11 Dynamic Analysis of Plane Frames

The dynamic analysis using the stiffness matrix method for structures modeled as beams was presented in Chap. 10. This method of analysis when applied to beams requires the calculation of element matrices (stiffness, mass, and damping matrices), the assemblage from these matrices of the corresponding system matrices, the formation of the force vector, and the solution of the resultant equations of motion. These equations, as we have seen, may be solved in general by the modal superposition method or by numerical integration of the differential equations of motion. In this chapter and in the following chapters, the dynamic analysis of structures modeled as frames is presented.

We begin in the present chapter with the analysis of structures modeled as plane frames and with the loads acting in the plane of the frame. The dynamic analysis of such structures requires the inclusion of the axial effects in the stiffness and mass matrices. It also requires a coordinate transformation of the nodal coordinates from element or local coordinates to system or global coordinates. Except for the consideration of axial effects and the need to transform these coordinates, the dynamic analysis by the stiffness method when applied to frames is identical to the analysis of beams as discussed in Chap. 10.

11.1 Element Stiffness Matrix for Axial Effects

The inclusion of axial forces in the stiffness matrix of a flexural beam element requires the determination of the stiffness coefficients for axial loads. To derive the stiffness matrix for an axially loaded member, consider in Fig. 11.1 a beam segment acted on by the axial forces P_1 and P_2 producing axial displacements δ_1 and δ_2 at the nodes of the element. For a prismatic and uniform beam segment of length L and cross-sectional A, it is relatively simple to obtain the stiffness relation for axial effects by the application of Hooke's law. In relation to the beam shown in Fig. 11.1, the displacements δ_1 produced by the force P_1 acting at node 1 while node 2 is maintained fixed ($\delta_2 = 0$) is given by

$$\delta_1 = \frac{P_1 L}{AE} \tag{11.1}$$

From Eq. (11.1) and the definition of the stiffness coefficient k_{11} (force at node 1 to produce a unit displacement, $\delta_1 = 1$), we obtain

M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_11

$$k_{11} = \frac{P_1}{\delta_1} = \frac{AE}{L} \tag{11.2a}$$

The equilibrium of the beam segment acted upon by the force k_{11} requires a force k_{21} at the other end of equal magnitude but in opposite direction, namely

$$k_{21} = -k_{11} = -\frac{AE}{L} \tag{11.2b}$$

Analogously, the other stiffness coefficients due to a unit displacement at node 2 ($\delta_2 = 1$) are:

$$k_{22} = \frac{AE}{L} \tag{11.2c}$$

and

$$k_{12} = -\frac{AE}{L} \tag{11.2d}$$

The stiffness coefficients as given by Eq. (11.2a) are the elements of the stiffness matrix relating axial forces and displacements for a prismatic beam segment, that is,

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} \tag{11.3}$$

The stiffness matrix corresponding to the nodal coordinates for the beam segment shown in Fig. 11.2 is obtained by combining in a single matrix the stiffness matrix for axial effects, Eq. (11.3), and the stiffness matrix for flexural effects, Eq. (10.20). The matrix resulting from this combination relates the forces P_i and the displacements δ_1 at the coordinates indicated in Fig. 11.2 as

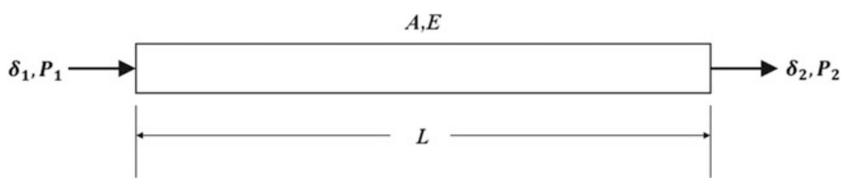

Fig. 11.1 Beam element showing nodal axial loads P_1, P_2, and corresponding nodal displacements δ_1, δ_2

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} AL^2/I & & & & & \\ 0 & 12 & & & & \\ 0 & 6L & 4L^2 & & & \\ -AL^2/I & 0 & 0 & AL^2/I & & \\ 0 & -12 & -6L & 0 & 12 & \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \tag{11.4}$$

or, in concise notation,

$$\{P\} = [K]\{\delta\} \tag{11.5}$$

11.2 Element Mass Matrix for Axial Effects

The determination of mass influence coefficients for axial effects of a beam element may be carried out by any of two methods indicated previously for the flexural effects: (1) the lumped mass method and (2) the consistent mass method. In the lumped mass method, the mass allocation to the nodes of the beam element is found from static considerations, which for a uniform beam gives half of the total mass of the beam segment allocated at each node. Then for a prismatic beam segment, the relation between modal axial forces and modal accelerations is given by

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{\bar{m}L}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{Bmatrix} \tag{11.6}$$

where $\bar{m}$ is the mass per unit of length. The combination of the flexural lumped mass coefficient and axial mass coefficients gives, in reference to the modal coordinates in Fig. 11.2, the following diagonal matrix:

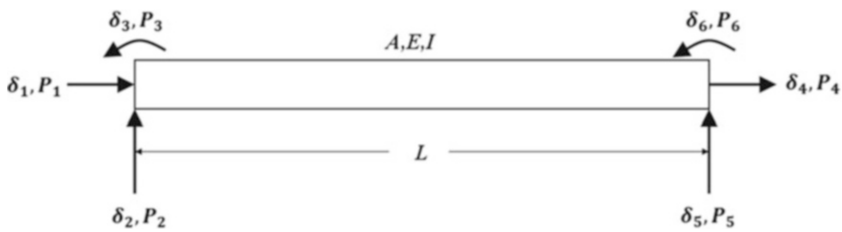

Fig. 11.2 Beam element showing flexural and axial nodal forces and displacements

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\bar{m}L}{2} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \\ \ddot{\delta}_5 \\ \ddot{\delta}_6 \end{Bmatrix} \tag{11.7}$$

To calculate the coefficients for the consistent mass matrix, it is necessary first to determine the displacement functions corresponding to a unit axial displacement at one of the modal coordinates. Consider in Fig. 11.3 an axial unit displacement $\delta_1 = 1$ of node 1 while the other node 2 is kept fixed so that $\delta_2 = 0$. If $u = u\,(x)$ is the displacement at section x, the displacement at section $x + dx$ will be $u + du$. It is evident then that the element dx in the new position has changed in length by an amount du, and thus, the strain is du/dx. Since from Hooke's law, the ratio of stress to strain is equal to the modulus of elasticity E, we can write

$$\frac{du}{dx} = \frac{P(x)}{AE} \tag{11.8}$$

Integration with respect to x yields

$$u = \frac{P(x)}{AE}x + C \tag{11.9}$$

in which C is a constant of integration. Introducing the boundary conditions, $u = 1$ at $x = 0$ and $u = 0$ at $x = L$, we obtain the displacement function u_1 (x) corresponding to a unit displacement $\delta_1 = 1$ as

$$u_1(x) = 1 - \frac{x}{L} \tag{11.10}$$

Analogously, the displacement function u_2 (x) corresponding to a unit displacement $\delta_2 = 1$ is found to be:

$$u_2(x) = \frac{x}{L} \tag{11.11}$$

The application of the principle of virtual work results in a general expression for the calculation of the stiffness coefficients. For example, consider the beam in Fig. 11.3, which is in equilibrium with the forces $P_1 = k_{11}$ and $P_2 = k_{21}$ at its two ends.

Assume that a virtual displacement $\delta_2 = 1$ takes place. Then, according to the principle of virtual work, during this virtual displacement, the work of the external and internal forces are equal. The external force k_{21} performs the work
or

$$W_E = k_{21} \tag{11.12}$$

since $\delta_2 = 1$. The internal force $P(x)$ at any section x is obtained from Eq. (11.8) as

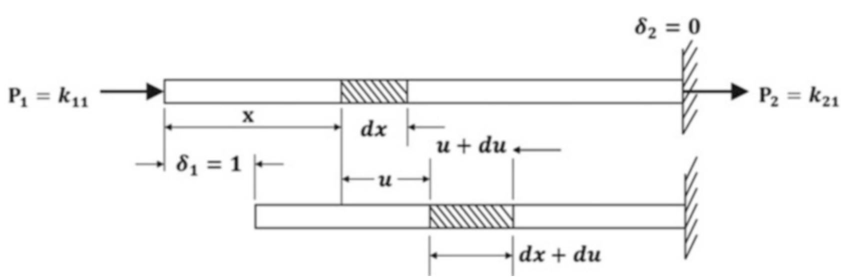

Fig. 11.3 Displacement at node 1 ($\delta_1 = 1$) of a beam element

$$P(x) = AEu_1^{'}(x) \tag{11.13}$$

in which u_1' $(x) = du_1/dx$.

The incremental displacement δu_2 of element dx during this virtual displacement may be expressed as

$$du_2 = \frac{du_2}{dx}dx \tag{11.14}$$

Hence the internal work for element dx is obtained from Eqs. (11.13) and (11.14) as

$$dW_I = AEu_1'(x)u_2'(x)dx$$

and for the beam segment of length L

$$W_I = \int_0^L AEu_1'(x)u_2'(x)dx \tag{11.15}$$

Finally, equating $W_E = W_I$ from Eqs. (11.12) and (11.15) gives the stiffness coefficient

$$k_{21} = \int_0^L AEu_1'(x)u_2'(x)dx \tag{11.16}$$

In general, the stiffness coefficient k_{ij} for axial effects may be obtained from

$$k_{ij} = \int_0^L AEu_i'(x)u_j'(x)dx \tag{11.17}$$

Using Eq. (11.17), the reader may check the results obtained in Eq. (11.3) for a uniform beam. However, Eq. (11.17) could as well be used for nonuniform beams in which the cross-sectional area A would in general be a function of x. In practice, the same displacement $u_1(x)$ and $u_2(x)$ obtained for a uniform beam, are also used in Eq. (11.17) for a non uniform member. The displacement u(x, t) at any section x of a beam element due to dynamic nodal displacements, $\delta_1(t)$ and $\delta_2(t)$ is obtained by superposition. Hence

$$u(x,t) = u_1(x)\delta_1(t) + u_2(x)\delta_2(t) \tag{11.18}$$

in which $u_1(x)$ and $u_2(x)$ are given by Eqs. (11.10) and (11.11) (Fig. 11.4).

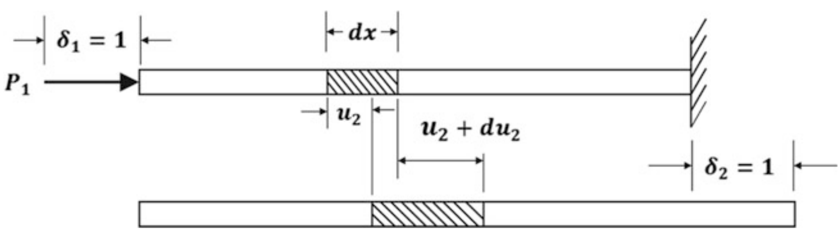

Fig. 11.4 Displacement along of a beam element subjected to axial loading that give a unit displacement ($\delta_2 = 1$)

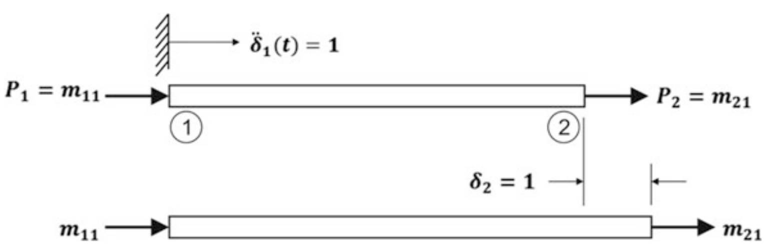

Fig. 11.5 Beam element with unit displacement at node 2($\delta_2 = 1$) undergoing a unit axial acceleration at node 1 $[\ddot{\delta}_1(t) = 1]$

Now consider the beam of Fig. 11.5 while undergoing a unit acceleration, $\ddot{\delta}_1(t) = 1$ which by Eq. (11.18) results in an acceleration at x given by

$$\ddot{u}_1(x,t) = u_1(x)\ddot{\delta}_1(t)$$

or

$$\ddot{u}_1(x,t) = u_1(x)$$

since $\ddot{\delta}_1(t) = 1$. The inertial force per unit length along the beam resulting from this unit acceleration is

$$f_I = \bar{m}(x)u_1(x) \tag{11.19}$$

where $\bar{m}(x)$ is the mass per unit length along the beam. Now, to determine the mass coefficient m_{21}, we give to the beam shown in Fig. 11.5 a virtual displacement $\delta_2 = 1$. The only external force doing work during this virtual displacement is the reaction m_{21}. This work is then

$$W_E = m_{21}\delta_2$$

or

$$W_E = m_{21} \tag{11.20}$$

since $\delta_2 = 1$. The internal work per unit length along the beam performed by the inertial force f_I during this virtual displacement is

$$\delta W_I = f_I u_2(x)$$

or, from Eq. (11.19)

$$\delta W_I = \bar{m}(x)\, u_1(x)\, u_2(x)$$

Hence the total internal work is

$$W_I = \int_0^L \bar{m}(x)u_1(x)u_2(x)dx \tag{11.21}$$

Finally, equating Eqs. (11.20) and (11.21) yields

$$m_{21} = \int_0^L \bar{m}(x)u_1(x)u_2(x)dx \tag{11.22}$$

or, in general,

$$m_{ij} = \int_0^L \bar{m}(x)u_i(x)u_j(x)dx \tag{11.23}$$

The application of Eq. (11.23) to the special case of a uniform beam results in

$$m_{11} = \int_0^L \bar{m}\left(1 - \frac{x}{L}\right)^2 dx = \frac{\bar{m}L}{3} \tag{11.24}$$

Similarly,

$$m_{22} = \frac{\bar{m}L}{3}$$

and

$$m_{12} = m_{21} = \int_0^L \bar{m}\left(1 - \frac{x}{L}\right)\left(\frac{x}{L}\right)dx = \frac{\bar{m}L}{6} \tag{11.25}$$

In matrix form, the axial inertial force relationship for a uniform beam may be written as

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{\bar{m}L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{Bmatrix} \tag{11.26}$$

Finally, combining the mass matrix Eq. (10.34) for flexural effects with Eq. (11.26) for the axial effects, we obtain the consistent mass matrix for a uniform element of a plane frame in reference to the modal coordinates shown in Fig. 11.2 as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\bar{m}L}{420}\begin{bmatrix} 140 & & & & & \\ 0 & 156 & & & & \\ 0 & 22L & 4L^2 & & & \\ 70 & 0 & 0 & 140 & & \\ 0 & 54 & 13L & 0 & 156 & \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix}\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \tag{11.27}$$

or, in condensed notation,

$$\{P\} = [M_c]\{\ddot{\delta}\}$$

in which $[M_c]$ is the consistent mass matrix for an element of a plane frame.

11.3 Coordinate Transformation

The stiffness matrix for an element of a plane frame in Eq. (11.4) as well as the mass matrix in Eq. (11.27) are in reference to nodal coordinates defined by coordinate axes fixed on the beam element. These axes are called *local* or *element coordinate axes* while the coordinate axes for the whole structure are known as *global* or *system coordinate axes*. Figure 11.6 shows a beam element with nodal forces $P_1, P_2, \ldots, P_6$ referred to the local coordinate axes x, y, z, and $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_6$ referred to global coordinate set of axes X, Y, Z. The objective is to transform the element matrices (stiffness, mass, etc.) from the reference of local coordinate axes to the global coordinate axes. This transformation is required in order that the matrices for all the elements refer to the same set of coordinates; hence, the matrices become compatible for assemblage into the system matrices for the structure. We begin by expressing the forces (P_1, P_2, P_3) in terms of the forces $(\bar{P}_1, \bar{P}_2, \bar{P}_3)$. Since these two sets of forces are equivalent, we obtain from Fig. 11.6 the following relationships:

$$\begin{aligned} P_1 &= \bar{P}_1 \cos\theta + \bar{P}_2 \sin\theta \\ P_2 &= -\bar{P}_1 \sin\theta + \bar{P}_2 \cos\theta \\ P_3 &= \bar{P}_3 \end{aligned} \tag{11.28}$$

The equations of Eq. (11.28) may be written in matrix notation as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \end{Bmatrix} \tag{11.29}$$

Analogously, we obtain for the forces on the other node the relationships:

$$\begin{aligned} P_4 &= \bar{P}_4 \cos\theta + \bar{P}_5 \sin\theta \\ P_5 &= -\bar{P}_4 \sin\theta + \bar{P}_5 \cos\theta \\ P_6 &= \bar{P}_6 \end{aligned} \tag{11.30}$$

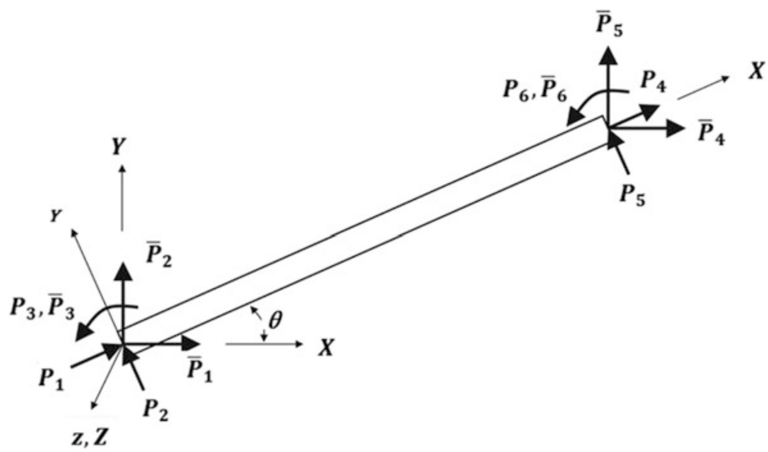

Fig. 11.6 Beam element showing nodal forces P_i in local (x, y, z) and nodal forces $\bar{P}_1$, in global coordinate axes (X, Y, Z)

Equations (11.28) and (11.30) may conveniently be arranged in matrix form as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \\ \bar{P}_4 \\ \bar{P}_5 \\ \bar{P}_6 \end{Bmatrix} \tag{11.31}$$

or in condensed notation

$$\{P\} = [T]\{\bar{P}\} \tag{11.32}$$

in which $\{P\}$ and $\{\bar{P}\}$ are, respectively, the vectors of the element nodal forces in local and global coordinates and $[T]$ is the transformation matrix given by the square matrix in Eq. (11.31).

Repeating the same procedure, we obtain the relation between nodal displacements $(\delta_1, \delta_2, \ldots, \delta_6)$ in local coordinates and the components of the nodal displacements in global coordinates $(\bar{\delta}_1, \bar{\delta}_2, \ldots, \bar{\delta}_6)$, namely

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \overline{\delta}_1 \\ \overline{\delta}_2 \\ \overline{\delta}_3 \\ \overline{\delta}_4 \\ \overline{\delta}_5 \\ \overline{\delta}_6 \end{Bmatrix} \tag{11.33}$$

or

$$\{\delta\} = [T]\{\overline{\delta}\} \tag{11.34}$$

Now, the substitution of $\{P\}$ from Eq. (11.32) and $\{\delta\}$ from Eq. (11.34) into the stiffness equation referred to local axes $\{P\} = [\mathrm{K}]\ \{\delta\}$ results in

$$[T]\{\bar{P}\} = [K][T]\{\overline{\delta}\}$$

or

$$\{\bar{P}\} = [T]^{-1}[K][T]\{\overline{\delta}\} \tag{11.35}$$

where $[T]^{-1}$ is the inverse of matrix $[T]$. However, as the reader may verify, the transformation matrix $[T]$ in Eq. (11.31) is an orthogonal matrix, that is, $[T]^{-1} = [T]^T$. Hence

$$\{\bar{P}\} = [T]^{\mathrm{T}}[K][T]\{\overline{\delta}\} \tag{11.36}$$

or, in a more convenient notation,

$$\{\bar{P}\} = [\bar{K}]\{\overline{\delta}\} \tag{11.37}$$

in which

$$\{\bar{K}\} = [T]^{\mathrm{T}}[K][T] \tag{11.38}$$

is the stiffness matrix for an element of a plane frame in reference to the global system of coordinates.

Repeating the procedure of transformation as applied to the stiffness matrix for the lumped mass, Eq. (11.7), or the consistent mass matrix, Eq. (11.27), we obtain in a similar manner

$$\{\bar{P}\} = [\bar{M}]\{\overline{\delta}\}$$

in which

$$\{\bar{M}\} = [T]^{\mathrm{T}}[K][T] \tag{11.39}$$

is the mass matrix for an element of a plane frame in reference to the global system of coordinates and $[T]$ is the transformation matrix given by the square matrix in Eq. (11.33).

Illustrative Example 11.1

Consider in Fig. 11.7 a plane frame having two prismatic beam elements and three degrees of freedom as indicated in the figure. Using the consistent mass formulation, determine the three natural frequencies and corresponding normal modes for this discrete model of the frame.

Solution:
The stiffness matrix for element 1 or 2 in local coordinates by Eq. (11.4) is.

$$[K_1] = [K_2] = \begin{bmatrix} 600 & & & & \text{Symmetric} & \\ 0 & 12 & & & & \\ 0 & 600 & 40,000 & & & \\ -600 & 0 & 0 & 600 & & \\ 0 & -12 & -600 & 0 & 12 & \\ 0 & 600 & 20,000 & 0 & -600 & 40,000 \end{bmatrix}$$

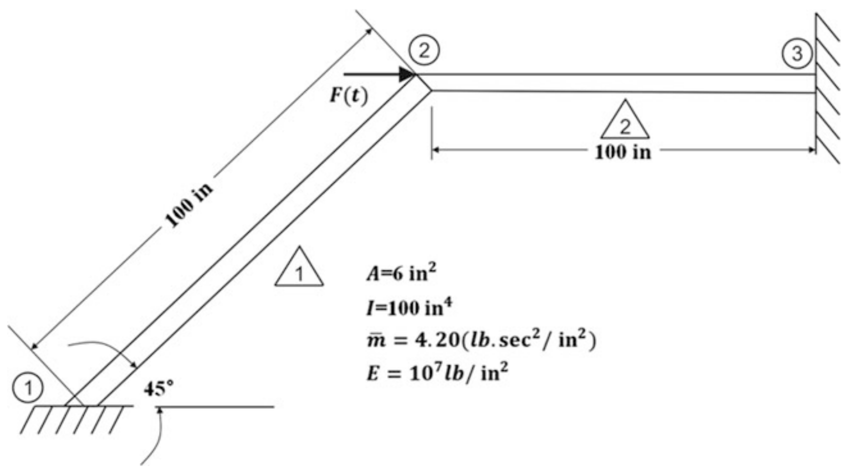

Fig. 11.7 Plane Frame of Illustrative Example 11.1

The transformation matrix for element 1 by Eq. (11.31) with $\theta = 45°$ is

$$[T_1] \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$$

and for element 2 ith $\theta = 0°$ is the identity matrix:

$$[T_2] = [I]$$

The mass matrix in local coordinates for either of the two elements of this frame from Eq. (11.27) is

$$[M_1] = [M_2] = \begin{bmatrix} 140 & & & & \text{Symmetric} & \\ 0 & 156 & & & & \\ 0 & 2200 & 40,000 & & & \\ 70 & 0 & 0 & 140 & & \\ 0 & 54 & 1300 & 0 & 156 & \\ 0 & -1300 & -30,000 & 0 & -2200 & 40,000 \end{bmatrix}$$

The element stiffness and mass matrices in reference to the global system of coordinates, are, respectively, calculated by Eqs. (11.38) and (11.39). For element 1 the stiffness matrix is

$$\begin{matrix} & 4 & 5 & 6 & 1 & 2 & 3 \end{matrix}$$
$$[\bar{K}_1] = 10^6 \begin{bmatrix} 0.306 & & & & \text{Symmetric} & \\ 0.294 & 0.306 & & & & \\ -0.424 & 0.424 & 40,000 & & & \\ -0.306 & -0.294 & 0.424 & 0.306 & & \\ -0.294 & -0.306 & -0.424 & 0.294 & 0.306 & \\ -0.424 & 0.424 & 20,000 & 0.424 & -0.424 & 40,000 \end{bmatrix} \begin{matrix} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{matrix}$$

and the mass matrix

$$\begin{matrix} & 4 & 5 & 6 & 1 & 2 & 3 \end{matrix}$$
$$[\bar{M}_1] = \begin{bmatrix} 148 & & & & \text{Symmetric} & \\ -8 & 148 & & & & \\ -1556 & 1556 & 40,000 & & & \\ 62 & 8 & -919 & 148 & & \\ 8 & 62 & 919 & -8 & 148 & \\ 919 & -919 & -30,000 & 1556 & -1556 & 40,000 \end{bmatrix} \begin{matrix} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{matrix}$$

For element 2:

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}$$
$$[\bar{K}_2] = 10^6 \begin{bmatrix} 0.600 & & & & \text{Symmetric} & \\ 0 & 0.012 & & & & \\ 0 & 0.600 & 40,000 & & & \\ -0.600 & 0 & 0 & 0.600 & & \\ 0 & -0.012 & -0.600 & 0 & 0.012 & \\ 0 & 0.600 & 20,000 & 0 & -0.600 & 40,000 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

and

$$\left[\bar{M}_2\right] = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 140 & & & & \text{Symmetric} & \\ 0 & 156 & & & & \\ 0 & 2200 & 40,000 & & & \\ 70 & 0 & 0 & 140 & & \\ 0 & 54 & 1300 & 0 & 156 & \\ 0 & -1300 & -30,000 & 0 & -2200 & 40,000 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

The system stiffness and mass matrices are assembled by the direct method. As was mentioned before, it is expedient for hand calculation of these matrices to indicate the corresponding system nodal coordinates at the top and right of the element matrices. We thus obtain, considering only the free coordinates, the system stiffness matrix:

$$\left[\bar{K}\right] = 10^6 \begin{bmatrix} 0.906 & 0.294 & 0.424 \\ 0.294 & 0.318 & 0.176 \\ 0.424 & 0.176 & 80.000 \end{bmatrix}$$

and the system mass matrix as

$$\left[\bar{M}\right] = \begin{bmatrix} 288 & -8 & 1556 \\ -8 & 304 & 644 \\ 1556 & 644 & 80,000 \end{bmatrix}$$

The natural frequencies are found as the roots of the characteristic equation

$$\left|[K] - \omega^2 [M]\right| = 0$$

which, upon substituting the numerical values given for this example, yields

$$10^3 \begin{vmatrix} 906 - 0.288\omega^2 & 294 + 0.008\omega^2 & 424 - 1.556\omega^2 \\ 294 + 0.008\omega^2 & 318 - 0.304\omega^2 & 176 - 0.644\omega^2 \\ 424 - 1.556\omega^2 & 176 - 0.644\omega^2 & 80,000 - 80\omega^2 \end{vmatrix} = 0$$

The roots then are found to be:

$$\omega_1{}^2 = 638.5, \omega_2{}^2 = 976.6, \omega_3{}^2 = 4211.6,$$

and the natural frequencies are

$$\omega_1 = 25.26\,\text{rad/sec}\,, \;\; \omega_2 = 31.24\,\text{rad/sec}\,, \;\; \text{and} \;\; \omega_3 = 64.90\,\text{rad/sec}\,,$$

or

$$f_1 = 4.02\ \text{cps},\ f_2 = 4.97\ \text{cps},\ \ \text{and}\ f_3 = 10.33\ \text{cps},$$

The normal modes are given as the nontrivial solution of the eigenproblem

$$\left([K] - \omega^2 [M]\right)\{a\} = \{0\}$$

Substituting $\omega_1{}^2 = 638.5$ and setting $a_{11} = 1.0$, we obtain the first mode shape as

$$\{a_1\} = \begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ -2.38 \\ 0 \end{Bmatrix}$$

which is normalized with the factor

$$\sqrt{\{a_1\}^T[M]\{a_1\}} = 45.81$$

The normalized eigenvector is then

$$\{\phi_1\} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} = \begin{Bmatrix} 0.0218 \\ -0.0527 \\ 0 \end{Bmatrix}$$

Analogously, for the other two modes, we obtain:

$$\{\phi_2\} = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} = \begin{Bmatrix} 0.00498 \\ 0.00206 \\ 0.00341 \end{Bmatrix} \quad \text{and} \quad \{\phi_3\} = \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} = \begin{Bmatrix} 0.0583 \\ 0.0241 \\ -0.0016 \end{Bmatrix}$$

arranging these modal vectors into columns of the modal matrix, we obtain:

$$[\Phi] = \begin{bmatrix} 0.0218 & 0.00498 & 0.0583 \\ -0.0527 & 0.00206 & 0.0241 \\ 0 & 0.00341 & -0.0016 \end{bmatrix}$$

Illustrative Example 11.2

Determine the maximum displacement at the nodal coordinates of the frame in Fig. 11.7 when a force of magnitude 100,000 lb. is suddenly applied at nodal coordinate 1. Neglect damping.

Solution:

From Illustrative Example 11.1, the natural frequencies are $\omega_1 = 25.26$ rad/sec, $\omega_2 = 31.24$ rad/sec, and $\omega_3 = 64.90$ rad/sec; and the modal matrix is.

$$[\Phi] = \begin{bmatrix} 0.0218 & 0.00498 & 0.0583 \\ -0.0527 & 0.00206 & 0.0241 \\ 0 & 0.00341 & -0.0016 \end{bmatrix}$$

The modal equations have the form of

$$\ddot{z}_i + \omega_i^2 z_i = P_i \tag{a}$$

where

$$P_i = \sum_j \phi_{ji} F_j \tag{b}$$

In this example, the nodal applied forces are

$$F_1 = 100,000\ \text{lb},\ \ F_2 = 0,\ \ F_3 = 0$$

We thus obtain, after substituting numerical values into Eqs. (a) and (b), the modal equations as

$$\begin{aligned} \ddot{z}_1 + 638.5z_1 &= 2180 \\ \ddot{z}_2 + 976.6z_2 &= 498 \\ \ddot{z}_3 + 4211.6z_3 &= 5830 \end{aligned} \tag{c}$$

The solutions of these equations by (4.5) are of the form

$$z_i = \frac{P_i}{\omega_i^2}(1 - \cos\omega_i t)$$

Substitution for P_i and ω_i yields

$$\begin{aligned} z_1 &= 3.414(1 - \cos 25.2685t) \\ z_2 &= 0.510(1 - \cos 31.2506t) \\ z_3 &= 1.384(1 - \cos 64.8970t) \end{aligned} \tag{d}$$

The nodal displacements are obtained from

$$\{u\} = [\Phi]\{z\}$$

which results in

$$\begin{aligned} u_1 &= 0.1577 - 0.0744\cos 25.26t - 0.00254\cos 31.25t - 0.0807\cos 64.9t\ (\text{in}) \\ u_2 &= -0.1455 + 0.1800\cos 25.26t - 0.00105\cos 31.25t - 0.0333\cos 64.9t\ (\text{in}) \\ u_3 &= -0.000475 + 0\cos 25.26t - 0.00174\cos 31.25t + 0.0022\cos 64.9t\ \ (\text{radian}) \end{aligned} \tag{e}$$

The maximum possible displacements at the nodal coordinates may then be estimated as the summation of the absolute values of the coefficients in the above expressions. Hence

$$u_{1\max} = 0.3177\ \text{in} \qquad u_{2\max} = 0.3589\ \text{in} \qquad u_{3\max} = 0.0044\ \text{radian}$$

11.4 Modeling Structures as Plane Frames Using MATLAB

MATLAB program is used to determine the stiffness and the mass matrices for a plane frame.

Illustrative Example 11.3
Use MATLAB to determine the stiffness and mass matrices for the plane frame shown in Fig. 11.7.

Solution:
The following MATLAB file is used to compute stiffness and mass matrices using the system matrix. Two function files, FrameElement.m and FrameConMass.m are needed. After assembling matrices, the system matrix can be found using System.m file (Chap.10).

```
clc
close all
clear all

%_____________________________________________________________________________
%  Determine System Matrices/Determine Force
%_____________________________________________________________________________

%%%%-GIVEN VALUES-%%%%

E=10^7; A=6;                  % E (psi), A (in.^2)
inertia = 100;                %Second Moment of Inertia (in^4)

%%%Create frame model (ith row of nodes is ith node)
nodes = [0, 0; 70.71,70.71; 170.71,70.71];
%%%Element number (ith row = ith element with two nodes)
conn=[1,2; 2,3];
%%%Dofs for ith element (ith row)
lmm=[1:3,4:6; 4:6,7:9];
%%%Dofs are elimniated at supports for system matrix
debc=[1:3,7:9];

m_bar = 4.2;                  %Mass per unit length (lb-sec^2/in/in)

dof = 3*length(nodes);        % Total No. dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1:2
    lm=lmm(i,:);
    con=conn(i,:);
    ke = PlaneFrameElement(E, inertia, A, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke
end

%%%Generate mass matrix for each element and assemble them.
for i=1:2
    lm=lmm(i,:);
    con=conn(i,:);
    m=FrameConsMass(m_bar, nodes(con,:));
    M(lm, lm) = M(lm, lm) + m;
end

K;

M;

%%%Define the load vector
F = zeros(dof,1);                %Applied force at specific dofs

%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F, [1:3,7:9]);

Kf

Mf

Rf

save ('temp0.mat', 'Mf', 'Kf' ,'Rf');
```

The function file of MATLAB is used to assemble the stiffness matrix of each beam element for global stiffness matrix.

```
function ke = PlaneFrameElement(modulus, inertia, A, coord)
% ke = PlaneFrameElement(modulus, inertia, A, coord)
% Generates equations for a plane frame element
% modulus = modulus of elasticity
% inertia = moment of inertia
% A = area of cross-section
% coord = coordinates at the element ends

EI=modulus*inertia; EA = modulus*A;
x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
C=(x2-x1)/L; S=(y2-y1)/L;

ke = [(EA*L^2*C^2 + 12*EI*S^2)/L^3, ((-12*EI + EA*L^2)*C*S)/L^3, ...
        (-6*EI*S)/L^2, -((EA*L^2*C^2 + 12*EI*S^2)/L^3), ...
        ((12*EI - EA*L^2)*C*S)/L^3, (-6*EI*S)/L^2;
    ((-12*EI + EA*L^2)*C*S)/L^3, (12*EI*C^2 + EA*L^2*S^2)/L^3, ...
        (6*EI*C)/L^2, ((12*EI - EA*L^2)*C*S)/L^3, ...
        -((12*EI*C^2 + EA*L^2*S^2)/L^3), (6*EI*C)/L^2;
    (-6*EI*S)/L^2, (6*EI*C)/L^2, (4*EI)/L, ...
        (6*EI*S)/L^2, (-6*EI*C)/L^2,(2*EI)/L;
    -((EA*L^2*C^2 + 12*EI*S^2)/L^3), ((12*EI -EA*L^2)*C*S)/L^3, ...
        (6*EI*S)/L^2, (EA*L^2*C^2 + 12*EI*S^2)/L^3, ...
        ((-12*EI + EA*L^2)*C*S)/L^3, (6*EI*S)/L^2;
    ((12*EI - EA*L^2)*C*S)/L^3, -((12*EI*C^2 + EA*L^2*S^2)/L^3),...
        (-6*EI*C)/L^2, ((-12*EI + EA*L^2)*C*S)/L^3, ...
        (12*EI*C^2 + EA*L^2*S^2)/L^3, (-6*EI*C)/L^2;
    (-6*EI*S)/L^2, (6*EI*C)/L^2, (2*EI)/L, (6*EI*S)/L^2, ...
        (-6*EI*C)/L^2, (4*EI)/L];
```

The function file of MATLAB is used to assemble the mass matrix of each beam element for global mass matrix.

```
function m = FrameConsMass(m_bar, coord)
% FrameConsMass(m_bar, nodes(con,:))
% Generates mass matrix for a plane frame element
% m = mass (lb.sec^2/in.^2)
% L = length
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
ls=(x2-x1)/L; ms=(y2-y1)/L;
T = [ls ms 0 0 0 0;
    -ms ls 0 0 0 0;
       0 0 1 0 0 0;
     0 0 0 ls ms 0;
     0 0 0 -ms ls 0;
     0 0 0 0 0 1];

m = m_bar*L/420*T.'*[140  0    0     70   0       0  ;
                    0  156    22*L  0    54      -13*L;
                    0  22*L  4*L^2  0    13*L    -3*L^2;
                    70  0      0   140   0       0  ;
                     0  54     13*L  0   156     -22*L;
                     0 -13*L -3*L^2 0   -22*L   4*L^2]*T;
```

We obtain:

$$[\Phi] = \begin{bmatrix} 0.0218 & 0.0050 & -0.0583 \\ -0.0527 & 0.0021 & -0.0242 \\ 0 & 0.0034 & 0.0016 \end{bmatrix}$$

This is identical to modal matrix in Illustrative Example 11.2.

11.5 Dynamic Analysis of Plane Frames Using MATLAB

Illustrative Example 11.4

For the structure shown in Fig. 11.7 which was modeled in Illustrative Example 11.3, determine: (a) natural frequencies and modal shapes; (b) the response to a force of magnitude 100,000 lb. suddenly applied at nodal coordinate 2.

Solution:

This MATLAB file is to yield the results of section (a). Two function files, FrameElement.m and FrameConMass.m are needed. After assembling matrices, the system matrix can be found using System.m file (Chap.10). Using system matrices, the natural frequencies and mode shapes will be found.

```
clc
close all
clear all

%___________________________________________________________________________
%  Determine System Matrices/Determine Force
%___________________________________________________________________________

%%%%-GIVEN VALUES-%%%%

E=10^7; A=6;                % E (psi), A (in.^2)
inertia = 100;              %Second Moment of Inertia (in^4)

%%%Create frame model (ith row of nodes is ith node)
nodes = [0, 0; 70.71,70.71; 170.71,70.71];
%%%Element number (ith row = ith element with two nodes)
conn=[1,2; 2,3];
%%%Dofs for ith element (ith row)
lmm=[1:3,4:6; 4:6,7:9];
%%%Dofs are elimniated at supports for system matrix
debc=[1:3,7:9];

m_bar = 4.2;                %Mass per unit length (lb-sec^2/in/in)

dof = 3*length(nodes);      % Total No. dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1:2
    lm=lmm(i,:);
    con=conn(i,:);
    ke = PlaneFrameElement(E, inertia, A, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke
end

%%%Generate mass matrix for each element and assemble them.
for i=1:2
    lm=lmm(i,:);
    con=conn(i,:);
    m=FrameConsMass(m_bar, nodes(con,:));
    M(lm, lm) = M(lm, lm) + m;
end

K;

M;
```

```
%%%Define the load vector
F = zeros(dof,1); F(4) = 100000;            %Applied force at specific dofs

%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F, [1:3,7:9]);

Kf

Mf

Rf
%______________________________________________________________________
% Solve the eigenvalue problem and normalized eigenvectors
%______________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(Kf, Mf);

[omegas,ii] = sort(sqrt(diag(D)));         %Natural Frequencies

omegas

a = a(:,ii)                                %Mode Shapes

T = 2*pi./omegas;                          %Natural Periods

save ('temp0.mat', 'Mf', 'Kf' ,'Rf');
```

This MATLAB file is to yield the results of section (b).

After running the program above, the following MATLAB program will yield the response. The duration of response is from 0 to 5 seconds with the interval of 0.01 second.

```
clear all
close all

%______________________________________________________________________________
% Inputs:
%   M, K
%   F = forcing function
%   t = Time period
%   u0 = initial displacement
%   v0 = initial velocity
%______________________________________________________________________________
 t = 0:0.01:0.5;

load ('temp0.mat', 'Mf', 'Kf', 'Rf')

%%%Deifne Mass Matrix

M = Mf

%%%Deifne Stiffness Matrix

K = Kf

[n,n]= size(M);

F = Rf;

u0 = zeros(n,1); u0(1) =0;
v0 = zeros(n,1); v0(1) =0;
[n,n]= size(M);
```

```
%______________________________________________________________________________
% Solve the eignevalue problem and normalized the eigenvectors
%______________________________________________________________________________
[a, D] = eig(K, M)                  % Solve for eigenvalues (D) and eigenvectors (a)

[omegas,k] = sort(sqrt(diag(D)));   % Natrual Frequencies
a = a(:,k)

T = 2*pi./omegas;                   % Natural Periods

aMa = diag(a'*M*a)                  % aMa = {a}'*[M]*(a}

nom_phi = (a)*inv(sqrt(diag(aMa))) % Normalized modal matrix

%______________________________________________________________________________
% Initial conditions
%______________________________________________________________________________

P = nom_phi'*F;                     % Normalized force, P = nom_F
q0 = nom_phi'*M*u0
dq0 = nom_phi'*M*v0
%______________________________________________________________________________
% Damping matrix using the proportional damping matrix
% [C] = a[M]+b[K]
% zetas = damping ratios
%______________________________________________________________________________
a = 0;
b = 0;
nom_C = nom_phi'*(a*M+b*K)*nom_phi;
zetas = diag((1/2)*nom_C*inv(diag(omegas)));

save ('temp1.mat', 'omegas', 'P' ,'zetas');
q = [];
r = [];
for i=1:n
q0_i = q0(i,:);
dq0_i = dq0(i,:);

load temp1.mat
omega = omegas(i,:);
P = P(i,:);
m = M(i,i);
zeta = zetas(i,:);

save ('temp2.mat', 'omega', 'P', 'm', 'zeta');

[t,q] = ode45(@MDOFP, t, [q0_i dq0_i]',[]);

r(:,i) = q(:,1);
save ('temp3.mat', 'r')

end

load ('temp3.mat', 'r');

yim = nom_phi*[r'];

save ('response.mat','yim');

figure
subplot(3,1,1);         % Node 2: x displ (in.).

xlabel ('Time (sec)'); ylabel ('u_2(in.)'); grid on

subplot(3,1,2);         % Node 2: y displ (in.).
plot(t, yim(2,:))
xlabel ('Time (sec)'); ylabel ('v_2(in.)'); grid on

subplot(3,1,3);         % Node 2: rotation (radian).
plot(t, yim(3,:))
xlabel ('Time (sec)'); ylabel ('\phi_2(radian)'); grid on

umax_1=max(abs(yim(1,:)))
umax_2=max(abs(yim(2,:)))
umax_3=max(abs(yim(3,:)))
```

The function file of MATLAB is used to determine the function of the applied force and solve the uncoupled equation (Fig. 11.8).

```
function   q = MDOFP(t, q)
load ('temp2.mat', 'omega', 'P', 'm' ,'zeta')

    P = P;

q = [q(2); -omega*omega*q(1)-2*zeta*omega*q(2)+P];
```

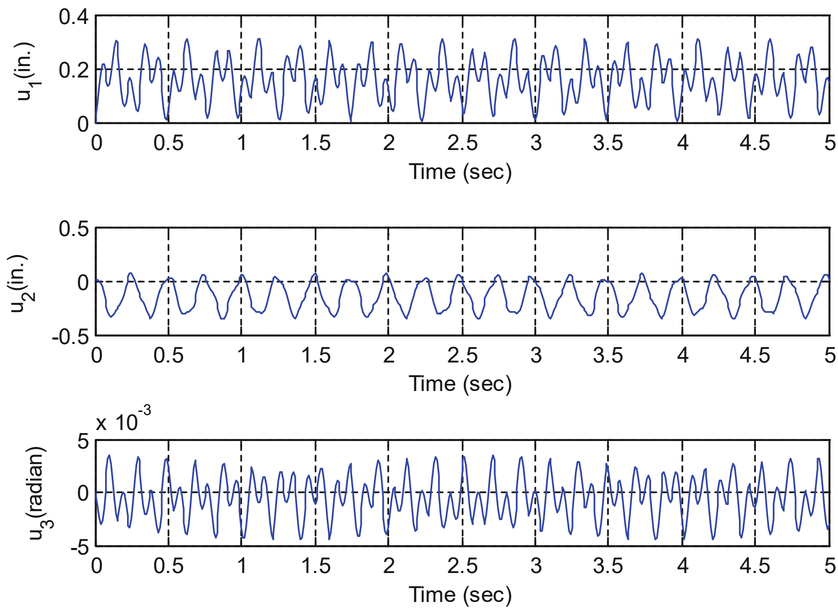

Fig. 11.8 Response of Illustrative Example 11.4

The maximum displacements at the nodal coordinates were estimated using MATLAB. These are

very close to the estimation of the summation of the absolute values obtained from Illustrative Example 11.2.

$$u_{1\max} = 0.3119 \text{ in} \quad u_{2\max} = 0.3590 \text{ in} \quad u_{3\max} = 0.0045 \text{ radian}$$

Illustrative Example 11.5

Use MATLAB to model the plane frame shown in Fig. 11.7 using a total of two beam elements and to calculate the response due to the force $F(t)$ shown in Fig. 11.9.

Solution:

The same files will be used in illustrative example 11.3 and 11.4 with the change of MDOFP function as bellow. The plot is generated from the range of t = 0 to 0.5 seconds.

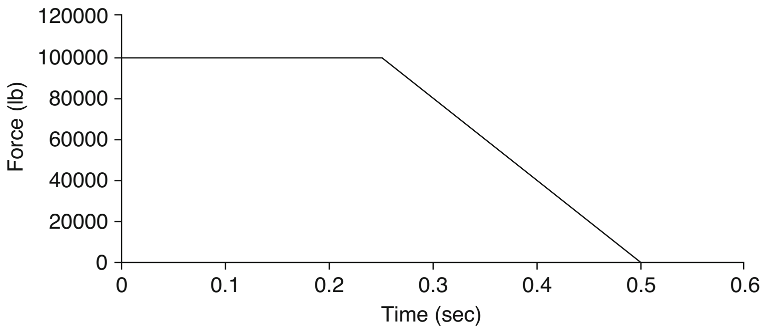

Fig. 11.9 Forcing function applied at joint 2 of the plane frame of Example 11.5

The function file of MATLAB is used to determine the function of the applied force and solve the uncoupled equation (Fig. 11.10).

```
function   q = MDOFP(t, q)
load ('temp2.mat', 'omega', 'P', 'm' ,'zeta')

if t <= 0.25
    P = P;

elseif t <0.5

   P = 2*P*(1-2*t);

else
   P = 0;

end

q = [q(2); -omega*omega*q(1)-2*zeta*omega*q(2)+P];
```

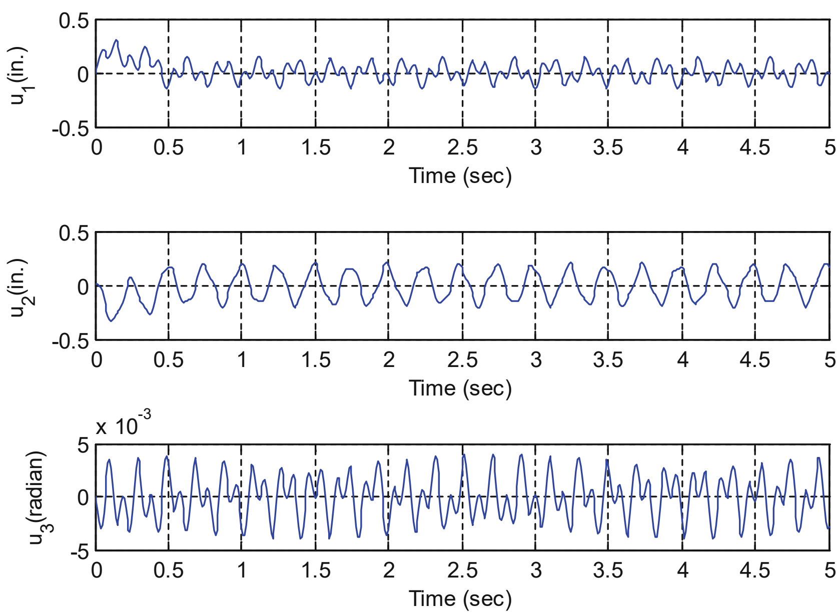

Fig. 11.10 The maximum displacements at the nodal coordinates were estimated using MATLAB

$u_{1\max} = 0.3037$ in. $u_{2\max} = 0.3352$ in. $u_{3\max} = 0.0040$ radian

11.6 Summary

The dynamic analysis of plane frames by the stiffness method requires the inclusion of the axial effects in the system matrices (stiffness, mass, etc.). It also requires a transformation of coordinates in order to refer all the element matrices to the same coordinate system, so that the appropriate superposition can be applied to assemble the system matrices.

The required matrices for consideration of axial effects as well as the matrix required for the transformation of coordinates are developed in this chapter. A computer program for modeling structures as plane frames is also presented. This program is organized following the pattern of the BEAM program of the preceding chapter.

11.7 Problems

The following problems are intended for hand calculation, though it is recommended that whenever possible solutions should also be obtained using Program 14 to model the structure as a plane frame, Program 8 to determine natural frequencies and modal shapes, and Program 9 to calculate the response using modal superposition method.

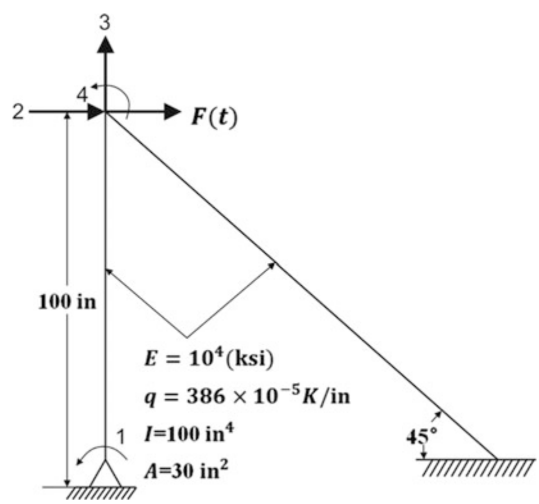

Fig. P11.1

Problem 11.1
For the plane frame shown in Fig. P11.1 determine the system stiffness and mass matrices. Base the analysis on the four free nodal coordinates indicated in the figure. Use consistent mass method.

Problem 11.2

Use the results obtained in Problem 11.1 in performing the static condensation to eliminate the rotational degree of freedom at the support to determine the transformation matrix and the reduced stiffness and mass matrices.

Problem 11.3

Determine the natural frequencies and corresponding normal modes for the reduced system in Problem 11.2.

Problem 11.4

Determine the response of the frame shown in Fig. P11.1 when it is acted upon by a force $F(t) = 1.0$ Kip suddenly applied at nodal coordinate 2 for 0.05 sec. Use results of Problem 11.3 to obtain the modal equations. Neglect damping in the system.

Problem 11.5

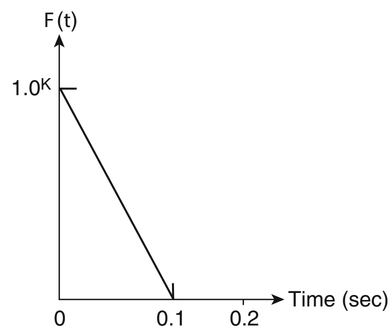

Fig. P11.5

Determine the maximum response of the frame shown in Fig. P11.1 when subjected to the triangular impulsive load (Fig. P11.5) along the nodal coordinate 2. Use results of Problem 11.3 to obtain the modal equations and use the appropriate response spectrum to find maximum modal response (Fig. 4.5). Neglect damping in the system.

Pit).

Problem 11.6

Determine the steady-state response of the frame shown in Fig. P11.1 when subjected to harmonic force $F(t) = 10 \sin 30$ t (Kip) along nodal coordinate 2. Neglect damping in the system.

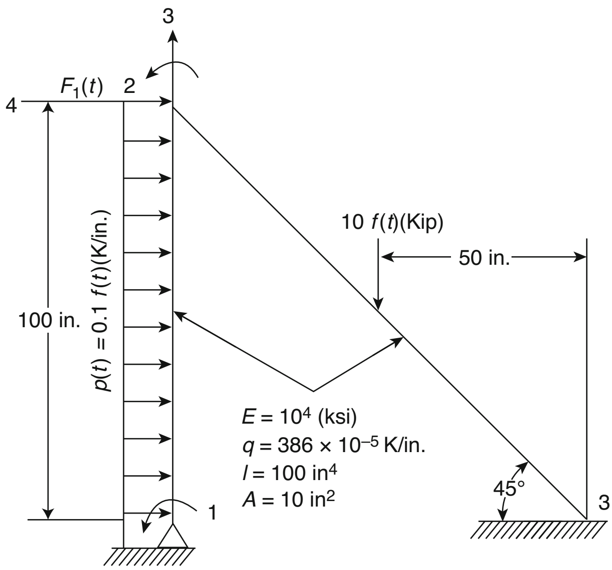

Fig. P11.8

Problem 11.7
Repeat Problem 11.6 assuming that the damping is proportional to the stiffness matrix of the system, $[C] = a_0\,[K]$, where $a_0 = 0.2$.

Problem 11.8
The frame shown in Fig. P11.8 is acted upon by the dynamic forces shown in he figure. Determine the equivalent nodal forces corresponding to each member of the frame.

Problem 11.9
Assemble the system equivalent nodal forces $\{F_e\}$ from equivalent member nodal forces which were calculated in Problem 11.8.

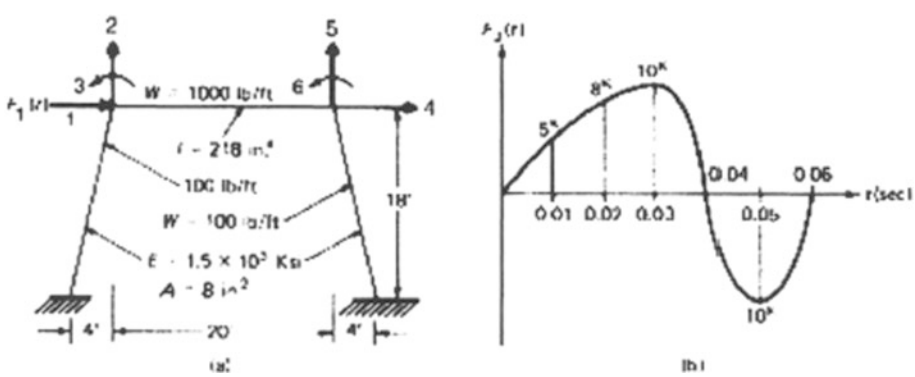

Fig. P11.11

Problem 11.10
Determine the natural frequencies and corresponding normal modes for the frame shown in Fig. P11.8.

Problem 11.11
Determine the response for the frame shown in Fig. P11.11 (a) when subjected to the force $F_3(t)$ [Fig. 11.11(b)] acting along nodal coordinate 1. Assume 5% damping in all the modes.

Problem 11.12
Determine the steady-state response of the frame in Fig. P11.11 acted upon harmonic force $F_1(t) = 10\cos 50\,t$ (Kip) as indicated in the figure. Neglect damping in the system.

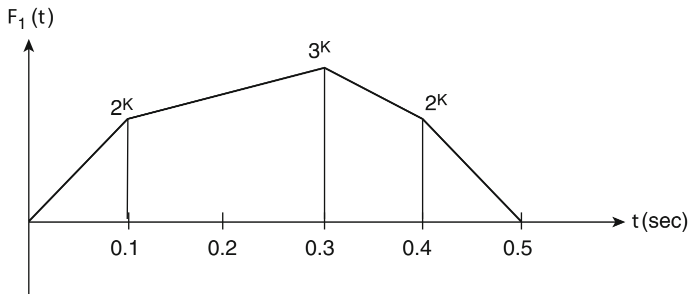

Fig. P11.14

Problem 11.13
Solve Problem 11.11 using step-by-step linear acceleration method (Program 19). Neglect damping.

Problem 11.14
Determine the response of the frame shown in Fig. P11.1 when acted upon by the force $F_1(t)$ (depicted in Fig. P11.14 applied at nodal coordinate 1. Assume 10% damping in all the modes. Use modal superposition method.

Problem 11.15
Find the response in Problem 11.14 using step-by step linear acceleration method (Program 19). Assume damping proportional to stiffness by a factor $a_0 = 0.01$.