Chapter 14
Postulates of Quantum Mechanics

> In this chapter, we compile our findings about quantum mechanics from the preceding chapters, insofar as they concern its structure, and we formulate some basic rules for its application. They make up the general framework for our further considerations.

In previous chapters, we have frequently mentioned structural elements of quantum mechanics, which we now summarize and present systematically, namely in the form of *postulates* or *rules*. These rules map the behavior of physical systems, or more precisely, our methods for describing that behavior.[1] In fact, there are basically only three questions to which the physical description of a system must provide answers:

1. How can we describe the state of the system at a given time?
2. Which variables of the system are measurable and how can we predict the results of measurements?
3. How is the state of the system obtained at time t from its known initial state at time t_0?

The answers to these questions differ, of course, depending on the field (classical mechanics, quantum mechanics, hydrodynamics, quantum electrodynamics, ...). In the following, we will clothe them in the form of *postulates for quantum mechanics*, whereby the word 'postulate' in this context is equivalent to a thesis, a principle or a rule which is not proven, but is quite plausible and evident. We do not aim at an absolutely rigorous system of axioms (in the sense of a minimal set of statements). It is rather a question of creating a viable and practical set of rules (also called 'quasi-axiomatic'), and for the sake of practicality we also take into account the fact that

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[1] "The human understanding is of its own nature prone to suppose the existence of more order and regularity in the world than it finds." Francis Bacon (1561–1626), English philosopher and statesman, in *New Organon*. "Rain, snow, winds follow each other so that we do not animadvert a clear law in their order, but laws again are only conceived by us to facilitate comprehending a thing, as we create species." Georg Christoph Lichtenberg, *Scrap Books*, Vol. A (192).

one postulate might be derivable from others.[2] Moreover, the set of rules established here is only one of many possible such sets. Other formulations of the postulates can be found in Appendix R, Vol. 1.[3]

In introductions to quantum mechanics, the postulates are often presented right at the beginning, so to speak as the basis for the further development of quantum mechanics. This proceeding has the immediate advantage of conceptual clarity, since e.g. borrowing from classical mechanics (the correspondence principle, etc.) is not necessary. On the other hand, for the 'uninitiated', the postulates somehow seem to fall from the heavens—without background information, it is probably quite difficult to understand how formulations like these were arrived at in the first place.[4]

Our access to quantum mechanics was a two-pronged one in the first chapters. In the analytical approach, we began with the dynamics (SEq, question 3) and subsequently took up questions 1 and 2. In the algebraic approach, we tried to make the postulates plausible (or to anticipate their statements) in the order given here and on the basis of simple physical systems.

Finally, we remark that the postulates do not raise any new difficulties of comprehension, but nevertheless they surely sharpen our view of open problems. In previous chapters, we have already mentioned questions of this kind several times, and we will do so again at the end of this chapter.

14.1 Postulates

The numbering of the postulates refers to the numbers of the questions given above.

14.1.1 States, State Space (Question 1)

We have seen that both the solutions of the SEq as well as vectors, e.g. polarization vectors, can be linearly superposed and satisfy the axioms of a vector space. The first postulate summarizes this situation.

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[2]Moreover, one cannot of course exclude with certainty today that the rules established in the following will be (or will have to be) modified sometime later.

[3]Indeed, there is no overall agreement about the basic facts. For instance, some of the authors listed in Appendix R, Vol. 1 treat the indistinguishability of identical quantum objects as a postulate of quantum mechanics, but others do not.

[4]The (quasi-) axiomatic approach has the great advantage that it does not need false analogies and does not implant false images into the minds of students; thus, it has been proposed as a possible way to teach quantum mechanics in schools. This is feasible with a (suitably adapted) form of the algebraic approach. With the exception of Postulate 3, the postulates can be deduced, or at least motivated, if one confines oneself essentially to the two-dimensional case. It is not least for this reason that the algebraic approach is of great didactic interest.

Postulate 1: The state of a quantum system at a given moment is completely defined by giving its *state vector* (ket), $|\varphi\rangle$. The state vector is an element of the *Hilbert space* $\mathcal{H}$, which also is called the *state space*.

Remarks:

1. In contrast to e.g. classical mechanics, quantum mechanics describes the states by elements of a *vector space*, i.e. by vectors. The abstract state vector or ket, $|\varphi\rangle$, is the mathematical representation of the information we have about the physical state of the system.
2. Because $\mathcal{H}$ is a vector space, the *superposition principle* holds; it is characteristic of the linearity of the theory. As a dominant principle of quantum mechanics, it is responsible for many of those phenomena of quantum mechanics which seem so strange to our everyday understanding.
3. Because of the linearity of the theory, we can always assume that the state vectors are *normalized*. If this is not the case, we must, where necessary, normalize them *post hoc* by dividing by the norm.
4. In anticipation, we note that only eigenvalues and absolute values of scalar products such as $|\langle\varphi|\,\psi\rangle|$ are relevant to a measurement. This means that the states $|\varphi\rangle$ and $|\varphi'\rangle = e^{i\alpha}|\varphi\rangle$ with $\alpha \in \mathbb{R}$, differing only in their phase, are physically equivalent (which fact we have used in considering the infinite potential well). Strictly speaking, a (normalized) physical state is therefore not represented by a vector, but rather by a *ray* in $\mathcal{H}$, i.e. the set $\left\{e^{i\alpha}|\varphi\rangle, \alpha \in \mathbb{R}\right\}$. This fact is called the 'independence of the physics from the *global phase*'. Changing the *relative phases* naturally leads to different states; $c_1|\varphi\rangle + c_2|\psi\rangle$ and $c_1 e^{i\alpha}|\varphi\rangle + c_2|\psi\rangle$ are physically different for $\alpha \neq 2n\pi$.

   As we shall see, the difference between a ray and a vector is hereafter (luckily) only once of real importance, namely in the consideration of symmetry under time reversal (see Chap. 21, Vol. 2). Apart from this exception, we can work with state vectors (and ignore rays).
5. In the examples considered so far, any vector in $\mathcal{H}$ is a *physically realizable* state. This is not necessarily the case in all situations, as we shall see later in the treatment of identical particles, where there is no superposition of the states of fermions and bosons. The nonexistence of such (superposed) states is reflected in *superselection rules*.
6. It is still controversial just what the state vector 'really' means. The opinion that the state vector describes the physical reality of an *individual* quantum system is shared by many (and is also the position taken in this book), but it is by far not the only one. For more on this issue see Chap. 28, Vol. 2 on the interpretations of quantum mechanics. We stress once again that the state vector does not have a direct and concrete (everyday) meaning.

14.1.2   *Probability Amplitudes, Probability (Question 2)*

We have shown, e.g. by considering the polarization, that the absolute square of an amplitude gives the probability of finding the system in the respective state. This fact is generalized in Postulate 2.1.

Postulate 2.1: If a system is described by the vector $|\varphi\rangle$, and $|\psi\rangle$ is another state, then a probability amplitude exists for finding the system in state $|\psi\rangle$, and it is given by the scalar product $\langle\psi\,|\varphi\rangle$ in $\mathcal{H}$. The probability that the system is in the state $|\psi\rangle$ is the absolute square $|\langle\psi\,|\varphi\rangle|^2$ of the probability amplitude.

Remarks:

1. The vectors must be normalized to ensure that the probability concept is inherently consistent.
2. Using the projection operator $P_\psi = |\psi\rangle\,\langle\psi|$, the term $|\langle\psi\,|\varphi\rangle|^2$ can be written as $\langle\varphi|\,P_\psi\,|\varphi\rangle$.
3. The probability statements of this postulate provide a direct link to the term 'expectation value' or 'mean value.'
4. Probabilities usually indicate that the necessary information is not completely available. Hence, the idea arose quite early that quantum mechanics is not a complete theory, and hidden variables (hidden to us) must be added. But this is not the case according to present knowledge, at least not in the sense that the hidden variables have the simple and familiar properties of classical physics. We will address this issue in later chapters in Vol. 2.
5. This postulate is also called *Born's rule*.

14.1.3   *Physical Quantities and Hermitian Operators (Question 2)*

We have seen that a measurable physical quantity such as the momentum is represented by a Hermitian operator. The next postulate generalizes this relationship.

Postulate 2.2: Every measurable *physical quantity* is described by a *Hermitian operator A* acting in $\mathcal{H}$; this operator is an observable.[5] If a physical quantity is measured, the result can be only one of the eigenvalues of the corresponding observable A.

Remarks:

1. Quantum mechanics describes physical quantities by *operators* (in contrast to classical mechanics).
2. These operators are *observables*, i.e. Hermitian operators that represent a consistently measurable physical quantity.[6] In this way, we take into account that

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[5]Recall that we defined 'observables' as those Hermitian operators which represent a consistent measurable physical quantity, cf. Sect. 13.1.

[6]We note again that the word *observable* does not imply the existence of a (human) observer.

not every self-adjoint operator (with reasonable eigenfunctions) must necessarily represent a physical observable.[7] More on this topic in Vol. 1 Appendix I.

3. Because the operators are Hermitian, measurements always yield *real* values.
4. Not all physically measurable quantities are associated with non-trivial operators. Mass and charge, for example, are and remain simple numbers.

14.1.4 Measurement and State Reduction (Question 2)

If a right circular-polarized photon $|r\rangle = (|h\rangle + i\,|v\rangle)/\sqrt{2}$ passes through a PBS, we obtain with a probability $1/2$ either a horizontally or a vertically linear-polarized photon, i.e. $|h\rangle$ or $|v\rangle$. The next postulate formalizes and generalizes this fact.

Postulate 2.3: Suppose that the measurement of A on a system which was originally in the state $|\varphi\rangle$ yielded the value a_n. Then, immediately after the measurement, the state of the system is the normalized projection of $|\varphi\rangle$ onto the eigenspace belonging to a_n (see Chap. 13)[8]

$$|\varphi\rangle \to |\psi\rangle = \frac{P_n\,|\varphi\rangle}{\sqrt{\langle\varphi|\,P_n\,|\varphi\rangle}} \tag{14.1}$$

The state $|\psi\rangle$ is normalized:

$$\|P_n\,|\varphi\rangle\|^2 = \langle\varphi|\,P_n^\dagger P_n\,|\varphi\rangle = \langle\varphi|\,P_n\,|\varphi\rangle \tag{14.2}$$

Remarks:

1. This postulate assumes an *ideal measurement*; meaning, among other things, that further measurements on the quantum object must be possible.[9] Immediately after measuring, the state of the system is always an eigenvector of A with eigenvalue a_n. Any immediately following further measurement must, of course, yield the same result.[10] The transition from a superposition state to a single state is called *state reduction* or *collapse of the wavefunction*. It is an *irreversible* evolution which marks a direction in time.[11]

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[7]By the way, the practical implementation of arbitrary operators often raises some difficulties. However, for us this is not a strong constraint, because we need essentially only the well-known operators such as position, momentum, etc., or combinations of them.

[8]Any remaining phase plays no physical role; see the remark following Postulate 1 about states and rays.

[9]The only change in the measured system is the collapse of the wavefunction. In particular, the spectrum remains unchanged. One speaks in this context also of 'recoilless'. See also Appendix S, Vol. 1, where several remarks pertinent to the topic of 'measurement' can be found.

[10]Thereby one can prevent a change of state by measuring it repeatedly. The associated keyword is 'quantum Zeno effect'. The concept is summarized by the handy phrase 'a watched pot never boils'. More about this topic in Appendix L, Vol. 1.

[11]This applies only if the initial state is not already an eigenstate of the operator.

2. The position underlying this postulate is not sensitive to the details of the measurement process, but rather assumes the measuring apparatus to be a kind of black box. A more detailed analysis of the measurement process, including interactions of the quantum system with the measuring apparatus and the environment, shows that one can interpret Postulate 2.3 as a consequence of Postulates 2.1 and 2.2. However, this postulate is *fapp*, i.e. a useful 'working tool' for all the usual applications of quantum mechanics. We will address this question again in Chap. 24 (decoherence) and in Chap. 28 (interpretations), both in Vol. 2.

3. Measurement in quantum mechanics is obviously something very different than in classical physics. Classically, a (single) value of a physical quantity is measured, which already existed before the measurement (pre-existence). In quantum mechanics, this is the case only when the system is initially in an eigenstate of the measured observable; otherwise, there is no well-defined measurement value before the measurement.[12] This fact is also called the *eigenvector-eigenvalue rule*: A state has the value a of a property represented by the operator A if and only if the state is an eigenvector of A with eigenvalue a.[13] In this case, we can say that the system has the property a (For more cautious formulations of this relation, see Chap. 13.).

4. The spreading of the measured values is sometimes attributed to the fact that the measurement disturbs the measured quantity (e.g. the spin) uncontrollably. But this is wrong from the perspective of Postulate 2.3. For if the system is in an eigenstate of the measured quantity before the measurement, it will not be disturbed by the measurement. If it is not in an eigenstate, the measured value does not exist as such before the measurement—and what does not exist, cannot be disturbed.[14]

5. For degenerate and continuous cases, (14.1) must be appropriately modified.

6. This postulate is also called the projection postulate, the Neumann projection postulate, the postulate of Neumann–Lüders, etc.

14.1.5 Time Evolution (Question 3)

So far, our discussion was limited to a fixed time—now we start the clock. We recall that we have restricted ourselves to time-independent interactions.

Postulate 3: The temporal evolution of the state vector $|\psi(t)\rangle$ of an *isolated* quantum system is described by the equation (evolution equation, *Schrödinger equation*):

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle = H\,|\psi(t)\rangle. \tag{14.3}$$

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[12]See also the corresponding remarks in Chap. 13 (projection operators).

[13]Thus we have here a translation rule which connects physical quantities to mathematical objects.

[14]This remark is of course a bit shortened and flippant. The point is that the value of a variable is determined by the measurement, in general. Before the measurement, the value does not exist and thus can not be disturbed, for perturbation means changing an *existing* value into another one. More on this issue e.g. in Chap. 20, Vol. 2.

The Hermitian operator H which is associated with the total energy of the system is called the *Hamiltonian*.[15]

Remarks:

1. We consider isolated systems which do not interact with their environments. Their realization is anything but trivial, which is one of the obstacles to the rapid development of quantum computers. If, on the other hand, there is a coupling (observed or unobserved) to the degrees of freedom of the environment, one speaks of an *open* quantum system. More on this issue is to be found in Chap. 24, Vol. 2 (decoherence) and in Appendix S, Vol. 1.
2. The postulate tells us nothing about the specific form of the Hamiltonian. This is determined by the physical problem and the accuracy with which one wants to describe it. Further considerations are found below in the 'Concluding Remarks' section.
3. We had already stated the main characteristics of the SEq: it is among other things (a) complex, (b) linear, (c) of first order with respect to time. Stochastic components do not occur, hence the SEq is deterministic. Stationary states (eigenstates of energy E) have the time behavior of $|\varphi(t)\rangle = e^{-i\frac{tE}{\hbar}}|\varphi(0)\rangle$.
4. Since H is Hermitian, the time evolution is unitary and thus *reversible* and norm-preserving:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)\,|\psi(t)\rangle = \langle\dot{\psi}(t)\,|\psi(t)\rangle + \langle\psi(t)\,|\dot{\psi}(t)\rangle = 0. \qquad (14.4)$$

In contrast, the measurement process according to Postulate 2.3 is in general not unitary and is therefore irreversible; the norm is not conserved, but instead, one has to normalize the new measurement result. In order that the SEq $i\hbar\,|\dot{\psi}(t)\rangle = H\,|\psi(t)\rangle$ holds, e.g. between two measurements, the system must be isolated. During the measurement, however, the system is not isolated.

We can also formulate the time evolution with the help of the propagator rather than the differential form (14.3), and thus express Postulate 3 in another form:

Postulate 3′: The state vector at the initial time $|\varphi(t_0)\rangle$ is transferred into the state $|\varphi(t)\rangle$ at time t by a *unitary operator* $U(t, t_0)$, called the *time evolution operator* or *propagator*:

$$|\varphi(t)\rangle = U(t, t_0)\,|\varphi(t_0)\rangle. \qquad (14.5)$$

Remarks:

1. The unitarity of the propagator ensures the conservation of the norm.
2. For time-independent H, the propagator can be represented as $U = e^{-i\frac{Ht}{\hbar}}$.[16]

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[15]We note that (14.3) applies also to a time-dependent $H(t)$. But since we restrict ourselves in this whole book to the consideration of time-independent H, we formulate this postulate only for that case.

[16]For the formulation of the propagator as an integral operator, see Appendix I, Vol. 1; an example for the case of free motion is found in Chap. 5, Exercise 11.

3. With the propagator, the reversibility of the time evolution can be seen in a particularly simple manner, since we have $|\varphi(t_0)\rangle = U^{-1}(t, t_0)|\varphi(t)\rangle$.
4. Postulates 3 and 3′ are equivalent for our purposes. Strictly speaking, however, there is a difference, since U is bounded even if H is not bounded. In this respect, the propagator U appears to be more fundamental than the Hamiltonian H.

We want to stress here again a fundamental difference between classical mechanics and quantum mechanics. While classical mechanics describes the time evolution of the *factual*, quantum mechanics (or the SEq) describes the time evolution of the *possible*. In other words, the possibility structure of our universe is not fixed, but is a dynamically evolving structure.

14.2 Some Open Problems

As stated in the introduction to this chapter, we summarize here once again problems of comprehension which are essentially centered around the concept of measurement, a completely innocuous notion in classical physics.[17] In contrast, measurement seems to play a very special role in quantum mechanics. This was already clear in the early days of quantum mechanics, and even today the problem is not solved in depth, but remains the subject of current discussions.

One can of course avoid all these problems by adopting the *instrumentalist* or *pragmatic* point of view; namely, that we live in a classical world, and that the postulates are simply computational tools or instructions that work well without asserting the claim of representing reality. Niels Bohr put it this way: "There is no quantum world. There is only an abstract physical description. It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature." With this position (also called the *minimal interpretation*), one need not worry about the problems listed below, let alone about trying to solve them.

However, many people are dissatisfied with the idea that the fundamental description of the world should be a handful of rules which are closed to debate. The *realistic* position assumes that the quantum systems of the theory have real counterparts in one way or the other. The postulates, together with this point of view, are called the *standard interpretation* (or standard representation) by many authors.

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[17]In classical mechanics, the properties of a system are always well defined (where we always assume a non-pathological phase space). They can be described as functions of the phase space variables (i.e. points in phase space) and thus always have a direct three-dimensional spatial significance. In quantum mechanics, properties are not always well defined and we cannot represent them mathematically as functions of point sets. Small table:

	Classical mechanics	Quantum mechanics
State space	Phase space (point set)	Hilbert space
States	Points	Vectors
Properties	Functions of points	Eigenvalues of operators

Regardless of the question of pragmatic *vs.* realistic, the problematic concepts play a fundamental role in the formulation of quantum mechanics (or its postulates), and therefore certainly deserve a deeper understanding, or at least a deeper awareness. That is why we want to address briefly the key issues in the following.[18]

To avoid misunderstandings, we emphasize one remark: On the formal level—technically, so to speak—quantum mechanics works perfectly, with often impressively accurate results. In fact, quantum mechanics is one of the most carefully examined and well-tested physical theories in existence; it has yet to be falsified experimentally.

So the problem has to do with the level of *understanding* of quantum mechanics. What does all this imply, what does it mean? If we take the above postulates as a basis, some open and interrelated questions arise concerning the measurement process, which we summarize here briefly. The discussion is continued in Chap. 24 (decoherence) and Chap. 28 (interpretations), both in Vol. 2.[19]

1. **Status of the Measurement Process**: It is perhaps time to define more precisely the concept of *measurement*. By measurement, we understand the performance of an irreversible operation on a system which determines the status of one (or more) physical quantities, namely as a storable number. More remarks on the term 'measurement' (or on different but related concepts such as preparation, testing, maximum test etc.) can be found in Appendix S, Vol. 1.

   The special status of the measurement has nothing to do with whether one assumes pragmatically that values of a physical quantity have a meaning only as the result of a measurement, or if one asserts that the postulates are valid also for individual systems (realistic position). With regard to this relationship between the measurement and the values of physical quantities, the pragmatically-oriented must explain why the concept of 'measurement' plays such a fundamental role in quantum mechanics.[20] But also the realistically-oriented assign a prominent role to the measurement, since it causes the transition from the possible to the actual. It remains to investigate when an *interaction* between two systems *A* and *B* is the *measurement* of a physical quantity of *A* by *B*, and, in that context, whether and how one can describe the measurement in a quantum-mechanical way, i.e. including the measuring apparatus.

2. **Probability**: This term enters into the theory through Postulate 2.1. Before the measurement, one can in general specify only a probability that a particular

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[18]"But our present (quantum-mechanical) formalism is not purely epistemological; it is a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature - all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble. Yet we think that the unscrambling is a prerequisite for any further advance in basic physical theory. For, if we cannot separate the subjective and objective aspects of the formalism, we cannot know what we are talking about; it is just that simple." E.T. Jaynes in: *Complexity, Entropy and the Physics of Information* (ed. Zurek, W.H.) 381 (Addison-Wesley, 1990).

[19]In anticipation of our further discussion, we want to point out here that there are not answers to all the questions, at least not unique answers.

[20]This is especially true in the traditional view, according to which the measurement apparatus is to be regarded as a classical system.

measurement result will appear. This is true even if the maximum information about the system is known. Through the measurement, one and only one of the available options is realized. This means, in other words, that an observable generally does not have a definite value *before* a measurement. Correspondingly, measurement does not determine the value of an observable which it already has, but rather this value is created by the measurement itself—the measurement determines the reality, and not *vice versa*.

As explained in Chap. 2 and later on several times, the occurrence of probabilities in classical physics means that we do not have sufficient information at our disposal in order to calculate certain properties explicitly. In quantum mechanics, the situation is different. Here, the term 'probability' or 'objective chance' is literally a structural element of the theory and stands for the fact that quantum mechanics is concerned with possibilities, one of which is then realized by the measurement process. From the classical point of view, one would assume that beneath our level of formulation there are further, thus far hidden variables, knowledge of which would allow us to avoid the use of probabilities. But this assumption has been disproved experimentally (at least in its local or non-contextual, i.e. intuitively plausible form, see Chaps. 20 and 27, Vol. 2); apparently, we cannot avoid the concept of objective chance.

3. **Collapse**: How can we explain the change from a superposition to a single state as described in Postulate 2.3? What is the mechanism, what is its time frame? Is it a fundamental effect or only a pragmatic approximation to the description of a quantum system and a measuring apparatus which can be derived in principle from the existing formalism? That would obviously be particularly important if one does not want to attribute an essential meaning to the 'measurement', but would rather like to see it as simply one of many possible interactions.

This is not an exotic, constructed effect. We think e.g. of the right circular-polarized photon, already invoked several times, which we sent through a linear analyzer, only to find it afterwards in the state of e.g. horizontal linear polarization. In this process of measurement, the superposition $\frac{|h\rangle+i|v\rangle}{\sqrt{2}}$ is collapsed into the state $|h\rangle$.

Of course the answer to the question also depends on what we mean by 'state'.[21] Does it entail a direct description of the system, or only of our knowledge of the system? In the latter case, the collapse would represent a change in our knowledge, by adding more information. Otherwise, there would have to be a way to formulate the state reduction in direct physical terms.

As we will see in the discussion on entangled systems later on, the collapse of states is a non-local (i.e. superluminal) effect.

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[21]For example, 'state' can have the meanings:—an *individual* quantum system A,—our *knowledge* of the properties of the system A,—the result of a measurement that has been or could be performed on system A,—an *ensemble* E (real or hypothetical) of identically prepared copies of a system,—our *knowledge* of the properties of the ensemble E,—the results of *repeated* measurements that have been or could be performed on the ensemble E.

4. **Two Time Evolutions**: Through the measurement (observation), the wave function collapses. This change of state is discontinuous, irreversible, not deterministic in principle and therefore is in contrast to the continuous and reversible time evolution of the SEq. Accordingly, we have two very different processes or dynamics. This raises the following questions: Are the rules of quantum mechanics really different for observed and for unobserved systems? If so, why? What is an observer, must it be a human observer? Is the observer also subject to the laws of quantum mechanics? If so, how there can be irreversible evolutions that contradict the SEq? We will later find answers to some of these questions, especially in Chap. 24, Vol. 2.

5. **The Boundary Between Classical Mechanics and Quantum Mechanics**: If the measurement is not included in the SEq, does this mean that measurement is a non-quantum-mechanical process, so that the measuring apparatus is not subject to the rules of quantum mechanics? If that is the case—which rules apply instead? It is a common notion that the measuring apparatus obeys classical rules. Thus there would be two areas, the classical domain and quantum mechanics. But where is the cut (also called the *Heisenberg cut*) between quantum mechanics and classical mechanics, what begins where, what ends where? From which size, from which particle number on is a system no longer described by quantum mechanics, but instead by classical mechanics?

Of course, the idea of starting with small quantum mechanical systems and studying larger and larger ones to see whether and if so, how, they become 'more classical' is obvious. This fails, however, due to the fact that the description of larger quantum-mechanical systems with an increasing number of degrees of freedom becomes very quickly enormously complicated, so that a clear relationship between classical mechanics and quantum mechanics is difficult to elucidate.

One can in principle think of three possibilities: (1) classical mechanics includes quantum mechanics; (2) classical mechanics and quantum mechanics are on equal footing; and (3) quantum mechanics includes classical mechanics; see Fig. 14.1. The majority of the physics community favors the third option, i.e. that classical mechanics is based on quantum mechanics. But here, also, it has to be clarified
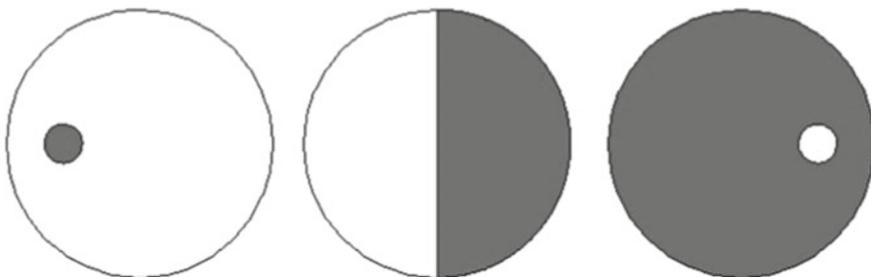

Fig. 14.1 Schematic figure of possible boundaries quantum mechanics–classical mechanics. *White*: classical mechanics, *black*: quantum mechanics

where the cut is (if it exists at all). In addition, there is the question: If quantum mechanics is the basic theory—why do we never see certain quantum effects in the macroscopic world, such as the superposition of states?

Let us consider the problem again from a different angle. The measuring apparatus consists of atoms and is thus itself a quantum system. In fact, it can interact only in this way with the measured quantum object. On the other hand, the measuring apparatus has to react as a classical system when it finally returns a result. Quantum system *and* classical system—these are two requirements for the measuring apparatus, which seem difficult to reconcile.

These problems of demarcation are particularly clear-cut in quantum cosmology, which attempts to describe the entire universe as a *single* quantum system. If one assumes that the universe is isolated and its dynamics (as a giant quantum system) are described by a *single* SEq (which is deterministic), then this poses the question of how measurement can be a process that is performed on a quantum system from the *outside*.

In any case, in practice, the distinction between a quantum-mechanical system and a classical measurement apparatus has been very successful (*fapp*). Whoever works only in a results-oriented manner may be satisfied with the minimal interpretation, i.e. with the argument that the wavefunction is not a description of real objects, but only a tool by which one can obtain the relevant results.[22]

We repeat that we will take up these issues again in later chapters. Especially the theory of decoherence (Chap. 24, Vol. 2) will alleviate most of these problems.

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[22]Discussing the nature of the wavefunction is not an ivory-tower topic, but rather the subject of ongoing research. For instance, as said above there is the view that the wavefunction reflects the partial knowledge which an experimenter has about the system. But such a view is wrong if one follows a recently published theorem; it is a no-go theorem which states that if the quantum state represents merely information about the real physical state of a system, then experimental predictions will be obtained which contradict those of quantum theory. However, this theorem depends on the crucial assumption that quantum systems have an objective underlying physical state—an assumption which is controversial. See e.g. Matthew F. Pusey et al., 'On the reality of the quantum state', *Nature Physics 8*, 475–478 (2012) or S. Mansfield, 'Reality of the quantum state: Towards a stronger ψ-ontology theorem', *Phys. Rev. A* 94, 042124 (Oct 2016).

Apart from that, we note that even though there is no explanation of the wavefunction in everyday terms, it is measurable. To date, the experimental determination of wavefunctions (i.e. modulus and phase or real and imaginary parts) has been accomplished by means of certain indirect methods (called tomographic methods). However, recently a method was presented for measuring wavefunctions *directly*. In it, a special technique is used, called *weak measurement*. See Jeff S. Lundeen et al., 'Direct measurement of the wave function', Nature 474, 188–191 (2011). We note that these measurements are performed on an ensemble; it is impossible to determine the completely unknown wavefunction of a *single* system. For other experimental methods see e.g. G.C. Knee, 'Towards optimal experimental tests on the reality of the quantum state', *New Journal of Physics* 19, 023004, https://doi.org/10.1088/1367-2630/aa54ab (Feb 2017).

14.3 Concluding Remarks

14.3.1 Postulates of Quantum Mechanics as a Framework

We have distilled out the postulates from considerations of simple example systems. This is possible precisely because the postulates do not depend on the specific system, but constitute something like the general framework or the general rules of quantum mechanics. In other words, the postulates are valid for all possible systems (referred to our frame of consideration), simple as well as complicated ones.

Hence it is clear that the postulates cannot act as instructions for the practical calculation of a physical problem. In order to do this, one has to determine the state space $\mathcal{H}$ and the Hamiltonian H for the system under consideration. Of course, it then becomes relevant which concrete physical system is selected, how it is modelled physically, which precision is required, and so on.

We have described e.g. neutrino oscillations without further ado in a two-dimensional space. This is obviously a very crude model, but it is entirely adequate for the intended purpose. In the analytical approach, we obtained the Hamiltonian by using the correspondence principle,[23] i.e. by replacement of the classical quantities $(\mathbf{r}, \mathbf{p})$ which occur in the energy by $\left(\mathbf{r}, \frac{\hbar}{i}\nabla\right)$. This allows us to represent a non-relativistic quantum object in a scalar potential, while vector potentials, interactions between a number of quantum objects, relativistic effects such as spin, etc. are not considered. We can see our simple model as the beginning of a 'hierarchy of models'; we address this point briefly again in Chap. 17, Vol. 2.

In short, the choice of $\mathcal{H}$ and H always means that one operates with certain models and approximations.[24] The postulates, however, are strict. This is schematically indicated in Fig. 14.2.

14.3.2 Outlook

The postulates in the form in which we have presented them give the foundation of quantum mechanics, but there are some complements needed. In subsequent chapters in Vol. 2, we will become acquainted with the following three:

1. We will extend the concept of *state* and also look at states that are no longer represented as vectors of an (extended) Hilbert space (so-called mixed states, keyword *density operator*, Chap. 22, Vol. 2).

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[23]We repeat the remark that this principle has a mainly heuristic value. A more convincing method is e.g. the introduction of position and momentum operators by means of symmetry transformations (see Chap. 21, Vol. 2, and Appendix L, Vol. 2).

[24]"Although this may be seen as a paradox, all exact science is dominated by the idea of approximation." (Bertrand Russell).

Fig. 14.2 The postulates as framework for the quantum mechanical description of physical systems
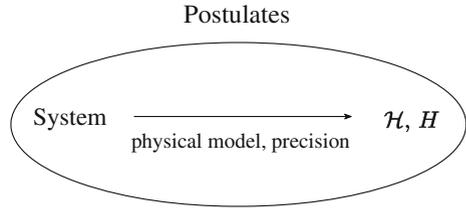
2. The systems considered so far are isolated, i.e. they are not coupled to any environment whatsoever. We will consider in the following also systems that are composed of several interacting subsystems (keyword *open systems*, Chap. 24, Vol. 2).
3. By an extension to open systems, we want to trace the separate role of the measurement (i.e. the projection postulate 2.3) back to the other postulates (keyword *decoherence*, Chap. 24, Vol. 2).

But first, in the initial chapters of Vol. 2, we will fill in the conceptual framework given above with some applications, some specialized subjects and extensions.

14.4   Exercises

1. Given are an observable A and a state $|\varphi\rangle$. Show by means of Postulates (2.1) and (2.2) that the expected result of a measurement of A is given by $\langle A \rangle = \langle\varphi| A |\varphi\rangle$. To simplify the discussion, we consider an observable A whose eigenvalues are discrete and nondegenerate and whose eigenvectors form a CONS, $A |n\rangle = a_n |n\rangle$.
2. Show that the operator $s_x + s_z$ is Hermitian, but does not represent a measurable physical quantity if understood literally, i.e., as the instruction to measure the x-component plus (and) the z-component of the spin. The spin matrices s_i are related to the Pauli matrices σ_i by $s_i = \frac{\hbar}{2}\sigma_i$.
3. (An example concerning projections, probabilities and expectation values.) The angular momentum operator $\mathbf{L}$ for angular momentum 1 can be represented in the vector space $\mathbb{C}^3$ by the following matrices (see Chap. 16, Vol. 2):

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad L_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad L_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$(14.6)$$

   (a) Which measured results are possible in a measurement of L_i ($i = x, y, z$)?
   (b) What are the corresponding eigenvectors for L_z?
   (c) What are the probabilities of measuring the results $+\hbar, 0, -\hbar$ on the state

$$|\psi\rangle = \begin{pmatrix} 1 \\ i \\ -2 \end{pmatrix}?$$
$$(14.7)$$

4. Given the state

$$|\psi\rangle_v = \frac{|x_1\rangle \, e^{-i\omega t} + |x_2\rangle \, e^{-2i\omega t}}{\sqrt{2}} \tag{14.8}$$

with normalized and mutually orthogonal states $|x_i\rangle$. We measure the x_1 component of $|\psi\rangle_v$. After the measurement, we have

$$|\psi\rangle_n = |x_1\rangle \, e^{-i\omega t} \tag{14.9}$$

Illustrate this state reduction by considering the change in the real or imaginary part of $|\psi\rangle$.