

# Chapter 8

## Neutrino Oscillations



Thus far, in the algebraic approach we have not considered the question of the time evolution of a system. We now want to tackle this topic on the basis of a problem of current interest. In addition, we meet up with Hermitian operators, and we address once again the problem of measurement.

### 8.1 The Neutrino Problem

As is well known, the *neutrino*  $\nu$  was originally postulated by Wolfgang Pauli in order to ‘save’ the conservation of energy in beta decay. As it turned out later after careful examination. Each of the three elementary particles, the electron  $e$ , the muon  $\mu$  and the tauon  $\tau$  has its ‘own’ neutrino, i.e.  $\nu_e$ ,  $\nu_\mu$ , and  $\nu_\tau$ .<sup>1</sup> The rest mass of all three neutrinos seemed to be vanishingly small, and it was generally assumed to be zero.

Change of scene: We consider now the sun and the particles which it emits. Among them are the three neutrino species, and those in a certain ratio, which can be determined reasonably reliably on the basis of current solar models. But measurements on earth yielded a rather different value for this ratio. The question was: Are the solar models incorrect, or is something wrong with our description of neutrinos?

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<sup>1</sup>Wolfgang Pauli in 1930 initially chose the name ‘neutron.’ The term ‘neutrino’ was introduced later by Enrico Fermi. In 1956, the electron neutrino was detected experimentally for the first time, and in 1962, the muon neutrino. The tauon was observed in 1975, but the corresponding neutrino only in 2000. There may be still other types of neutrinos. These (as yet hypothetical) *sterile neutrinos* interact only via gravity and not—like the other neutrinos—through the weak interaction (hence the adjective ‘sterile’). See e.g. D. Castolvecchi, Icy telescope throws cold water on sterile neutrino theory, *Nature*, <https://doi.org/10.1038/nature.2016.20382> (Aug 2016), and literature referenced there.

There were good arguments to regard the solar models as correct. So something had to be changed in the description of the neutrinos. And it was this: If one assumes that the rest masses of the neutrinos are not exactly zero, the three neutrino species can change into each other over the course of time (*neutrino oscillations*); that is, on the way from the sun to the earth. In this way it could be explained that on earth, we measure a different relative abundance of the three neutrinos than is predicted by the solar models.

## 8.2 Modelling the Neutrino Oscillations<sup>2</sup>

We will now describe the process of neutrino oscillations, as simply as possible. In order to make clear the principle, we confine ourselves to a simpler model with only two neutrinos, since the computations for three neutrinos are more complicated. A few words about the three-dimensional case can be found at the end of this chapter.

In this chapter, we will for once visit the field of relativistic phenomena. We can do so because we need only the statement that there is a Hamiltonian (and in particular its energy eigenvalues) for the physical problem, without having to worry about its specific form or any details of the interaction.

### 8.2.1 States

We start with the production of neutrinos (e.g. in the sun or in an accelerator) as a superposition of two states  $|\nu_1\rangle$  and  $|\nu_2\rangle$  with *well-defined, different rest masses*  $m_{01}$  and  $m_{02}$ , called mass (eigen-)states. The momenta are equal, but the total energies  $E_1$  and  $E_2$  are therefore different.<sup>3</sup> Without loss of generality, we can set  $\Delta m := m_{01} - m_{02} > 0$  and hence  $\Delta E := E_1 - E_2 > 0$  or  $\Delta\omega = \omega_1 - \omega_2$  (with  $\omega = E/\hbar$ ). The states form a CONS;  $\langle\nu_i|\nu_j\rangle = \delta_{ij}$  and  $|\nu_1\rangle\langle\nu_1| + |\nu_2\rangle\langle\nu_2| = 1$ .<sup>4</sup>

For certain reasons, one cannot measure the states  $|\nu_1\rangle$  and  $|\nu_2\rangle$  directly, but only superpositions of these states, which we call (referring to the actual situation) the electron neutrino and the muon neutrino,  $|\nu_e\rangle$  and  $|\nu_\mu\rangle$  (also termed *flavor states*). We have

$$\begin{aligned} |\nu_e\rangle &= \cos\vartheta |\nu_1\rangle + \sin\vartheta |\nu_2\rangle \\ |\nu_\mu\rangle &= -\sin\vartheta |\nu_1\rangle + \cos\vartheta |\nu_2\rangle. \end{aligned} \tag{8.1}$$

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<sup>2</sup>The importance of the issue can be seen e.g. from the fact that the Nobel Prize in Physics 2015 was awarded jointly to Takaaki Kajita (born 1959, Japanese physicist) and Arthur B. McDonald (born 1943, Canadian physicist) “for the discovery of neutrino oscillations, which shows that neutrinos have mass”.

<sup>3</sup>We remind the reader:  $E^2 = m_0^2c^4 + p^2c^2$ .

<sup>4</sup>Call the Hamiltonian for free neutrino motion  $H$ . We have  $H|\nu_1\rangle = E_1|\nu_1\rangle$  and  $H|\nu_2\rangle = E_2|\nu_2\rangle$  with  $\Delta E = E_1 - E_2 > 0$ .

Here,  $\vartheta$  is an (abstract) angle, called the mixing angle. The states  $|\nu_e\rangle$  and  $|\nu_\mu\rangle$  also form a CONS. Therefore, we can represent  $|\nu_1\rangle$  and  $|\nu_2\rangle$  as a superposition of  $|\nu_e\rangle$  and  $|\nu_\mu\rangle$ . We then have

$$\begin{aligned} |\nu_1\rangle &= \cos \vartheta |\nu_e\rangle - \sin \vartheta |\nu_\mu\rangle \\ |\nu_2\rangle &= \sin \vartheta |\nu_e\rangle + \cos \vartheta |\nu_\mu\rangle. \end{aligned} \quad (8.2)$$

In fact, these transformations are nothing more than rotations by the angle  $\pm\vartheta$  within a two-dimensional space, or equivalently, a change of basis, which is described by the well-known transformation

$$\begin{pmatrix} \cos \vartheta & \pm \vartheta \sin \vartheta \\ \mp \sin \vartheta & \cos \vartheta \end{pmatrix}. \quad (8.3)$$

It represents a particularly simple example of a unitary matrix.

### 8.2.2 Time Evolution

Next, we want to investigate the time evolution of the states  $|\nu_e\rangle$  and  $|\nu_\mu\rangle$ . To this end we use the fact, found from the analytical approach, that the time evolution of a state with well-defined energy  $E$  is described by the factor  $e^{-iEt/\hbar}$ . Although this requirement suggests itself, it is not self-evident that it must be satisfied here. If we accept that it holds true (or regard it as an axiom for the moment), we find: If at time zero an initial state  $|z(t=0)\rangle = |z(0)\rangle$  exists with the *well-defined energy*  $E = \hbar\omega$ , its time evolution is described by

$$|z(t)\rangle = |z(0)\rangle e^{-iEt/\hbar}. \quad (8.4)$$

This is a very important and universally valid fact in quantum mechanics. It follows that

$$i\hbar \frac{d}{dt} |z(t)\rangle = E |z(t)\rangle. \quad (8.5)$$

If we assume that  $E$  is an eigenvalue of an operator  $H$ , we have essentially ‘recovered’ the free SEq.<sup>5</sup>

We see that the time evolution (8.4) is a unitary process that conserves the norm:

$$\langle z(t) | z(t) \rangle = \langle z(0) | e^{i\omega t} e^{-i\omega t} | z(0) \rangle = \langle z(0) | z(0) \rangle. \quad (8.6)$$

We now take a muon-neutrino as the initial state  $|\nu(0)\rangle$ , i.e.  $|\nu(0)\rangle = |\nu_\mu\rangle$ . Then it follows with (8.1) for the time evolution:

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<sup>5</sup>Here,  $H$  denotes a (still) unknown operator and *not* the well-known operator  $-\frac{\hbar^2}{2m}\nabla^2 + V$ . Double meanings of this type are quite common in quantum mechanics. We will learn the reason for this in later chapters.

$$|\nu(t)\rangle = -\sin\vartheta|\nu_1\rangle e^{-i\omega_1 t} + \cos\vartheta|\nu_2\rangle e^{-i\omega_2 t}. \quad (8.7)$$

Evidently, we have  $|\langle\nu_1|\nu\rangle|^2 = |-\sin\vartheta e^{-i\omega_1 t}|^2 = \sin^2\vartheta$ —this would be the probability of obtaining  $|\nu_1\rangle$  in a measurement. But since we can measure only the states  $|\nu_e\rangle$  and  $|\nu_\mu\rangle$ , we have to project the corresponding portions out of  $|\nu(t)\rangle$ , by means of the projection operators  $|\nu_e\rangle\langle\nu_e|$  and  $|\nu_\mu\rangle\langle\nu_\mu|$ . With (8.2) we find, for example,  $\langle\nu_e|\nu_1\rangle = \cos\vartheta$  and  $\langle\nu_e|\nu_2\rangle = \sin\vartheta$ . It follows for the electron neutrino that

$$|\nu_e\rangle\langle\nu_e|\nu(T)\rangle = [-\sin\vartheta\cos\vartheta e^{-i\omega_1 T} + \cos\vartheta\sin\vartheta e^{-i\omega_2 T}]|\nu_e\rangle. \quad (8.8)$$

It is seen that this term includes both frequencies  $\omega_1$  and  $\omega_2$  and thus displays a very different behavior from the mass states. We obtain the probability of measuring  $|\nu_e\rangle$  by the usual application of the absolute square of the prefactor (see the exercises):

$$p_e(T) = |-\sin\vartheta\cos\vartheta e^{-i\omega_1 T} + \cos\vartheta\sin\vartheta e^{-i\omega_2 T}|^2 = \sin^2 2\vartheta \cdot \sin^2\left(\frac{\Delta\omega}{2}T\right). \quad (8.9)$$

### 8.2.3 Numerical Data

Equation (8.9) shows that the probability to find the neutrino in the state  $|\nu_e\rangle$  depends periodically on time, where the period is  $\tau = \frac{2\pi}{\Delta\omega}$ . The neutrino oscillates between the states  $|\nu_e\rangle$  and  $|\nu_\mu\rangle$ ; see Fig. 8.1. This is quite similar to two coupled pendulums which show beats, in which the energy flows periodically from one pendulum to the other.

To get a feeling for the order of magnitudes, we perform a rough calculation. We can assume in good approximation that the neutrinos, due to their low mass, are moving with nearly the speed of light. In space, we have a period of length  $L = c\tau = c\frac{2\pi}{\Delta\omega}$ . We approximate the difference  $\Delta\omega$  by (see the exercises)

$$\hbar\Delta\omega = \frac{c^4}{2pc}(m_1^2 - m_2^2) := \frac{c^4\Delta m^2}{2pc}. \quad (8.10)$$

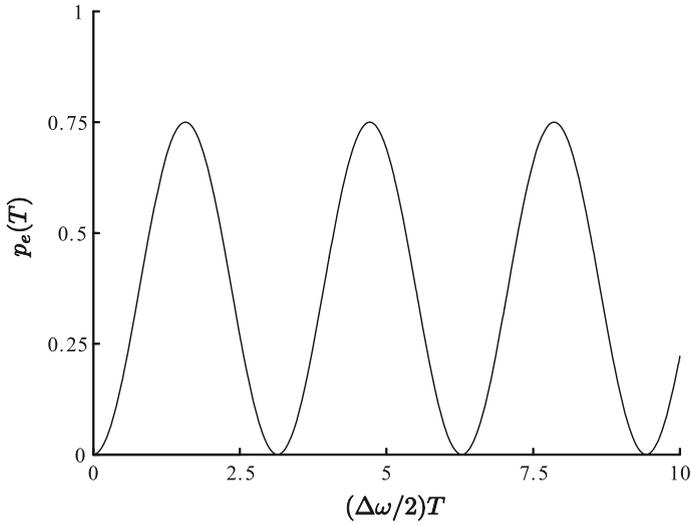
It follows

$$L = c\frac{2\pi}{\Delta\omega} = \frac{4\pi\hbar}{c^2} \frac{p}{\Delta m^2}. \quad (8.11)$$

This term is most easily evaluated in the theoretical unit system in which  $\hbar = c = 1$  and energies and masses are measured in eV, see Appendix B, vol. 1.<sup>6</sup> The

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<sup>6</sup>Numerical examples: the electron in this system of units has a rest mass of about 0.5 MeV. The accelerator LHC operates with protons of energies of up to 7 TeV.



**Fig. 8.1**  $p_e(T)$  of (8.9) for  $\vartheta = \pi/6$

mass difference between neutrinos<sup>7</sup> is about  $\Delta m^2 \approx 10^{-3} \text{ (eV)}^2$ , the momentum is  $10 \text{ GeV} = 10^{10} \text{ eV}$ . We then find

$$L \hat{=} 4\pi \frac{10^{10}}{10^{-3} \text{ eV}} = 4\pi \frac{10^{19}}{\text{MeV}}, \tag{8.12}$$

and with the conversion of units of length  $\frac{1}{\text{MeV}} \hat{=} 0.1973 \times 10^{-12} \text{ m}$  it follows finally

$$L = 4\pi \times 10^{19} \times 0.1973 \times 10^{-12} \text{ m} \approx 25000 \text{ km}. \tag{8.13}$$

Of course we should not take the numerical value too seriously—we have considered only two instead of three neutrinos, and just the uncertainty regarding the mass difference leaves a wide margin for error. What is instead important is that we can at least qualitatively describe an effect such as neutrino oscillations, and this with only the simplest of formal means.

### 8.2.4 Three-Dimensional Neutrino Oscillations

The neutrino question remains an issue of ongoing research, since there are still some unresolved problems.<sup>8</sup> We will not go into this more deeply, but give only a very

<sup>7</sup>Of course, this is a key parameter—if it is  $10^{-6} \text{ eV}$  instead of  $10^{-3} \text{ eV}$ , then the length increases correspondingly by a factor of 1000.

<sup>8</sup>A recent review which also contains the values cited in Table 8.1 is given by G.L. Fogli et al., ‘Global analysis of neutrino masses, mixings and phases: entering the era of leptonic CP violation searches’, <http://arXiv.org/abs/1205.5254v3> (2012).

**Table 8.1** Values of the mixing angles

$s_{12}^2 = 0.307$	or	$\theta_{12} \approx 34^\circ$
$s_{23}^2 = 0.5$	or	$\theta_{23} \approx 45^\circ$
$s_{13}^2 = 0.021$	or	$\theta_{13} \approx 8^\circ$

brief comment on the three-dimensional problem. One assumes three flavor states and three mass states:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \quad (8.14)$$

With the abbreviations  $s_{ij} = \sin \theta_{ij}$  and  $c_{ij} = \cos \theta_{ij}$ , the transformation matrix  $U$  is written as

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \cdot \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \cdot \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (8.15)$$

The first three of these four unitary matrices describe (from left to right) the changes  $\nu_\mu \leftrightarrow \nu_\tau$ ,  $\nu_e \leftrightarrow \nu_\tau$ , and  $\nu_e \leftrightarrow \nu_\mu$ . The phases  $\delta$  (Dirac phase) and  $\alpha_i$  (Majorana phase) are introduced as a result of further considerations.<sup>9</sup> As a product of unitary matrices, the matrix  $U$  is itself again unitary (see the exercises).

For the angles, current values are given in Table 8.1.

For the mass differences, one finds  $\delta m^2 = \Delta m_{21}^2 = m_2^2 - m_1^2 = 7.5 \times 10^{-5} \text{ eV}^2$  and  $\Delta m^2 = \Delta m_{23}^2 = m_3^2 - \frac{m_1^2 + m_2^2}{2} = 2.5 \times 10^{-3} \text{ eV}^2$ . The unit eV is defined in Appendix B, Vol. 1.<sup>10</sup>

For several reasons, new discoveries of the neutrino's properties are expected to change our understanding of the universe. Thus, neutrinos are a topic of ongoing research, see e.g. E. Gibney, Morphing neutrinos provide clue to antimatter mystery, *Nature* <https://doi.org/10.1038/nature.2016.20405> (Aug 2016). A most important

<sup>9</sup>The first three matrices are (except for the phase shift  $\delta$ ) the rotation matrices  $D_x(\theta_{23}) D_y(\theta_{13}) D_z(\theta_{12})$ . The first matrix describes e.g. a rotation by the angle  $\theta_{23}$  around the  $x$  axis.

<sup>10</sup>The values for the mixing angles and the mass differences are from Neutrino Mixing - Particle Data Group, [pdg.lbl.gov/2017/listings/rpp2017-list-neutrino-mixing.pdf](http://pdg.lbl.gov/2017/listings/rpp2017-list-neutrino-mixing.pdf) (30. 5. 2017). The precise determination of these angles is a current topic; see for instance Eugenie S. Reich, 'Neutrino oscillations measured with record precision', *Nature* 08 March 2012, where the measurement of the angle  $\theta_{13}$  is discussed, or P. Adamson et al. (NOvA Collaboration), Measurement of the Neutrino Mixing Angle  $\theta_{23}$  in NOvA, *Phys. Rev. Lett* 118, 151802 (10. 4. 2017).

open issue is the absolute mass scale of neutrinos. This question will be investigated e.g. by the KATRIN experiment, launched in June 2018 (KARlsruhe TRItium Neutrino, Karlsruhe, Germany).

## 8.3 Generalizations

### 8.3.1 Hermitian Operators

In this section, we want to generalize the findings obtained on the basis of the neutrino problem. First, we extend the formulation (8.5) to a ‘proper’ SEq

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle. \quad (8.16)$$

Apart from the mere analogy to the SEq in the analytical approach, the motivation for this step is that we want to find a linear differential equation of first order in time also for the algebraic approach. It is clear that we have at this point no information about the operator  $H$  which appears in (8.16)—neither how it is constructed internally (spatial derivatives as in the Laplace operator cannot occur here), nor about its relation to the Hamiltonian used in the analytical approach. These points will be discussed in later chapters.

Here we want to clarify which properties  $H$  must have in order that the evolution of  $|\psi(t)\rangle$  be unitary, which means that the scalar product  $\langle\psi(t)|\psi(t)\rangle$  must be constant for all times. With this in mind, we write (8.16) and the adjoint equation in compact form

$$i\hbar |\dot{\psi}(t)\rangle = H |\psi(t)\rangle; \quad -i\hbar \langle\dot{\psi}(t)| = \langle\psi(t)| H^\dagger. \quad (8.17)$$

If  $\langle\psi(t)|\psi(t)\rangle$  does not depend on time, it follows that

$$i\hbar \frac{d}{dt} \langle\psi(t)|\psi(t)\rangle = i\hbar \langle\dot{\psi}(t)|\psi(t)\rangle + i\hbar \langle\psi(t)|\dot{\psi}(t)\rangle = 0. \quad (8.18)$$

We insert (8.17) and obtain

$$-\langle\psi(t)| H^\dagger |\psi(t)\rangle + \langle\psi(t)| H |\psi(t)\rangle = \langle\psi(t)| H - H^\dagger |\psi(t)\rangle = 0. \quad (8.19)$$

Since this equation holds for every  $|\psi(t)\rangle$ , it follows that  $H^\dagger = H$ .

In general, an operator  $A$  is called *self-adjoint* or *Hermitian* if  $A = A^\dagger$ . The importance of such operators in quantum mechanics lies in the fact that all physically-measurable quantities are represented by self-adjoint operators. Indeed, Hermitian

operators have real eigenvalues<sup>11</sup> as we want to show now. Let  $A$  be a Hermitian operator,  $A = A^\dagger$ . Then the eigenvalue problem and its adjoint version read:

$$A |a_n\rangle = \lambda_n |a_n\rangle \quad \text{and} \quad \langle a_n| A^\dagger = \lambda_n^* \langle a_n|. \quad (8.20)$$

Multiplication of the first equation from the left by  $\langle a_n|$  and of the second equation from the right by  $|a_n\rangle$  leads, due to  $A = A^\dagger$ , to

$$\langle a_n| A |a_n\rangle = \lambda_n \langle a_n| a_n\rangle \quad \text{and} \quad \langle a_n| A^\dagger |a_n\rangle = \langle a_n| A |a_n\rangle = \lambda_n^* \langle a_n| a_n\rangle. \quad (8.21)$$

The comparison shows  $\lambda_n = \lambda_n^*$ , i.e.  $\lambda_n \in \mathbb{R}$ . Other properties of Hermitian operators are discussed in the following chapters.

In Chap. 4, we made the acquaintance of projection operators, and in Chap. 6 of unitary operators, and now Hermitian operators join in.<sup>12</sup> The good news is that the zoo of operators<sup>13</sup> of quantum mechanics is complete—we will be concerned *only* (to be exact, with *one* exception) with these three types of operators (or the corresponding matrices or other representations):

$$\begin{array}{ll} A = A^\dagger & \text{Hermitian operator} \\ AA^\dagger = A^\dagger A = 1 & \text{unitary operator} \\ A^2 = A & \text{projection operator.} \end{array} \quad (8.22)$$

The names are used also for the corresponding matrices and representations. We outline in brief form the applications of these operators: We can represent physically-measurable quantities by Hermitian operators; the unperturbed time evolution of a system is described by a unitary operator; and the measurement process can be modelled with the help of projection operators.

### 8.3.2 Time Evolution and Measurement

We denote the eigenvalues and eigenvectors of  $H$  in (8.16) by  $E_n$  and  $|\varphi_n\rangle$ . The general solution as a generalization of (8.7) is a superposition of the eigenvectors:

$$|\psi(t)\rangle = \sum_n c_n |\varphi_n\rangle e^{-iE_n t/\hbar} \quad (8.23)$$

where the integration constants  $c_i$  are determined by the initial conditions (see the exercises).

<sup>11</sup>Since measured values are real, we can interpret them as eigenvalues of Hermitian operators.

<sup>12</sup>These properties are not mutually exclusive: A unitary operator or a projection operator can also be e.g. Hermitian.

<sup>13</sup>Since these operators exhibit only a few species and are fairly well-behaved, one could also speak of a ‘pet zoo’.

A measurement interrupts the time evolution of  $|\psi(t)\rangle$  as described in (8.23). If we want to measure e.g. the state  $|\chi\rangle$ , then we can describe it by projecting onto  $|\chi\rangle$ ; that is by the term  $|\chi\rangle\langle\chi|$  corresponding to (8.8). Here,  $|\langle\chi|\psi\rangle|^2$  is the probability that we actually obtain  $|\chi\rangle$  from a measurement:

$$\begin{aligned} |\langle\chi|\psi\rangle|^2 &= \sum_n c_n \langle\chi|\varphi_n\rangle e^{-iE_n t/\hbar} \sum_m c_m^* \langle\varphi_m|\chi\rangle e^{iE_m t/\hbar} \\ &= \sum_{n,m} c_n c_m^* \langle\chi|\varphi_n\rangle \langle\varphi_m|\chi\rangle e^{-i(E_n - E_m)t/\hbar}. \end{aligned} \tag{8.24}$$

After or due to the measurement, we have the state  $|\chi\rangle$  instead of  $|\psi\rangle$ .

We remark that all these considerations hold for systems of arbitrary dimensions.

### 8.4 Exercises

1. Given that  $|\nu_1\rangle\langle\nu_1| + |\nu_2\rangle\langle\nu_2| = 1$ , show:  $|\nu_e\rangle\langle\nu_e| + |\nu_\mu\rangle\langle\nu_\mu| = 1$ .
2. Show that the matrices  $\begin{pmatrix} c & 0 & s e^{-i\delta} \\ 0 & 1 & 0 \\ -s e^{i\delta} & 0 & c \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}$  with  $\delta \in \mathbb{R}$  are unitary. The abbreviations  $s$  and  $c$  stand for  $\sin \alpha$  and  $\cos \alpha$ .
3. Show that the product of two unitary matrices is also unitary.
4. Is the beam splitter operator  $T$  from Chap. 6,

$$T = \frac{1+i}{2} [1 + i|H\rangle\langle V| + i|V\rangle\langle H|], \tag{8.25}$$

a Hermitian, a unitary or a projection operator?  $\{|H\rangle, |V\rangle\}$  is a CONS.

5. Given  $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ :
  - (a) Show that  $A$  is Hermitian, but not unitary.
  - (b) Calculate  $e^{cA}$ .
6. Given the operators<sup>14</sup>

$$\begin{aligned} L_1 &= \frac{|v\rangle(\langle u| + \langle w|) + (|u\rangle + |w\rangle)\langle v|}{\sqrt{2}} \\ L_2 &= \frac{-|v\rangle(\langle u| - \langle w|) + (|u\rangle - |w\rangle)\langle v|}{i\sqrt{2}} \\ L_3 &= |u\rangle\langle u| - |w\rangle\langle w|. \end{aligned} \tag{8.26}$$

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<sup>14</sup>These are essentially the three components of the orbital angular momentum operator for angular momentum 1; see Chap. 16, Vol. 2.

- (a) Are these Hermitian, unitary or projection operators?  
 (b) Calculate  $[L_1, L_2]$ .

7. Show that the time evolution

$$|\nu(t)\rangle = -\sin \vartheta |\nu_1\rangle e^{-i\omega_1 t} + \cos \vartheta |\nu_2\rangle e^{-i\omega_2 t} \quad (8.27)$$

is unitary.

8. Determine explicitly  $\langle \nu_e | \nu(t) \rangle$  in (8.8), and  $\langle \nu_\mu | \nu(t) \rangle$ .  
 9. Determine explicitly  $p_e$  in (8.9), and  $p_\mu$ .  
 10. Prove (8.10); find an approximation for  $\Delta E$  in the case of very small rest masses.  
 11. Given the state

$$|\psi(t)\rangle = \sum_n c_n |\varphi_n\rangle e^{-iE_n t/\hbar} \quad (8.28)$$

with the initial condition  $|\psi(0)\rangle$ .  $\{|\varphi_n\rangle\}$  is a CONS. How are the constants  $c_n$  related to the initial conditions?

12. Given two CONS  $\{|\varphi_i\rangle\}$  and  $\{|\psi_i\rangle\}$ . A quantum system is in the superposition  $|z\rangle = \sum_i d_i |\psi_i\rangle$ .

- (a) Calculate the probability of measuring the quantum system in the state  $|\varphi_k\rangle$ .  
 (b) Show that  $\sum_k p_k = 1$ .

13. Given the model system

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle \quad \text{with } H = 1 + A\sigma_y; A > 0, \quad (8.29)$$

where  $\sigma_y$  is the  $y$ -Pauli matrix:

- (a) Determine the eigenvalues and eigenvectors of  $H$ ;  
 (b) How does the general expression  $|\psi(t)\rangle$  read for a time-dependent state?  
 (c) How is  $|\psi(t)\rangle$  expressed for the initial state  $|\psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ?  
 (d) Assume that we measure  $|\psi(t)\rangle$  from part c. With which probability will we find the state  $|\chi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  (i.e. the initial state)?