Chapter 7
Position Probability

We establish the concept of probability within the analytical approach to quantum mechanics in the form of the position probability density and its associated probability current density

In the algebraic approach to quantum mechanics, we introduced early on the notion of probability. Now we want to develop this concept in the analytical approach, as well, and furthermore we aim at merging the two approaches gradually. The problem is as follows: In the algebraic approach, probabilities appear rather naturally (due to the plausible redefinition of intensity → probability). The SEq, however, is deterministic. An initial state fixes the time evolution of the wavefunction for all times, and clearly this leaves no room for chance.

Therefore, probabilities cannot come into play from the SEq itself, but only through the wavefunction $\Psi(x,t)$. As we already briefly mentioned in Chap. 5, the absolute square of $\Psi(x,t)$ can be regarded as the *position probability density*,[1] usually denoted by the letter ρ:

$$\rho(x,t) = \Psi^*(x,t)\,\Psi(x,t) = |\Psi(x,t)|^2 \tag{7.1}$$

The interpretation of ρ as a probability density is not at all obvious, and in the early days of quantum mechanics it took some time until Max Born arrived at this concept. At that point (and still for us at present), it was a hypothesis or conjecture which had to prove itself by leading to consistent results and conclusions (which of course it did).[2]

We will develop this concept in the following and will discuss its consequences.

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[1] The probability w (probability of finding the quantum object in a given region of space) is obtained by integrating the probability density ρ, as in $w = \int \rho \mathrm{d}V$, over the spatial region of interest. Analogously, the mass m is given as an integral over the mass density ρ as $m = \int \rho \mathrm{d}V$.

[2] Especially when one is speaking to lay people about probabilities in quantum mechanics, one should always keep in mind that this is a conceptually difficult notion. On the one hand, there is the wavefunction with its abstractness, not understandable in everyday terms. On the other hand, it is just this wavefunction which allows us to determine concrete values of probabilities. The How

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7.1 Position Probability and Measurements

7.1.1 Example: Infinite Potential Wall

This section is intended to serve primarily as a brief motivation.

We want to calculate the probability of finding an object with well-defined energy in the infinite potential well within the interval $0 < x_1 < x_2 < a$. Classically this is quite simple[3]; the probability is evidently given by

$$w^{cl}_{x_1,x_2} = \frac{x_2 - x_1}{a}. \tag{7.2}$$

For the quantum-mechanical analysis, we assume a state with the given energy E_n (see Chap. 5):

$$\Psi(x,t) = e^{-iE_nt/\hbar}\sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x;\ E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2;\ n = 1, 2, \ldots \tag{7.3}$$

We consider the expression

$$w^{qm}_{x_1,x_2} = \int_{x_1}^{x_2} \Psi^*(x,t)\,\Psi(x,t)\,\mathrm{d}x. \tag{7.4}$$

As outlined in Chap. 5, we choose as the first factor under the integral not $\Psi(x,t)$, but rather the complex conjugate wavefunction $\Psi^*(x,t)$. This guarantees that we always obtain *positive* expressions for the probability, since $\Psi^*(x,t)\,\Psi(x,t) \geq 0$; this is as required of a probability. A simple calculation leads us to

$$w^{qm}_{x_1,x_2} = \frac{x_2 - x_1}{a} - \frac{\sin\left(n\pi\frac{x_2-x_1}{a}\right)\cos\left(n\pi\frac{x_2+x_1}{a}\right)}{n\pi}. \tag{7.5}$$

The comparison of (7.2) and (7.5) suggests the interpretation of $w^{qm}_{x_1,x_2}$ as the probability of finding the object in the interval $[x_1, x_2]$. This has the consequence that we can interpret $\Psi^*(x,t)\,\Psi(x,t) = |\Psi|^2$ as a probability *density*. We see (compare also Fig. 7.1) that the quantum-mechanical probability becomes increasingly similar to the classical one with increasing n, i.e. with increasing energy. This behavior is typical of many quantum-mechanical phenomena: The quantum character becomes all the more clearer, the lower the energies (low with respect to the energy scale of the system under consideration), and vice versa.

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and Why are certainly not intuitively obvious and cannot be formulated convincingly with the aid of familiar everyday ideas.

[3]The velocity is constant between the turning points.

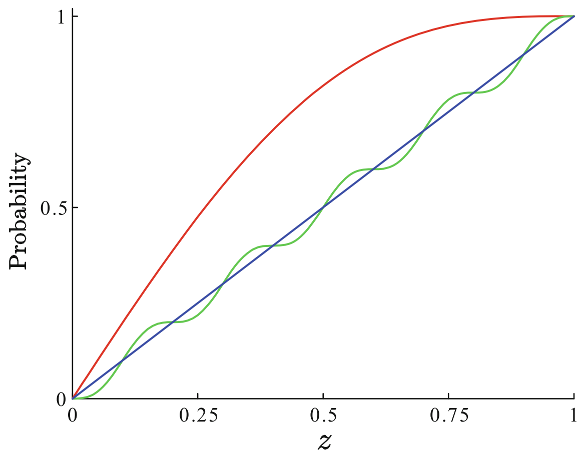

Fig. 7.1 Position probability (7.5) as a function of $z = \frac{x_2 - x_1}{a}$ for $x_2 = a\frac{1+z}{2}$ and $x_1 = a\frac{1-z}{2}$. The situation is shown for $n = 1$ (*red*), $n = 10$ (*green*) and $n = 1000$ (*blue*). The latter case is graphically indistinguishable from the classical straight line $w^{cl}_{x_1,x_2}$ given in (7.2)

7.1.2 Bound Systems

We start with the time-dependent SEq:

$$i\hbar\dot{\Psi}(x,t) = H\Psi(x,t). \tag{7.6}$$

Using the separation *ansatz*

$$\Psi(x,t) = e^{-i\frac{Et}{\hbar}}\varphi(x), \tag{7.7}$$

we obtain the time-independent SEq:

$$H\varphi(x) = E\varphi(x). \tag{7.8}$$

In this paragraph we assume that there are only discrete and no continuous eigenvalues, as discussed in Sect. 5.3. The eigenvalues and the eigenfunctions are given by $E_n = \hbar\omega_n$ and $\varphi_n(x)$, and the total solution reads

$$\Psi(x,t) = \sum_n c_n\varphi_n(x)\, e^{-i\frac{E_n t}{\hbar}};\ c_n \in \mathbb{C}, \tag{7.9}$$

with the initial state

$$\Psi(x,0) = \sum_n c_n\varphi_n(x). \tag{7.10}$$

The *orthonormality* of the eigenfunctions, already mentioned in Chap. 5, is important for the following discussion (as we will show later on, this is a common feature of the eigenfunctions of *all* the Hamiltonians we will deal with in this book):

$$\int_{-\infty}^{\infty} \varphi_n^* (x) \varphi_l (x) \, \mathrm{d}x = \delta_{nl}. \tag{7.11}$$

Since the total wavefunction Ψ is a solution of a linear differential equation, multiples of it are also solutions. We choose a multiple so that Ψ is normalized, i.e.

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, \mathrm{d}x = \int_{-\infty}^{\infty} \rho(x,t) \mathrm{d}x = 1. \tag{7.12}$$

In short, we can always assume Ψ to be normalized.

We now interpret $|\Psi(x,t)|^2$ as a (position) probability density. Thus, the last equation implies that the quantum object is located with probability 1 (i.e. with certainty) *somewhere* in space, as it must be. The probability that the object is localized in a particular region, say $a \leq x \leq b$ at time t, is given by (as in (7.4)):

$$w(a \leq x \leq b, t) = \int_a^b |\Psi(x,t)|^2 \, \mathrm{d}x. \tag{7.13}$$

Clearly, this probability is always positive definite, and the total probability $\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, \mathrm{d}x$ equals one.[4] Thus we have found for the wavefunction not an immediate, but at least an indirect physical significance,[5] in that its absolute square can be viewed as the position probability density.[6]

The extension of these considerations to three dimensions causes no problems.

7.1.2.1 Conclusions

What are the conclusions one can draw? We insert the total wavefunction (7.9) into (7.12) and obtain in the first step

$$1 \stackrel{!}{=} \int_{-\infty}^{\infty} \sum_n c_n^* \varphi_n^* (x) \, e^{i\omega_n t} \sum_l c_l \varphi_l (x) \, e^{-i\omega_l t} \mathrm{d}x. \tag{7.14}$$

[4] The fact that we can actually interpret this as a probability is shown by the general definition. A probability measure μ on $\mathbb{R}$ is a mapping μ from the set of intervals (which are given here by the integration intervals) into the unit interval [0, 1] which meets the following requirements: (i) $\mu(I) = 1 \geq 0$ for all intervals I (positive definite), (ii) $\mu(\mathbb{R}) = 1$ (normalized), (iii) $\mu(I_1 \cup I_2) = \mu(I_1) + \mu(I_2)$ for all pairwise disjoint intervals I_1 and I_2 (additivity property or σ additivity).

[5] The wavefunction itself is non-intuitive—it is just a complex-valued field of possibilities, as mentioned above.

[6] Since the concept is unique, one often omits the term 'position' and uses the more compact 'probability density'. We will do so also, for the most part.

Under the usual assumption that we can interchange the sum and the integral, and with the notation of the two sums as a double sum, we obtain

$$\begin{aligned}1 &= \sum_{n,l} c_n^* e^{i\omega_n t} c_l e^{-i\omega_l t} \int_{-\infty}^{\infty} \varphi_n^* (x) \, \varphi_l (x) \, \mathrm{d}x \\ &= \sum_{n,l} c_n^* c_l e^{i(\omega_n - \omega_l)t} \delta_{n,l} = \sum_n c_n^* c_n = \sum_n |c_n|^2. \end{aligned} \tag{7.15}$$

In other words: the fact that Ψ is normalized is equivalent to

$$\sum_n |c_n|^2 = 1. \tag{7.16}$$

This equation is valid independently of time, so we can limit ourselves in our further considerations to $t = 0$.

With (7.16), we have found the same relation as in the algebraic approach: The absolute squares of the coefficients give the probabilities of finding the corresponding states or quantum numbers. We illustrate this point by means of two concepts: mean value and collapse.

Mean value. We consider an ensemble of identically-prepared systems (7.10), where the measured quantity is the energy.[7] If we measure N members of the ensemble, we observe the state $\varphi_n(x)$ (or the energy E_n) r_n times, where of course $N = \sum_n r_n$. As usual, the *mean value* of the energy is found to be

$$E_{\text{mean value}} = \sum\nolimits_n h_n E_n \tag{7.17}$$

with the relative frequencies of occurrence $h_n = r_n/N$. For $N \to \infty$, the relative frequencies h_n become the probabilities $|c_n|^2$ of measuring the state $\varphi_n(x)$ or the energy E_n, and we obtain the *expectation value*

$$E_{\text{expectation value}} = \sum\nolimits_n |c_n|^2 \, E_n. \tag{7.18}$$

These concepts and the question of how they can be extended to continuous variables will be discussed further in Chap. 9.

Collapse. We can apply the concept of probability to individual systems, with which one mainly deals in practice. *Before* a single measurement, we can say that by measuring (7.10) we will obtain *one* of the states φ_n, say $\varphi_j(x)$, with the probability $w_j = |c_j|^2$. *After* the measurement, the system is in a well-defined state, let us say $\varphi_l(x)$. Thus we know immediately *after* the measurement the state of the system with certainty:

[7] Those who wish may consider the infinite potential well as a concrete example.

$$c_n = 0 \text{ for } n \neq l \text{ directly after measurement} \tag{7.19}$$

or, formulated explicitly,[8]

$$\Psi_{\text{before}}(x,t) = \sum_n c_n \varphi_n(x)\, e^{-i\frac{E_n t}{\hbar}} \underset{\text{measurement}}{\rightarrow} \Psi_{\text{after}}(x,t) = \frac{c_l}{|c_l|}\varphi_l(x)\, e^{-i\frac{E_l t}{\hbar}}. \tag{7.20}$$

We see that the measurement has forced the system into a unique state. We have already met up with this process of *state reduction* in the algebraic approach. That it now also occurs here is *not* due to the SEq. There is an additional element, namely, our interpretation of the wavefunction as a *probability amplitude* or a complex-valued field of possibilities.

The following picture emerges: The SEq describes the unperturbed time evolution of a quantum system. This evolution is interrupted by the measurement process, which changes the wavefunction. One also speaks of the *collapse of the wavefunction*. After the measurement, the system is again subject to the time evolution described by the SEq.[9]

As in the algebraic approach, there are open questions concerning the measurement. For example, if the measurement process is not included in the SEq, does this mean that measurement is not a quantum-mechanical process? Or is our description by means of the SEq plus measurement process insufficient? Or is it simply the best we can ever achieve, because nature is in reality not as simple as described by our theories? In short, what does 'to measure' actually mean in quantum mechanics?

7.1.3 Free Systems

In the case of free, unbounded systems, we have seen that an initial situation of the form (we limit ourselves to one dimension)

$$\rho(x,0) = |\Psi(x,0)|^2 = \frac{1}{\sqrt{\pi} b_0} \exp\left(-\frac{x^2}{b_0^2}\right) \tag{7.21}$$

evolves in the course of time into

$$\rho(x,t) = |\Psi(x,t)|^2 = \frac{1}{\sqrt{\pi} b(t)} \exp\left(-\frac{\left(x - \frac{\hbar K}{m}t\right)^2}{b^2(t)}\right), \tag{7.22}$$

with $b(t)$ given by

[8] The state must be normalized after the measurement; this is expressed by the factor $\frac{c_l}{|c_l|}$.

[9] We note again that the exact process of measurement itself is not described.

$$b(t) = \sqrt{b_0^2 + \left(\frac{\hbar t}{b_0 m}\right)^2}. \tag{7.23}$$

We repeat the following remark (cf. Sect. 5.2.2): We model here the quantum-like behavior of *material bodies* ($m \neq 0$) such as electrons. The spreading of $\rho(x, t)$ does not mean that the electron itself is 'smeared out' in space (in that case, the smearing would also apply to the electron's properties such as its mass and charge), like a mound of honey which flattens and spreads. It is the *wavefunction*, which determines the position probability, that disperses, rather than the object itself. In other words, the uncertainty with which we can determine the location of a quantum object increases in the course of time: $\Delta x \approx b(t)$.

Again the question arises: What happens when we perform a measurement? Let us assume that we have arrayed detectors along the entire x axis, each of length a. Now we release a free quantum object; the detectors are still switched off. We wait sufficiently long to be sure that $\Delta x \approx b(t) \gg a$ holds. Then we measure the position of the object by activating the detectors — one of them will respond. At that moment, the spatial uncertainty, which had grown steadily before our measurement, shrinks abruptly to a, and the wavefunction is correspondingly modified.[10] This means that we again observe the connection between the measurement process and the collapse of the wavefunction (or state reduction).

The considerations about the mean value which were outlined above for discrete eigenvalues cannot readily be applied to continuous measurements of quantities such as the position or the momentum. We will address this issue again in Chap. 9, and will formulate it so generally that the nature of the eigenvalue spectrum will no longer be relevant.

7.2 Real Potentials

The probability density ρ is positive definite, which follows directly from the definition (7.1). The probability of localizing the quantum object at time t in the interval $[x_1, x_2]$ is given by $W(x_1 < x < x_2; t) = \int_{x_1}^{x_2} \rho(x, t)\, dx$. In order to indeed interpret ρ as a probability density, the equation[11]

$$\int_{-\infty}^{\infty} \rho(x, t)\, dx \overset{!}{=} 1 \ \ \forall t \tag{7.24}$$

must hold. In words: The quantum object must be located somewhere in space, and this must be true *at all times*. Therefore, two requirements must be met:

[10] We can regard this state as the initial condition for a new cycle of free propagation, in which case one refers to the measurement process as a (state-)preparation.

[11] We assume that there are neither creation nor annihilation processes.

1. The integral $\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, \mathrm{d}x$ has to exist, at least at a certain time t. If it does, we can normalize the wave function so that at this time t, $\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, \mathrm{d}x = 1$ holds.
2. In addition, we must show that the normalization constant does not change, so that $\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, \mathrm{d}x = 1$ is valid for *all times*.

Can we always satisfy these two requirements?

The first requirement means that $\Psi(x,t)$ must be square integrable in view of the interpretation of $|\Psi(x,t)|^2$ as a probability density. This is certainly the case over a finite interval of space when the wavefunction is sufficiently smooth or 'well-behaved' (i.e. does not have singularities, etc.), which we always assume in the following. In order that the integral from $-\infty$ to ∞ exists, the condition

$$\Psi \underset{|x|\to\infty}{\sim} |x|^{\alpha} \, ; \, \alpha < -\frac{1}{2} \tag{7.25}$$

must be fulfilled in addition, at least at some time t. One often describes this condition by saying that the wavefunction must *approach zero rapidly enough* at infinity.[12] We note in this context that there may be correct mathematical solutions of differential equations that must still be excluded for physical reasons. More about this issue is included in some of the following chapters and in Appendix E, Vol. 1.

As to the second requirement[13]: We have to show that $\int_{-\infty}^{\infty} \rho(x,t) \, \mathrm{d}x = 1$ holds at all times. This means that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} \rho(x,t) \, \mathrm{d}x \overset{!}{=} 0, \tag{7.26}$$

and it follows[14] that

$$0 \overset{!}{=} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \Psi^* \Psi \mathrm{d}x = \int_{-\infty}^{\infty} \left(\dot{\Psi}^* \Psi + \Psi^* \dot{\Psi} \right) \mathrm{d}x. \tag{7.27}$$

We replace the time derivatives making use of the SEq

[12]In three dimensions, the condition is slightly different. Because of $\int \mathrm{d}V = \int r^2 \mathrm{d}r \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi$, the wavefunction has to go to zero as r^{α} with $\alpha < -\frac{3}{2}$.

[13]With the conceptual framework derived in later chapters, the proof may be formulated in a considerably shorter way.

[14]As always, we assume the commutability of differentiation and integration. See Appendix D, Vol. 1.

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m}\Psi'' + V\Psi \tag{7.28}$$

and find, assuming a *real potential* $V \in \mathbb{R}$,[15]

$$\begin{aligned} 0 &\stackrel{!}{=} \int\limits_{-\infty}^{\infty} \left[\left(\frac{\hbar}{2mi}\Psi^{*\prime\prime} - \frac{V}{i\hbar}\Psi^*\right)\Psi + \Psi^*\left(-\frac{\hbar}{2mi}\Psi'' + \frac{V}{i\hbar}\Psi\right)\right]\mathrm{d}x \\ &= \frac{\hbar}{2mi}\int\limits_{-\infty}^{\infty}\left(\Psi^{*''}\Psi - \Psi^*\Psi''\right)\mathrm{d}x. \end{aligned} \tag{7.29}$$

We transform the second derivatives w.r.t. spatial coordinates by partial integration. It follows that

$$0 \stackrel{!}{=} \frac{\hbar}{2mi}\left[\left(\Psi^{*\prime}\Psi\right)_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty}\Psi^{*\prime}\Psi'\mathrm{d}x\right] - \frac{\hbar}{2mi}\left[\left(\Psi^*\Psi'\right)_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty}\Psi^{*\prime}\Psi'\mathrm{d}x\right]. \tag{7.30}$$

The integrals cancel each other and we finally obtain

$$\left(\Psi^{*\prime}\Psi - \Psi^*\Psi'\right)\Big|_{-\infty}^{\infty} \stackrel{!}{=} 0. \tag{7.31}$$

This condition is fulfilled due to (7.25), since for $\alpha < -\frac{1}{2}$, we have $\Psi'\Psi \underset{|x|\to\infty}{\sim} |x|^{2\alpha-1} \to 0$.

We see that the probability concept is inherently consistent, if the wavefunction vanishes at infinity rapidly enough and if the potential is real. These are very important properties, which we assume to be *always* fulfilled from now on.[16]

7.3 Probability Current Density

In the following, an expression for the (position) probability current density is derived. We rely on the *continuity equation*[17]

$$\frac{\partial\rho}{\partial t} + \nabla\mathbf{j} = \mathbf{0}. \tag{7.32}$$

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[15] Here, the potential may depend on the time t.

[16] Complex potentials are required when one wants to describe e.g. absorption processes. These potentials are also called *optical potentials* (referring to the complex optical refractive index whose imaginary part describes absorption). An example is found in the exercises.

[17] The derivation of the continuity equation is given in Appendix N, Vol. 1.

This equation is a differential formulation of a global conservation law. It is valid not only for the mass density, but in fact applies to all densities (e.g. the charge density) for which integral conservation laws hold (e.g. global conservation of charge).

In particular, we assume the validity of the continuity equation for the probability density of quantum mechanics. Thus, we can calculate the probability current density **j**. For the sake of simplicity, we consider only the one-dimensional problem and extend the result at the end to three dimensions.

In one dimension, the continuity equation reads

$$\dot{\rho}(x,t) + \frac{\partial}{\partial x} j(x,t) = 0. \tag{7.33}$$

To derive the relationship between j and Ψ, we insert $\rho = |\Psi|^2$. With $\dot{\rho} = \dot{\Psi}^*\Psi + \Psi^*\dot{\Psi}$ and the SEq

$$\dot{\Psi} = -\frac{\hbar}{2mi}\Psi'' + \frac{V\Psi}{i\hbar}, \tag{7.34}$$

we can rewrite the continuity equation as

$$\left(\frac{\hbar}{2mi}\Psi^{*\prime\prime} - \frac{V^*\Psi^*}{i\hbar}\right)\Psi + \Psi^*\left(-\frac{\hbar}{2mi}\Psi'' + \frac{V\Psi}{i\hbar}\right) + \frac{\partial}{\partial x} j = 0. \tag{7.35}$$

Since we assume $V \in \mathbb{R}$, the potential terms cancel. It follows that

$$\begin{aligned} \frac{\partial}{\partial x} j &= \frac{\hbar}{2mi}\left(\Psi^*\Psi'' - \Psi^{*\prime\prime}\Psi\right) \\ &= \frac{\hbar}{2mi}\Psi^*\Psi'' - \Psi^{*\prime\prime}\Psi + \Psi^{*\prime}\Psi' - \Psi^{*\prime}\Psi') \\ &= \frac{\hbar}{2mi}\left(\frac{\partial}{\partial x}\Psi^*\Psi' - \frac{\partial}{\partial x}\Psi^{*\prime}\Psi\right). \end{aligned} \tag{7.36}$$

Integration gives[18]:

$$j(x,t) = \frac{\hbar}{2mi}(\Psi^*\Psi' - \Psi^{*\prime}\Psi). \tag{7.37}$$

We have thus found an expression for the probability current density. We already know that it vanishes at infinity; see (7.31).

The extension of the probability current density to three dimensions yields in a straightforward manner:

$$\mathbf{j}(\mathbf{r},t) = \frac{\hbar}{2mi}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*). \tag{7.38}$$

[18] Actually, there could still be a constant of integration on the right-hand side, but it is set equal to zero due to the requirement $j = 0$ for $\Psi = 0$.

As an (unphysical, but familiar[19]) example, we consider a plane wave

$$\Psi(\mathbf{r}, t) = Ae^{i(\mathbf{kr}-\omega t)}. \tag{7.39}$$

With

$$\nabla\Psi(\mathbf{r}, t) = Ai\mathbf{k}e^{i(\mathbf{kr}-\omega t)}, \tag{7.40}$$

it follows that

$$\mathbf{j}(\mathbf{r}, t) = \frac{\hbar}{2mi}\left(i\mathbf{k}AA^* + i\mathbf{k}AA^*\right) = \frac{\hbar\mathbf{k}}{m}|A|^2. \tag{7.41}$$

Because of $\rho = \Psi^*\Psi = |A|^2$, we obtain the well-known relationship

$$\mathbf{j} = \frac{\hbar\mathbf{k}}{m}\rho = \frac{\mathbf{p}}{m}\rho := \mathbf{v}\rho \tag{7.42}$$

where $\mathbf{v}$ is the velocity of e.g. a maximum of the wave.[20]

We make some general remarks on the one-dimensional probability current density $j = \frac{\hbar}{2im}\left(\varphi^*\varphi' - \varphi^{*\prime}\varphi\right)$:

1. For $\varphi(x) = Ae^{\alpha x}$ $(\alpha \in \mathbb{R},\ A \in \mathbb{C})$, we have

$$j = \frac{\hbar}{2im}\left(\alpha\,|A|^2\,e^{2\alpha x} - \alpha\,|A|^2\,e^{2\alpha x}\right) = 0. \tag{7.43}$$

With real exponents, j disappears. To put it graphically, this does not mean that nothing flows into the region or out of it, but rather that whatever flows in must also flow out again.

2. For $\varphi(x) = Ae^{i\gamma x}$ $(\gamma \in \mathbb{R},\ A \in \mathbb{C})$, we have

$$j = \frac{\hbar}{2im}\left(i\gamma\,|A|^2 + i\gamma\,|A|^2\right) = \frac{\hbar}{m}\gamma|A|^2. \tag{7.44}$$

So there is a 'net flow', that is, something is actually transported.

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[19] Unphysical, because the infinitely-extended plane wave, whose magnitude is one everywhere, does not represent a physical object. The fact that we can still make use of plane waves in quantum mechanics is due to the linearity of quantum mechanics, which allows us to construct wave packets with physically reasonable behavior by superposition of plane waves.

[20] We note that the 'velocity' of a quantum object is a seldom-used notion in quantum mechanics. The momentum is the central quantity. 'Velocity' will appear only in the context of Galilean transformations (relative motion of inertial frames, Chap. 21, Vol. 2) and in the Bohmian interpretation of quantum mechanics (see Chap. 28, Vol. 2), which is based on classical mechanics.

7.4 Exercises

1. Show for $\rho = |\psi(x,t)|^2$ that:

$$\int_{-\infty}^{\infty} \rho(x,t)\,\mathrm{d}x = 1 \;\forall t. \tag{7.45}$$

Here, we assume that (i) the potential is real, and (ii) $\Psi \underset{x\to\infty}{\sim} x^a$, with $a < -\frac{1}{2}$.

2. Infinite potential well: Given the wave functions

   (a) $\Psi(x,t) = e^{-i\omega_n t}\sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x$

   (b) $\Psi(x,t) = c_n e^{-i\omega_n t}\sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x + c_m e^{-i\omega_m t}\sqrt{\frac{2}{a}}\sin\frac{m\pi}{a}x,$

   Calculate for both cases the probability of finding the quantum object in the interval (x_1, x_2)

$$w^{qm}_{x_1,x_2} = \int_{x_1}^{x_2} \Psi^*(x,t)\,\Psi(x,t)\,\mathrm{d}x. \tag{7.46}$$

3. Given the SEq $i\hbar\dot{\psi} = H\psi$ with a real potential, derive from the continuity equation constructively (i.e. not just proving by insertion) that $\mathbf{j}$ is given by

$$\mathbf{j} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*). \tag{7.47}$$

4. Calculate j (one-dimensional) for $\psi = Ae^{\gamma x}$ and $\psi = Ae^{i\gamma x}$, with $\gamma \in \mathbb{R}$ and $A \in \mathbb{C}$.
5. Calculate $\mathbf{j}(\mathbf{r},t)$ for $\Psi(\mathbf{r},t) = Ae^{i(\mathbf{kr}-\omega t)}$.
6. Given a modification of the infinite potential well, namely the potential

$$V(x) = \begin{cases} iW \text{ for } 0 < x < a \\ \infty \text{ otherwise} \end{cases} ;\; W \in \mathbb{R}, \tag{7.48}$$

calculate the energy spectrum and show that the norm of the (time-dependent) total wavefunction is independent of time only for $W = 0$.