12 Dynamic Analysis of Grid Frames

In Chap. 11 consideration was given to the dynamic analysis of the plane frame when subjected to forces acting on the plane of the structure. When the planar structural system is subjected to loads applied normally to its plane, the structure is referred to as a grid frame. This structure can also be treated as a special case of the three-dimensional frame to be presented in Chap. 13. The reason for considering the planar frame, whether loaded in its plane or normal to its plane, as a special case, is the immediate reduction of unknown nodal coordinates for an element of these special structures, hence a considerable reduction in the number of unknown displacements for the structural system.

When analyzing the planar frame under action of loads in the plane, the possible components of joint displacements that had to be considered were translations in the X and Y directions and rotation about the Z axis. However, if a plane frame is loaded normal to the plane of the structure, the components of joint displacements required to describe the displacements of a joint are a translation in the Z direction and rotations about the X and Y axes. Thus treating the planar grid structure as a special case, it will be necessary to consider only three components of nodal displacements at each end of a typical element of a grid frame.

12.1 Local and Global Coordinate Systems

For an element of a grid frame, the local orthogonal axes will be established such that the x defines the longitudinal centroidal axis of the member and the x-y plane will coincide with the plane of the structural system. In this case, the z axis will define the minorprincipal axis of the cross section while the y axis will define the major axis of the cross section. It will be assumed that the shear center of the cross section coincides with the centroid of the cross section. The grid member may have either a variable or constant cross section along its length.

The possible nodal displacements with respect to the local or to the global systems of coordinates are identified in Fig. 12.1. It can be seen that the linear displacements along the z direction for local axes and along the Z direction for the global system are identical since the two axes coincide. However, in general, rotational components at the nodal coordinates differ for these two coordinate systems. Hence, a transformation of coordinates will be required to transform the element matrices from the local to the global coordinates.

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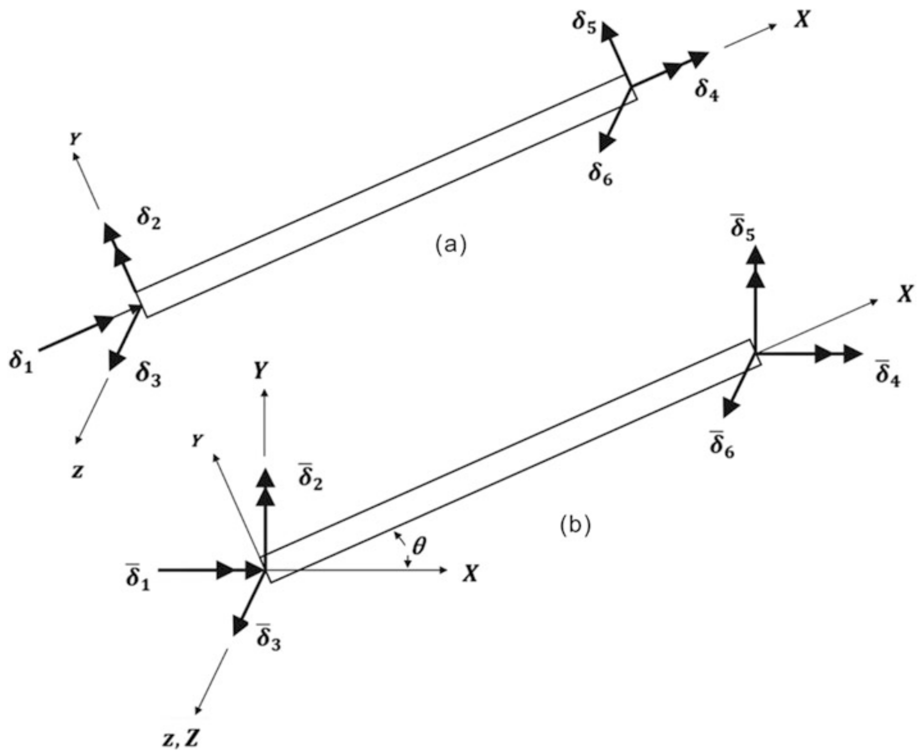

Fig. 12.1 Components of nodal displacements for a grid member. (**a**) Local coordinate system. (**b**) Global coordinate system

12.2 Torsional Effects

The dynamic analysis by the stiffness method for grid frames, that is, for plane frames subjected to normal loads, requires the determination of the torsional stiffness and mass coefficients for a typical element of the grid frame. The derivation of these coefficients is essentially identical to the derivation of the stiffness and mass coefficients for axial effects on a beam element. Similarity between these two derivations occurs because the differential equations for both problems have the same mathematical form. For the axial problem, the differential equation for the displacement function is given by Eq. (11.8) as

$$\frac{du}{dx} = \frac{P}{AE} \text{ (repeated)} \tag{11.8}$$

Likewise, the differential equation for torsional displacement is

$$\frac{d\theta}{dx} = \frac{T}{JG} \tag{12.1}$$

in which θ is the angular displacement, T is the torsional moment, G is the modulus of elasticity in shear, and J is the torsional constant of the cross section (polar moment of inertia for circular sections).

As a consequence of the analogy between Eqs. (11.8) and (12.1), we can write the following results already obtained for axial effects. The displacement functions for the torsional effects are the same as the corresponding functions giving the displacements for axial effects; hence by analogy to Eqs. (11.10) and (11.11) and in reference to the nodal coordinates of Fig. 12.2, we obtain

$$\theta_1(x) = \left(1 - \frac{x}{L}\right) \tag{12.2}$$

and

$$\theta_2(x) = \frac{x}{L} \tag{12.3}$$

in which the angular displacement function $\theta_1(x)$ corresponds to a unit angular displacement $\delta_1 = 1$ at nodal coordinate 1 and $\theta_2\,(x)$ corresponds to the displacement.

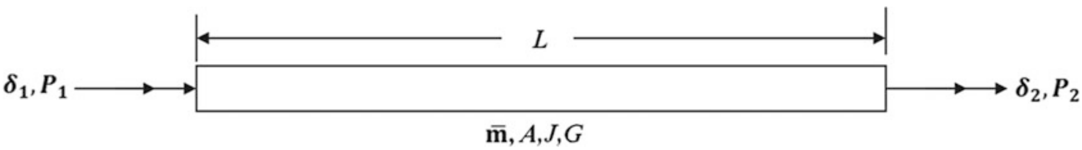

Fig. 12.2 Nodal torsional coordinates for a beam element

function resulting from a unit angular displacement $\delta_2 = 1$ at nodal coordinate 2. Analogous to Eq. (11.17), the stiffness coefficients for torsional effects may be calculated from

$$k_{ij} = \int_0^L JG\theta_i'(x)\theta_j'(x)dx \tag{12.4}$$

in which θ_1'(x) and θ_2'(x) are the derivatives with respect to x of the displacement functions $\theta_1(x)$ and $\theta_2(x)$. Also analogous to Eq. (11.23), the consistent mass matrix coefficients for torsional effects are given by

$$m_{ij} = \int_0^L I_{\bar{m}}\theta_i(x)\theta_j'(x)dx \tag{12.5}$$

in which $I_{\bar{m}}$ is the polar mass moment of inertia, per unit length along the beam element. This moment of inertia may conveniently be expressed as the product of the mass $\bar{m}$ per unit length times the radius of gyration squared, k^2. The radius of gyration may, in turn, be calculated as the ratio I_0/A. Therefore, the mass polar moment of inertia per unit length $I_{\bar{m}}$ is given by

$$I_{\bar{m}} = \bar{m}\frac{I_0}{A} \tag{12.6}$$

in which I_0 is the polar moment of inertia of the cross-sectional area and A the cross-sectional area.

The application of Eqs. (12.4) and (12.5) for a uniform beam yields the stiffness and mass matrices for torsional effects as

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{JG}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} \tag{12.7}$$

and

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{I_{\bar{m}} L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{Bmatrix} \tag{12.8}$$

in which $I_{\bar{m}}$ is given by Eq. (12.6), and T_1, T_2 are torsional moments at the ends of the beam element labeled in Fig. 12.2 as P_1 and P_2.

12.3 Stiffness Matrix for a Grid Element

The torsional stiffness matrix, Eq. (12.7), is combined with the flexural stiffness matrix, Eq. (10.20), to obtain the stiffness matrix for a typical element of a grid frame. In reference to the local coordinate system indicated in Fig. 12.1a, the stillness equation for a uniform element is then

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{EI}{L^3} \begin{Bmatrix} JGL^2/EI & & & & & \\ 0 & 4L^2 & & & & \\ 0 & -6L & 12 & & & \\ -JGL^2/EI & 0 & 0 & JGL^2/EI & & \\ 0 & 2L^2 & -6L & 0 & 4L^2 & \\ 0 & 6L & -12 & 0 & 6L & 12 \end{Bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \tag{12.9}$$

or in condensed form

$$\{P\} = [K]\{\delta\} \tag{12.10}$$

12.4 Consistent Mass Matrix for a Grid Element

The combination of the consistent mass matrix for flexural effects (10.34) with the consistent mass matrix for torsional effects (12.8) results in the consistent mass matrix for a typical member of a grid, namely

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\bar{m}L}{420} \begin{bmatrix} 140I_0/A & & & \textit{Symmetric} & & \\ & 4L^2 & & & & \\ 0 & 22L & 156 & & & \\ 70I_0/A & 0 & 0 & 140I_0/A & & \\ 0 & -3L^2 & -13L & 0 & 4L^2 & \\ 0 & 13L & 54 & 0 & -22L & 156 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \tag{12.11}$$

or in concise notation

$$\{P\} = [M_c]\{\delta\} \tag{12.12}$$

in which $[M_c]$ is the mass matrix for a typical uniform member of a grid structure.

12.5 Lumped Mass Matrix for a Grid Element

The lumped mass allocation to the nodal coordinates of a typical grid member is obtained from static considerations. For a uniform member having a uniform distributed mass along its length, the nodal mass is simply one-half of the total rotational mass $I_{\bar{m}}L$. The matrix equation for the lumped mass matrix corresponding to the torsional effects is then

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{I_{\bar{m}}L}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{bmatrix} \tag{12.13}$$

in which $I_{\bar{m}}$ is given by Eq. (12.6). The combination of the lumped torsional mass and the lumped translatory mass results in the diagonal matrix which is the lumped mass matrix for the grid element. This matrix, relating forces and accelerations at nodal coordinates, is given by the following equation:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\bar{m}L}{2} \begin{bmatrix} I_0/A & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & I_0/A & & \\ & & & & 0 & \\ & & & & & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \\ \ddot{\delta}_5 \\ \ddot{\delta}_6 \end{Bmatrix} \tag{12.14}$$

or briefly

$$\{P\} = \lceil M_L \rfloor \{\delta\} \tag{12.15}$$

in which $\lceil M_L \rfloor$ is, in this case, the diagonal lumped mass matrix for a grid element.

12.6 Transformation of Coordinates

The stiffness matrix, Eq. (12.9), as well as the consistent and the lumped mass matrices in Eqs. (12.11) and (12.14), respectively, are in reference to the local system of coordinates. Therefore, it is necessary to transform the reference of these matrices to the global system of coordinates before their assemblage in the corresponding matrices for the structure. As has been indicated, the z axis for the local coordinate system coincides with the Z axis for the global system. Therefore, the only step left to perform is a rotation of the coordinates in the x-y plane. The corresponding matrix for this transformation may be obtained by establishing the relationship between components of the moment at the nodes expressed in these two systems of coordinates. In reference to Fig. 12.3, these relations when written for node ① are

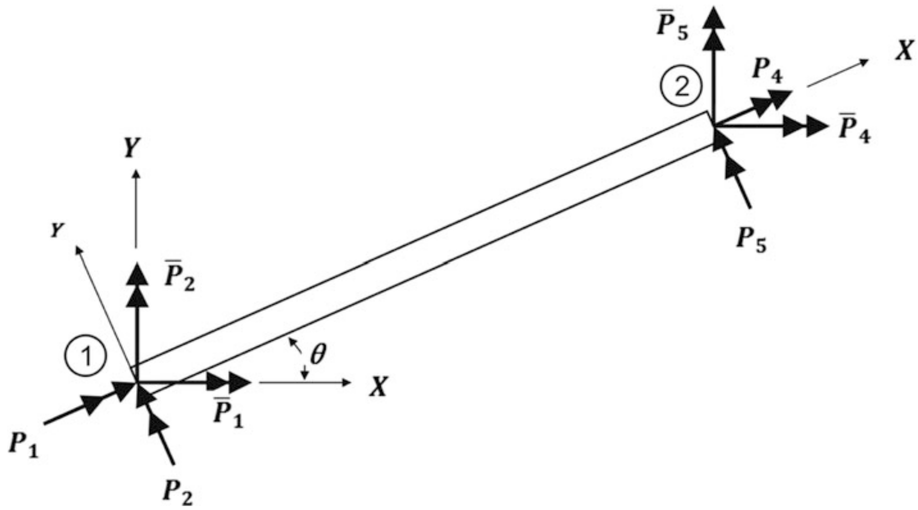

Fig. 12.3 Components of the nodal moments in local and global coordinates

$$\begin{aligned} P_1 &= \bar{P}_1 \cos\theta + \bar{P}_2 \sin\theta \\ P_2 &= -\bar{P}_1 \sin\theta + \bar{P}_2 \sin\theta \\ P_3 &= \bar{P}_3 \end{aligned} \tag{12.16a}$$

and for node ②

$$\begin{aligned} P_4 &= \bar{P}_4 \cos\theta + \bar{P}_5 \sin\theta \\ P_5 &= -\bar{P}_4 \sin\theta + \bar{P}_5 \sin\theta \\ P_6 &= \bar{P}_6 \end{aligned} \tag{12.16b}$$

The identical form of these equations with those derived for the transformation of coordinates for nodal forces of an element of a plane frame, Eqs. (11.28) and (11.30), should be noted. Equations (12.16) may be written in matrix notation as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \\ \bar{P}_4 \\ \bar{P}_5 \\ \bar{P}_6 \end{Bmatrix} \tag{12.17}$$

or in short notation

$$\{P\} = [T]\{\bar{P}\} \tag{12.18}$$

in which $\{P\}$ and $\{\bar{P}\}$ are, respectively, the vectors of the nodal forces of a typical grid member in local and global coordinates and $[T]$ the transformation matrix. The same transformation matrix $[T]$

serves also to transform the nodal components of the displacements from a global to a local system of coordinates. In condensed notation, this relation is given by

$$\{\delta\} = [T]\{\bar{\delta}\} \tag{12.19}$$

where $\{\delta\}$ and $\{\bar{\delta}\}$ are, respectively, the components of nodal displacements in local and global coordinates. The substitution of Eqs. (12.18) and (12.19) in the stiffness relation Eq. (12.10) yields the element stiffness matrix in reference to the global coordinate system, that is,

$$[T]\{\bar{P}\} = [K][T]\{\bar{\delta}\}$$

or, since $[T]$ is an orthogonal matrix $[(T)^{-1} = (T)^T]$, it follows that

$$\{\bar{P}\} = [T]^T[K][T]\{\bar{\delta}\}$$

or

$$\{\bar{P}\} = [\bar{K}]\{\bar{\delta}\} \tag{12.20}$$

in which

$$\{\bar{K}\} = [T]^T[K][T] \tag{12.21}$$

is the stiffness matrix of an element of a grid frame in reference to the global system of coordinates.

Analogously, for the mass matrix, we find

$$\{\bar{P}\} = [\bar{M}]\{\ddot{\bar{\delta}}\} \tag{12.22}$$

in which

$$\{\bar{M}\} = [T]^T[M][T] \tag{12.23}$$

is the transformed mass matrix.

Illustrative Example 12.1

Figure 12.4 shows a grid frame in a horizontal plane consisting of two prismatic beam elements with a total of three degrees of freedom as indicated. Determine the natural frequencies and corresponding mode shapes. Use the consistent mass formulation.

Solution:

The stiffness matrix for elements 1 or 2 of the grid frame in reference to the local system of coordinates, by Eq. (12.9), is

$$[K_1] = 10^6 \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 40 & 0 & 0 & -40 & 0 & 0 \\ 0 & 200 & -5 & 0 & 100 & 5 \\ 0 & -5 & 0.167 & 0 & -5 & -0.167 \\ -40 & 0 & 0 & 40 & 0 & 0 \\ 0 & 100 & -5 & 0 & 200 & 5 \\ 0 & 5 & -0.167 & 0 & 5 & 0.167 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

The transformation matrix for element ⚠ with $\theta = 0^0$ is simply the unit matrix $[T_1] = [I]$. Hence

$$\left[\bar{K}_1\right] = [T_1]^T [K_1][T_1] = [K_1]$$

and for element (2) with $\theta = 90°$.

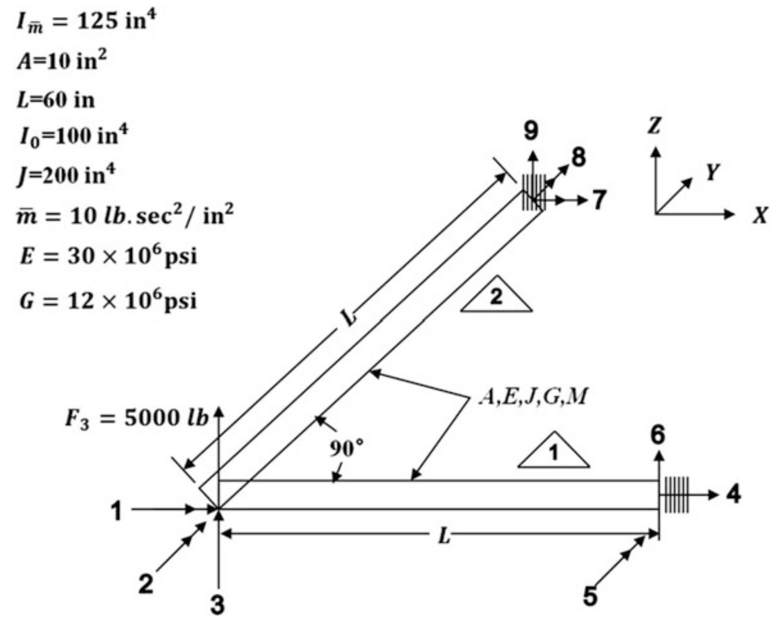

Fig. 12.4 Grid frame of Illustrative Example 12.1

$$[T_2] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that

$$\left[\bar{K}_2\right] = [T_2]^T [K_2][T_2]$$

$$\begin{array}{c} \\ [\bar{K}_2] = 10^6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 200 & 0 & 5 & 100 & 0 & 5 \\ 0 & 40 & 0 & 0 & -40 & 0 \\ 5 & 0 & 0.167 & -5 & 0 & 0.167 \\ 100 & 0 & 5 & 200 & 0 & -5 \\ 0 & -40 & 0 & 0 & 40 & 0 \\ -5 & 0 & -0.167 & -5 & 0 & 0.167 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

The system matrix $[K_s]$ assembled from $[\bar{K}_1]$ and $[\bar{K}_2]$ is

$$[K_s] = 10^6 \begin{bmatrix} 240 & 0 & 5 \\ 0 & 240 & -5 \\ 5 & -5 & 0.333 \end{bmatrix}$$

Analogously, for the mass, we have from Eq. (12.11)

$$[M_1] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 2500 & 0 & 0 & 1250 & 0 & 0 \\ 0 & 20{,}570 & 1886 & 0 & -15{,}430 & 1114 \\ 0 & 1886 & 223 & 0 & -1114 & 77 \\ 1250 & 0 & 0 & 2500 & 0 & 0 \\ 0 & -15{,}430 & -1114 & 0 & 20{,}570 & -1886 \\ 0 & 1114 & 77 & 0 & -1886 & 223 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

We then calculate using Eq. (12.23)

$$[\bar{M}_1] = [M_1]$$

since

$$[T_1] = [I]$$

and analogously

$$[\bar{M}_2] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 20{,}570 & 0 & -1886 & 15{,}430 & 0 & 1114 \\ 0 & 2500 & 0 & 0 & 1250 & 0 \\ -1886 & 0 & 223 & 1114 & 0 & 77 \\ 15{,}430 & 0 & 1114 & 20{,}570 & 0 & 1886 \\ 0 & 1250 & 0 & 0 & 2500 & 0 \\ 1114 & 0 & 77 & 1886 & 0 & 223 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

From $[\bar{M}_1]$ and $[\bar{M}_2]$ we assemble the system mass matrix and obtain

$$[M_s] = \begin{bmatrix} 23{,}070 & 0 & 1886 \\ 0 & 23{,}070 & -1886 \\ 1886 & -1886 & 446 \end{bmatrix}$$

The natural frequencies and mode shapes are obtained from the solution of the eigenproblem

$$\left([K_s] - \omega^2 [M_s]\right)\{a\} = \{0\}$$

which gives the eigenvalues (squares of the natural frequencies)

$$\omega_1^2 = 198 \qquad \omega_2^2 = 10{,}402, \quad \text{and} \qquad \omega_3^2 = 47{,}849$$

Then

$$\omega_1 = 14.06 \text{ rad/ sec}, \qquad \omega_2 = 101.99 \text{ rad/ sec}, \quad \text{and} \qquad \omega_3 = 218.74 \text{ rad/ sec}$$

and the eigenvectors ordered in the column of the modal matrix:

$$[a] = \begin{bmatrix} -1.000 & 1.000 & -1.000 \\ 1.000 & 1.000 & 1.000 \\ 43.82 & 0 & 9.072 \end{bmatrix}$$

The eigenvectors are conveniently normalized by dividing the columns of the modal matrix, respectively, by the factors

$$\sqrt{\{a_1\}^T[M_s]\{a_1\}} = 1,110$$
$$\sqrt{\{a_2\}^T[M_s]\{a_2\}} = 214.81$$
$$\sqrt{\{a_3\}^T[M_s]\{a_{31}\}} = 119.99$$

The normalized eigenvectors are arranged in columns of the modal matrix, so that

$$[\Phi] = \begin{bmatrix} -0.0009 & 0.0047 & 0.0083 \\ 0.0009 & 0.0047 & -0.0083 \\ 0.0395 & 0 & 0.0756 \end{bmatrix}$$

Illustrative Example 12.2

Determine the response of the grid frame shown in Fig. 12.4 when subjected to a suddenly applied force $F_3 = 5000$ lb. as shown in the figure.

Solution:

The natural frequencies and modal shapes for this structure were calculated in Example 12.1. The modal equation is given in general as

$$\ddot{z}_n + \omega_n^2 z_n = P_n \qquad (n = 1, 2, 3)$$

where

$$P_n = \sum_{i=1}^{3} \phi_{in} F_i$$

and F_i the external forces at the nodal coordinates which for this example are $F_1 = F_2 = 0$ and $F_3 = 5000$ lb. Hence, we obtain

$$\ddot{z}_1 + 198\text{z}_1 = 197.35$$
$$\ddot{z}_2 + 10,402\text{z}_2 = 0$$
$$\ddot{z}_3 + 47,849\text{z}_3 = 378.05$$

The solution of these equations for zero initial conditions is

$$\begin{aligned} z_1 &= \frac{197.35}{198}(1 - \cos\ 14.06t) \\ z_2 &= 0 \\ z_3 &= \frac{378.05}{47,849}(1 - \cos\ 218.74t) \end{aligned}$$

The displacements at the nodal coordinates are calculated from

$$\{u\} = [\Phi]\{z\}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} -0.0009 & 0.0047 & 0.0083 \\ 0.0009 & 0.0047 & -0.0083 \\ 0.0395 & 0 & 0.0756 \end{bmatrix} \begin{bmatrix} 0.9967 & (1-\cos 14.06t) \\ & 0 \\ 0.0079 & (1-\cos 218.74t) \end{bmatrix}$$

and finally

$$\begin{aligned} u_1 &= 10^{-3}(-0.8315 + 0.897\cos\ \ 14.06t - 0.0656\cos\ \ 218.74t)\,\text{in} \\ u_2 &= 10^{-3}(0.9626 - 0.897\cos\ \ 14.06t - 0.0656\cos\ \ 218.74t)\,\text{in} \\ u_3 &= 10^{-3}(39.97 - 39.37\cos\ \ 14.06t - 0.60\cos\ \ 154.49t)\,\text{radian} \end{aligned}$$

12.7 Modeling Structures as Grid Frames Using MATLAB

MATLAB calculates the stiffness and mass matrices for a grid frame and stores the coefficients of these matrices in a file. After calculating system matrices, the natural frequencies and mode shapes can be estimated. After solving modal equations, the MATLAB can plot the responses.

Illustrative Example 12.3
For the grid frame shown in Fig. 12.4 and analyzed in the previous examples, (a) model this structure, (b) calculate the natural frequencies and mode shapes, and (c) determine the response to a constant force of 5000 lb. suddenly applied for 0.1 s as indicated in the figure.

Solution:
This MATLAB file is to yield the results of sections (a) and (b). Two function files, GridFrameElement.m and GridConMass.m are needed. After assembling matrices, the system matrix can be found using System.m file. Using system matrices, the natural frequencies and mode shapes will be found.

```
clc
close all
clear all

%_____________________________________________________________________________
%  Determine System Matrices/Determine Force
%_____________________________________________________________________________

%%%%-GIVEN VALUES-%%%%

E=30*10^6; A=10;             %E (psi), A (in.^2)
Iz = 100;                    %Second Moment of Inertia (in^4)
J = 200;                     %Torsional constant
G = 12*10^6;                %Modulus of rigidity (psi)
I0 = 125;                    %Polar moment of inertia of cross sectional area(in^4)


%%%Create frame model (ith row of nodes is ith node)
nodes = [0, 0; 60,0; 0,60];
%%%Element number (ith row = ith element with two nodes)
conn=[1,2; 1,3];
%%%Dofs for ith element (ith row)
lmm=[1:3,4:6; 1:3,7:9];


m_bar = 10;                 %Mass per unit length (lb-sec^2/in/in)

dof = 3*length(nodes);      % Total No. dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1:2
lm=lmm(i,:);
con=conn(i,:);
ke = GridFrameElement(E, Iz, G, J, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke;
end

%%%Generate mass matrix for each element and assemble them.
for i=1:2
lm=lmm(i,:);
con=conn(i,:);
    m=GridConsMass(m_bar, I0, A, nodes(con,:));
    M(lm, lm) = M(lm, lm) + m;
end

K;

M;

%%%Define the load vector size
F = zeros(dof,1); F(3) = 5000;               %Applied force at specific dofs

%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F, [4:6,7:9]);

Kf

Mf

Rf
%______________________________________________________________________
% Solve the eigenvalue problem and normalized eigenvectors
%______________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(Kf, Mf);

[omegas,ii] = sort(sqrt(diag(D))); %Natural Frequencies

omegas

a = a(:,ii)                        %Mode Shapes

T = 2*pi./omegas;                  %Natural Periods

save ('temp0.mat', 'Mf', 'Kf' ,'Rf');
```

The function file of MATLAB is used to assemble the stiffness matrix of grid frame element for global stiffness matrix.

```
functionke = GridFrameElement(E, Iz, G, J, coord)
% ke = GridFrameElement(e, Iz, J, coord)
% Generates equations for a space frame element
% E = modulus of elasticity
% G = Modulus of rigidity (psi)
% Iz = moment of inertia about element z axe
% J = torsional constant
% coord = coordinates at the element ends

EI=E*Iz;  GJ=G*J;
x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
C=(x2-x1)/L; S=(y2-y1)/L;

T = [C S 0 0 0 0;
    -S C 0 0 0 0;
       0 0 1 0 0 0;
     0 0 0 C S 0;
     0 0 0 -S C 0;
     0 0 0 0 0 1];

ke = EI/L^3*T'*[GJ*L^2/EI, 0, 0, -GJ*L^2/EI, 0, 0;
         0, 4*L^2, -6*L, 0, 2*L^2, 6*L;
         0, -6*L, 12, 0, -6*L, -12;
         -GJ*L^2/EI, 0, 0, GJ*L^2/EI, 0, 0;
         0, 2*L^2, -6*L, 0, 4*L^2, 6*L;
         0, 6*L, -12, 0, 6*L, 12]*T;
```

The function file of MATLAB is used to assemble the mass matrix of each grid frame element for global mass matrix.

```
function m = GridConsMass(m_bar,I0, A, coord)
% FrameConsMass(m_bar, nodes(con,:))
% Generates mass matrix for a grid frame element
% m = mass (lb.sec^2/in.^2)
% L = length
% A = area of cross-section
% I0 = polar moment of inertia of cross sectional area(in^4)
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
C=(x2-x1)/L; S=(y2-y1)/L;


T = [C S 0 0 0 0;
    -S C 0 0 0 0;
       0 0 1 0 0 0;
     0 0 0 C S 0;
     0 0 0 -S C 0;
     0 0 0 0 0 1];

m = m_bar*L/420*T.'*[140*I0/A  0        0     70*I0/A     0          0  ;
                    0       4*L^2       22*L    0        -3*L^2      13*L;
                    0       22*L        156     0        -13*L       54;
                    70*I0/A     0        0       140*I0/A     0       0;
                    0          -3*L^2 -13*L        0        4*L^2     -22*L;
                    0       13*L        54         0          -22*L    156]*T;
```

This MATLAB file is to yield the results of section (c).

After running the program above, the following MATLAB program will yield the response. The duration of response is from 0 to 5 s with the interval of 0.01 s (Fig. 12.5).

```
function   q = MDOFP(t, q)
load ('temp2.mat', 'omega', 'P', 'm' ,'zeta')

if t > 0.1
     P=0;
else
     P=P;
end

q = [q(2); -omega*omega*q(1)-2*zeta*omega*q(2)+P];
```

The maximum displacements at the nodal coordinates were estimated using MATLAB.

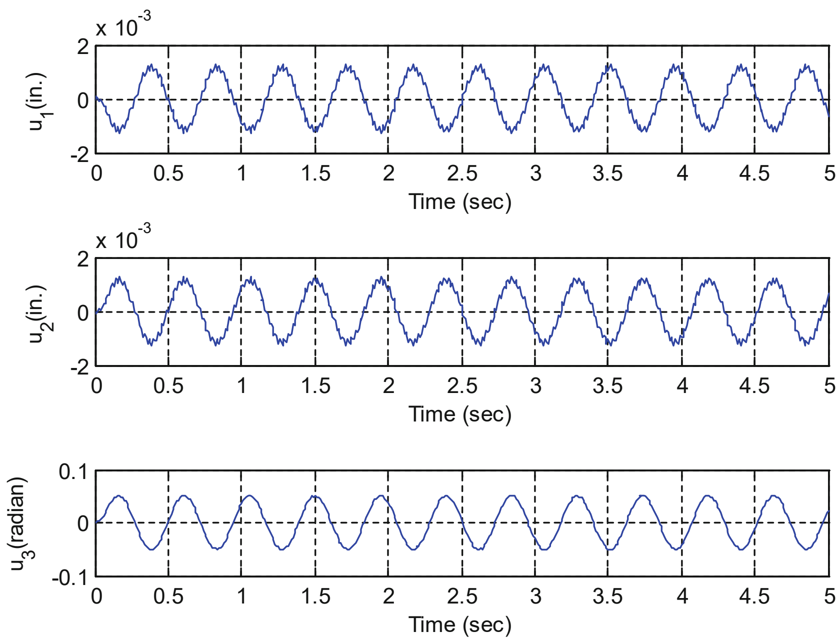

Fig. 12.5 Response of Illustrative Example 12.3

$$u_{1\max} = 0.0013 \text{ in} \quad u_{2\max} = 0.0013 \text{ in} \quad u_{3\max} = 0.0517 \text{ radian}$$

12.8 Summary

This chapter has presented the dynamic analysis of structures modeled as grid supporting loads applied normally to its plane. The dynamic analysis of grids requires the inclusion of torsional effects in the element stiffness and mass matrices. It also requires a transformation of coordinates of the element matrices previous to the assembling of the system matrix. The required matrices for torsional effects are developed and a computer program for the dynamic analysis of grids is presented. This program is also organized along the same pattern of the programs in the two preceding chapters for the dynamic analysis of beams and plane frames.

12.9 Problems

The following problems are intended for hand calculation, though it is recommended that whenever possible solutions should also be obtained using MATLAB to model the structure and calculate natural frequencies and to solve for the response.

Problem 12.1

Use MATLAB: (1) Model the plane grid frame shown in Fig. P12.1 using a total of four beam elements, (2) Determine the first three natural frequencies and corresponding mode shapes, and (3) Calculate the response due to the force $F(t) = 5000$ lb. applied in the Z direction at joint 1 suddenly for 0.1 sec and then decreasing linearly to zero at time 0.5 s.

Problem 12.2

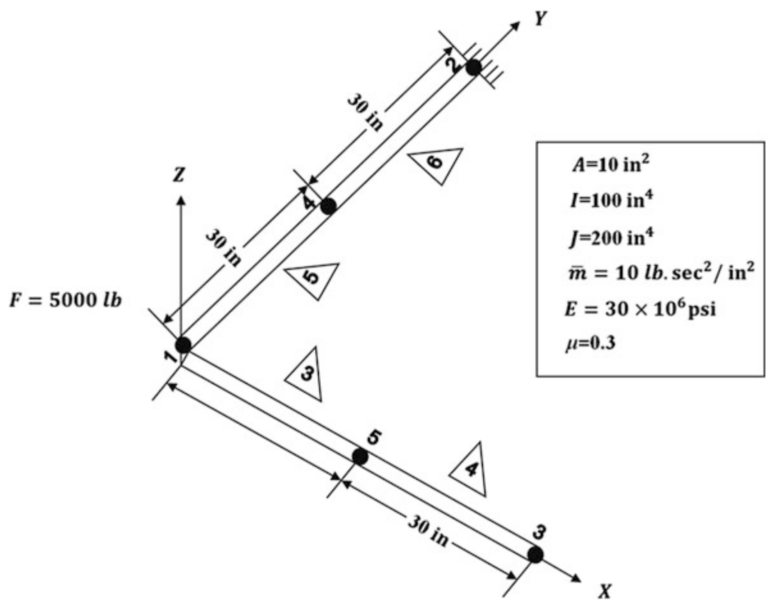

Fig. P12.1

For the grid shown in Fig. P12.2 determine the system stiffness and mass matrices. Base the analysis on the three nodal coordinates indicated in the figure. Use consistent mass method.

Problem 12.3

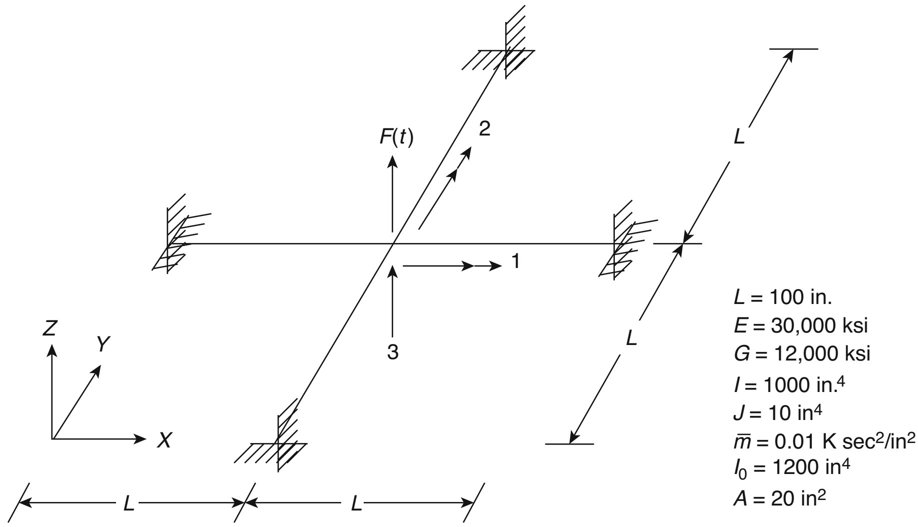

Fig. P12.2

Use static condensation to eliminate the rotational degrees of freedom and determine the transformation matrix and the reduced stiffness and mass matrices in Problem 12.2.

Problem 12.4
Determine the natural frequency for the reduced system in Problem 12.3.

Problem 12.5
Determine the natural frequencies and corresponding normal modes for the grid analyzed in Problem 12.2.

Problem 12.6
Determine the response of the grid shown in Fig. P12.1 when acted upon by a force $F(t) = 10$ Kip suddenly applied for 1 s at the nodal coordinate 3 as shown in the figure. Use results of Problem 12.2 to obtain the equation of motion for the condensed system. Assume 10% modal damping.

Problem 12.7
Use results from Problem 12.4 to solve Problem 12.5 on the basis of the three nodal coordinates as indicated in Fig. P12.2.

Problem 12.8
Determine the steady-state response of the grid shown in Fig. P12.2 when subjected to harmonic force $F(t) = 10 \sin 50\, t$ (Kip) along nodal coordinate 3. Neglect damping in the system.

Problem 12.9
Repeat Problem 12.8 assuming that the damping is proportional to the stiffness of the system, $[C] = a_0\,[K]$, where $a_0 = 0.3$.

Problem 12.10

Determine the equivalent nodal forces for a member of a grid loaded with a dynamic force, $P(t) = P_0 f(t)$, uniformly distributed along its length.

Problem 12.11

Determine the equivalent nodal forces for a member of a grid supporting a concentrated dynamic force $F(f)$ as shown in Fig. P12.11.

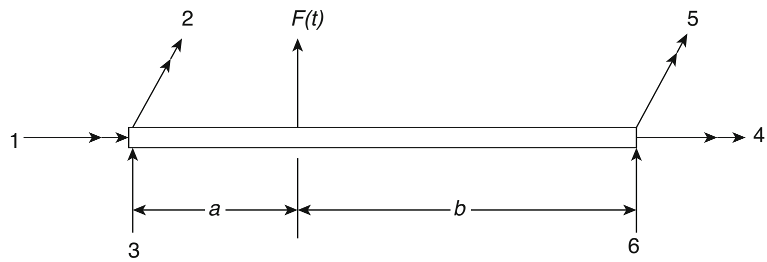

Fig. P12.11

The following problems are intended for computer solution using MATLAB, to model the structure and to determine natural frequencies and nodal shapes and to calculate the response.

Problem 12.12

Determine the natural frequencies and corresponding normal modes for the grid shown in Fig. P12.2.

Problem 12.13

Determine the response of the grid shown in Fig. P12.2 when acted upon by the force depicted in Fig. P12.2 acting along nodal coordinate 3. Neglect damping in the system.

Problem 12.14

Repeat Problem 12.13 for 15% damping in all the modes. Use modal super-position method.

Problem 12.15

Repeat Problem 12.13. Use step-by-step linear acceleration method. Neglect damping Fig. P12.15.

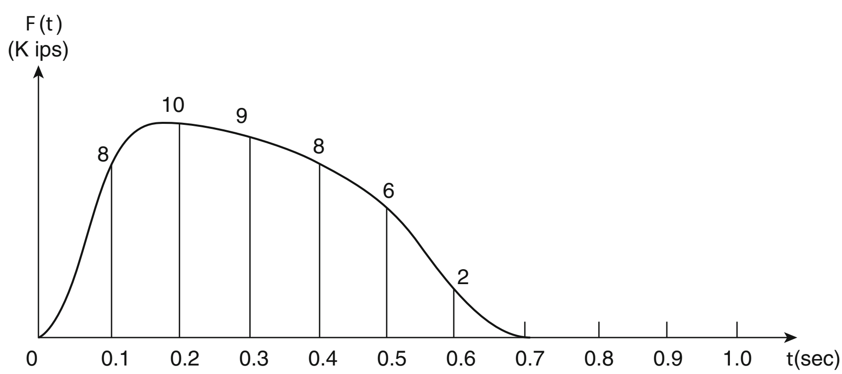

Fig. P12.15