Chapter 27
Is Quantum Mechanics Complete?

We delve further into the question of whether quantum mechanics is complete. As the Kochen–Specker theorem and the GHZ states show, a realistic representation of quantum mechanics is compatible neither with non-contextuality nor with locality.

In the context of the EPR paradox and Bell's inequality (see Chap. 20), we came across the question of whether quantum mechanics as a physical theory is complete.

If we assume that it is, and trust the formalism of quantum mechanics developed thus far, then we must also accept e.g. that objective chance exists, and that properties are not necessarily fixed a priori, but are only 'created' by a measurement.[1] This contradicts classical physics, where properties are presumed to exist independently of measurements (pre-existence), and where measurement does not mean the creation of a property, but rather the reduction of our ignorance about this property.

In contrast, if we do not accept this contradiction, and therefore (or for other reasons) take the view that quantum mechanics is *in*complete, we have to introduce the additional variables that make it a complete theory. These postulated other quantities are usually called *hidden variables* (HV). They would ensure that all aspects of probability can be removed from quantum mechanics (at least in principle), and that its predictions generally are deterministic. Hence, one speaks also of *cryptodeterminism* and the sub-quantum world.

So if we want to postulate that a realistic view of the world as is commonly accepted in classical physics also applies to quantum mechanics, then this means that (1) quantum states refer to individual systems, not just to the results of repeated

[1]In any case, this is apparently the prevailing view, e.g.: "Values cannot be ascribed to observables prior to measurement; such values are only the outcomes of measurement." K. Gottfried and T.-M. Yan, *Quantum Mechanics: Fundamentals*, 2nd Edition, 2003, p. 42.

J. Pade, *Quantum Mechanics for Pedestrians 2*, Undergraduate Lecture Notes in Physics, https://doi.org/10.1007/978-3-030-00467-5_27

measurements,[2] and in particular that (2) a measurement determines the value of a physical quantity which that quantity had immediately before and independently of the measurement.

For example, if a circularly-polarized photon impinges on an analyzer sensitive to linear polarization, we obtain with probability 1/2 either a horizontally or vertically linear-polarized photon; more cannot be said according to the rules of quantum mechanics. If we now demand that it must be certain *before* the measurement whether the photon will be polarized vertically or horizontally, we cannot avoid introducing additional variables which contain this information.

The key question is the following[3]: Let us suppose that a quantum system is in the state $|\psi\rangle$. Does then every observable A have an objectively pre-existing value $A(\psi, \lambda_1, \lambda_2, \ldots)$, determined by $|\psi\rangle$ and a set of hidden variables $(\lambda_1, \lambda_2, \ldots)$? If we could answer this question with 'yes', then the values of all observables would be elements of physical reality; we would have a realistic theory.

But our considerations based on Bell's inequality have shown that naive realism collides with quantum mechanics. As we have seen, it is an experimentally testable and confirmed statement that realism and locality are not at the same time compatible with quantum mechanics.

Before we take up this issue again, we will examine another combination of conditions: Is it compatible with quantum mechanics that all properties of a quantum system (a) are defined at all times (value-definiteness) and in addition (b) do not depend on the context of the measurement (non-contextuality)? The answer is 'no', as we will see in the following on the basis of the Kochen–Specker theorem.[4]

Finally a point of clarification: The fact that we take up the question of hidden variables again and in a wider context here does not mean that we aim to introduce them (through the back door, so to speak) at the end of this book. In fact, according to present knowledge, hidden variables are fighting rather a losing battle. The question of interest is instead: Why does the introduction of hidden variables fail? In trying to answer this question, we can learn more about the way quantum mechanics 'works'.

27.1 The Kochen–Specker Theorem

It is a seemingly innocuous assumption that everything that exists in the physical world is 'really there' and, furthermore, exists independently of our measurements. We substantiate this idea in two terms or conditions:

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[2]However, also in approaches based on objective chance or tending towards the many-worlds interpretation, one assumes that states refer to individual systems and not merely to ensembles. The crucial element for the following considerations is requirement (2).

[3]As a Hamlet-style question, so to speak: 'To be' or 'to be found'?

[4]Contextuality states that the measurement outcome of an observable depends on the set of compatible observables that are measured at the same time. Thus, nonlocality can be considered as a reflection of contextuality in spatially separated systems.

1. *Value-definiteness* (VD, also called pre-assigned initial values): All properties of a quantum system are defined at all times, even when the system is for example in a superposition state.
2. *Non-contextuality* (NC): The properties of a quantum system do not depend on which quantities are measured in an experiment. They are thus independent of the measurement context, i.e. they are non-contextual.

The Kochen–Specker theorem (KST) shows that in quantum mechanics, these two demands for value definiteness and non-contextuality cannot be fulfilled simultaneously. It follows that there cannot exist realistic non-contextual models with hidden variables in quantum mechanics.

Basically, this theorem from 1967 is a mathematical result about the nature of Hilbert spaces; it can be reduced to the purely geometric problem that it is not possible to color the surface of a three-dimensional sphere in a certain way.[5]

27.1.1 Value Function

In order to quantify the concepts, we introduce a value function $V_{|\psi\rangle}(A)$. It denotes the value of the physical quantity A when the system is in state $|\psi\rangle$. If $|\psi\rangle$ is an eigenvector of A with eigenvalue a_n, then we can assume $V_{|\psi\rangle}(A) = a_n$. But if $|\psi\rangle$ is not an eigenvector, we need to ask for additional properties of V in order to arrive at a reasonable statement. It seems natural and intuitively obvious to require

$$V_{|\psi\rangle}(F(A)) = F\left(V_{|\psi\rangle}(A)\right) \tag{27.1}$$

Therefore, for each condition of a quantum system, the value function of the function of a physical quantity equals the function of the value function—or in brief, the value of a function equals the function of the value. A simple example: The value of L_x is $m\hbar$; then the value of L_x^2 equals the square of the value of L_x, i.e., it equals $(m\hbar)^2$ and not $(m\hbar)^{3/2}$ or the like.

Requiring (27.1) has the consequence that if $[A, B] = 0$, it holds that:

$$\begin{aligned} V_{|\psi\rangle}(A + B) &= V_{|\psi\rangle}(A) + V_{|\psi\rangle}(B) : \text{sum rule} \\ V_{|\psi\rangle}(A \cdot B) &= V_{|\psi\rangle}(A) \cdot V_{|\psi\rangle}(B) : \text{product rule,} \end{aligned} \tag{27.2}$$

and that in addition, $V_{|\psi\rangle}(1) = 1$. For the proofs see the exercises. Note that $[A, B] = 0$ is a precondition for (27.2). With noncommuting observables, there is generally no consistent way to assign values. As an example, we consider $A = \sigma_x$, $B = \sigma_y$; the assigned value is an eigenvalue of each operator. The eigenvalues

[5] In this context, Gleasons's theorem is of interest (see Appendix T, Vol. 2). It deals in fact with the question of how to introduce probabilities into quantum mechanics, but also refers to a contradiction in the assignment of properties of a quantum system. This contradiction is addressed by the Kochen–Specker theorem.

of σ_x and σ_y are ± 1, but those of $\sigma_x + \sigma_y$ are $\pm\sqrt{2}$, so that the requirement $V_{|\psi\rangle}(A + B) = V_{|\psi\rangle}(A) + V_{|\psi\rangle}(B)$ cannot be fulfilled.

With the value function, the above assumptions can now be rewritten as follows:

1. VD: Each set of physical properties which is represented by corresponding operators $A, B, C, \ldots$ in $\mathcal{H}$, has well-defined values $V_{|\psi\rangle}(A), V_{|\psi\rangle}(B), V_{|\psi\rangle}(C), \ldots$
2. NC: For commuting operators A, B, the rules $V_{|\psi\rangle}(A + B) = V_{|\psi\rangle}(A) + V_{|\psi\rangle}(B)$ and $V_{|\psi\rangle}(AB) = V_{|\psi\rangle}(A)\, V_{|\psi\rangle}(B)$ apply.

As we have already discussed in Chap. 13, Vol. 1, the question of whether a quantum system has a property or not may be conveniently formulated by means of projection operators. We briefly review: The basis of the N-dimensional Hilbert space is the CONS $\{|a_n\rangle, n = 1, 2, \ldots\}$. Because of the completeness of this basis, the projection operators $P_{|a_n\rangle} = |a_n\rangle\langle a_n|$ satisfy

$$\sum\nolimits_n P_{|a_n\rangle} = 1. \tag{27.3}$$

The projection operators act on the basis states according to

$$P_{|a_n\rangle}|a_m\rangle = |a_n\rangle\langle a_n|\,a_m\rangle = \delta_{nm}\cdot|a_m\rangle\,. \tag{27.4}$$

Hence, $|a_n\rangle$ is an eigenvector of $P_{|a_n\rangle}$ with eigenvalue 1; all other vectors $|a_m\rangle$ with $n \neq m$ are eigenvectors of $P_{|a_n\rangle}$ with eigenvalues 0.

The CONS $\{|a_n\rangle\}$ can be seen as the eigenvectors of an operator $A = \sum\nolimits_n a_n |a_n\rangle\langle a_n| = \sum\nolimits_n a_n P_{|a_n\rangle}$ acting in $\mathcal{H}$ (spectral representation), where the eigenvalue equation is $A|a_n\rangle = a_n|a_n\rangle$. Thus, we can understand the spectral operators (projection operators) as a representation of yes-no observables, i.e. as a response to the question as to whether a quantum-mechanical system has a property a_n (1, yes) or not (0, no) with respect to the physical quantity A.

27.1.2 From the Value Function to Coloring

We now take advantage of this connection between projection operators P and properties. With the product rule and because of $P^2 = P$, it generally follows that $V_{|\psi\rangle}(P^2) = V^2_{|\psi\rangle}(P) = V_{|\psi\rangle}(P)$, and thus

$$V_{|\psi\rangle}(P) = 0 \text{ or } 1. \tag{27.5}$$

If we think of P as a statement, then the value function gives an assignment of 'true' (equal to 1) or 'false' (equal to 0) in the state $|\psi\rangle$.

For the spectral operators $P_{|a_n\rangle}$, we have[6]:

[6]We note that these projectors commute.

$$P_{|a_i\rangle} P_{|a_j\rangle} = \delta_{ij} P_{|a_i\rangle}; \sum\nolimits_n P_{|a_n\rangle} = 1. \tag{27.6}$$

Because of the sum rule, exactly one of the values of the set $\left\{V_{|\psi\rangle}\left(P_{|a_i\rangle}\right)\right\}$ is 1 (this statement is true), let us say $V_{|\psi\rangle}\left(P_{|a_m\rangle}\right)$, and all others are 0 (these statements are false). We therefore find for the CONS $\{|a_n\rangle\}$ the results:

$$\sum\nolimits_n V_{|\psi\rangle}\left(P_{|a_n\rangle}\right) = 1 \ \text{ with } \ V_{|\psi\rangle}\left(P_{|a_n\rangle}\right) = 1 \quad \text{or} \quad 0. \tag{27.7}$$

This means in other words that the operator A has a well-defined value, namely the eigenvalue a_m, for which $V_{|\psi\rangle}\left(P_{|a_m\rangle}\right) = 1$. At this point, we do not know m or which eigenvalue this is; it is sufficient that there is exactly one.

One can render these formulations a bit more intuitive. Namely, we color in such a way that the basis vector $|a_m\rangle$ with $V_{|\psi\rangle}\left(P_{|a_m\rangle}\right) = 1$ becomes black and all others $|a_n\rangle$ with $V_{|\psi\rangle}\left(P_{|a_n\rangle}\right) = 0$, $n \neq m$ become white. Then the two assumptions can be written as

1. NC: Given a CONS, the vector with $V_{|\psi\rangle} = 1$ is colored black and the other vectors white ($V_{|\psi\rangle} = 0$).[7]
2. VD: This coloring process must be performed for *all* basis systems (CONS) of the Hilbert space.

If condition 2 were not fulfilled, then not all the properties of the quantum system would be well defined.

We note that the physical quantity A has a value independent of the measurement. Thus, if a vector is colored black in a certain basis and also appears in a different basis, it is black there, too (non-contextuality).

27.1.3 Coloring

The KST now states that the last two assumptions cannot be met in Hilbert spaces (i.e. in quantum mechanics) of dimension $\geq$3. One can divide the proof into several steps. First, one proves that the existence of a value function for a Hilbert space of dimension N implies that there is a value function for all spaces with dimension less than N. Next, one proves that the existence of a value function for a complex Hilbert space implies that there also a value function for the real Hilbert space of the same dimension. For simplicity, we accept these results. Finally, one must still show that there are no such value functions in the three-dimensional real Euclidean space.

Thus one has boiled down the initial question to a three-dimensional geometrical problem. The question is now whether one can mark *all* basis systems (more precisely, all CONS) in a 3-dimensional real space so that one vector is always black ($V_{|\psi\rangle} = 1$)

[7] Strictly speaking, the statement does not apply to the states $|a_n\rangle$, but to the corresponding rays; the phase factors cancel each other in the expression $|a_n\rangle \langle a_n|$. For reasons of clarity, we accept this imprecision.

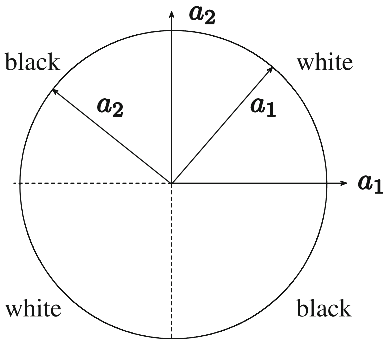

Fig. 27.1 Coloring for dim = 2

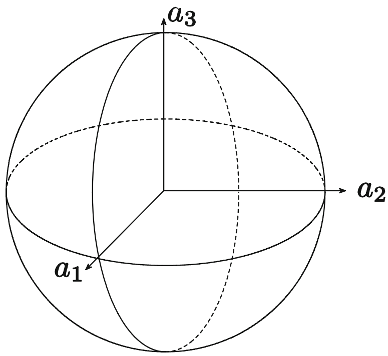

Fig. 27.2 Coloring for dim = 3

and the other two are white ($V_{|\psi\rangle} = 0$). Or perhaps more intuitively: Can a spherical surface be colored by means of orthogonal point triples, where one point is black and the two others white?

In fact, for dim = 2 (a circular surface), the assumptions of NC and VD are easily met, for example by coloring the four quadrants (or the corresponding circular segments) alternately black and white; see Fig. 27.1.

But for dim = 3, things look different Fig. 27.2. While we can color the equatorial plane as in two dimensions, the additional third dimension makes a consistent coloring of the entire surface impossible. To prove this, we need to find only *one* suitable CONS which cannot be colored accordingly. In other words, we need only to falsify the statements (VD + NC) once, and the simplest example will suffice. But this is surprisingly complicated, given the clarity of the question. In 1967, Kochen and Specker needed 117 vectors to demonstrate the theorem named after them.

We demonstrate their approach explicitly using the example of a set of vectors which is at the moment probably the smallest known set. It was published 1997 by A. Cabello and comprises 18 vectors in an albeit *four-dimensional* Hilbert space.[8]

[8] It is the set of these 18 basic vectors (or yes-no-tests) in a four-dimensional space with which recently the first experimental implementation of a Kochen–Specker set was performed; see

Table 27.1 Four orthogonal vectors (i.e., one column) form nine different bases

(0, 0, 0, 1)	(0, 0, 0, 1)	(1, −1, 1, −1)	(1, −1, 1, −1)	(0, 0, 1, 0)	(1, −1, −1, 1)	(1, 1, −1, 1)	(1, 1, −1, 1)	(1, 1, 1, −1)
(0, 0, 1, 0)	(0, 1, 0, 0)	(1, −1, −1, 1)	(1, 1, 1, 1)	(0, 1, 0, 0)	(1, 1, 1, 1)	(1, 1, 1, −1)	(−1, 1, 1, 1)	(−1, 1, 1, 1)
(1, 1, 0, 0)	(1, 0, 1, 0)	(1, 1, 0, 0)	(1, 0, −1, 0)	(1, 0, 0, 1)	(1, 0, 0, −1)	(1, −1, 0, 0)	(1, 0, 1, 0)	(1, 0, 0, 1)
(1, −1, 0 ,0)	(1, 0, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(1, 0, 0, −1)	(0, 1, −1, 0)	(0, 0, 1, 1)	(0, 1, 0, −1)	(0, 1, −1, 0)

They are listed in Table 27.1, where each vector occurs twice. Each of the nine columns of the table represents a CONS.

At this point, the reasoning is very simple: For each CONS of 4 vectors (i.e. for each column in the table), one vector has to be colored black, so it has the value 1. Since in the table each vector occurs exactly twice, and the values assigned to the vectors are either 0 or 1, the sum of these values over the entire table is always an *even* number. On the other hand, the sum of these values in each column must be 1 (only one vector is black, the other three are white), so that the sum over the entire table must be 9, which is *not* even.

We have thus shown that quantum mechanics (which operates in Hilbert spaces) is not compatible with the requirements of value definiteness and non-contextuality. So there is no realistic non-contextual hidden-variables theory.

27.1.4 Interim Review: The Kochen–Specker Theorem

The KST shows that there is a contradiction between quantum mechanics and the pair value definiteness/non-contextuality (which is in essence due to the fact that quantum mechanics operates in a Hilbert space). Logically, we must abandon one of the two assumptions, or both. But at present it is not clear where the correct path may lead.

As in Bell's inequality, the KST is independent of the physics of the quantum systems (because it ultimately is a statement about Hilbert spaces). Its role in the discussion of hidden variables is based on the following points:

1. The KST has nothing to do with the uncertainty principle, etc.; it is based on the vector-space structure of the state space.
2. The KST requires only a *finite* set of *discrete* commuting observables, and thus avoids the problems that arise when considering a continuum of quantum-mechanical statements.
3. In contrast to Bell's inequality, the KST has nothing to do with statistical correlations of an ensemble. It compares the results of various measurements that can be performed on a *single* system. Assumptions about locality and separability are

Vincenzo D'Ambrosio et al., 'Experimental Implementation of a Kochen–Specker Set of quantum Tests', *Phys. Rev.* X 3, 011012 (2013).

not needed. In this way, it is rather similar to the GHZ considerations discussed below.

Finally, we add a few remarks:

1. The theorem shows that there is no consistent value assignment for a sufficiently large but finite set of observables. The possibility of such an assignment disappears somewhere between the consideration of *one* and of *all* the observables. (We emphasize again that one cannot attribute the values 0 or 1 to all properties). How to construct the smallest non-colorable configuration is an open problem (i.e. finding the largest set of observables for which one can still make a value assignment).
2. Numbers: While in the original work of Kochen and Specker, 117 rays were used for $\mathbb{R}^3$, later on configurations of 33 and 31 rays were found. For $\mathbb{R}^4$, there exist configurations between 33 rays and the 18 given explicitly above. More numbers: in $\mathbb{R}^5$, 29; in $\mathbb{R}^6$, 31; in $\mathbb{R}^7$, 34; and in $\mathbb{R}^8$, 36 rays are the current minimal numbers.
3. The KST does not generally exclude hidden variables, but only those that are not contextual.
4. Experimentally, the KST has been confirmed impeccably.[9] Thus it was shown that the measurement of a property of a quantum system (two laser-cooled calcium ions in an electromagnetic trap) depends on other measurements on the system. By the way, techniques were used in this experiment that were originally developed for building a quantum computer.

For a recent review of the topic see e.g. D. Rajan, M. Visser, Kochen–Specker theorem revisited, arXiv:1708.01380v1 [quant-ph] (4.8.2017).

27.2 GHZ States

We have seen in Chap. 20 on the basis of Bell's inequality that quantum mechanics is incompatible with a local-realistic hidden variable theory. Now, one might see a certain disadvantage of Bell's argument in the fact that it is based on a statistical treatment, i.e. that it requires validation by means of an ensemble. But there is a possibility to test the compatibility of local-realistic theories with quantum mechanics which is independent of Bell's inequality and does without this statistical argument. It was proposed by Greenberg, Horne and Zeilinger[10] (GHZ) in 1989 and it involves entangled states of *three* quantum objects (GHZ states).[11]

[9] G. Kirchmair et al., 'State-independent experimental test of quantum contextuality', *Nature* 460, 494–497 (2009).

[10] In fact, in this paper *four* spin-1/2 systems are used. The simplification to the three quantum objects considered here was introduced some time later by Mermin.

[11] An attempt at a treatment of this topic suitable for schools was given for example by : 'EPR Paradoxon in school—Absolute and relative, and Bertelsmann's socks', by K. Jaeckel and J. Pade, in: H. Fischler (Ed.), *Quantum physics in school*, IPN 133, (1992) (text in German).

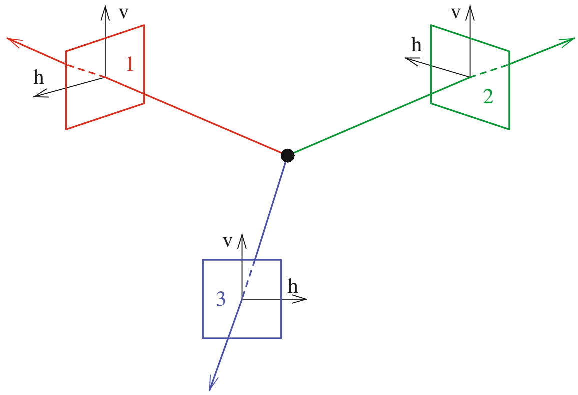

Fig. 27.3 GHZ display for three photons

Using GHZ states, it can be shown in a way independent of Bell's inequality that quantum mechanics and local realism are not compatible. The GHZ argument has the advantage that it does not require the measurement of a whole ensemble to determine probabilities, because it does not involve statistical correlations, but rather a perfect anti-correlation. Four measurements suffice, while for the verification of Bell's inequality one needs a large number of measurements to obtain reasonable statistics.

We discuss the situation for the case that a total system decays from a certain initial state into three photons. The three photons move in a plane towards three observers, separated in each case by an angular distance of 120°; see Fig. 27.3. They can measure the following states of polarization: The linear polarizations $|h\rangle$ / $|v\rangle$, the states $|h'\rangle$ / $|v'\rangle$ rotated by 45°, and the circular polarizations $|r\rangle$ / $|l\rangle$. Thus, we have:

$$\left|h'\right\rangle = \frac{|h\rangle + |v\rangle}{\sqrt{2}} \;\; ; \; \left|v'\right\rangle = \frac{-|h\rangle + |v\rangle}{\sqrt{2}} \; ; \;\; |r\rangle = \frac{|h\rangle + i\,|v\rangle}{\sqrt{2}} \; ; \; |l\rangle = \frac{|h\rangle - i\,|v\rangle}{\sqrt{2}}. \tag{27.8}$$

The measurements are carried out simultaneously at the three stations, so that a 'communication' among the photons could occur only superluminally.

The system is prepared in such a way that it is in a special entangled overall state (the GHZ state) prior to measurement[12]:

$$|\psi\rangle = \frac{|h, h, h\rangle + |v, v, v\rangle}{\sqrt{2}}. \tag{27.9}$$

Obviously, this state is invariant under any permutation of the three observers.

We will use the *polarization operators* already introduced in Chap. 4, Vol. 1:

[12]Since we have only three observers, we use the shorthand notation $|\psi\rangle = \frac{|h,h,h\rangle+|v,v,v\rangle}{\sqrt{2}} \equiv \frac{|1{:}h,2{:}h,3{:}h\rangle+|1{:}v,2{:}v,3{:}v\rangle}{\sqrt{2}}$.

$$P_L = |h\rangle\langle h| - |v\rangle\langle v| \ ; P_{L'} = |h'\rangle\langle h'| - |v'\rangle\langle v'| \ ; P_C = |r\rangle\langle r| - |l\rangle\langle l| . \quad (27.10)$$

(The indices L and C signify, of course, longitudinal and circular). In the usual representation $|h\rangle \cong \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|v\rangle \cong \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, one immediately sees the connection of these polarization operators to the Pauli matrices:

$$P_L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z; P_{L'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x; P_C = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y. \quad (27.11)$$

Accordingly, the $LL'C$ measurement is also called a zxy measurement.

The three observers measure the polarizations, namely two observers measure the circular $(|r\rangle / |l\rangle)$ and one observer the rotated linear polarization $\left(|h'\rangle / |v'\rangle\right)$. We consider as an example the measurement of $(|r\rangle / |l\rangle)_1 (|r\rangle / |l\rangle)_2 \left(|h'\rangle / |v'\rangle\right)_3$, or for short the CCL' measurement. With the help of the inverse transforms of (27.8) (see exercises), we can write the state (27.9) in this case (see exercises) as:

$$\begin{aligned}
|\psi\rangle_{CCL'} &= \frac{|r, l, h'\rangle + |l, r, h'\rangle - |r, r, v'\rangle - |l, l, v'\rangle}{2} \\
|\psi\rangle_{L'CC} &= \frac{|h', r, l\rangle + |h'l, r\rangle - |v'r, r\rangle - |v'l, l\rangle}{2} \\
|\psi\rangle_{CL'C} &= \frac{|l, h', r\rangle + |r, h', l\rangle - |r, v', r\rangle - |l, v', l\rangle}{2},
\end{aligned} \quad (27.12)$$

where by the last two states follow by cyclic permutation.

In all cases, we can predict with certainty the outcome for the third photon if the results for two of the photons are known. If e.g. in the CCL' measurement (27.12) photon 1 and 2 are right-handed circularly polarized (i.e. they are in the state $|r\rangle$), then photon 3 is in the state $|v'\rangle$ with certainty, without the need of a further measurement. Consequently, the local realism sees elements of reality in the individual measurement results.

We assign values to these elements which we call $L'_i = \pm 1$ for h'/v' polarizations and $C_i = \pm 1$ for r/l polarizations, $i = 1, 2, 3$. (These are the eigenvalues of the polarization operators in (27.11)). We choose $+1$ for h' and r and -1 for v' and l. Hence, we assign the value $(+1)(-1)(+1) = -1$ to the state $|r, l, h'\rangle$. In this way we obtain for (27.12) and its cyclic permutations the relations

$$C_1 C_2 L'_3 = -1; C_1 L'_2 C_3 = -1; L'_1 C_2 C_3 = -1. \quad (27.13)$$

As a fourth measurement, we consider the case that all three observers measure the linear rotated polarization (for short, an $L'L'L'$ measurement). Here, the conversion of (27.9) leads to the state (see exercises):

$$|\psi\rangle_{L'L'L'} = \frac{|h', h', h'\rangle + |h', v', v'\rangle + |v', h', v'\rangle + |v', v', h'\rangle}{2}. \quad (27.14)$$

The argument against local realism now runs as follows: Because of the locality, each measurement result h'/v' of a photon is independent of the measurements for the other two photons; this applies correspondingly to the values L'_i and C_i. Because of $C_i^2 = (\pm 1)^2 = +1$, we can with (27.13) write $L'_1 L'_2 L'_3 = (L'_1 C_2 C_3)(C_1 L'_2 C_3)(C_1 C_2 L'_3)$ and thus obtain

$$L'_1 L'_2 L'_3 = -1. \quad (27.15)$$

This being the case, for the local realism only the following h'/v' measurements are possible: $v'v'v'$, $h'h'v'$, $h'v'h'$, $v'h'h'$. In other words, an *odd* number of photons is in the state v'.

But as we see from (27.14), according to quantum mechanics the possible results are of the form $h'h'h'$, $h'v'v'$, $v'h'v'$, $v'v'h'$; therefore, an *even* number of photons is in the state v', and we have

$$L'_1 L'_2 L'_3 = 1. \quad (27.16)$$

What is the reason for this contradiction between (27.15) and (27.16)? The essential point is this: The assumption that e.g. the two terms Z_1 occurring in (27.13) are identical, is wrong. In fact, these values are not fixed from the outset, but are contextual, i.e. they depend on which other variables are measured simultaneously.[13] We can therefore not assume $C_i^2 = (\pm 1)^2 = +1$ and, consequently, cannot derive the (27.15).

As always, measurements have the last word. The corresponding experiment was performed for the first time in the year 2000.[14] It clearly demonstrated that the quantum-mechanical result is correct. Local-realistic hidden variables have no place in quantum mechanics.

Interim review: GHZ

With the help of GHZ states, it can be shown in a way independent of Bell's inequality that quantum mechanics and local realism are not compatible. The GHZ argument has the advantage that it does not require the measurement of an ensemble to determine probabilities, because it does not make use of statistical correlations, but instead of a perfect anti-correlation. Four measurements suffice, while it requires a large number of measurements for reasonable statistics in order to prove Bell's inequality.

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[13] As in the case of Bell's inequality, the problem is that one cannot measure the six quantities $L'_1, L'_2, L'_3, C_1, C_2, C_3$ simultaneously; these are the eigenvalues of operators that do not all commute with each other. The measurement of the six variables is counterfactual: in one experiment, one cannot measure more than three of them.

[14] Jian-Wei Pan et al., 'Experimental test of quantum nonlocality in three-photon Greenberger–Horne–Zeilinger entanglement', *Nature*, Vol. 403, pp. 515–519 (2000).

27.3 Discussion and Outlook

As we have seen, a realistic representation of quantum mechanics is compatible neither with non-contextuality nor with locality, two properties which in the context of a realistic approach are generally taken for granted. This leaves us with the possibilities of a realistic contextual nonlocal description, or of simply non-realistic theories.

In recent years, the debate has concentrated on the pair realism/locality. Should we abandon locality, or instead the notion of physical reality—or both concepts? This question cannot be answered solely by means of logic.

In 2003, Anthony Leggett[15] explored one of the options, considering a certain class of physically plausible theories which are nonlocal but realistic (called *crypto-nonlocal* by Leggett). He noted that these theories are incompatible with quantum mechanics, and expressed this in terms of new inequalities. The inequalities were investigated experimentally in 2006 and 2007 and confirmed,[16] which would imply that one should rather question realism instead of locality.

However, these considerations are not without controversy. One objection is, for example, that the violation of Leggett's inequality just means that realism and a certain type of nonlocality are incompatible, while there are other types of nonlocality that are not addressed by Leggett's inequality.[17] And Leggett himself acknowledged[18] in 2008 that certain local elements have some influence in his problem: "A critic may argue that . . . we have in effect smuggled the concept of locality back in again." Perhaps, he continues, the message is that although the concept of local realism is clearly defined, it might not be a particularly useful exercise to analyze this concept separately in terms of its two main components.

In any case, it is currently not clear in which direction to travel. In addition, still other conditions, currently generally taken for granted, could be violated.[19] We have for example always tacitly assumed causality, i.e. the fact that an event cannot be influenced by other events that lie in the future (the arrow of time). This assumption is made explicitly in the *objective local theories* (OLT), based on the three postulates of locality, realism, and induction (causality). It can be shown quite generally that these

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[15] A.J. Leggett, 'Nonlocal hidden-variable theories and quantum mechanics: an incompatibility theorem', *Found. Phys*. 33 (2003) 1469–1493. was awarded the 2003 Nobel Prize for his work in the field of superfluidity.

[16] S. Gröblacher et al., 'An experimental test of non-local realism', *Nature* 446 (2007), pp. 871–875.

[17] M. Socolovsky, 'Quantum mechanics and Leggett's inequalities', *Int. J. Theor. Phys*. 48 (2009), pp. 3303–3311.

[18] A.J. Leggett, 'Realism and the physical world', *Rep. Prog. Phys*. 71 (2008), 022001.

[19] In principle, one cannot exclude for example that the rules of ordinary logic do not apply and/or need to be expanded in the realm of quantum mechanics—rules which are tacitly applied in the derivation of Bell's inequality and the other arguments. Here, Gödel's theorem comes into play, according to which, roughly speaking, any theory that is proposed as the basis for mathematics, including logic, is necessarily inadequate, incomplete or contradictory. It is not clear at this point, however, what should be changed in conventional logic in order that Bell's inequalities not be violated by quantum mechanics.

theories are incompatible with quantum mechanics.[20] In principle, we can also (or perhaps we must?) sacrifice the arrow of time. Thus, a measurement of an entangled quantum object at a given time would determine the properties of the other one at the moment of its emission from the same source, hence in the past.[21]

The idea of giving up the familiar notion of 'cause and effect' is more than unusual, of course.[22] It would certainly be very difficult to continue doing physics the way we are accustomed without it. However, there are very reputable and respected physicists who not only think that this sacrifice is possible, but expect that the next major revolution in physics will force us to do just that.[23]

Be that as it may—we can state that we must take leave of one or several plausible ideas in order to maintain the classical notion of realism.[24] For this reason, the expectation is often expressed that any future extension of quantum mechanics, compatible with experiments, must give up certain features of realism.[25]

Of course there are alternatives allowing one to avoid the whole discussion about Realism and Co.: For example, one can argue that quantum mechanics is just a set of calculation rules for the determination of measured values and has no intrinsic meaning beyond that. We will meet up with more viewpoints in Chap. 28, but we can already state here that in all cases, the results of the discussion about hidden variables have no direct influence on the practical usefulness of quantum mechanics. In view of this, one may ask of course why this topic should be of general interest. The answer is perhaps more a matter of personal preference; but it must be emphasized that this

[20] This is done by comparing experimental results with an extension of Bell's inequality, the *CSCH inequality*, proposed in 1969 by Clauser, Horne, Shimony and Holt.

[21] In connection with delayed-choice experiments (see Appendix M, Vol. 1) also, the idea of a time-reversed effect is discussed. In fact, the fundamental laws of physics are time-reversal invariant, i.e. time-symmetrically causal, and do not reflect the time-asymmetrical idea of cause and effect.

[22] Though the idea that events obey a definite causal order is deeply rooted in our understanding of the world, causal order needs not be a required property of nature. For instance, it was recently shown that in quantum mechanics, there are correlations that cannot be understood in terms of definite causal order; see Ognyan Oreshkov et al., 'Quantum correlations with no causal order', Nature Comm. 3, 1092 (2012), doi: 10.1038/ncomms2076.

[23] "I believe that in our present picture of physical reality, especially regarding the nature of time, a huge upheaval is imminent, it may be even greater than the revolution that has already been triggered by relativity theory and quantum mechanics." Roger Penrose, British mathematician and physicist, in *New Mind. The emperor's new clothes or the debate over artificial intelligence, consciousness and the laws of nature*.

[24] In Appendix U, Vol. 2, some quotes from philosophers, artists, etc. are compiled; they show illustratively that *the* classical notion of 'reality' does not exist and has never existed in the past.

[25] A certain skepticism about the concept of 'reality' is in the tradition of modern science. More than 200 years ago, Georg Christoph Lichtenberg stated in one of his physics lectures: "We care little about whether the bodies have an objective reality apart from us or not. It would always be possible that at least some would not have one. We have to imagine the things; the idea does not depend on us, but of those things that make an impression on us, the impression cannot act on us as in another way than our abilities admit. At least is that what we feel of the bodies apart from us, not always objectively real." Gottlieb Gamauf, in *Physics lectures, from the memoirs of Gottlieb Gamauf*.

debate has allowed us to look deeper into the mysteries of quantum mechanics.[26] Whether the hope will be fulfilled that this may contribute towards leading quantum information[27] from its current state of basic research to a full-blown technical revolution remains to be seen.

In any case, the debate has forced us to question our notions of 'self-evident' and to obtain in this way new insights into the world. And that is one of the main tasks of science.

27.4 Exercises

1. A system is in the polarization state $|r\rangle$. Using $w_P = tr(\rho P)$, calculate the probability of measuring the system in the state $|h\rangle$.
2. A mixture is described by $\rho = \sum p_n |\varphi_n\rangle \langle\varphi_n|$, where $\{|\varphi_n\rangle\}$ is a CONS. Using $w_P = tr(\rho P)$, calculate the probability of measuring the system in the state $|\varphi_N\rangle$.
3. The value function $V_{|\psi\rangle}$ is defined by $V_{|\psi\rangle}(F(A)) = F\left(V_{|\psi\rangle}(A)\right)$.
   (a) Prove for $[A, B] = 0$ the sum rule $V_{|\psi\rangle}(A + B) = V_{|\psi\rangle}(A) + V_{|\psi\rangle}(B)$.
   (b) Prove for $[A, B] = 0$ the product rule $V_{|\psi\rangle}(A \cdot B) = V_{|\psi\rangle}(A) \cdot V_{|\psi\rangle}(B)$.
   (c) Show that $V_{|\psi\rangle}(1) = 1$.
4. Given the polarization operators P_L, $P_{L'}$ and P_C (or the corresponding Pauli matrices, see (27.11)):
   (a) Determine (once more) their eigenvalues and eigenvectors.
   (b) Express the eigenvectors of P_C and $P_{L'}$ in terms of those of P_L.
5. Given the GHZ state
$$|\psi\rangle_\pm = \frac{|h, h, h\rangle \pm |v, v, v\rangle}{\sqrt{2}} \tag{27.17}$$
   corresponding to an LLL measurement; rewrite this for a CCL' measurement (plus $CL'C$ and $L'CC$) (27.12) and for an $L'L'L'$ measurement (27.14).

[26]The topic is far from complete and the subject of current research. This is shown by conferences (e.g. 'The Nature of Quantum Reality', One-Day Conference, 10th June 2017, St Cross College, University of Oxford), by reviews (e.g. the very readable article by Z. Merali, 'Quantum physics: What is really real?', *Nature* 521, 278–280, (May 2015) doi:10.1038/521278a), and by a number of scientific papers (e.g. G.C. Krizek, 'The conception of reality in Quantum Mechanics', arXiv:1708.02148v1 [quant-ph] (Aug 2017)).

[27]"The development of quantum mechanics early in the twentieth century obliged physicists to change radically the concepts they used to describe the world. The main ingredient of the first quantum revolution, wave-particle duality, has led to inventions such as the transistor and the laser that are at the root of the information society. Thanks to ideas developed by Albert Einstein and John S. Bell, another essential quantum ingredient, entanglement, is now leading us through the conceptual beginnings of a second quantum revolution—this time based on quantum information." Alain Aspect, 'Quantum mechanics: To be or not to be local', *Nature* 446, pp. 866–867 (19th April, 2007).

6. The following combinations of the polarization operators (27.10) are given:

$$Q_1 = P_{1L'}P_{2C}P_{3C}; \quad Q_2 = P_{1C}P_{2L'}P_{3C}$$
$$Q_3 = P_{1C}P_{2C}P_{3L'}; \quad Q = P_{1L'}P_{2L'}P_{3L'}. \tag{27.18}$$

The numerical index denotes the space in which the particular polarization operator acts. We use in the following the fact that operators from different spaces commute, e.g. $P_{1L'}P_{2C} = P_{2C}P_{1L'}$. In addition, we have $P_{nL'}P_{nC} = -P_{nC}P_{nL'}$ as well as $P_{nC}^2 = P_{nL'}^2 = 1$.

(a) Show that the three operators Q_i have the eigenvalues ± 1.
(b) Show that the three operators Q_i commute pairwise.
(c) Show that the states

$$|\psi\rangle_\pm = \frac{|h, h, h\rangle \pm |v, v, v\rangle}{\sqrt{2}} \tag{27.19}$$

are common eigenstates of the three operators Q_i with the eigenvalues ∓ 1, as well as eigenstates of the operator Q with the eigenvalues ± 1.