al., ‘Polarization-Entangled Light Pulses of 105 Photons’, *Phys. Rev. Lett.* 109, 150502 (2012).

Entanglement is not restricted to photons, see e.g. F. Fröwis et al., Experimental certification of millions of genuinely entangled atoms in a solid, *Nature Communications* 8, Article number: 907 (2017) <https://doi.org/10.1038/s41467-017-00898-6> (Oct 2017) where the entanglement of 16 million atoms in a one-centimeter crystal is reported. Stationary objects can also be entangled, or even stationary and propagating objects; see e.g. W.B. Gao et al., ‘Observation of entanglement between a quantum dot spin and a single photon’, *Nature* 491, 426–430 (2012). Moreover, a scheme was proposed to entangle the motion of two macroscopically separated objects: C. Gneiting and K. Hornberger, ‘Bell test for the free motion of material particles’, *Phys. Rev. Lett.* 101, 260503 (2008).

Fig. 20.1 Two entangled photons



$$|h_1\rangle \langle h_1| \otimes I_2 |\Psi\rangle = \frac{|h_1\rangle \langle h_1| h_1\rangle |v_2\rangle - |h_1\rangle \langle h_1| v_1\rangle |h_2\rangle}{\sqrt{2}} = \frac{|h_1\rangle |v_2\rangle}{\sqrt{2}}. \quad (20.13)$$

This means the following:

1. On measuring, we observe a well-defined state—quantum object 1 is horizontally, quantum object 2 is vertically polarized. Before the measurement, we could not make this statement. This means that a measurement forces the measured system from the realm of *possibility* into the realm of *actuality* or, viewed in the product Hilbert space, into the subspace of factorized states.
2. After the measurement on quantum object 1, we have two possible states, each with probability 1/2: After the measurement per $|h_1\rangle \langle h_1|$, the state $|h_1\rangle |v_2\rangle$; and after the measurement per $|v_1\rangle \langle v_1|$, the state $|v_1\rangle |h_2\rangle$.
3. The new aspect of entangled states is this: The measurement of *one* quantum object defines the properties of *both* quantum objects—and this fact has far-reaching consequences.

We discuss this using the example of two entangled photons, for instance in the state (20.12), moving in different directions, say photon 1 to the left and photon 2 to the right; cf. Fig. 20.1.¹³ If we now measure the polarization of photon 1 as in (20.13), the polarization of photon 2 is automatically determined—regardless of their mutual distance.¹⁴ It is not hard to imagine experimental arrangements in which this ‘collapse’ of the state from (20.12) to (20.13) *must* be faster than the speed of light. In other words, the entanglement leads to the *non-locality*¹⁵ of quantum mechanics—a

¹³This case (or the similar one using electrons instead of photons) is the illustrative ‘standard scenario’ for this topic.

¹⁴For example, in an experiment performed in 1997, entangled photons were sent in opposite directions in glass fiber cables. When they were far apart (about 10 km), they were detected. Despite the distance, a measurement on one photon impacted on the other photon as expected. Some 10 years later, a record for distance propagation in the open air was established between two of the Canary Islands. A Bell state of two photons of the form $|\Psi^-\rangle = \frac{|hv\rangle - |vh\rangle}{\sqrt{2}}$ was produced at La Palma; one photon was measured there, the other on Tenerife, 144km away. Details in R. Ursin et al., ‘Entanglement-based quantum communication over 144 km’, *Nature Physics* 3 (2007), 481–486. Meanwhile it is about other distances, see e.g. Juan Yin et al., ‘Satellite-based entanglement distribution over 1200 kilometers’, *Science* 356 (2017), 1140–1144 16 Jun 2017: Vol. 356, Issue 6343, pp., <https://doi.org/10.1126/science.aan3211>

Moreover, the ‘speed’ of entanglement as a quantum mechanical, nonlocal connection was experimentally determined to be at least 4 orders of magnitude greater than the speed of light; see D. Salart et al., ‘Testing spooky action at a distance’, *Nature* 454 (2008), 861–864.

¹⁵Locality refers to the following requirement: when two (sub-) systems *A* and *B* cannot interact with each other (for example due to a correspondingly large spatiotemporal separation), then modifications of *A* cannot lead to changes in *B*. See also the section ‘EPR’, below.

fact which was not accepted by many physicists for a long time. Einstein was one of them; he regarded non-locality as a hardly-credible ‘spooky action at a distance’. His objections and those of others are summarized in the EPR paradox, which we discuss below.

To sum up: With entangled states, one of the peculiarities of quantum mechanics, it does not make sense to speak of well-defined properties of one of the two partners. Only when a measurement is performed do the states of the *single* quantum objects become defined and the entanglement is broken. Thus, the measurement of *one* of the two partners also fixes the corresponding properties of the *other* partner, regardless of their mutual distance. Mathematically, the special point of entangled states manifests itself in the fact that they do not factorize, i.e. they cannot be written as products of the individual states (they cannot be ‘separated’, being instead ‘entangled’).

So we have found another ‘special’ concept of quantum mechanics. To those that we have already considered, such as the necessity of probability, the effect of measurements, and state reduction, we now must add entanglement. We note that, despite the difficulties that they may cause for our understanding, entangled states are mathematically just ‘normal’ states. They can, for example, form a basis of the state space, for instance in the form of *Bell states*:

$$\begin{aligned} |\Psi^+\rangle &= \frac{|hv\rangle + |vh\rangle}{\sqrt{2}}; & |\Psi^-\rangle &= \frac{|hv\rangle - |vh\rangle}{\sqrt{2}} \\ |\Phi^+\rangle &= \frac{|hh\rangle + |vv\rangle}{\sqrt{2}}; & |\Phi^-\rangle &= \frac{|hh\rangle - |vv\rangle}{\sqrt{2}}. \end{aligned} \quad (20.14)$$

It is a basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$, like any other.¹⁶

Finally, we want to point out that an entangled state cannot be ‘disentangled’ by a subtle change of the basis of the single quantum objects; the property of entanglement is preserved even in a different basis (see exercises).

20.2.3 Schrödinger’s Cat

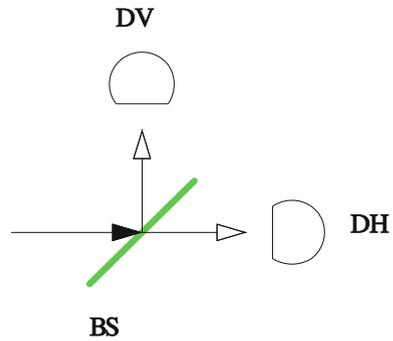
This is a famous example that shows what problematic consequences may result from entanglement—at least when one transfers it unthinkingly to macroscopic systems.

We first consider a physically harmless case; cf. Fig. 20.2.

A photon is incident horizontally, i.e. in the state $|H\rangle$, on a beam splitter BS. After passing through the beam splitter, it is in the state $\frac{(1+i)}{2} [|H\rangle + i|V\rangle]$ (see Chap. 6, Vol. 1). In shorthand notation, we write this as $|H\rangle \rightarrow \frac{(1+i)}{2} [|H\rangle + i|V\rangle]$. Now we incorporate—this is the new feature—the two detectors in the description also, namely by a ‘detector’-ket with the following properties: If neither detector

¹⁶ $|\Psi^-\rangle = \frac{|hv\rangle - |vh\rangle}{\sqrt{2}}$ is a singlet state (the global sign changes on interchange of 1 and 2), the other form is a triplet (the global sign is invariant on interchange of 1 and 2).

Fig. 20.2 A photon is incident on a beam splitter BS and is detected in one of the detectors DH or DV



clicks, we have $|00\rangle$; if DH or DV clicks, the state reads $|10\rangle$ or $|01\rangle$. Then we can divide the process into three stages as follows:

$$|H\rangle |00\rangle \rightarrow \frac{(1+i)}{2} [|H\rangle + i|V\rangle] |00\rangle \rightarrow \frac{(1+i)}{2} [|H\rangle |10\rangle + i|V\rangle |01\rangle]. \quad (20.15)$$

In words: At the beginning, the photon is in the state $|H\rangle$, no detector is activated and the total state is factorizable. This is initially also the case when the photon has passed through the beam splitter; eventually, with probability 50%, it impinges on one of the two detectors and activates it. As we can see directly, the final state in (20.15) is entangled.

This may not seem particularly remarkable—but what happens if we choose a cat as detector? Indeed, in the course of the debate about the EPR paradox (see below), Erwin Schrödinger¹⁷ published a thought experiment in 1935 pointing out the deficiencies which in his view existed in quantum mechanics. To illustrate these, he introduced a very special measuring apparatus:

One can even set up quite ridiculous cases. A cat is trapped in a steel chamber, along with the following device (which must be secured against direct interference by the cat): adjacent to a Geiger counter, there is a tiny bit of a radioactive substance, so little that in the course of an hour, perhaps only one of its atoms decays; but, with equal probability, perhaps none decays. If a decay occurs, the counter tube discharges, and through a relay a hammer is released and shatters a small flask of hydrocyanic acid. If this entire system is left to itself for an hour, one would say that the cat is still alive if meanwhile no atom has decayed. The psi-function of the entire system would express this by containing both the live and the dead cat (s.v.v.)¹⁸ mixed or smeared out in two equal parts. It is typical of such examples that an indeterminacy originally restricted to the atomic domain has been transferred to a macroscopic indeterminacy, which can then be resolved by direct observation. This prevents us from naively accepting a ‘fuzzy model’ as valid for representing reality.

We look at this in more detail, using the notation based on the example of the beam splitter. For this purpose we describe the states of the radioactive atom by $|A \rightarrow d\rangle$

¹⁷Erwin Schrödinger, *Naturwissenschaften* 23 (1935), p. 812.

¹⁸Latin ‘*sit venia verbo*’: pardon the expression.

(not decayed)¹⁹ and $|Ad\rangle$ (decayed) and the states of the cat by $|Ca\rangle$ (alive) and $|Cd\rangle$ (dead). We start at time $t = 0$ with the state $|A\neg d\rangle |Ca\rangle$, which changes to a linear combination of the form:

$$|A\neg d\rangle |Ca\rangle \rightarrow |A\neg d\rangle |Ca\rangle + |Ad\rangle |Cd\rangle \quad (20.16)$$

(the kets are time dependent, and the normalization does not matter here). Obviously, we have here a normal entangled state like the one in the example of the beam splitter (20.15). The difference is just that in this state, a cat appears simultaneously to be dead and alive, which contradicts our everyday experience.

We are therefore faced with the situation that on the one hand, entanglement occurs in microscopic systems, and *must* occur—for instance, the Pauli principle for two electrons necessarily requires entanglement (see Chap. 23). On the other hand, entanglement is never observed in cats, chess boards, socks or grand pianos. In the macroscopic view, the problem is not only with macroscopic superpositions, but also with the fact that entangled states obtain objective properties only through a measurement. Entanglement suggests a holistic structure of the world. This is in conflict with our everyday experience and the (reductionist) method of the natural sciences based on it, whose success is partly due to the fact that one can examine individual subsystems of large integrated systems.²⁰

A way out of this tricky situation is offered for example by the theory of decoherence, which we will examine in more detail in Chap. 24. According to it, state superpositions collapse due to interactions with the *environment*—not only through (possibly man-made) measurements. But interactions with the environment cannot be excluded in practice for a macroscopic system—it would have to be totally shielded from the outside world. This is extremely difficult even for microscopic systems, as the slow pace of development of quantum computers clearly illustrates. So decoherence effects prevent paradoxical mixed states containing both life and death, not only for cats.

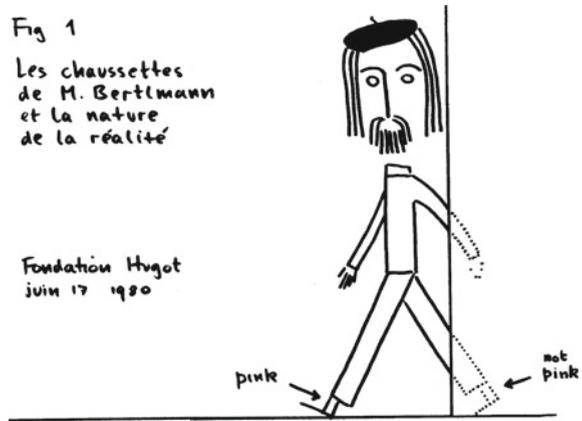
20.2.4 A Misunderstanding

We want to warn of an obvious misunderstanding, which we illustrate with an example of suitcases and socks. Suppose we put a yellow and a blue sock into each of two suitcases, outwardly indistinguishable, so that we cannot say after closing the suitcases which one contains which sock. Then we send one suitcase to Greenland and take the other one with us to Tasmania. Here, we open the suitcase that we brought

¹⁹The symbol $\neg a$ is the (logical) negation of the property a . We assume that the relevant variables are dichotomous or binary, i.e. they take on one of two values, either a or $\neg a$. Intermediate forms cannot occur, a third possibility does not exist (*tertium non datur*). This applies of course also to the pair (a, d) ; states *between* ‘alive’ and ‘dead’ are excluded.

²⁰To describe the motion of a simple pendulum in the lab, we do not need any information about, for example, solar flares, or the total number of penguins in the world.

Fig. 20.3 Bertlmann's socks. From John Stewart Bell, 'Bertlmann's socks and the nature of reality', in *J. Phys. Colloq.* 42, C22 (1981) pp. C2.41–C2.62



along, see a yellow sock and know at the same moment with certainty that the blue sock is in Greenland (classically correlated events). In this case, the measurement (=opening of the suitcase + looking) annuls our lack of knowledge of the system—and this is *quite a different matter* from the assignment of values by measurement in the case of entangled quantum objects. In the example of the suitcases, entanglement would mean that the two socks do not *have* a well-defined color²¹; instead, they would *obtain* it (either yellow or blue) only when the suitcase is opened (quantum-mechanically correlated events). So in the quantum-mechanical case, the problem is not that we do not know the color of the sock in a particular suitcase; but rather the fact that the color ‘emerges’ only due to the opening of the suitcase—only then, that is at the moment of ‘measurement’, do the socks receive (and show) their colors.

Apropos, the use of socks as examples in this type of problem has a long tradition. In 1981, John Bell wrote the article ‘Bertlmann’s socks and the nature of reality’. Reinhold Bertlmann, then working at CERN, always wore two different-colored socks. When one saw a pink sock, it was known with certainty that the other sock was *not* pink. Bell contrasted this with entanglement in quantum mechanics. Compare Fig. 20.3.

20.3 The EPR Paradox

The view that one cannot always assign well-defined properties to objects, or that there are non-local processes, was the subject of serious discussions even in the early days of quantum mechanics. It contrasted with the belief that quantum objects

²¹The allocation of the mixed color green (=yellow + blue) before the measurement would also not be correct, because then the color of the socks would be a mixture of two definite states. At best, in a universe in which there are only yellow and blue and no other colors, i.e. in which green is an unknown color, one could argue using green socks (however, precisely this color does not exist in that universe).

must have an autonomous reality that is independent of measurements—i.e. that an electron has spin, position and so on, even if it is not observed (boldly transferred to macroscopic conditions in Einstein’s question, “Do you really think the moon isn’t there if you aren’t looking at it?”).²²

This dissatisfaction with the unusual perspective of quantum mechanics was focussed by the famous ‘EPR paper’.²³ In 1935, Albert Einstein, Boris Podolsky and Nathan Rosen published an article intending to show that quantum mechanics does not satisfy the requirements which an acceptable physical theory generally has to fulfill. Essentially, these requirements are the following:

- *Reality*: A physical quantity whose value can be predicted with certainty is a property of a physical system A . In other words, the system ‘has’ or ‘owns’ this property, independently of measurements. This property is therefore an element of physical reality (Einstein reality).
- *Locality*: The result of a measurement on a system A is not influenced by manipulations of other systems B which are space-like separated²⁴ from A —or, in another formulation: What exists in B does not depend on what is measured in A (Einstein locality). Realism and locality together are also referred to as *local realism*. Of course, in order for the term locality to be meaningful, the system A must be characterizable in terms of its own intrinsic properties, regardless of the properties of the other systems B (*separability*).
- *Completeness*: A physical theory must be complete within the limits of its validity. This means that there must be a corresponding theoretical counterpart (within the frame of the model) for each element of physical reality. If quantum mechanics were not complete, then there must exist in addition to the state $|\psi\rangle$ further (albeit unknown) variables Λ , the so-called *hidden variables*. Knowledge of those variables would allow a complete description, since then each observable A has an *objectively existing* value $A(\psi, \Lambda)$.

The EPR paper formulated its objections on the basis of the non-commuting variables *momentum* and *position*. A simpler design of the relevant (at that time) *thought* experiment was introduced in 1952 by the U.S. physicist D. Bohm. We consider a system with a total spin of zero, which decays into two *entangled* quantum objects, each with spin $\frac{1}{2}$ (e.g. an electron—positron pair). One of them (Q1) moves off to the left, the other one (Q2) to the right; cf. Fig. (20.4).²⁵ When the quantum objects are so far apart that they no longer interact with each other, we measure the

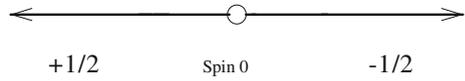
²²The (current, i.e. 1981) response of the U.S. physicist David Mermin: “We now know that the moon is demonstrably not there when nobody looks”. This answer is correct under microscopic conditions (at least, according to the majority view of the physics community), but not under macroscopic ones (due to decoherence, see Chap. 24). In this sense it is a witty, eye-catching answer to a striking question, a *bon mot* with a certain truth content. And if the question of the moon appears too banal—is the rainbow also there when nobody looks?

²³EPR is an acronym for the last names of the three authors.

²⁴This means that the one system is not within the light cone of the other.

²⁵Since this experiment is discussed generally with electrons/spin, we present it here in this way. Of course, one could instead use photons/polarization. For this, recall the polarization operators,

Fig. 20.4 Decay of a singlet state into two oppositely-oriented spins



orientation of one spin, for instance that of Q1. Suppose we first measure s_x and obtain $+\frac{\hbar}{2}$. Then we know for sure that Q2 is in the state $-\frac{\hbar}{2}$. We also know that the two quantum objects do not affect each other, because of their sufficiently large spatial separation. Then it follows as a consequence that the state $-\frac{\hbar}{2}$ of Q2 is an element of physical reality in the above sense.

The same reasoning can now be applied to the other two spin components s_y and s_z . Thus, the *three* spin components of Q2 are elements of reality, as we can predict their values with certainty without measuring them. But this is contrary to quantum mechanics, which states that only *one* spin component can be determined, since the three spatial components of the angular momentum do not commute. So EPR concludes that quantum mechanics is not complete.²⁶

Niels Bohr then responded by noting that the term ‘physical reality’ can refer only to situations in which the experimental setup is completely determined; but this was not the case here, because the system is perturbed by the decision of the experimenter to measure e.g. the spin component along the x -axis instead of the z -axis. Therefore, only the specific experimental setup, i.e. the context of measurement (contextuality), determines to which quantity physical reality can be attributed.

Are the observed correlations thus classical or quantum mechanical? In the classical case, the spin components of Q1 and Q2 would be equal and opposite, because the total spin is zero. But that would result from the fact that the spin vectors have well-defined values and directions *from the beginning*, and the process of measuring Q1 would not disturb Q2 in any way. In this case (and the incompleteness of quantum mechanics in the view of EPR suggests this assumption), there would therefore have to be a theory ‘underlying’ quantum mechanics, i.e. a theory with hidden variables.

On the other hand, one can argue quite clearly that the individual components of entangled states are not in a well-defined state from the beginning. For this, we consider two polarization-entangled photons moving apart in the state

$$|\Phi^+\rangle = \frac{|hh\rangle + |vv\rangle}{\sqrt{2}}. \tag{20.17}$$

Each photon is incident on an analyzer. With the same, otherwise arbitrary orientation α of the two analyzers, either *both* photons pass through the analyzers (with probability $\cos^2 \alpha$), or they are *both* absorbed. If now the polarization direction of the two photons were determined from the start, then $\cos^2 \alpha$ would be the probability for each photon to pass its analyzer, *independently* of one another. Consequently it would be observed from time to time that e.g. only *one* photon is absorbed. Since this

defined in Chap. 4, Vol. 1, for linear, circular, and 45°-rotated linear polarization, i.e. $P_L = |h\rangle\langle h| - |v\rangle\langle v| = \sigma_z$, $P_C = |r\rangle\langle r| - |l\rangle\langle l| = \sigma_y$ and $P_{L'} = |h'\rangle\langle h'| - |v'\rangle\langle v'| = \sigma_x$.

²⁶However, whether a complete description exists at all is not discussed in the EPR paper.

is never observed, the assumption seems natural that the photons have *no* definite polarization before the measurement, but that it is determined only by the measurement.

However, as we said above, this is just an (albeit very plausible) assumption. We can directly confirm it in this experiment only if we show for *all* the photon pairs that they both pass or are both absorbed. But it is obviously impossible to verify this assumption. To proceed, we need a falsifiable statement. This is Bell's inequality, which we will discuss in the next section.

20.4 Bell's Inequality

How can we distinguish experimentally between classical and quantum correlations? It was John Bell who gave an answer to this question in 1964 by formulating a general inequality.^{27,28} There are two remarkable points about this inequality: on the one hand, it is very simple, and on the other hand, it can be *checked experimentally*. The results of the experiments show that quantum mechanics and local realism are not compatible.

20.4.1 Derivation of Bell's Inequality

We consider a set whose elements can be characterized by three dichotomous properties, namely a , b , c . For example, $n(a, \neg b)$ means the set of elements that have the property a , but not b (nothing is said about c); $n(a, b, \neg c)$ is the set of all elements with the properties a and b , but not c . We emphasize that it is quite essential that the elements have these properties, regardless of whether we measure (or inquire into) them or not. Then we have

$$\begin{aligned} n(a, c) &= n(a, \neg b, c) + n(a, b, c) \\ n(b, \neg c) &= n(a, b, \neg c) + n(\neg a, b, \neg c). \end{aligned} \tag{20.18}$$

Since all n are positive, it follows that

$$\begin{aligned} n(a, c) &\geq n(a, b, c) \\ n(b, \neg c) &\geq n(a, b, \neg c). \end{aligned} \tag{20.19}$$

The addition of the last two inequalities leads to

²⁷In fact, there are several ways to establish such inequalities, which is why often the plural 'Bell's inequalities' is used.

²⁸John Stewart Bell, 1928–1990, Northern Irish physicist.

$$n(a, c) + n(b, \neg c) \geq n(a, b, c) + n(a, b, \neg c) = n(a, b), \quad (20.20)$$

or, compactly,

$$n(a, b) \leq n(a, c) + n(b, \neg c). \quad (20.21)$$

This is *Bell's inequality* (or one possible formulation thereof).²⁹

A concrete example: In a population, we distinguish the categories of female–male (f and $\neg f$), eye color blue–not blue (b and $\neg b$), and size short–not short (≤ 170 cm and > 170 cm; s and $\neg s$). Then (20.21) states that the number of women with blue eyes is less than or equal to the number of short women plus the number of tall persons with blue eyes. Even if this fact perhaps is not obvious at first glance, it is still correct.

It should be emphasized once more that Bell's inequality (20.21) is based exclusively on the fact that objects *have* (in the sense of possess or own) uniquely fixed characteristics or properties. In this context, the inequality applies generally and is not restricted to the realm of quantum mechanics.

One of the simplest possibilities for testing the inequality (20.21) is given by considering the polarization of entangled photons. In this case, the characteristics a , b and c are represented by three different polarization (or analyzer) settings.

20.4.2 EPR Photon Pairs

To begin with, we consider two entangled photons which are emitted by a source and move apart in opposite directions, whereby each photon impinges on a polarization analyzer. The state of the system is

$$|\Phi^+\rangle = \frac{|hh\rangle + |vv\rangle}{\sqrt{2}}. \quad (20.22)$$

We use *different* orientations of the analyzers 1 and 2, i.e. $\alpha \neq \beta$, see Fig. 20.5. To avoid a 'collusion' of the two photons, the analyzer settings are chosen only after both photons are underway so that agreement would require a superluminal exchange of information. If the analyzer settings are different, the two photons will not necessarily suffer the same fate (passing or being absorbed), but of course, probability statements are also possible for this setup.

For reasons of clarity, we assume that photon 1 arrives first at its analyzer. Since the state $|\Phi^+\rangle$ includes any direction, so to speak, photon 1 passes its analyzer with a probability of $\frac{1}{2}$ (and it is absorbed with the same probability).³⁰ If it was not absorbed, then photon 1 has a well-defined polarization, namely α . Thus, the polarization of photon 2 is also fixed—it is also α . Consequently, the angle between the analyzer

²⁹A derivation of this inequality based on set theory is given in Appendix K, Vol. 2.

³⁰For the explicit calculation of the probabilities used in this paragraph, see the exercises.

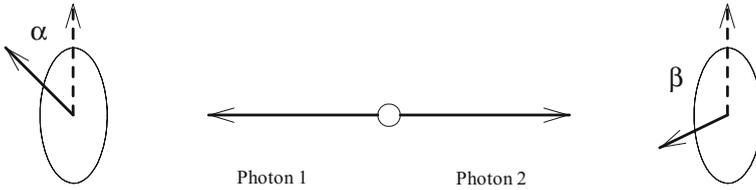


Fig. 20.5 Two entangled photons are incident on differently-oriented analyzers

setting and the polarization direction is $\alpha - \beta$, and we can write the probability for the passage of photon 2 with the help of Malus' law as $\cos^2(\alpha - \beta)$. The probability that both photons pass their analyzers is thus given by

$$p(\alpha, \beta) = \frac{1}{2} \cos^2(\alpha - \beta). \tag{20.23}$$

This probability depends on the difference of the angles and is symmetric w.r.t. reversal of the angles, as it indeed should be.

Analogously, we obtain the probability that one photon is absorbed. Photon 1 passes with probability $\frac{1}{2}$, photon 2 is absorbed with probability $1 - \cos^2(\alpha - \beta) = \sin^2(\alpha - \beta)$. With the notation $\neg\beta$, meaning absorption at the angle setting β , it follows that

$$p(\alpha, \neg\beta) = \frac{1}{2} \sin^2(\alpha - \beta). \tag{20.24}$$

To verify experimentally these two equations, one has to measure many pairs of photons (strictly speaking, an ensemble), $N \gg 1$. Then we have

$$n(\alpha, \beta) = \frac{N}{2} \cos^2(\alpha - \beta); \quad n(\alpha, \neg\beta) = \frac{N}{2} \sin^2(\alpha - \beta). \tag{20.25}$$

20.4.3 EPR and Bell

We can now apply the form of Bell's inequality derived above if we assume that it is an objective property of the photon (i.e. determined even before the measurement) to pass an analyzer at a given orientation (or not to pass). We need *three*³¹ analyzer settings, α, β and γ , where the three differences $\alpha - \beta, \alpha - \gamma, \beta - \gamma$ should occur on the average with equal frequencies (so that the frequencies n in (20.25) for the three different angle differences each are the same). The directions of the analyzers are chosen only when the photons are already underway.

³¹Note that this consideration is not based on two parameters (or one relative parameter) as in Bohm's thought experiment, but rather on three (or two relative) parameters.

For the triple of analyzer settings (α, β, γ) , Bell's inequality reads

$$n(\alpha, \beta) \leq n(\alpha, \gamma) + n(\beta, \neg\gamma). \quad (20.26)$$

With (20.25), this can be written as

$$\cos^2(\alpha - \beta) \leq \cos^2(\alpha - \gamma) + \sin^2(\beta - \gamma). \quad (20.27)$$

In order to show more clearly that this inequality is not fulfilled for certain angles, and which angles those are, we now simplify, setting $\alpha = 0$ without loss of generality. It follows that

$$\cos^2 \beta \leq \cos^2 \gamma + \sin^2(\beta - \gamma). \quad (20.28)$$

We can transform this to give³²:

$$\sin(\gamma + \beta) \sin(\gamma - \beta) \leq \sin^2(\gamma - \beta). \quad (20.29)$$

Rearranging this last expression yields

$$\sin(\gamma - \beta) \cos \gamma \sin \beta \leq 0. \quad (20.30)$$

Since the orientation is not fixed, we can set $0 < \beta < \pi$. This means $\sin \beta > 0$; therefore the inequality

$$\sin(\gamma - \beta) \cos \gamma \leq 0. \quad (20.31)$$

must hold if our assumption is valid that there exist hidden variables. In other words: if there are angles β and γ for which the inequality

$$0 < \sin(\gamma - \beta) \cos \gamma \quad (20.32)$$

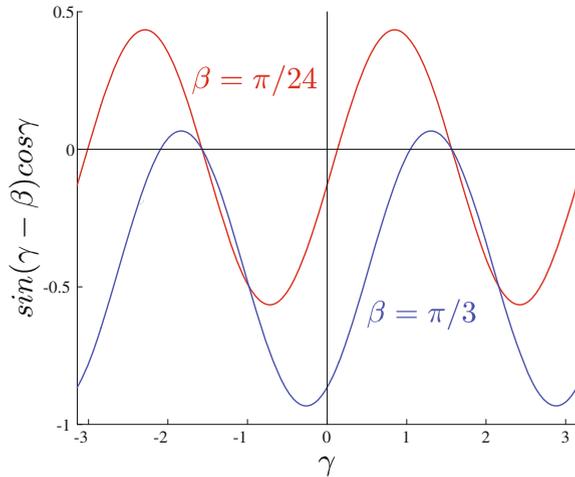
is fulfilled, that assumption is *not* valid.

Now one sees directly that the last inequality is satisfied for $0 < \gamma - \beta < \pi$ and $0 < \gamma < \pi/2$. This proves that the postulated hidden variables do not exist. In fact, the function $f(\beta, \gamma) = \sin(\gamma - \beta) \cos \gamma$ always has positive ranges for $\beta \neq \pi/2$ (see exercises), as shown by an example in Fig. 20.6 for two different values of β . Hence, the view of EPR is not sustainable; the results of quantum mechanics cannot be explained by a local-realistic theory.

What do experiments say about Bell's inequalities? Their violation by quantum mechanics has been studied experimentally since the late 1960s; however, at first the results seemed not to be taken very seriously. The breakthrough came with an

³²For the transformations, see the exercises.

Fig. 20.6 The function $\sin(\gamma - \beta) \cos \gamma$ for $\beta = \frac{\pi}{24}$ (red) and $\beta = \frac{\pi}{3}$ (blue). The positive components shown here demonstrate that quantum mechanics violates local realism



experiment conducted in 1981 in Paris.³³ It convinced (almost) all the doubters. Since then, the facts have been confirmed in a series of experiments. Among others, in an experiment performed in 1998, the polarizing filters were 400 m apart, so that a ‘collusion’ of the entangled photons was impossible due to the finite speed of light.³⁴

An addendum: In 1989, an experiment was presented with three quantum objects, entangled in a certain way (Green, Horne, Zeilinger, GHZ). Here, a *single* measurement is sufficient to refute local realism, while for Bell’s inequalities, one needs a large number of measurements to check the inequalities. The GHZ states are discussed in more detail in Chap. 27.

³³Alain Aspect, Philippe Grangier and Gérard Roger, ‘Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell’s Inequalities’, *Phys. Rev. Lett.* 49, 91–94 (1982).

³⁴It is not surprising that the experimental reality is more complicated than described here. Upon closer inspection it turned out, for example, that the conditions of the relevant experiments did not fully satisfy the hypothesis underlying the Bell inequality. This means that there are so-called loopholes, i.e. possibilities of explaining the experimental results by means of local-hidden-variable theories. These loopholes are related (i) to the separation between the local measurements and (ii) to the detection efficiency. For instance, in certain experiments, not all the photons were detected. Thus, one can argue (at least in principle) that only the detected photons agree with quantum mechanics, while the entire ensemble satisfies Bell inequalities. Of course one may assume, in contrast, that each sample of pairs of photons detected is representative of all the pairs emitted (the ‘fair sampling assumption’); or, in other words, that nature is not malicious, but this does not refute the argument. However, in the meantime a series of ‘loophole-free’ experiments have been carried out, for instance: M. Giustina et al., ‘Bell violation with entangled photons, free of the fair-sampling assumption’, [arXiv:1212.0533](https://arxiv.org/abs/1212.0533) (2012); A. Cabello and F. Sciarrino, ‘Loophole-Free Bell Test Based on Local Precertification of Photon’s Presence’, *Phys. Rev. X* 2, 021010 (2012); B. Wittman et al., ‘Loophole-free Einstein-Podolsky-Rosen experiment via quantum steering’, *New J. of Phys.* 14, 053030 (2012); W. Rosenfeld et al., ‘Event-Ready Bell Test Using Entangled Atoms Simultaneously Closing Detection and Locality Loopholes’, *Phys. Rev. Lett.* 119, 010402 (Jul 2017). The current state of discussion is found in ‘The BIG Bell Test Collaboration’, *Nature* 557, 212 (2018).

20.5 Conclusions

We note once again that our derivation of Bell's inequalities is independent of quantum mechanics and is based upon the assumptions of local realism: (1) measurement values 'really' exist, regardless of whether they are measured or not; (2) properties of a system are not directly influenced by other space-like separated systems.³⁵

The violation of Bell's inequality³⁶ by quantum mechanics can therefore in principle have three causes: Either not all measurement values are fixed prior to the measurement; or the measurement results depend non-locally on (arbitrarily) distant random decisions; or quantum mechanics is neither realistic nor local (in the Einsteinian sense). At this point, we cannot discern what is really the cause. But in any case, it is certain that either quantum mechanics is complete, or the hidden variables have to have very exotic properties. A return to the concrete world of classical mechanics is ruled out.

We will take up these considerations again in Chap. 27. Here, we want to note in anticipation that it is now largely accepted that reality simply is not independent of measurement. All known facts seem to indicate that we must definitely abandon the idea that the properties of quantum objects exist independently of our observations or, more generally, of their environment (as we have indeed always assumed).³⁷

Finally, a word about non-locality. Locality is apparently violated in the measurement of entangled states, since we then have a nonlocal (i.e. faster than light) change

³⁵In principle, one could explain an experimental violation of the Bell inequality by models based on hidden influences propagating at a finite (and sufficiently high) speed $v > c$. But as may be shown, such models predict, for any finite speed v with $c < v < \infty$, correlations that can be exploited for faster-than-light communication. Thus, assuming the impossibility of using nonlocal correlations for superluminal communication, any possible explanation of quantum correlations in terms of influences propagating at any finite speed can be excluded. See J-D. Bancal et al., 'Quantum non-locality based on finite-speed causal influences leads to superluminal signalling', *Nature Physics* 8 867–870 (2012), <https://doi.org/10.1038/nphys2460>.

³⁶In addition to the Bell inequalities, there are other inequalities based on certain classical properties which can be proven wrong in quantum mechanics. An example are the Leggett–Garg inequalities (named for A.J. Leggett & A. Garg, 'Quantum Mechanics versus macroscopic realism: is the flux there when nobody looks?', *Phys. Rev. Lett.* 54, 857 (1985)); for details see e.g. A.J. Leggett, 'Testing the limits of quantum mechanics: motivation, state of play, prospects', *J. Phys. Condens. Matter* 14 (2002) R415–R451. An experimental realization was reported by George C. Knee et al., 'Violation of a Leggett-Garg inequality with ideal non-invasive measurements', *Nature Communications* 3, Article Number 606; <https://doi.org/10.1038/ncomms1614> (2012). The results demonstrate clearly the necessity of a non-classical picture.

³⁷The question 'What is real?' has been discussed in philosophy for thousands of years—in this sense, one can indeed say with some justification that this basic research into quantum mechanics is a kind of experimental philosophy.

The subject has also fascinated writers over and over. One of countless examples: "Imagine a man in a library. The books are all empty, until he pulls one out. Then the simulators—or whoever—fill the book with print. But only as long as he is leafing through it, and with a minimum of words, just in time, just as his eye turns to the page. If he returns the book and looks for a different one, the simulators make that book exist. Most of the library is a bluff, just a lot of book covers, which don't even have titles if you aren't looking too closely." Jonathan Lethem: *Chronic City* (2011), p. 281.

of state. However, locality is not violated in the sense that using entangled photons would make a superluminal *information exchange* possible. If one for example looks only at analyzer 2, half of the photons pass, half are absorbed. But this behavior does not permit conclusions about the setting of analyzer 1. Only in retrospect can the comparison (which can be carried out at most with the speed of light) tell us whether or not entanglement existed. In short, there are instantaneous correlations, but no instantaneous interactions.³⁸

20.6 Exercises

1. Given two matrices A and B with

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}. \tag{20.33}$$

Determine $A \otimes B$.

2. Represent the Bell states (20.14) as column vectors. Show in this representation that the Bell states are entangled and that they form a CONS.
3. Two photons are in the state

$$|\Psi\rangle = \frac{|hv\rangle - |vh\rangle}{\sqrt{2}}. \tag{20.34}$$

- (a) Show explicitly that it is an entangled state.
 - (b) Photon 1 passes an analyzer for right-handed circular polarization (the corresponding state reads $\frac{|h\rangle + i|v\rangle}{\sqrt{2}}$). Show that through a measurement, the state $|\Psi\rangle$ is changed into a product state.
4. Show that the Bell states can be transformed into each other by applying the Pauli matrices to a subsystem.
 5. Show that the Bell states are eigenvectors of products of the same Pauli matrices.
 6. Transform the inequality (20.27)

$$\cos^2(\alpha - \beta) \leq \cos^2(\alpha - \gamma) + \sin^2(\beta - \gamma) \tag{20.35}$$

for $\alpha = 0$ and $0 < \beta < \pi$ to give

$$\sin(\gamma - \beta) \cos \gamma \leq 0. \tag{20.36}$$

7. Given the function

$$f(\gamma, \beta) = \sin(\gamma - \beta) \cos \gamma; \tag{20.37}$$

³⁸A very comprehensive and readable overview is found in A. Zeilinger, ‘Light for the quantum. Entangled photons and their applications: a very personal perspective’, *Physica Scripta* 92 072501 (2017).

determine the position of its zeros and the positions and values of its maxima with respect to γ .

8. A system of two photons is in one of the Bell states. The photon Q1 is incident on an analyzer for horizontal polarization, rotated by an angle α . What is the probability that Q1 passes the analyzer?
9. Two photons in the state $|h_0\rangle$ are rotated by the angles α and β to give the states $|h_\alpha\rangle$ and $|h_\beta\rangle$. How does the projection operator referring to $|h_\alpha h_\beta\rangle$ act on the Bell states?
10. Given two quantum objects Q1 and Q2, with an N -dimensional CONS $\{|\varphi_i\rangle\}$ for Q1 and $\{|\psi_j\rangle\}$ for Q2 (due to this notation we can omit the index for the number of the quantum object). The initial state is

$$|\chi\rangle = \sum_{ij} c_{ij} |\varphi_i\rangle |\psi_j\rangle. \quad (20.38)$$

What is the probability of measuring Q1 in some state $|\lambda\rangle$ (no matter which state)?

11. Show that entangled states such as the Bell states cannot be ‘disentangled’ by a reversible transformation of the single-quantum-object basis; entanglement is preserved even in a different basis.
12. Determine the behavior of the Bell states under reversible transformations. Consider the case of rotations.