Discretization of Continuous Systems 18

The modal superposition method of analysis was applied in the preceding chapter to some simple structures having distributed properties. The determination of the response by this method requires the evaluation of several natural frequencies and corresponding mode shapes. The calculation of these dynamic properties is rather laborious, as we have seen, even for simple structures such as one-span uniform beams. The problem becomes increasingly more complicated and unmanageable as this method of solution is applied to more complex structures. However, the analysis of such structures becomes relatively simple if for each segment or element of the structure the properties are expressed in terms of dynamic coefficients much in the same manner as done previously when static deflection functions were used as an approximation to dynamic deflections in determining stiffness, mass, and other coefficients.

In this chapter the dynamic coefficients relating harmonic forces and displacements at the nodal coordinates of a beam segment are obtained from dynamic deflection functions. These coefficients can then be used to assemble the dynamic matrix for the whole structure by the direct method as shown in the preceding chapters for assembling the system stiffness and mass matrices. Also, in the present chapter, the mathematical relationship between the dynamic coefficients based on dynamic displacement functions and the coefficients of the stiffness and consistent mass matrices derived from static displacement functions is established.

18.1 Dynamic Matrix for Flexural Effects

As in the case of static influence coefficients (stiffness coefficients, for example), the dynamic influence coefficients also relate forces and displacements at the nodal coordinates of a beam element. The difference between the dynamic and static coefficients is that the dynamic coefficients refer to nodal forces and displacements that vary harmonically while the static coefficients relate static forces and displacements at the nodal coordinates. The dynamic influence coefficient S_{ij} is then defined as the harmonic force of frequency ϖ at nodal coordinates i, due to a harmonic displacement of unit amplitude and of the same frequency at nodal coordinate j.

To determine the expressions for the various dynamic coefficients for a uniform beam element as shown in Fig. 18.1, we refer to the differential equation of motion, Eq. (17.5), which in the absence of external loads in the span, that is, $p(x, t) = 0$, is

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$$EI\frac{\partial^4 y}{\partial x^4} + \bar{m}\frac{\partial^2 y}{\partial t^2} = 0 \tag{18.1}$$

For harmonic boundary displacements of frequency ϖ, we introduce in Eq. (18.1) the trial solution

$$y(x,t) = \Phi(x)\sin\varpi t \tag{18.2}$$

Substitution of Eq. (18.2) into Eq. (18.1) yields

$$\Phi^{IV}(x) - \bar{a}^4\Phi(x) = 0 \tag{18.3}$$

where

$$\bar{a}^4 = \frac{\bar{m}\varpi^2}{EI} \tag{18.4}$$

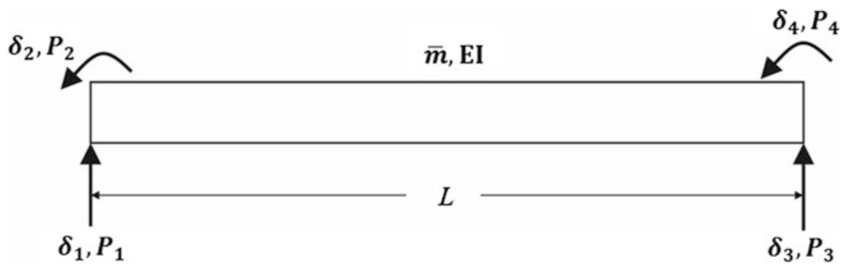

Fig. 18.1 Nodal coordinates of a flexural beam segment

We note that Eq. (18.3) is equivalent to Eq. (17.10), which is the differential equation for the shape function of a beam segment in free vibration. The difference between these two equations is that Eq. (18.3) is a function of the parameter $\bar{a}$ which, in turn, is a function of the forcing frequency ϖ while "a" in (17.10) depends on the natural frequency ω. The solution of Eq. (18.3) is of the same form as the solution of Eq. (17.10). Thus by analogy with Eq. (17.20), we can write.

$$\Phi(x) = C_1\sin\bar{a}x + C_2\cos\bar{a}x + C_3\sinh\bar{a}x + C_4\cosh\bar{a}x \tag{18.5}$$

Now, to obtain the dynamic coefficient for the beam segment, boundary conditions indicated by Eqs. (18.6) and (18.7) are imposed:

$$\begin{aligned} \Phi(0) &= \delta_1, & \Phi(L) &= \delta_3 \\ \Phi'(0) &= \delta_2, & \Phi'(L) &= \delta_4 \end{aligned} \tag{18.6}$$

Also

$$\begin{aligned} \Phi'''(0) &= \frac{P_1}{EI}, & \Phi'''(L) &= -\frac{P_3}{EI} \\ \Phi''(0) &= -\frac{P_2}{EI}, & \Phi''(L) &= \frac{P_4}{EI} \end{aligned} \tag{18.7}$$

In Eqs. (18.6), δ_1, δ_2, δ_3, and δ_4 are amplitudes of linear and angular harmonic displacements at the nodal coordinates while in Eqs. (18.7) P_1, P_2, P_3, and P_4 are the corresponding harmonic forces and moments as shown in Fig. 18.1. The substitution of the boundary conditions, Eqs. (18.6) and (18.7), into Eq. (18.5) results in

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \bar{a} & 0 & \bar{a} & 0 \\ s & c & S & C \\ \bar{a}c & -\bar{a}s & \bar{a}C & \bar{a}S \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{18.8}$$

and

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = EI \begin{bmatrix} -\bar{a}^3 & 0 & \bar{a}^3 & 0 \\ 0 & \bar{a}^2 & 0 & -\bar{a}^2 \\ \bar{a}^3 c & -\bar{a}^3 s & -\bar{a}^3 C & -\bar{a}^3 S \\ \bar{a}^2 s & -\bar{a}^2 c & \bar{a}^2 S & \bar{a}^2 C \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{18.9}$$

in which

$$\begin{aligned} s &= \sin \bar{a}L, \qquad S = \sinh \bar{a}L \\ c &= \cos \bar{a}L, \qquad C = \cosh \bar{a}L \end{aligned} \tag{18.10}$$

Next, Eq. (18.8) is solved for the constants of integration C_1, C_2, C_3, and C_4, which are subsequently substituted into Eq. (18.9). We thus obtain the dynamic matrix relating harmonic displacements and harmonic forces at the nodal coordinate of the beam element, namely.

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = B \begin{bmatrix} \bar{a}^2(cS + sC) & & \text{Symmetric} & \\ \bar{a}sS & sC - cS & & \\ -\bar{a}^2(s + S) & \bar{a}(c - C) & \bar{a}^2(cS + sC) & \\ \bar{a}(C - c) & S - s & -\bar{a}sS & sC - cS \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \tag{18.11}$$

where

$$B = \frac{\bar{a}EI}{1 - cC} \tag{18.12}$$

We require the denominator to be different from zero, that is,

$$1 - \cos \bar{a}L \cosh \bar{a}L \neq 0 \tag{18.13}$$

The element dynamic matrix in Eq. (18.11) can then be used to assemble the system dynamic matrix for a continuous beam or a plane frame in a manner entirely analogous to the assemblage of the system stiffness matrix from element stiffness matrices.

18.2 Dynamic Matrix for Axial Effects

The governing equation for axial vibration of a beam element is obtained by establishing the dynamic equilibrium of a differential element dx of the beam, as shown in Fig. 18.2. Thus

$$\begin{aligned} &\left(P + \frac{\partial P}{\partial x} dx\right) - P - (\bar{m}\, dx) \frac{\partial^2 u}{\partial t^2} = 0 \\ &\frac{\partial P}{\partial x} = \bar{m} \frac{\partial^2 u}{\partial t^2} \end{aligned} \tag{18.14}$$

where u is the displacement at x. The displacement at $x + dx$ will then be $u + (\partial u/\partial x)\,dx$. It is evident that the element dx in the new position has changed length by an amount $(\partial u/\partial x)\,dx$, and thus the strain is $\partial u/\partial x$. Since from Hooke's law the ratio of stress to strain is equal to the modulus of elasticity E, we can write

$$\frac{\partial u}{\partial x} = \frac{P}{AE} \tag{18.15}$$

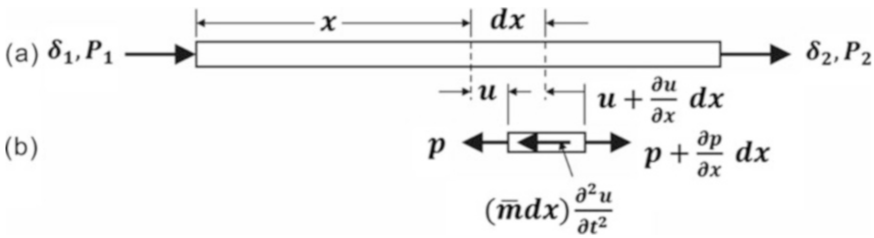

Fig. 18.2 Axial effects on a beam. (**a**) Nodal axial coordinates. (**b**) Forces acting on a differential element

where A is the cross-sectional area of the beam. Differentiating with respect to x results in

$$AE\frac{\partial^2 u}{\partial x^2} = \frac{\partial P}{\partial x} \tag{18.16}$$

and combining Eqs. (18.14) and (18.16) yields the differential equation for axial vibration of a beam segment, namely,

$$\frac{\partial^2 u}{\partial x^2} - \frac{\bar{m}}{AE}\frac{\partial^2 u}{\partial t^2} = 0 \tag{18.17}$$

A solution of Eq. (18.17) of the fond

$$u(x,t) = U(x)\sin\overline{\omega}t \tag{18.18}$$

will result in a harmonic motion of amplitude

$$U(x) = C_1 \sin bx + C_2 \cos bx \tag{18.19}$$

where

$$b = \sqrt{\frac{\bar{m}\overline{\omega}^2}{AE}} \tag{18.20}$$

and C_1 and C_2 are constants of integration.

To obtain the dynamic matrix for the axially vibrating beam segment, boundary conditions indicated by Eqs. (18.21) and (18.22) are imposed, namely.

$$U(0) = \delta_1, \qquad U(L) = \delta_2 \tag{18.21}$$

$$U'(0) = -\frac{P_1}{AE}, \qquad U'(L) = \frac{P_2}{AE} \tag{18.22}$$

where δ_1 and δ_2 are the displacements and P_1 and P_2 are the forces at the nodal coordinates of the beam segment as shown in Fig. 18.2.

Substitution of the boundary conditions, Eqs. (18.21) and (18.22), into Eq. (18.19) results in

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sin bL & \cos bL \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{18.23}$$

and

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = AEb \begin{bmatrix} -1 & 0 \\ \cos bL & -\sin bL \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{18.24}$$

Then, solving Eq. (18.23) for the constants of integration, we obtain

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -\cot bL & \operatorname{cosec} bL \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \tag{18.25}$$

subject to the condition

$$\sin bL \neq 0 \tag{18.26}$$

Finally, the substitution of Eq. (18.25) into Eq. (18.24) results in Eq. (18.27) relating harmonic forces and displacement at the nodal coordinates through the dynamic matrix for an axially vibrating beam segment. Thus we have.

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = EAb \begin{bmatrix} \cot bL & -\operatorname{cosec} bL \\ -\operatorname{cosec} bL & \cot bL \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \tag{18.27}$$

18.3 Dynamic Matrix for Torsional Effects

The equation of motion of a beam segment in torsional vibration is similar to that of the axial vibration of beams discussed in the preceding section. Let x (Fig. 18.3) be measured along the length of the beam. Then the angle of twist for any element of length dx of the beam due to a torque T is

$$d\theta = \frac{T\,dx}{J_T G} \tag{18.28}$$

where $J_T G$ is the torsional stiffness given by the product of the torsional constant J_T (J_T is the polar moment of inertial for circular sections) and the shear modulus of elasticity G. The torque applied on the faces of the element are T and $T + (\partial T/\partial x)\, dx$ as shown in Fig. 18.3. From Eq. (18.28), the net torque is then

$$\frac{\partial T}{\partial x} dx = J_T G \frac{\partial^2 \theta}{\partial x^2} dx \tag{18.29}$$

Equating this torque to the product of the mass moment of inertia $I_{\bar{m}}$ of the element dx and the angular acceleration $\partial^2\theta/\partial t^2$, we obtain the differential equation of motion

$$J_T G \frac{\partial^2 \theta}{\partial x^2} dx = I_{\bar{m}} \frac{\partial^2 \theta}{\partial t^2} dx$$

or

$$\frac{\partial^2\theta}{\partial x^2} - \frac{I_{\bar{m}}}{J_T G}\frac{\partial^2\theta}{\partial t^2} = 0 \tag{18.30}$$

where $I_{\bar{m}}$ is the mass moment of inertia per unit length about the longitudinal axis x given by

$$I_{\bar{m}} = \bar{m}\frac{I_0}{A} \tag{18.31}$$

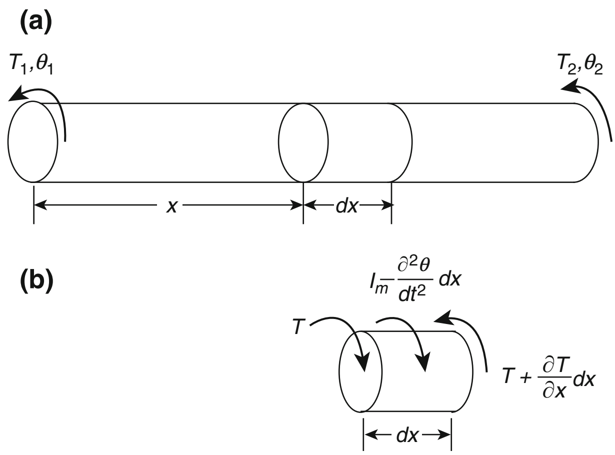

Fig. 18.3 Torsional effects on a beam. (**a**) Nodal torsional coordinates. (**b**) Moments acting on a differential element

in which I_0 is the polar moment of inertia of the cross-sectional area A.

We seek a solution of Eq. (18.30) in the form

$$\theta(x,t) = \theta(x)\sin\overline{\omega}t$$

which, upon substitution into Eq. (18.30), results in a harmonic torsional motion of amplitude

$$\theta(\mathrm{x}) = C_1 \sin cx + C_2 \cos cx \tag{18.32}$$

in which

$$c = \sqrt{\frac{I_{\bar{m}}\overline{\omega}^2}{J_T G}} \tag{18.33}$$

For a circular section, the torsional constant J_T is equal to the polar moment of inertia I_0. Thus Eq. (18.33) reduces to

$$c = \sqrt{\frac{\bar{m}\overline{\omega}^2}{AG}} \tag{18.34}$$

Since $I_{\bar{m}} = I_0\bar{m}/A$ as indicated by Eq. (18.31).

We note that Eq. (18.30) for torsional vibration is analogous to Eq. (18.17) for axial vibration of beam segments. It follows that by analogy to Eq. (18.27) we can write the dynamic relation between torsional moments and rotations in a beam segment. Hence

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = J_T \mathrm{Gc}\begin{bmatrix} \cot cL & -\operatorname{cosec} cL \\ -\operatorname{cosec} cL & \cot cL \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \tag{18.35}$$

18.4 Beam Flexure Including Axial-Force Effect

When a beam is subjected to a force along its longitudinal axis in addition to lateral loading, the dynamic equilibrium equation for a differential element of the beam is affected by the presence of this force. Consider the beam shown in Fig. 18.4 in which the axial force is assumed to remain constant during flexure with respect to both magnitude and direction. The dynamic equilibrium for a differential element dx of the beam (Fig. 18.4b) is established by equating to zero both the sum of the forces and the sum of the moments.

Summing forces in the y direction, we obtain

$$V + p(x,t)dx - \left(V + \frac{\partial V}{\partial x}dx\right) - (\bar{m}\ \ dx)\frac{\partial^2 y}{\partial t^2} = 0 \tag{18.36}$$

which, upon reduction, yields

$$\frac{\partial V}{\partial x} + \bar{m}\frac{\partial^2 y}{\partial t^2} = p(x,t) \tag{18.37}$$

The summation of moments about point 0 gives

$$M + Vdx - \left(M + \frac{\partial M}{\partial x}\right) + \frac{1}{2}\left(p(x,t) - \bar{m}\frac{\partial^2 y}{\partial t^2}\right)dx^2 - N\frac{\partial y}{\partial x}dx = 0 \tag{18.38}$$

Discarding higher order terms, we obtain for the shear force the expression

$$V = N\frac{\partial y}{\partial x} + \frac{\partial M}{\partial x} \tag{18.39}$$

Then using the familiar relationship from bending theory,

$$M = EI\frac{\partial^2 y}{\partial x^2} \tag{18.40}$$

and combining Eqs. (18.37), (18.39), and (18.40), we obtain the equation of motion of a beam segment including the effect of the axial forces, that is,

$$EI\frac{\partial^4 y}{\partial x^4} + N\frac{\partial^2 y}{\partial x^2} + \bar{m}\frac{\partial^2 y}{\partial t^2} = p(x,t) \tag{18.41}$$

A comparison of Eqs. (18.41) and (17.5) reveals that the presence of the axial force gives rise to an additional transverse force acting on the beam. As indicated previously in Sect. 17.1, in the derivation of Eq. (18.41) it has been assumed that the deflections are small and that the deflections due to shear forces or rotary inertia are negligible.

In the absence of external loads applied to the span of the beam, Eq. (18.41) reduces to

$$EI\frac{\partial^4 y}{\partial x^4} + N\frac{\partial^2 y}{\partial x^2} + \bar{m}\frac{\partial^2 y}{\partial t^2} = 0 \tag{18.42}$$

The solution of Eq. (18.42) is found as before by substituting

$$y(x,t) = \Phi(x)\sin\bar{\omega}t \tag{18.43}$$

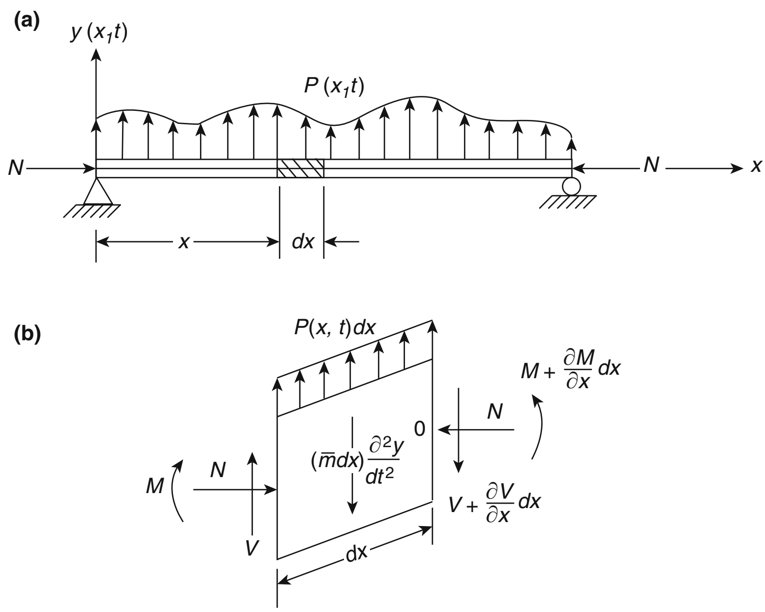

Fig. 18.4 Beam supporting constant axial force and lateral dynamic load. (**a**) Loaded beam. (**b**) Forces acting on a differential element

We thereby obtain the ordinary differential equation

$$\frac{d^4\Phi}{dx^4} + \frac{N}{EI}\frac{d^2\Phi}{dx^2} - \frac{\bar{m}\bar{\omega}^2}{EI}\Phi = 0 \tag{18.44}$$

The solution of Eq. (18.44) is

$$\Phi(x) = A\sin p_2 x + B\cos p_2 x + C\sinh p_1 x + D\cosh p_1 x \tag{18.45}$$

Where A, B, C, and D are constants of integration and

$$p_1 = \sqrt{\frac{-\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + \beta}}$$

$$p_2 = \sqrt{\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + \beta}} \tag{18.46}$$

$$\alpha = \frac{N}{EI} \tag{18.47}$$

$$\beta = \frac{\bar{m}\bar{\omega}^2}{EI} \tag{18.48}$$

To obtain the dynamic matrix (which in this case includes the effect of axial forces) for the transverse vibration of the beam element, the boundary conditions, Eqs. (18.49), are imposed, namely

$$\Phi(0) = \delta_1, \qquad \Phi(L) = \delta_3$$
$$\frac{d\Phi(0)}{dx} = \delta_2, \qquad \frac{d\Phi(L)}{dx} = \delta_4$$

$$\frac{d^3\Phi(0)}{dx^3} = \frac{P_1}{EI} - \frac{N}{EI}\delta_2, \qquad \frac{d^3\Phi(L)}{dx^3} = -\frac{P_3}{EI} - \frac{N}{EI}\delta_4$$
$$\frac{d^3\Phi(0)}{dx^3} = -\frac{P_2}{EI}, \qquad \frac{d^2\Phi(L)}{dx^2}\frac{P_4}{EI} \tag{18.49}$$

In Eqs. (18.49) δ_1, δ_3 and δ_2, δ_4 are, respectively, the transverse and angular displacements at the ends of the beam, while P_1, P_3 and P_2, P_4 are corresponding forces and moments at these nodal coordinates. The substitution into Eq. (18.45) of the boundary conditions given by Eqs. (18.49) results in a system of eight algebraic equations which upon elimination of the four constants of integration A, B, C, and D yields the dynamic matrix (including the effect of axial forces) relating harmonic forces and displacements at the nodal coordinates of a beam segment. The final result is

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} S_{11} & & Symmetric & \\ S_{21} & S_{22} & & \\ S_{31} & S_{32} & S_{33} & \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \tag{18.50}$$

where

$$\begin{aligned}
S_{11} &= S_{33} = B\left[\left(p_1^2 p_2^3 + p_1^4 p_2\right)cS + \left(p_1 p_2^4 + p_1^3 p_2^2\right)sC\right] \\
S_{21} &= -S_{43} = B\left[\left(p_1 p_2^3 - p_1^3 p_2\right) + \left(p_1^3 p_2 - p_1 p_2^3\right)cC + 2p_2^2 p_2^2 sS\right] \\
S_{22} &= S_{44} = B\left[\left(p_2^2 p_1 + p_1^3\right)sC - \left(p_2^3 + p_1^2 p_2\right)cS\right] \\
S_{41} &= -S_{32} = B\left[\left(p_1 p_2^3 + p_1^3 p_2\right)(C - c)\right] \\
S_{31} &= B\left[\left(-p_1^2 p_2^3 - p_1^4 p_2\right)S - \left(p_1^3 p_2^2 + p_1 p_2^4\right)s\right] \\
S_{42} &= B\left[\left(p_1^2 p_2 + p_2^3\right)S - \left(p_1 p_2^2 + p_1^3\right)s\right]
\end{aligned} \tag{18.51}$$

In the above, the letters s, c, S, and C denote

$$s = \sin p_2 L, \qquad S = \sinh p_1 L$$
$$c = \cos p_2 L, \qquad C = \cosh p_1 L$$

and the letter B denotes

$$B = \frac{EI}{2p_1 p_2 - 2p_1 p_2 cC + \left(p_1^2 - p_2^2\right)sS} \tag{18.52}$$

Furthermore, Eq. (18.50) is subject to the condition

$$2p_1 p_2 - 2p_1 p_2 cC + \left(p_1^2 - p_2^2\right)sS \neq 0 \tag{18.53}$$

18.5 Power Series Expansion of the Dynamic Matrix for Flexural Effects

It is of interest to demonstrate that the influence coefficients of the stiffness matrix, Eq. (10.20), and of the consistent mass matrix, Eq. (10.34), may be obtained by expanding the influence coefficients of the dynamic matrix in a Taylor's series (Paz 1973). For the sake of the discussion, we consider the dynamic coefficient from the second row and first column of the dynamic matrix, Eq. (18.11),

$$S_{21} = \frac{a^2 EI \sin \bar{a}L \sinh \bar{a}L}{1 - \cos \bar{a}L \cosh \bar{a}L} \tag{18.54}$$

In the following derivation, operations with power series, including addition, subtraction, multiplication, and division, are employed. The validity of these operations and convergence of the resulting series is proved in Knopp.[1] In general, convergent power series may be added, subtracted, or multiplied and the resulting series will converge at least in the common interval of convergence of the two original series. The operation of division of two power series may be carried out formally; however, the determination of the radius of convergence of the resulting series is more complicated. It requires the use of theorems in the field of complex variables and it is related to analytical continuation. Very briefly, it can be said that the power series obtained by division of two convergent power series about a complex point Z_0 will be convergent in a circle with center Z_0 and of radius given by the closest singularity to Z_0 of the functions represented by the series in the numerator and denominator.

The known expansions in power series about the origin of trigonometric and hyperbolic functions are used in the intermediate steps in expanding the function in Eq. (18.54), namely,

$$\cos x \cosh x = 1 - \frac{x^4}{6} + \frac{x^8}{2520} - \frac{x^{12}}{7,484,400} + \dots$$
$$(1 - \cos x \cosh x)^{-1} = \frac{6}{x^4} + \frac{1}{70} + \frac{85x^4}{2,910,600} + \dots$$
$$\sin x \sinh x = x^2 - \frac{x^6}{90} + \frac{x^{10}}{113,400} - \dots$$

where $x = \bar{a}L$. Substitution of these series equations in the dynamic coefficient, Eq. (18.54), yields

$$S_{21} = \frac{\bar{a}^2 EI \sin \bar{a}L \sinh \bar{a}L}{1 - \cos \bar{a}L \cosh \bar{a}L} = \frac{6EI}{L^2} - \frac{11\bar{m}L^2\bar{\omega}^2}{210} - \frac{223\bar{m}^2L^6\bar{\omega}^4}{2,910,600EI} \tag{18.55}$$

The first term on the right-hand side of Eq. (18.55) is the stiffness coefficient k_{21} in the stiffness matrix, Eq. (10.20), and the second term, the consistent mass coefficient m_{21} in the mass matrix, Eq. (10.34). The series expansion, Eq. (18.55), is convergent in the positive real field for

$$0 < \bar{a}L < 4.73 \tag{18.56}$$

or from Eq. (18.4)

$$0 < \bar{\omega} < (4.73)^2 \sqrt{\frac{EI}{\bar{m}L^4}} \tag{18.57}$$

In Eq. (18.56) the numerical value 4.73 is an approximation of the closest singularity to the origin of the functions in the quotient expanded in Eq. (18.54).

The series expansions for all the coefficients in the dynamic matrix, Eq. (18.11), are obtained by the method explained in obtaining the expansion of the coefficient S_{21}. These series expansions are:

[1] Knopp, K., Theory and Application of Infinite Series, Blackie, London, 1963.

$$
\begin{aligned}
S_{33} = S_{11} &= \frac{12EI}{L^3} - \frac{13L\bar{m}\bar{\omega}^2}{35} - \frac{59L^5\bar{m}^2\bar{\omega}^4}{161,700EI} - \ldots \\
S_{21} = -S_{43} &= \frac{6EI}{L^2} - \frac{11L^2\bar{m}\bar{\omega}^2}{210} - \frac{223L^6\bar{m}^2\bar{\omega}^4}{2,910,600EI} - \ldots \\
S_{41} = -S_{32} &= \frac{6EI}{L^2} - \frac{13L^2\bar{m}\bar{\omega}^2}{420} + \frac{1681L^6\bar{m}^2\bar{\omega}^4}{23,284,800EI} - \ldots \\
S_{22} = -S_{44} &= \frac{4EI}{L} - \frac{L^3\bar{m}\bar{\omega}^2}{105} - \frac{71L^7\bar{m}^2\bar{\omega}^4}{4,365,800EI} - \ldots \\
S_{31} &= -\frac{12EI}{L^3} - \frac{9L\bar{m}\bar{\omega}^2}{70} - \frac{1279L^5\bar{m}^2\bar{\omega}^4}{3,880,800EI} - \ldots \\
S_{42} &= \frac{2EI}{L} - \frac{L^3\bar{m}\bar{\omega}^2}{140} - \frac{1097L^7\bar{m}^2\bar{\omega}^4}{69,854,400EI} - \ldots
\end{aligned}
\qquad (18.58)
$$

18.6 Power Series Expansion of the Dynamic Matrix for Axial and for Torsional Effects

Proceeding in a manner entirely analogous to expansion of the dynamic coefficients for flexural effects, we can also expand the dynamic coefficients for axial and for torsional effects. The Taylor's series expansions, up to three terms, of the coefficients of the dynamic matrix in Eq. (18.27) (axial effects) are.

$$
\begin{aligned}
AEb\cot bL &= \frac{AE}{L} - \frac{\bar{m}\bar{\omega}^2 L}{3} - \frac{L^3\bar{m}^2\bar{\omega}^4}{45AE} - \ldots \\
-AEb\,\mathrm{cosec}\,bL &= -\frac{AE}{L} - \frac{\bar{m}\bar{\omega}^2 L}{6} - \frac{7L^3\bar{m}^2\bar{\omega}^4}{300AE} - \ldots
\end{aligned}
\qquad (18.59)
$$

It may be seen that the first term in each series of Eq. (18.59) is equal to the corresponding stiffness coefficient of the matrix in Eq. (11.3), and the second term to the consistent mass coefficient of the matrix in (11.26). Similarly, the Taylor's series expansions of the coefficients of the dynamic matrix for torsional effects, Eq. (18.35), are

$$
\begin{aligned}
J_T Gc\cot cL &= \frac{J_T G}{L} - \frac{LI_{\bar{m}}\bar{\omega}^2}{3} - \frac{L^3 I_{\bar{m}}^2\bar{\omega}^4}{45GJ_T} - \ldots \\
-J_T Gc\,\mathrm{cosec}\,cL &= -\frac{J_T G}{L} - \frac{LI_{\bar{m}}\bar{\omega}^2}{6} - \frac{7L^3 I_{\bar{m}}^2\bar{\omega}^4}{300GJ_T} - \ldots
\end{aligned}
\qquad (18.60)
$$

Comparing the first two terms of the above series with the stiffness and mass influence coefficients of the matrices in Eqs. (12.7) and (12.8), we find that for torsional effects the first term is also equal to the stiffness coefficient, and the second term to the consistent mass coefficient.

18.7 Power Series Expansion of the Dynamic Matrix Including the Effects of Axial Forces

The series expansions of the coefficients of the dynamic matrix, Eq. (18.50) (with axial effects), are obtained by the method described in the last two sections. Detailed derivation of these expansions are given by Paz and Dung (1975). The series expansion of the dynamic matrix, Eq. (18.50), is

$$[S] = [K] - [G_0]N - [M_0]\overline{\omega}^2 - [G_1]N^2 - [M_1]\overline{\omega}^4 - \ldots \tag{18.61}$$

where the first three matrices in this expansion [K], $[G_0]$, and $[M_0]$ are, respectively, the stiffness, geometric, and mass matrices which were obtained in previous chapters on the basis of static displacement functions. These matrices are given, respectively, by Eqs. (10.20), (10.45), and (10.34). The other matrices in Eq. (18.61) corresponding to higher order terms are represented as follows. The second-order mass-geometrical matrix

$$[A_1] = \frac{\bar{m}L^3}{EI}\begin{bmatrix} \frac{1}{3150} & & \text{Symmetric} & \\ \frac{L}{1360} & \frac{L^2}{3150} & & \\ -\frac{1}{3150} & \frac{L}{1680} & \frac{1}{3150} & \\ -\frac{L}{1680} & \frac{L^2}{3600} & -\frac{L}{1260} & \frac{L^2}{3150} \end{bmatrix}$$

The second-order geometrical matrix:

$$[G_1] = \frac{1}{EI}\begin{bmatrix} \frac{L}{700} & & \text{Symmetric} & \\ \frac{L^2}{1400} & \frac{11L^2}{6300} & & \\ -\frac{L}{700} & -\frac{L^2}{1400} & \frac{L}{700} & \\ \frac{L^2}{1400} & -\frac{13L^2}{12600} & -\frac{L^2}{1400} & \frac{11L^3}{6300} \end{bmatrix}$$

The second-order mass matrix:

$$[M_1] = \frac{\bar{m}^2L^3}{1000EI}\begin{bmatrix} \frac{59}{161.7} & & \text{Symmetric} & \\ \frac{223L}{2910.6} & \frac{71L^2}{4365.9} & & \\ \frac{1279}{3880.8} & \frac{1681L}{23284.8} & \frac{59}{161.7} & \\ -\frac{1681L}{23284.8} & -\frac{1097L^2}{69854.4} & -\frac{223L}{2910.6} & \frac{71L^2}{4365.9} \end{bmatrix}$$

18.8 Summary

The dynamic coefficients relating harmonic forces and displacements at the nodal coordinates of a beam segment were obtained from dynamic deflection equations. These coefficients can then be used in assembling the dynamic matrix for the entire structure by the same procedure (direct method) employed in assembling the stiffness and mass matrices for discrete systems.

In this chapter it has been demonstrated that the stiffness, consistent mass, and other influence coefficients may be obtained by expanding the dynamic influence coefficients in Taylor's series. This mathematical approach also provides higher order influence coefficients and the determination of the radius of convergence of the series expansion.