15 Dynamic Analysis of Structures Using the Finite Element Method

In the preceding chapters, we considered the dynamic analysis of structures modeled as beams, frames, or trusses. The elements of all these types of structures are described by a single coordinate along their longitudinal axis; that is, these are structures with unidirectional elements, called, "skeletal structures." They, in general, consist of individual members or elements connected at points designated as "nodal points" or "joints." For these types of structures, the behavior of each element is first considered independently through the calculation of the element stiffness matrix and the element mass matrix. These matrices are then assembled into the system stiffness matrix and the system mass matrix in such a way that the equilibrium of forces and the compatibility of displacements are satisfied at each nodal point. The analysis of such structures is commonly known as the Matrix Structural Method and could be applied equally to static and dynamic problems.

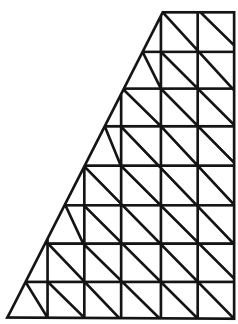

Fig. 15.1 Finite element modeling of thin plate with triangular elements

The structures presented in this chapter are continuous structures which are conveniently idealized as consisting of two-dimensional elements connected only at the selected nodal points. For example, Fig. 15.1 shows a thin plate idealized with plane triangular elements. The static or dynamic analysis of such idealized structures is known as the Finite Element Method (FEM). This is a powerful method for the analysis of structures with complex geometrical configurations, material properties or loading conditions. This method is entirely analogous to Matrix Structural Analysis for skeletal structures (beams, frames, and trusses) presented in the preceding chapters. The Finite Element Method differs from the Matrix Structural Method only in two respects: (1) the selection of elements and nodal points

M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_15

are not naturally or clearly established by the geometry as it is for skeletal structures and (2) the displacements at internal points of an element are expressed by approximate interpolating functions and not by an exact analytical relationship as it is in the Matrix Structural Method. Furthermore, for skeletal structures, the displacement of an interior point of an element is governed by an ordinary differential equation, while for a continuous two-dimensional element it is governed by a partial differential equation of much greater complexity.

15.1 Plane Elasticity Problems

Plane elasticity problems refer to plates that are loaded in their own planes. Out-of-plane displacements are induced when plates are loaded by normal forces that are perpendicular to the plane of the plate, such problems are generally referred to as plate bending. (Plate bending is considered in Sect. 15.2.) There are two different types of plane elasticity problems: (1) plane stress and (2) plane strain. In the plane stress problems, the plate is thin relative to the other dimensions and the stresses normal to the plane of the plate are not considered. Figure 15.2 shows a perforated strip-plate in tension as an example of a plane stress problem. For plane strain problems, the strain normal to the plane of loading is suppressed and assumed to be zero. Figure 15.3 shows a transverse slice of a retaining wall as an example of a plane strain problem.

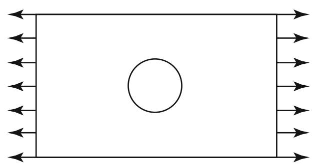

Fig. 15.2 Perforated plate tension element as an example of a structural member load in plane stress

In the analysis of plane elasticity problems, the continuous plate is idealized as finite elements interconnected at their nodal points. The displacements at these nodal points are the basic unknowns as are the displacements at the joints in the analysis of beams, frames or trusses. Consequently, the first step in the application of the FEM is to model the continuous system into discrete elements. The most common geometric elements used for plane elasticity problems are triangular, rectangular or quadrilateral, although other geometrical shapes could be used as well.

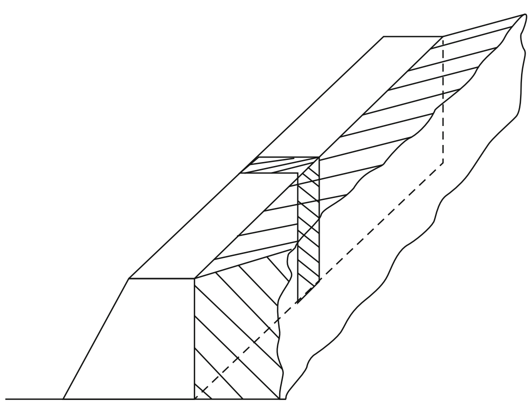

Fig. 15.3 Retaining wall showing a plate slice as an example of plane strain conditions

15.1.1 Triangular Plate Element for Plane Elasticity Problems

The following steps are used in the application of the FEM for the analysis of structural problems:

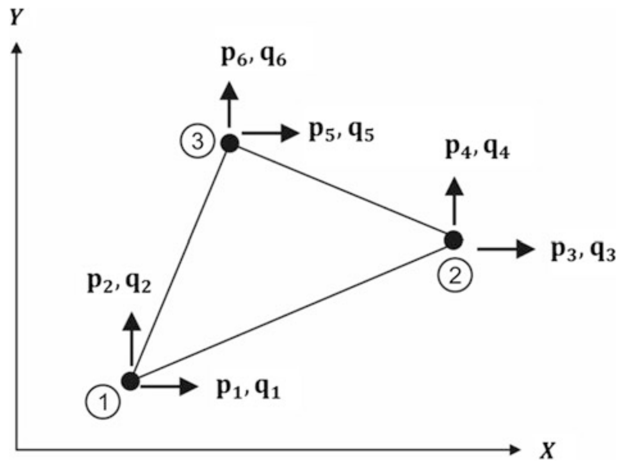

Fig. 15.4 Triangular plate element showing nodal forces P_i and corresponding nodal displacements q_i at the three nodes.

Step I: Modeling the Structure

Figure 15.4 Shows a triangular plate element with nodal forces $\{P\}_e$ and corresponding nodal displacements $\{q\}_e$ with components in the x and y directions at the three nodes of this element. In dynamic analysis, the plate element is assumed to be loaded by external forces distributed in the plane of the plate (Fig. 15.5). These external forces are: (1) body forces with components of forces per unit of volume in the x and y directions, b_x and b_y, conveniently arranged in the vector $\{\boldsymbol{b}\}_e$ and 2) inertial forces, $\rho\{\ddot{q}\}$, per unit of volume in which ρ is the mass density of the plate and $\{\ddot{q}\}$ is the acceleration of a point in the plate element with components $\ddot{q}_x$ and $\ddot{q}_y$ along the coordinates x and y. In addition to these two types of forces, other external forces may also be considered in the analysis.

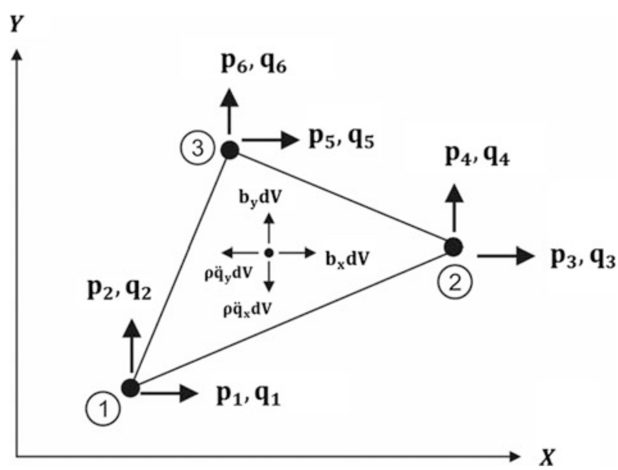

Fig. 15.5 Triangular plate element showing external forces

In a plane elasticity problem, the triangular element with three nodes has two degrees of freedom at each node, as shown in Fig. 15.5, resulting in a total of six degrees of freedom. Thus, the nodal displacement vector $\{\boldsymbol{q}\}_e$ and corresponding nodal force vector $\{\boldsymbol{P}\}_e$ for the plane triangular element have six terms. These vectors may be written as:

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \quad \{\mathrm{P}\}_e = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} \tag{15.1}$$

Therefore, for this plane triangular element, the element stiffness matrix $[K]_e$ relating nodal forces and nodal displacements and the element mass matrix $[M]_e$ relating nodal forces and nodal accelerations, are of dimension 6×6.

Step II: Selection of a Suitable Displacement Function

The displacements, $u = u\ (x, y)$ and $v = v\ (x, y)$, respectively in the x and y directions at any interior point $P\ (x, y)$ of the triangular element, are expressed approximately by polynomial functions with a total of six coefficients equal to the number of possible nodal displacements. In this case, the simplest expressions for the displacement functions, $u\ (x, y)$ and $v\ (x, y)$, at an interior point of the triangular element are:

$$\begin{aligned} u(x, y) &= c_1 + c_2 x + c_3 y \\ v(x, y) &= c_4 + c_5 x + c_6 y \end{aligned} \tag{15.2}$$

or

$$\{q(x, y)\} = [g(x, y)]\{c\} \tag{15.3}$$

where $\{c\}$ is a vector containing the six coefficients c_i, $[g\ (x, y)]$ a matrix function of the position coordinates (x, y) and $\{q\ (x, y)\}$ a vector with the displacement components $u(x, y)$ and $v(x, y)$ at an interior point along x and y directions, respectively.

Step III: Displacements {q (x, y)} at a Point in the Element Are Expressed in Terms of the Nodal Displacements, $\{\mathbf{q}\}_\mathbf{e}$

The evaluation of Eq. (15.3) for the displacements of the three nodal joints of the triangular element followed by the solution of the coefficients c_i $(i = 1, 2\ldots6)$ results in

$$\{q\}_e = [A]\{c\} \tag{15.4}$$

and

$$\{c\} = [A]^{-1}\{q\}_e \tag{15.5}$$

The subsequent introduction of $\{c\}$ from Eq. (15.5) into Eq. (15.3) gives the displacements at any interior point of the element in terms of the nodal displacements $\{q\}_e$ as

$$\{q(x,y)\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [g(x,y)][A]^{-1}\{q\}_e \tag{15.6}$$

or

$$\{q(x,y)\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [f(x,y)]\{q\}_e \tag{15.7}$$

where the matrix $[f(x, y)] = [A]^{-1}\,[g(x, y)]$ is only a function of the coordinates x, y of a point in the triangular element. Because the displacement functions $u\,(x, y)$ and $v(x, y)$ in Eq. (15.7) are both linear in x and y, displacement continuity is ensured along the interface adjoining elements for any value of nodal displacements.

Step IV: Relation between Strain, ε{(x, y)} at any Point within the Element to the Displacements {q (x, y)} and Hence to the Nodal Displacements {q}$_\mathbf{e}$.

In Theory of Elasticity (Timoshenko and Goodier, 1970), it is shown that the linear strain vector, $\{\varepsilon(x, y)\}$, with axial strain components along the x, y directions, ε_x, ε_y and shearing strain γ_{xy} is given by differentiation of the displacement functions $u(x, y)$ and $v(x, y)$ as follows:

$$\{\varepsilon(x,y)\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{15.8}$$

The strain components may then be expressed in terms of the nodal displacements $\{q\}_e$ by substituting from Eq. (15.7) the derivatives of $u(x, y)$ and $v\,(x, y)$ into Eq. (15.8):

$$\{\varepsilon(x,y)\} = [B]\{q\}_e \tag{15.9}$$

in which the matrix $[\boldsymbol{B}]$ is solely a function of the coordinates (x, y) at an interior point of the element.

Step V: Relationship between Internal Stresses {σ (x, y)} to Strains {ε (x, y)} and Hence to the Nodal Displacements {q}$_\mathbf{e}$

For plane elasticity problems, the relationship between the normal stresses σ_x, σ_y and shearing stress τ_{xy} to the corresponding strains ε_x, ε_y, and γ_{xy} may be expressed, in general as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{15.10}$$

or in a short matrix notation as:

$$\{\sigma(x,y)\} = [D]\{\varepsilon(x,y)\} \tag{15.11}$$

The substitution of $\{\varepsilon\ (x, y)$ from Eq. (15.9) into Eq.(15.11) gives the desired relationship between stresses $\{\sigma\ (x, y)\}$ at an interior point in the element and the displacements $\{q\}_e$ at the nodes as:

$$\{\sigma(x,y)\} = [D][B]\{q\}_{\mathrm{e}} \tag{15.12}$$

The terms d_{ij} of the matrix $\boldsymbol{D}$ in Eq. (15.10) have different expressions for plane stress problem than for plane strain problem. These expressions as given by the theory of elasticity are:

For plane stress problems:

$$\begin{aligned} d_{11} &= d_{22} = \frac{E}{1\text{-}v^2} \\ d_{12} &= d_{21} = \frac{vE}{1\text{-}v^2}\ 1 \\ d_{33} &= \frac{E}{2(1+v)} \end{aligned} \tag{15.13}$$

For plane strain problems:

$$\begin{aligned} d_{11} &= d_{22} = \frac{E}{1-v^2} \\ d_{12} &= d_{21} = \frac{vE}{1-v^2} \\ d_{33} &= \frac{E}{2(1+v)} \end{aligned} \tag{15.14}$$

in which E is the modulus of elasticity and v is the Poisson's ratio.

Step VI: Element Stiffness and Mass Matrices

Use is made of the Principle of Virtual Work to establish the expressions for the element stiffness matrix $[K]_e$ and the element mass matrix $[M]_e$ This principle states that for structures in dynamic equilibrium subjected to small, compatible virtual displacements, $\{\delta q\}$, the virtual work, δW_E, of the external forces is equal to the virtual work of internal stresses, δW_I, that is.

$$\delta W_I = \delta W_E \tag{15.15}$$

In applying this principle to a finite element, we assume a vector of virtual displacements $\delta q\{(x,y)\}$. Hence, by Eq. (15.7).

$$\delta\{q(x,y)\} = [f(x,y)]\delta\{q\}_e \tag{15.16}$$

Using the strain-displacement relationships, Eq. (15.9), we obtain.

$$\delta\{\varepsilon\} = [B]\delta\{q\}_e \tag{15.17}$$

The internal work over the volume of the element during this virtual displacement is then given by the product of the virtual strain and the stress integrated over the volume of the element:

$$\delta W_I = \int_v \delta\{\varepsilon\}^T\{\sigma(x,y)\}dV \tag{15.18}$$

The external virtual work W_E includes the work of the applied body forces $\{b\}_e\, dV$; the work of inertial forces $\rho\{\ddot{q}\}dV$; and the work of the nodal forces P_i $(i = 1, 2\ldots 6)$, shown in Fig. 15.5. The total external virtual work is equal to the product of these forces times the corresponding virtual displacements integrated over the volume of the element, namely.

$$\delta W_e = \int_v \delta\{q\}^T\{b\}_e dV \text{-} \int_v \delta\{q\}^T \rho\{\ddot{q}\}dV + \delta\{q\}_e^T\{P\}_e \tag{15.19}$$

The second term of Eq. (15.19) is negative because the inertial forces act in directions that are opposite to the positive sense of the accelerations. The substitution of Eqs. (15.18) and (15.19) into Eq. (15.15) results in

$$\int_v \delta\{\varepsilon\}^T\{\sigma(x,y)\}dV = \int_v \delta\{q\}^T\{b\}_e dV - \int_v \delta\{q\}^T\rho\{\ddot{q}\}dV + \delta\{q\}_e^T\{P\}_e \tag{15.20}$$

By substituting into Eq. (15.20) $\sigma(x, y) = [\boldsymbol{D}][B]\{q\}_e$ from Eq. (15.12), and the transposes of $\delta\{q\}^T$ and $\delta\{\varepsilon\}^T$ from Eqs. (15.16) and (15.17) respectively, we obtain, after cancellation of the common factor $\delta\{q\}^T$, the equation of motion for the element:

$$[\boldsymbol{M}]_e\{q\}_e + [\boldsymbol{K}]_e\{q\}_e = \{\boldsymbol{P}\}_e + \{\boldsymbol{P}_b\}_e \tag{15.21}$$

where

$$[K]_e = \int_v [B]^T[D][B]dV \tag{15.22}$$

$$[\boldsymbol{M}]_e = \int_v \rho[f(x,y)]^T[f(x,y)]dV \tag{15.23}$$

and

$$\{\boldsymbol{P}_b(t)\}_{\mathrm{e}} = \int_v [f(x,y)]\{b\}_e dV^2 \tag{15.24}$$

Matrix, $[\boldsymbol{K}]_e$, in Eq. (15.22) is the element stiffness matrix with the terms expressing the force for unit displacement at a nodal coordinate of the element. Eq. (15.23) gives the consistent mass matrix for the element with components expressing forces due to a unit value of the acceleration at a nodal coordinate. Finally, the vector $\{\boldsymbol{P}\}_e$ contains the external forces applied to the nodes of the element and the vector $\{\boldsymbol{P}_b(t)\}_e$ is the consistent nodal forces vector due to the body forces $\{b\}_e$ on the element.

The element stiffness matrix, $[\boldsymbol{K}]_e$, and the element mass matrix, $[\boldsymbol{M}]_e$, as well as the equivalent vector of the applied body forces, $\{\boldsymbol{P}_t\}_e$, may be readily obtained explicitly from Eqs.(15.22), (15.23), and (15.24) respectively, for the simple plane elasticity element. However, computer codes are generally written to calculate these matrices by numerical methods, particularly when the structure is modeled using more advanced elements developed from higher order polynomials.

Step VII: Assemblage of the System Stiffness Matrix [K], the System Mass Matrix [M], and the System Equivalent Nodal Force Vector $\{P_b\}_s$ Due to the Body Forces.

The system stiffness matrix $[\boldsymbol{K}]$ and the system mass matrix $[\boldsymbol{M}]$ are assembled from the appropriate summations of the corresponding element matrices, by exactly the same process used in the previous chapters to assemble these matrices for skeletal-type structures; the system force vector is assembled from the element equivalent nodal forces. Hence, we may symbolically write

$$[\boldsymbol{K}] = \boldsymbol{\Sigma}[\boldsymbol{K}]_e \tag{15.25}$$

$$[\boldsymbol{M}] = \boldsymbol{\Sigma}[\boldsymbol{M}]_e \tag{15.26}$$

and

$$\{\boldsymbol{P}_b\} = \boldsymbol{\Sigma}\{\boldsymbol{P}_b\}_e \tag{15.27}$$

The system of differential equations of motion is then given by

$$[\boldsymbol{M}]\{\ddot{u}\} + [\boldsymbol{K}\}\{\mathrm{u}\} = \{\boldsymbol{F}\} \tag{15.28}$$

in which $\{\boldsymbol{F}\}$ is the vector of the external forces at the system nodal coordinates $\{\mathrm{u}\}$ which includes the equivalent nodal forces for the body forces and for any other forces distributed over the structural element.

Step VIII: Solution of the Differential Equations of Motion
The solution of the system of differential equations of motion in terms of the nodal displacements $\{\mathrm{u}\}$ is usually obtained by the modal superposition method presented in Chap. 8. Damping in the system can readily be included in the analysis by the simple addition of the modal damping term to the modal equation. For systems with nonlinear behavior, the modal superposition is not valid and the solution must be obtained using a numerical method such as the step-by-step integration method presented for a single-degree-of-freedom system in Chap. 6 and for a multiple-degree-of freedom system in Chap. 16.

Step IX: Determination of Nodal Stresses
The final step is the calculation of stresses at the nodal points. These stresses can be calculated from the element nodal displacements $\{q\}_e$ selected from the system nodal displacements $\{\mathrm{u}\}$ already determined. The element nodal stresses, $\{\sigma\{x_j,\ y_j)\}_\mathrm{e}$, for node j of an element, are given from Eq. (15.12) by.

$$\left\{\sigma\left(x_j, y_j\right)\right\}_e = [\boldsymbol{D}]_j[\boldsymbol{B}]_j\{q\}_e \tag{15.29}$$

in which the matrices $[\boldsymbol{D}\}_j$ and $[\boldsymbol{B}]_j$ are evaluated for the coordinates of the node j of the element.

15.2 Plate Bending

The application of the finite element method is now considered for the analysis of plate bending, that is, plates loaded by forces that are perpendicular to the plane of the plate. The presentation that follows is based on two assumptions: (1) the thickness of the plate is assumed to be small compared to other dimensions of the plate and (2) the deflections of the loaded plate under the load are assumed to be small relative to its thickness. These assumptions are not particular to the application of finite element method; they are also made in the classical theory of elasticity for bending of thin plates. These two assumptions are necessary because if the thickness of the plate is large, then the plate has to be analyzed as a three-dimensional problem and if the deflections are also large, then in-plane membrane forces are developed and should be accounted for in the analysis. The analysis of plates can be undertaken by the finite element method without these two assumptions. The computer program such as SAP2000 used in this book is general and may be applied for the analysis of either thin plates undergoing small deflections or to thick plates in which these two assumptions are not required. However, the presentation in this section is limited to thin plates that undergo small deflections.

15.2.1 Rectangular Element for Plate Bending

The derivation of the stiffness and mass matrices as well as the vector of equivalent nodal forces for body forces, inertial forces or any other forces distributed on the plate elements is obtained following the same steps used for derivation presented for a triangular element subjected to in-plane loads in the preceding section.

Step I: Modeling the Structure

A suitable system of coordinates and node numbering is defined in Fig. 15.6a with *x, y* axes along the continuous sides of the rectangular plate element, and the *z* axis normal to the plane of the plate, completing a right-hand system of Cartesian coordinates. This rectangular element has three nodal coordinates at each of its four nodes, a rotation about the *x*-axis (θ_x), a rotation about the *y*-axis (θ_y), and a normal displacement (w) along the *z*-axis transverse to the plane of the plate. These nodal displacements are shown in their positive sense and labeled q_i (i = 1, 2... 12) in Fig. 15.6b.

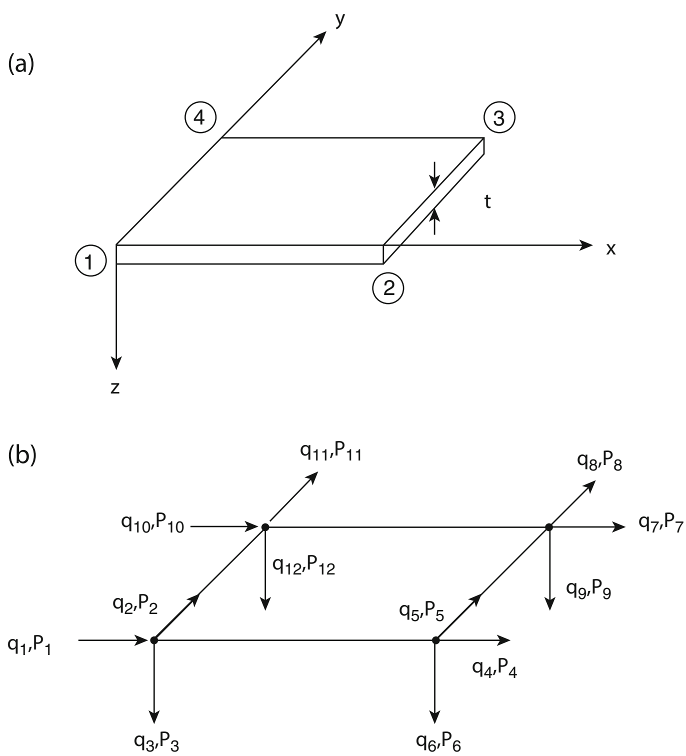

Fig. 15.6 Rectangular plate bending element. (**a**) Coordinate system and node numbering. (**b**) Nodal coordinates q_i and corresponding nodal forces p_i

Corresponding to the three nodal displacements at each node, two moments and a force which are labeled $P_{i,}$ are also shown in this figure. These 12 element nodal displacements and 12 nodal forces are conveniently arranged in two vectors of 12 components, $\{\boldsymbol{q}\}_e$ and $\{\boldsymbol{P}\}_e$. Therefore, the stiffness matrix or the mass matrix, respectively, relating the nodal forces and the nodal displacements, or nodal forces and nodal accelerations for this rectangular plate bending element with four nodes are of dimension 12 × 12. The angular displacements θ_x and θ_y at any point (x, y) of the plate element are related to the normal displacement w by the following expressions:

$$\theta_x = -\frac{\partial w}{\partial y} \quad \theta_y = \frac{\partial w}{\partial x} \tag{15.30}$$

The positive directions of θ_x and θ_y, are chosen to agree with the angular nodal displacement q_1, q_2, q_4 q_5, etc. is selected in Fig. 15.6b. Therefore, after a function, $w = w\,(x, y)$, is chosen for the lateral displacement w, the angular displacements are determined through the relations in Eq. (15.30).

Step II: Selection of a Suitable Displacement Function

Since the rectangular element in plate bending has twelve degrees of freedom, the polynomial expression chosen for the normal displacements w, must contain 12 constants. A suitable polynomial function is given by.

$$w = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 + c_7x^3 + c_8x^2y + c_9xy^2 + c_{10}y^3 + c_{11}x^3\mathrm{y} + c_{12}xy^3 \tag{15.31}$$

The displacement function for the rotations θ_x and θ_y are then obtained from Eqs. (15.30) and (15.31) as

$$\theta_x = -\frac{\partial w}{\partial y} = -\left(c_3 + c_5x + 2c_6y + c_8x^2 + 2c_9xy + 3c_{10}y^2 + c_{11}x^3 + 3c_{12}xy^2\right) \tag{15.32}$$

and

$$\theta_y = \frac{\partial w}{\partial x} = c_2 + 2c_4x + c_5y + 3c_7x^2 + 2c_8xy + c_9y^2 + 3c_{11}x^2y + c_{12}y^3 \tag{15.33}$$

By considering the displacements at the edge of one element, that is, in the boundary between adjacent elements, it may be demonstrated that there is continuity of normal lateral displacements and of the rotational displacement in the direction of the boundary line, but not in the direction transverse to this line as it is shown graphically in Fig. 15.7. The displacement function in Eq. (15.31) is called "a non-conforming function" because it does not satisfy the condition of continuity at the boundaries between elements for all three displacements w, θ_x, and θ_y.

Step III: Displacements {q (x, y)} at a Point within the Element Are Expressed in Terms of the Nodal Displacements $\{\mathbf{q}\}_\mathbf{e}$

Writing Eq. (15.31), (15.32), and (15.33) in matrix notation, evaluating the displacements at the nodal coordinates and solving for the unknown constants results in

$$\{q(x,y)\} = \begin{Bmatrix} \theta_x \\ \theta_y \\ w \end{Bmatrix} = [g(x,y)]\{c\} \tag{15.34}$$

$$\{q\}_e = [\boldsymbol{A}]\{c\} \tag{15.35}$$

and

$$\{c\} = [\boldsymbol{A}]^{-1}\{q\}_e \tag{15.36}$$

where $[\boldsymbol{A}]^{-1}$ is the inverse of the matrix $[\boldsymbol{A}]$ in Eq. (15.35), $[g\,(x, y)]$ is a function of the coordinates x, y at a point in the element, and $\{q\}_e$ is the vector of the 12 displacements at the nodal coordinates of the element (Fig. 15.7).

The substitution of the vector of constants $\{c\}$ from Eq. (15.36) into Eq. (15.34) provides the required relationship for displacements $\{q(x, y)\}$ at an interior point in the rectangular element and the displacements $\{q\}_e$ at the nodes.

$$\{q(x,y)\} = [g(x,y)][\boldsymbol{A}]^{-1}\{q\}_e \tag{15.37}$$

or using Eq. (15.30):

$$\{q(x,y)\} = \begin{Bmatrix} -\dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} \\ w \end{Bmatrix} = [f(x,y)]\{q\}_e \tag{15.38}$$

in which $[f(x, y)] = [g\,(x, y)][\boldsymbol{A}]^{-1}$ is solely a function of the coordinates x, y at a point within the element.

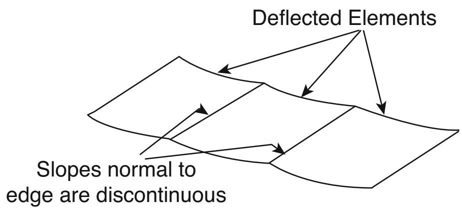

Fig. 15.7 Deflected continuous rectangular elements

Step IV: Relationship between Strains (ε (x, y) at any Point within the Element to Displacements {q (x, y)} and Hence to the Nodal Displacements $\{q\}_e$.

For plate bending, the state of strain at any point of the element may be represented by three components: the curvature in the x direction, the curvature in the y direction, and a component representing torsion in the plate. The curvature in the x direction is equal to the rate of change of the slope in that direction, that is, to the derivative of the slope,

$$-\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = -\frac{\partial^2 w}{\partial x^2} \tag{15.39}$$

Similarly, the curvature in the y direction is

$$-\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = -\frac{\partial^2 w}{\partial y^2} \tag{15.40}$$

Finally, the torsional strain component is equal to the rate of change, with respect to y, of the slope in the x direction, that is

$$\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial^2 w}{\partial x \partial y} \tag{15.41}$$

The bending moments M_x and M_y and the torsional moments M_{xy} and M_{yx} each act on two opposites sides of the element, but since M_{xy} is numerically equal to M_{yx}, one of these torsional moments, M_{xy}, can be considered to act in all four sides of the element, thus allowing for simply doubling the torsional strain component. Hence, the "strain" vector, $\{\varepsilon\ (x, y)\}$ for a plate bending element can be expressed by

$$\{\varepsilon(x,y)\} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{15.42}$$

The substitution in Eq. (15.42) of the second derivatives obtained by differentiation of Eq. (15.38) yields

$$\{\varepsilon(x,y)\} = [\boldsymbol{B}]\{q\}_e \tag{15.43}$$

in which $[\boldsymbol{B}]$ is a function of the coordinates (x, y) only.

Step V: Relationship between Internal {σ (x, y)} to Internal Strains {ε (x, y)} and Hence to Nodal Displacements $\{\mathbf{q}\}_\mathbf{e}$

In a plate bending, the internal "stresses" are expressed as bending and twisting moments, and the "strains" are the curvatures and the twist. Thus, for plate bending, the state of internal "stresses" can be represented by

$$\{\sigma(x,y)\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \tag{15.44}$$

where M_x and M_y are internal bending moments per unit of length and M_{xy} is the internal twisting moment per unit of length. For a small rectangular element of the plate bending, these internal moments are shown in Fig. 15.8 The moment-curvature relationships obtained from plate bending theory (Timoshenko and Goodier, 1970) are:

$$M_x = -\left(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right)$$
$$M_y = -\left(D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2}\right) \quad (15.45)$$
$$M_{xy} = 2D_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

These relations are written in general for an orthotopic plate, i.e. a plate which has different elastic properties in two perpendicular directions, in which D_x and D_y are flexural rigidities in the x and y directions, respectively, D_1 is a "coupling" rigidity coefficient representing the Poisson's ratio type of effect and D_{xy} is the torsional rigidity.

For an isotropic plate which has the same properties in all directions, the flexural and twisting rigidities are given by

$$D_x = D_y = D = \frac{Et^3}{12(1-\nu^2)} D$$
$$D_1 = \nu D \quad D_{xy} = \frac{(1-\nu)}{2} \quad (15.46)$$

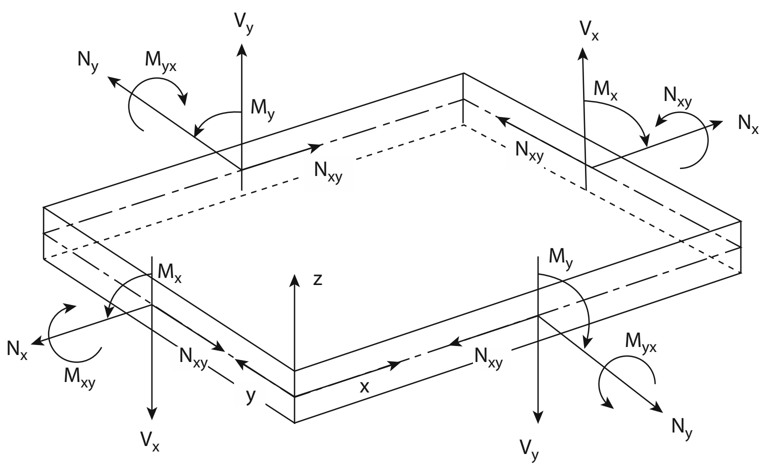

Fig. 15.8 Direction of force and moment per unit length as defined for thin shells

Equations (15.45) may be written in matrix form as

$$\{\sigma(x,y)\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \quad (15.47)$$

or symbolically as

$$\{\sigma(x,y)\} = [\boldsymbol{D}]\{\varepsilon(x,y)\} \tag{15.48}$$

where the matrix $[\boldsymbol{D}]$ is defined in Eq. (15.47). The substitution into Eq. (15.48) of $\{\varepsilon\ (x, y)\}$ from Eq. (15.43) results in the required relationship between element stresses and nodal displacements as

$$\{\sigma(x,y)\} = [\boldsymbol{D}][\boldsymbol{B}]\{q\}_e \tag{15.49}$$

Step VI: Element Stiffness and Mass Matrices

The stiffness matrix and the mass matrix for an element of plate bending obtained by applying the Principle of Virtual Work results in Eqs. (15.22) and (15.23), and for the equivalent forces due to the applied body forces on the element in Eq.(15.24). The matrices $[f\,(x, y)]$, $[\boldsymbol{B}]$, and $[\boldsymbol{D}]$ required in these equations are defined, respectively, in Eqs. (15.38), (15.43), and (15.47). The calculation of these matrices and also of the integral indicated in Eqs.(15.22), (15.23), and (15.24), is usually undertaken by numerical methods implemented in the coding of computer programs.

Step VII: Assemblage of the System Stiffness Matrix [K], the System Mass Matrix [M], and the Vector of the External Forces {F} at the System Nodal Coordinates {y}, which Includes the Equivalent Forces for the Body forces {P_b} and for any Other Forces Distributed over the Structural Element.

The system stiffness and mass matrices, as well as the nodal force vector, due to applied body forces, are assembled from the corresponding element matrices and vector as given in Eqs. (15.25), (15.26), and (15.27).

Step VIII: Solution of the Differential Equations of Motion

The system differential equation of motion is given by Eq. (15.28) in which $[\boldsymbol{M}]$ and $[\boldsymbol{K}]$ are the assembled mass and stiffness matrices, respectively and $\{F\}$ is the equivalent vector of the external forces. The solution of Eq. (15.28) thus provides the system vector of nodal displacements $\{y\}$.

Step IX: Determination of Nodal Stresses

The element nodal stresses, $\{\sigma\ (x_j, y_j)\}_e$, for node j of an element are given from Eq. (15.49) as

$$\left\{\sigma\left(x_j, y_j\right)\right\}_e = [\boldsymbol{D}]_j[\boldsymbol{B}]_j\{\boldsymbol{q}\}_e \tag{15.50}$$

in which the matrices $[\boldsymbol{D}]_j$ and $[\boldsymbol{B}]_j$ are evaluated for the coordinates of node j of the element.

15.3 Summary

In this chapter we have presented an introduction to the Finite Element Method (FEM) for the analysis of problems in Structural Dynamics. The theory of FEM in structural dynamics was formulated through the following steps:

1. Modeling of the entire structure into one-dimensional, two-dimensional, or three-dimensional beam, rod, triangular, quadrilateral, rectangular, or other types of structural elements.
2. Identifying nodes and nodal coordinates at joints between structural elements.

3. Selecting an interpolating function, which usually is a polynomial to express the displacements at an interior point in the element in terms of the
   (a) displacements at its nodal coordinates.
4. Establishing the relationships at the nodal coordinates of a structural element between forces and displacements (the element stiffness matrix) and between forces and accelerations (the element mass matrix.)
5. Obtaining the vector of the equivalent nodal forces for the body or other external forces acting on the element.
6. Assembling the system stiffness matrix, the system mass matrix, and the system vector of the equivalent nodal forces, respectively, from the element stiffness matrices, element mass matrices, and the element vectors of the equivalent nodal forces.
7. Establishing the dynamic equilibrium at the system nodal coordinates, among which [***M***] and [***K***] are, respectively, the system mass matrix and the system stiffness matrix, $\{F(t)\}$ is the system vector of the equivalent nodal forces, and $\{\mathrm{u}\}$ and $\{\ddot{u}\}$ are, respectively, the displacement and the acceleration vectors at the system nodal coordinates, the elastic forces, inertial forces and the external forces to obtain the system differential equation of motion:

$$[\boldsymbol{M}]\{\ddot{u}\} + [\boldsymbol{K}]\{\mathrm{u}\} = \{\boldsymbol{F}(t)\} \quad \text{(15.28 repeated)}$$

8. Introducing in the system differential equation the boundary conditions restricting displacements at specified nodal coordinates.

9. Solving of the system differential equation of motion, Eq. (15.28), to obtain the system nodal displacements $\{u\}$.
10. Determining the element nodal stresses $\{\sigma\ (x_j,\ y_j)\}$ from the calculated element nodal displacements $\{q\}_e$.

In engineering practice, the solution of problems by the Finite Element Method is obtained with the computer and appropriate software, such as the computer program such as SAP2000 used in this text. In the implementation of the computer program such as SAP2000, items are selected from menus presented by the program and data is supplied in response at the prompts in the program.

15.4 Problems

Problem 15.1
The steel plate shown in Fig. P15.1 of dimensions 20 in × 20 in and thickness 0.10 in with a circular hole of radius $r = 50$ in is subjected to a suddenly applied in-plane lateral compressive pressure along the edges *AD* and *BC* of magnitude $p = 100$ psi. Model the plate with elements PLANE2D on a 6 × 6 mesh in each quarter section of the plate. Determine the first five natural frequencies and plots of the response for displacements and stresses on the elements.

Problem 15.2
Solve Problem 15.1 taking advantage of the double symmetry of geometry and load introducing appropriate boundary conditions and analyzing only a quarter of the structure.

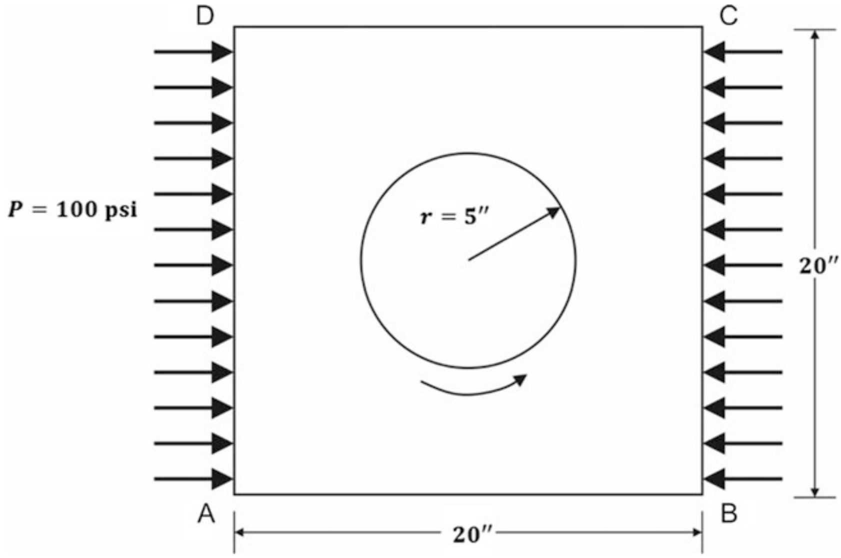

Fig. P15.1

Problem 15.3

A simple supported deep steel beam (Modulus of Elasticity, $E = 29E06$) shown in Fig. 15.6, with a distributed mass of 0.03 (lb.sec^2/in^2) is loaded at its center by the impulsive force F(t) depicted in Fig. 15.6b. Determine the first five natural periods and the time-displacement response at the center of the beam. Use computer program with rectangular shell elements to model this structure.

Problem 15.4

A square steel plate 40 in. by 40 in. and thickness 0.10 in., assumed to be fixed at the supports on its four sides (Fig. P15.4) is acted upon by a harmonic force F(t) = 0.1 sin 5.3 t (kip) applied normally at its center. Determine: (a) the first five natural frequencies, (b) the time-displacement function at the center of the plate, and (c) the time-stress function at the center of plate. Use time step $\Delta t = 0.01$ s. ($E = 29{,}500$ ksi, $\upsilon = 0.3$).

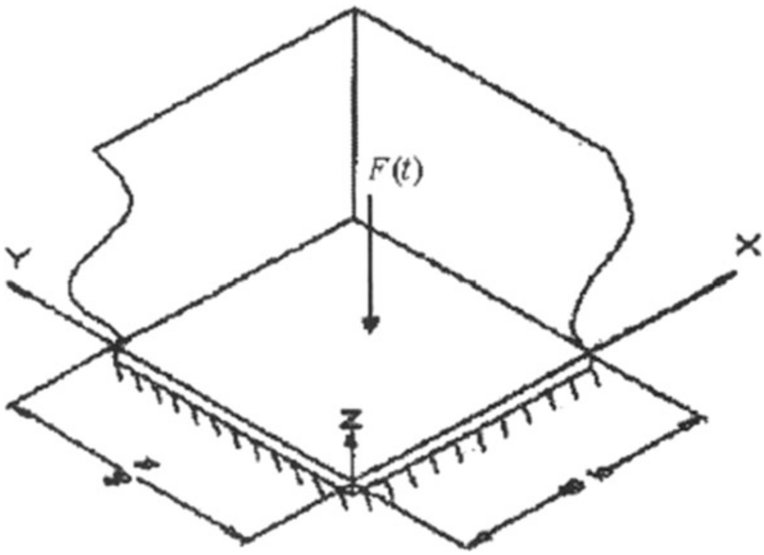

Fig P15.4 Square plate of Illustrative Example 15.2 supporting a normal harmonic force at the center.