13 Dynamic Analysis of Three-Dimensional Frames

The stiffness method for dynamic analysis of frames presented in Chap. 11 for plane frames and in Chap. 12 for grid frames can readily be expanded for the analysis of three-dimensional space frames. Although for the plane frame or for the grid there were only three nodal coordinates at each joint, the three-dimensional frame has a total of six possible nodal displacements at each unconstrained joint: three translation components along the x, y, z axes and three rotational components about these axes. Consequently, a beam element of a three-dimensional frame or a space frame has for its two joints a total of 12 nodal coordinates; hence the resulting element matrices will be of dimension 12×12.

The dynamic analysis of three-dimensional frames resulting in a comparatively longer computer program in general, requiring more input data as well as substantially more computational time. However, except for size, the analysis of three-dimensional frames by the stiffness method of dynamic analysis is basically identical to the analysis of plane frames or grid frames.

13.1 Element Stiffness Matrix

Figure 13.1 shows a beam segment of a space frame with its 12 nodal coordinates numbered consecutively. The convention adopted is to label first the three translatory displacements of the first joint followed by the three rotational displacements of the same joint, then to continue with the three translatory displacements of the second joint and finally the three rotational displacements of this second joint. The double arrows used in Fig. 13.1 serve to indicate rotational nodal coordinates; thus, these are distinguished from translational nodal coordinates for which single arrows are used.

The stiffness matrix for a three-dimensional uniform beam segment is readily written by the superposition of the axial stiffness matrix from Eq. (11.3), the torsional stiffness matrix from Eq. (12.6), and the flexural stiffness matrix from Eq. (10.20). The flexural stiffness matrix is used twice in forming the stiffness matrix of a three-dimensional beam segment to account for the flexural effects in the two principal planes of the cross section. Proceeding to combine in an appropriate manner these matrices, we obtain in Eq. (13.1) the stiffness equation for a uniform beam segment of a three-dimensional frame, namely

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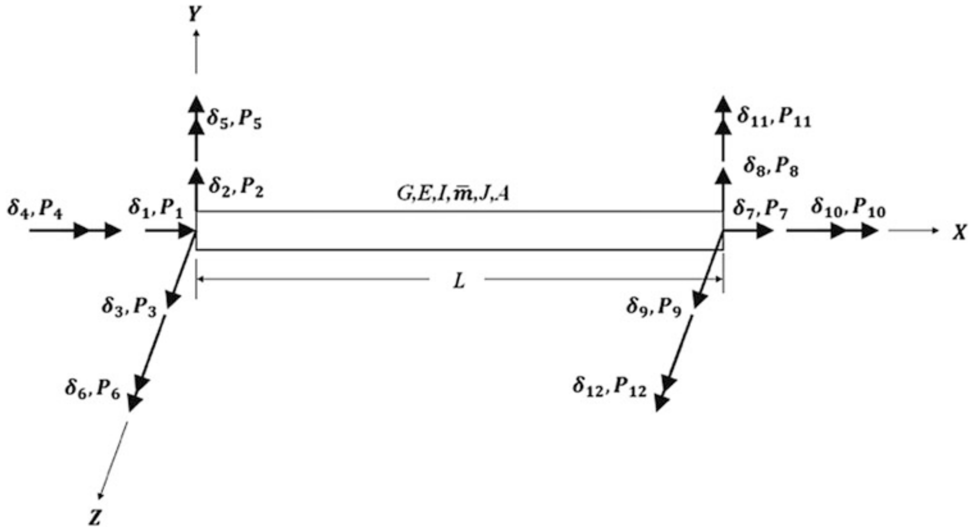

Fig. 13.1 Beam segment of a space frame showing forces and displacements at the nodal coordinates

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \\ P_{11} \\ P_{12} \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & & & & & & & & & & & \\ 0 & \frac{12EI_z}{L^3} & & & & & & & & & & \\ 0 & 0 & \frac{12EI_y}{L^3} & & & & & \text{Symmertic} & & & & \\ 0 & 0 & 0 & \frac{GI}{L} & & & & & & & & \\ 0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & & & & & & & \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_y}{L} & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & & & & & \\ \frac{-EA}{L} & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & & & & \\ 0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & & & \\ 0 & 0 & 0 & \frac{-GI}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI}{L} & & \\ 0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \\ \delta_{10} \\ \delta_{11} \\ \delta_{12} \end{bmatrix} \tag{13.1}$$

or in condensed notation

$$\{P\} = [K] = \{\delta\} \tag{13.2}$$

in which I_y and I_z are, respectively, the cross-sectional moments of inertia with respect to the principal axes labeled as y and z in Fig. 13.1, and L, A, and J are respectively the length, cross-sectional area, and torsional constant of the beam element.

13.2 Element Mass Matrix

The lumped mass matrix for the uniform beam segment of a three-dimensional frame is simply a diagonal matrix in which the coefficients corresponding to translatory and torsional displacements are equal to one-half of the total inertia of the beam segment while the coefficients corresponding to flexural rotations are assumed to be zero. The diagonal lumped mass matrix for the uniform beam of distributed mass $\bar{m}$ and polar mass moment $I_{\bar{m}} = \bar{m}I_o/A$ of inertia per unit of length may be written conveniently as

$$[M_L] = \frac{\bar{m}L}{2}\lceil 1\ \ 1\ \ 1\ \ I_o/A\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ I_o/A\ \ 0\ \ 0 \rfloor \tag{13.3}$$

in which I_o is the polar moment of inertia of the cross-sectional area A.

The consistent mass matrix for a uniform beam segment of a three-dimensional frame is readily obtained combining the consistent mass matrices, Eq. (11.26) for axial effects, Eq. (12.8) for torsional effects, and Eq. (10.34) for flexural effects. The appropriate combination of these matrices results in the consistent mass matrix for the uniform beam segment of a three-dimensional frame, namely,

$$\begin{bmatrix} P_1\\ P_2\\ P_3\\ P_4\\ P_5\\ P_6\\ P_7\\ P_8\\ P_9\\ P_{10}\\ P_{11}\\ P_{12} \end{bmatrix} = \frac{\bar{m}L}{420}\begin{bmatrix} 140 &&&&&&&&&&& \\ 0 & 156 &&&&&&&&&& \\ 0 & 0 & 156 &&&&&&&&& \\ 0 & 0 & 0 & \frac{140I_o}{A} &&&&&&&& \\ 0 & 0 & -22L & 0 & 4L^2 &&&&&&& \\ 0 & 22L & 0 & 0 & 0 & 4L^2 &&&&&& \\ 70 & 0 & 0 & 0 & 0 & 0 & 140 &&&&& \\ 0 & 54 & 0 & 0 & 0 & 13L & 0 & 156 &&&& \\ 0 & 0 & 54 & 0 & -13L & 0 & 0 & 0 & 156 &&& \\ 0 & 0 & 0 & \frac{70I_o}{A} & 0 & 0 & 0 & 0 & 0 & \frac{140I_o}{A} && \\ 0 & 0 & 13L & 0 & -3L^2 & 0 & 0 & 0 & 22L & 0 & 4L^2 & \\ 0 & -13L & 0 & 0 & 0 & -3L^2 & 0 & -22L & 0 & 0 & 0 & 4L^2 \end{bmatrix} \begin{bmatrix} \delta_1\\ \delta_2\\ \delta_3\\ \delta_4\\ \delta_5\\ \delta_6\\ \delta_7\\ \delta_8\\ \delta_9\\ \delta_{10}\\ \delta_{11}\\ \delta_{12} \end{bmatrix} \tag{13.4}$$

or in condensed notation

$$\{P\} = [M]\{\ddot{\delta}\} \tag{13.5}$$

13.3 Element Damping Matrix

The damping matrix for a uniform beam segment of a three-dimensional frame may be obtained in a manner entirely similar to those of the stiffness, Eq. (13.1), and mass, Eq. (13.4), matrices. Nevertheless, as was discussed in Sect. 10.5, in practice, damping is generally expressed in terms of damping ratios for each mode of vibration. Therefore, if the response is sought using the modal superposition method, these damping ratios are directly introduced in the modal equations. When the damping matrix is required explicitly, it may be determined from given values of damping ratios by the methods presented in Chap. 22.

13.4 Transformation of Coordinates

The stiffness and the mass matrices, respectively, given by Eqs. (13.1) and (13.4), are referred to local coordinates axes fixed on the beam segment. Inasmuch as the elements of these matrices corresponding to the same nodal coordinates of the structure should be added to obtain the system stiffness and mass matrices, it is necessary first to transform these matrices to the same reference system, the global system of coordinates. Figure 13.2 shows these two reference systems, the x, y, z axes representing the local system of coordinates and the X, Y, Z axes representing the global system of coordinates. Also shown in this figure is a general vector A with its components X, Y Z along the global coordinates. This vector $\boldsymbol{A}$ may represent any force or displacement at the nodal coordinates of one of the joints of the structure. To obtain the components of vector $\boldsymbol{A}$ along one of the local axes x, y, z, it is necessary to add the projections along that axis of the components X, Y, Z. For example, the component x of vector $\boldsymbol{A}$ along the x coordinate is given by

$$x = X_{\cos} xX + Y \cos xY + Z \cos xZ \tag{13.6a}$$

in which $\cos xY$ is the cosine of the angle between axes x and Y and corresponding definitions for other cosines. Similarly, the y and z components of A are

$$y = X_{\cos} yX + Y \cos yY + Z \cos yZ \tag{13.6b}$$

$$z = X_{\cos} zX + Y \cos zY + Z \cos zZ \tag{13.6c}$$

Equations (13.6a, 13.6b, and 13.6c) are conveniently written in matrix notation as

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} \cos xX & \cos xY & \cos xZ \\ \cos yX & \cos yY & \cos yZ \\ \cos zX & \cos zY & \cos zZ \end{Bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \tag{13.7}$$

or in short notation

$$\{A\} = \{T_1\}\{\bar{A}\} \tag{13.8}$$

in which $\{A\}$and $\{\bar{A}\}$ are, respectively, the components in the local and global systems of the general vector **A** and $[T_1]$ the transformation matrix given by

$$[T_1] = \begin{bmatrix} \cos xX & \cos xY & \cos xZ \\ \cos yX & \cos yY & \cos yZ \\ \cos zX & \cos zY & \cos zZ \end{bmatrix} \tag{13.9}$$

The cosines required in the transformation matrix $[T_1]$ are usually calculated in computer codes from the global coordinates of three points. The two points defining the two ends of the beam element along the local x axis and any third point located in x-y local plane in which y is one of the principal axes of the cross-sectional area of the member. The input data containing the global coordinates of these three points are sufficient for the evaluation of all the cosine terms in Eq. (13.9). To demonstrate

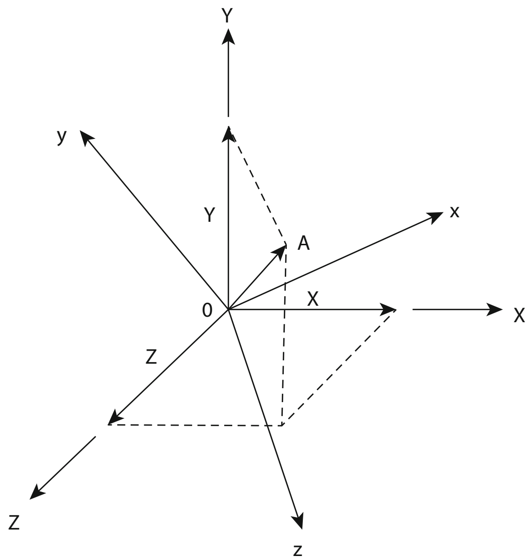

Fig. 13.2 Components of a general vector ***A*** in local and global coordinates

this fact let us designate by x_i, y_i, z_i, and x_j, y_j, z_j the coordinates of point Ⓘ and Ⓙ at the two ends of a beam element and by x_p, y_p, z_p, the coordinates of a point P placed on the local *x*-*y* plane. Then the direction cosines of local axis x along the beam element are given by

$$\cos xX = \frac{x_j - x_i}{L}, \quad \cos xY = \frac{y_j - y_i}{L}, \quad \cos xZ = \frac{z_j - z_i}{L} \tag{13.10}$$

where L is the length of the beam element given by

$$L = \sqrt{(x_j - x_i)^2 + \left(y_j - y_i\right)^2 + (z_j - z_i)^2} \tag{13.11}$$

The direction cosines of the *z* axis can be calculated from the condition that any vector ***Z*** along the *z* axis must be perpendicular to the plane formed by any two vectors in the local *x*-*y* plane. These two vectors could simply be the vector ***X*** from point Ⓘ to point Ⓙ along the *x* axis and the vector ***P*** from point Ⓘ to point ***P***. The orthogonality condition is then expressed by the cross product between vectors ***X*** and ***P*** as

$$\mathbf{Z} = \mathbf{X} \times \mathbf{P} \tag{13.12}$$

or substituting the components of these vectors as

$$z_x\widehat{i} + z_y\widehat{j} + z_z\widehat{k} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ x_j - x_i & y_j - y_i & z_j - z_i \\ x_p - x_i & y_p - y_i & z_p - z_i \end{vmatrix} \tag{13.13}$$

where $\widehat{i}$, $\widehat{j}$, and $\widehat{k}$ are unit vectors along the global coordinate axes *X*, *Y*, and *Z*, respectively. Consequently the direction cosines of axis z are given by

$$\cos zX = \frac{z_x}{|Z|}, \quad \cos zY = \frac{z_y}{|Z|}, \quad \cos zZ = \frac{z_x}{|Z|} \tag{13.14}$$

where

$$\begin{aligned} z_x &= \left(y_j - y_i\right)\ (z_p - z_i) - (z_j - z_i)\ (y_p - y_i) \\ z_y &= (z_j - z_i)\ (x_p - x_i) - (x_j - x_i)\ (z_p - z_i) \\ z_z &= (x_j - x_i)\ (y_p - y_i) - \left(y_j - y_i\right)\ (x_p - x_i) \end{aligned} \tag{13.15}$$

and

$$|Z| = \sqrt{z_x^2 + z_y^2 + z_z^2} \tag{13.16}$$

Analogously, the direction cosines of the local axis y are calculated from the condition of orthogonality between a vector $\boldsymbol{Y}$ along the y axis and the unit vectors $\boldsymbol{X_1}$ and $\boldsymbol{Z_1}$ along the x and z aces, respectively. Hence,

$$\boldsymbol{Y} = \boldsymbol{X}_1 \times \boldsymbol{Z}_1$$

or in expanded notation

$$y_x\widehat{i} + y_y\widehat{j} + y_z\widehat{k} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \cos xX & \cos xY & \cos xZ \\ \cos zX & \cos zY & \cos zZ \end{vmatrix} \tag{13.17}$$

Therefore,

$$\cos yX = \frac{y_x}{|Y|}, \quad \cos yY = \frac{y_y}{|Y|}, \quad \cos yZ = \frac{y_z}{|Y|}$$

where

$$\begin{aligned} y_x &= \cos xY \cos zZ - \cos xZ \cos zY \\ y_y &= \cos zX \cos zX - \cos xX \cos zZ \\ y_z &= \cos xX \cos zY - \cos xY \cos zX \end{aligned} \tag{13.18}$$

and

$$|Y| = \sqrt{y_x^2 + y_y^2 + y_z^2}$$

We have, therefore, shown that knowledge of the coordinates of points at the two ends of an element of a point P on the local plane x-y suffices to calculate the direction cosines of the transformation matrix $[T_1]$ in Eq. (13.9). The choice of point P is generally governed by the geometry of the structure and the orientation of the principal directions of the cross section of the member. Quite often point P is selected as a known point in the structure, which is placed on the local axis y, although, as it has been shown, the point P could be any point in the plane formed by the local x-y axes.

Alternatively, the direction cosines in the transformation matrix, Eq. (13.9) may be calculated from the nodal coordinates (x_{i}, y_{i}, z_{i},) and (x_{j}, y_{j}, z_{j}) at the two ends of the beam element and the knowledge of an angle known as the angle of rolling. This angle provides the rational information of the principal axes of the cross-sectional area with respect to an orientation defined as the standard orientation of these axes. The analytical development to implement the calculation of the direction cosines using the angle of rolling is presented at the end of this chapter as Problems 13.1 and 13.2.

We have, therefore, shown that the knowledge of the coordinates at the two ends of an element, together with the knowledge of either a third point located in the x-y local plane in which y is one of the principal axes of the cross-sectional area of the member, or alternatively, the knowledge of the angle of rolling, suffices to calculate the direction cosines of the transformation matrix $[T_1]$ in Eq. (13.9).

For the beam segment of a three-dimensional frame, the transformation of the nodal displacement vectors involve the transformation of linear and angular displacement vectors at each joint of the segment. Therefore, a beam element of a space frame requires, for the two joints, the transformation of a total of four displacement vectors. This transformation of the 12 nodal displacements $\{\bar{\delta}\}$ global coordinates to the displacement $\{\delta\}$ in local coordinates may be written in abbreviated form as

$$\{\delta\} = [T]\{\bar{\delta}\} \tag{13.19}$$

in which

$$[T] = \begin{bmatrix} [T_1] & & & \\ & [T_1] & & \\ & & [T_1] & \\ & & & [T_1] \end{bmatrix}$$

Analogously, the transformation from nodal forces $\{\bar{P}\}$ in global coordinates to nodal forces $\{P\}$ in local coordinates is given by

$$\{P\} = [T]\{\bar{P}\} \tag{13.20}$$

Finally, to obtain the stiffness matrix $[\bar{K}]$ and the mass matrix $\lfloor \bar{M} \rfloor$ in reference to the global system of coordinates, we simply substitute, into Eq. (13.2), {δ} from Eq. (13.19) and [P] from Eq. (13.21) to obtain

$$[T]\{\bar{P}\} = [K][T]\{\bar{\delta}\}$$

or

$$\{\bar{P}\} = [T]^T[K][T]\{\bar{\delta}\} \tag{13.21}$$

since $[T]$ is an orthogonal matrix. From Eq. (13.21), we may write

$$\{\bar{P}\} = [\bar{K}]\{\bar{\delta}\} \tag{13.22}$$

in which $\lfloor \bar{K} \rfloor$ is defined as

$$[\bar{K}] = [T]^T[K][T] \tag{13.23}$$

Analogously, the mass matrix in Eq. (13.5) is transformed from local to global coordinates by

$$[\bar{M}] = [T]^T[M][T] \tag{13.24}$$

and the damping matrix [C] by

$$[\bar{C}] = [T]^T[C][T] \tag{13.25}$$

13.5 Differential Equation of Motion

The direct method which was explained in detail in Chap. 10 may also be used to assemble the stiffness, mass, and damping matrices from the corresponding matrices for a three-dimensional beam segment, Eqs. (13.23), (13.24) and (13.25), which are referred to the global system of coordinates. The differential equations of motion referred to the global system of coordinates. The differential equations of motion which are obtained by establishing the dynamic equilibrium among the inertial, damping, and elastic forces with the external forces may be expressed in matrix notation as

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F(t)\} \tag{13.26}$$

in which $[M]$, $[C]$, and $[K]$ are, respectively, the system mass, damping, and stiffness matrices, $\{\ddot{u}\}$, $\{\dot{u}\}$, and $\{u\}$ are the system acceleration, velocity, and displacement vectors, and $\{F(t)\}$ is the force vector which includes the forces applied directly to the joints of the structure and the equivalent nodal forces for the forces not applied at the joints.

13.6 Dynamic Response

The integration of the differential equations of motion, Eq. (13.26), may be accomplished by any of the methods presented in previous chapters to obtain the response of structures modeled as beams, plane frames, or grids. The selection of the particular method of solution depends, as discussed previously, on the linearity of the differential equation, that is, whether the stiffness matrix $[K]$ or any other coefficient matrix is constant, and also depends on the complexity of the excitation as a function of time. When the differential equations of motion, Eq. (13.26), are linear, the modal superposition method is applicable. This method, as we have seen in the preceding chapters, requires the solution of an eigenproblem to uncouple the differential equations resulting in the modal equations of motion.

If the structure is assumed to follow an elastoplastic behavior or any other form of nonlinearity, it is necessary to resort to some kind of numerical integration in order to solve the differential equations of motion, Eq. (13.26). In Chap. 16, the linear acceleration method with a modification known as the Wilson-θ method is presented for analysis of linear structures with an elastic behavior.

13.7 Modeling Structures as Space Frames Using MATLAB

MATLAB calculates the stiffness and mass matrices for a three-dimensional frame and stores the coefficients of these matrices.

Illustrative Example 13.1

For the three-dimensional frame shown in Fig. 13.3, determine (a) the stiffness and mass matrices, (b) the natural frequencies and corresponding modal shapes. Model the structure with four beam elements and use consistent mass matrix formulation, and (c) determine the response to a constant force of 5000 lb. suddenly applied for 0.1 s at node 1 in z axis.

Solution:

As the first step in the analysis, the frame is divided in four elements, as indicated in Fig. 13.3. This division of the structure results in five nodes with a total of 30 nodal coordinates, of which 24 are fixed. The numerical values needed for analysis of this structure are given in Table 13.1.

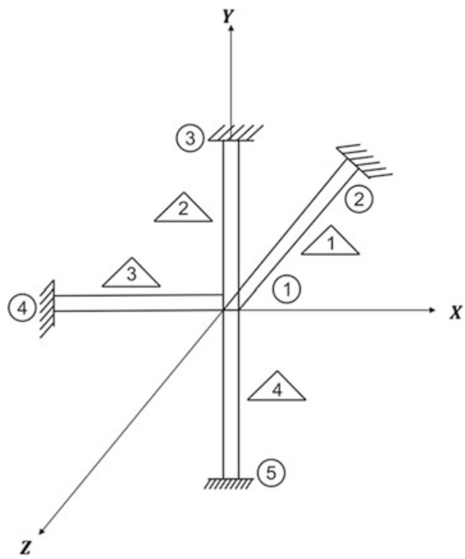

Fig. 13.3 Space Frame of Example 13.1

Table 13.1 Input Data for Illustrative Example 13.1

Quantity	Members 1,3	Members2,4
Modulus of elasticity (psi)	30×10^6	30×10^6
Modulus of rigidity (psi)	12×10^6	12×10^6
Distributed mass ($lb \cdot sec^2 /in^2$)	0.2	0.1
Cross-sectional *y* moment of inertia (in^4)	200	64
Cross-sectional *z* moment of inertia (in^4)	200	64
Torsional constant (in^4)	40.0	12.8
Cross-sectional area (in^2)	50	28
Cross-sectional moment of inertia (in^4)	205	68

This MATLAB file is to yield the results of sections (a) and (b). Two function files, SpaceFrameElement.m and SpaceConMass.m are needed. After assembling matrices, the system matrix can be found using System.m file. Using system matrices, the natural frequencies and mode shapes will be found.

```
clc
clear all
close all

%_____________________________________________________________________________
%  Determine System Matrices/Determine Force
%_____________________________________________________________________________

%%%%-GIVEN VALUES-%%%%

E=30*10^6;                   %E (psi)
G = 12*10^6;                 %G, Modulus of rigidity (psi)


% Members 1 & 3
A1 = 50;                     %Cross-sectional area A (in^2)
Iz1 = 200;                   %Second Moment of Inertia in z axes (in^4)
Iy1 = 200;                   %Second Moment of Inertia in y axes (in^4)
J1 = 40;                     %Torsional constant
m_bar1 = 0.2;                %Distributed mass (lb-sec^2/in/in)
I0_1 = 205;                  %Polar moment of inertia of cross sectional area (in^4)


% Members 2 & 4
A2 = 28;                     %Cross-sectional area A (in^2)
Iz2 = 64;                    %Second Moment of Inertia in z axes (in^4)
Iy2 = 64;                    %Second Moment of Inertia in y axes (in^4)
J2 = 12.8;                   %Torsional constant
m_bar2 = 0.1;                %Distribution mass (lb-sec^2/in^2)
I0_2 =  68;                  %Polar moment of inertia of cross sectional area (in^4)

%%%Create frame model (ith row of nodes is ith node)
nodes = [0, 0, 0; 0, 0, -200; 0, 200, 0; -200, 0, 0; 0, -200, 0];
%%%Element number (ith row = ith element with two nodes)
conn=[1,2,3; 1,3,2; 1,4,2; 1,5,2];
%%%Dofs for ith element (ith row)
lmm=[1:12; [1:6 13:18]; [1:6 19:24]; [1:6 25:30]];
dof=6*length(nodes);         % Total No. dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1
    lm=lmm(i,:);
    con=conn(i,:);
    ke = SpaceFrameElement(E, G, Iz1, Iy1, J1, A1, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke;
end

for i=2
    lm=lmm(i,:);
    con=conn(i,:);
    ke = SpaceFrameElement(E, G, Iz2, Iy2, J2, A2, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke;
end

for i=3
    lm=lmm(i,:);
    con=conn(i,:);
    ke = SpaceFrameElement(E, G, Iz1, Iy1, J1, A1, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke;
end

for i=4
    lm=lmm(i,:);
    con=conn(i,:);
    ke = SpaceFrameElement(E, G, Iz2, Iy2, J2, A2, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke
end
```

```
K;
M;
%%%Define the load vector
F = zeros(dof,1); F(3) = 5000;      %Applied force at specific dofs

%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F,[7:30]);

Kf
Mf
Rf
%_____________________________________________________________
% Solve the eigenvalue problem and normalized eigenvectors
%_____________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(Kf, Mf);

[omegas,ii] = sort(sqrt(diag(D))); %Natural Frequencies

omegas
a = a(:,ii)                        %Mode Shapes

T = 2*pi./omegas;                  %Natural Periods

save ('temp0.mat', 'Mf', 'Kf' ,'Rf');
```

The function file of MATLAB is used to assemble the stiffness matrix of space frame element for global stiffness matrix.

```
function ke = SpaceFrameElement(E, G, Iz, Iy, J, A, coord)
% ke = SpaceFrameElement(E, G, Iz, Iy, J, A, wz, wy, coord)
% Generates equations for a space frame element
% E = modulus of elasticity (psi)
% G = Modulus of rigidity (psi)
% Iz, Iy = moment of inertias about element z and y axes (in^4)
% J = torsional rigity (in^4)
% A = area of cross-section (in^2)
% coord = coordinates at the element ends

EIz=E*Iz; EIy=E*Iy; GJ=G*J; EA = E*A;
n1=coord(1,1:3); n2=coord(2,1:3); n3=coord(3,1:3);
L=sqrt(dot((n2-n1),(n2-n1)));
ex = (n2 - n1)/L;
eyy = cross(n3 - n1, n2 - n1);
ey = eyy/sqrt(dot(eyy,eyy));
ez = cross(ex, ey);
H = [ex; ey; ez];
T = zeros(12);
T([1, 2, 3], [1, 2, 3]) = H;
T([4,5,6], [4,5,6]) = H;
T([7,8,9], [7,8,9]) = H;
T([10,11,12], [10,11,12]) = H;
TT = T.';
ke = [EA/L, 0, 0, 0, 0, 0, -(EA/L), 0, 0, 0, 0, 0;
    0, (12*EIz)/L^3, 0, 0, 0, (6*EIz)/L^2, 0, -((12*EIz)/L^3), ...
        0,0, 0, (6*EIz)/L^2;
    0, 0, (12*EIy)/L^3, 0, -((6*EIy)/L^2), 0, ...
        0, 0, -((12*EIy)/L^3), 0, -((6*EIy)/L^2), 0;
    0, 0, 0, GJ/L, 0, 0, 0, 0, 0, -(GJ/L), 0, 0;
    0, 0, -((6*EIy)/L^2), 0, (4*EIy)/L, 0, 0, 0, (6*EIy)/L^2, ...
        0,(2*EIy)/L, 0;
    0, (6*EIz)/L^2, 0, 0, 0, (4*EIz)/L, 0, ...
        -((6*EIz)/L^2), 0, 0, 0, (2*EIz)/L;
    -(EA/L), 0, 0, 0, 0, 0, EA/L, 0, 0, 0, 0, 0;
    0, -((12*EIz)/L^3), 0, 0, 0, -((6*EIz)/L^2), ...
        0, (12*EIz)/L^3, 0, 0, 0, -((6*EIz)/L^2);
    0, 0, -((12*EIy)/L^3), 0, (6*EIy)/L^2, 0, 0, 0, ...
        (12*EIy)/L^3, 0, (6*EIy)/L^2, 0;
    0, 0, 0, -(GJ/L), 0, 0, 0, 0, 0, GJ/L, 0, 0;
    0, 0, -((6*EIy)/L^2), 0, (2*EIy)/L, 0, 0,0, (6*EIy)/L^2, ...
        0, (4*EIy)/L, 0;
    0, (6*EIz)/L^2, 0, 0, 0, (2*EIz)/L, 0, -((6*EIz)/L^2), 0, ...
        0, 0, (4*EIz)/L];
```

The function file of MATLAB is used to assemble the mass matrix of each space frame element for global mass matrix.

```
function m = SpaceFrameConsMass(m_bar,I0, A, coord)
% FrameConsMass(m_bar, nodes(con,:))
% Generates stiffness matrix for a space frame element
% m = distributed mass (lb.sec^2/in/in)
% L = length (in.)
% A = area of cross-section (in^2)
% I0 = Polar moment of inertia of cross sectional area (in^4)
% coord = coordinates at the element ends

n1=coord(1,1:3); n2=coord(2,1:3); n3=coord(3,1:3);
L=sqrt(dot((n2-n1),(n2-n1)));
ex = (n2 - n1)/L;
eyy = cross(n3 - n1, n2 - n1);
ey = eyy/sqrt(dot(eyy,eyy));
ez = cross(ex, ey);
H = [ex; ey; ez];
T = zeros(12);
T([4,5,6], [4,5,6]) = H;
T([7,8,9], [7,8,9]) = H;
T([10,11,12], [10,11,12]) = H;


m = m_bar*L/420*T.'*[140  0   0   0   0   0   70*I0/A  0   0   0   0   0 ;
                     0  156   0   0   0   22*L   0     54  0   0   0 -13*L;
                     0   0  156   0  22*L  0   0  0  54    0  -13*L  0 ;
                     0   0   0  140*I0/A  0    0   0  0  0  70*I0/A 0 0;
                     0   0  22*L  0   4*L^2   0   0  0  13*L 0 -3*L^2  0;
                     0  22*L  0    0   0  4*L^2  0  13*L 0   0   0 -3*L^2;
                     70  0    0    0    0  0     140  0   0  0  0  0;
                     0  54    0    0    0  13*L  0    156  0  0 0 -22*L;
                     0  0    54   0    13*L    0   0    0 156  0 -22*L  0;
                     0  0    0  70*I0/A  0   0  0  0  0  140*I0/A  0  0;
                     0  0  -13*L  0 -3*L^2   0  0  0 -22*L   0  4*L^2  0;
                     0 -13*L 0  0   0  -3*L^2  0 -22*L  0 0 0 4*L^2]*T;
```

This MATLAB file is to yield the results of section (c).

The response can be obtained from the similar manner in the previous chapters. The force is the same as the case of Example 12.3. The force is applied at node 1 in z axis with 5000 lb. for 0.1 s.

The maximum displacements at the nodal coordinates were estimated using MATLAB.

$$u_{1\max} = 8.814 \times 10^{-5} \text{ in.} \quad u_{2\max} = 2.6552 \times 10^{-7} \text{ in.} \quad u_{3\max} = 0.9827 \text{ radian}$$

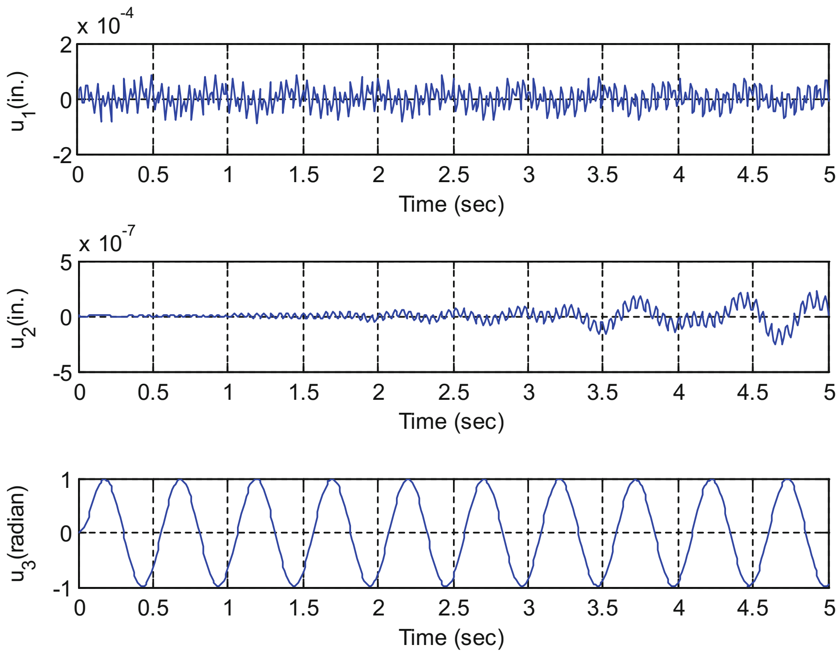

Fig. 13.4 Response of Illustrative Example 13.1

13.8 Summary

This chapter presented the formulation of the stiffness and mass matrices for an element of a space frame, as well as the transformation of coordinates required to refer these matrices to the global system of coordinates. Except for the larger dimensions of the matrices resulting from modeling a space frame, the procedure is identical to the case of a beam, plane frame, or grid frame described in the preceding chapters.

Problem 13.1

For the three-dimensional frame shown in Fig. 13.4a, determine: (a) The first 6 natural frequencies an corresponding modal shapes and (b) The dynamic response when the frame is acted upon by the impulsive force depicted in Fig. 13.4b applied at joint 8 of the frame along the Y- direction as shown in the figure (Fig. 13.5).

Problem Data (for all members):

Modulus of elasticity:	$E = 30 \times 106$ psi
Modulus of rigidity:	$G = 12 \times 106$ psi
Distributed mass:	$m = 0.2$ lb. · sec^2/in.
Concentrated masses:	$m = 10$ lb. · sec^2/in.
Cross-sectional y moment of inertia:	$I_y = 300$ in.^{4}
Cross-sectional z moment of inertia:	$I_z = 400$ in.^{4}
Torsional constant:	$J = 500$ in.^{4}
Cross-sectional area:	$A = 20$ in.^{2}

Problem 13.2

Determine the dynamic response of the three dimensional frame of Problem 13.1 subjected to the harmonic excitation of acceleration a(t) = 0.3 sin 5.3 t applied at support 3 in the Y-direction.

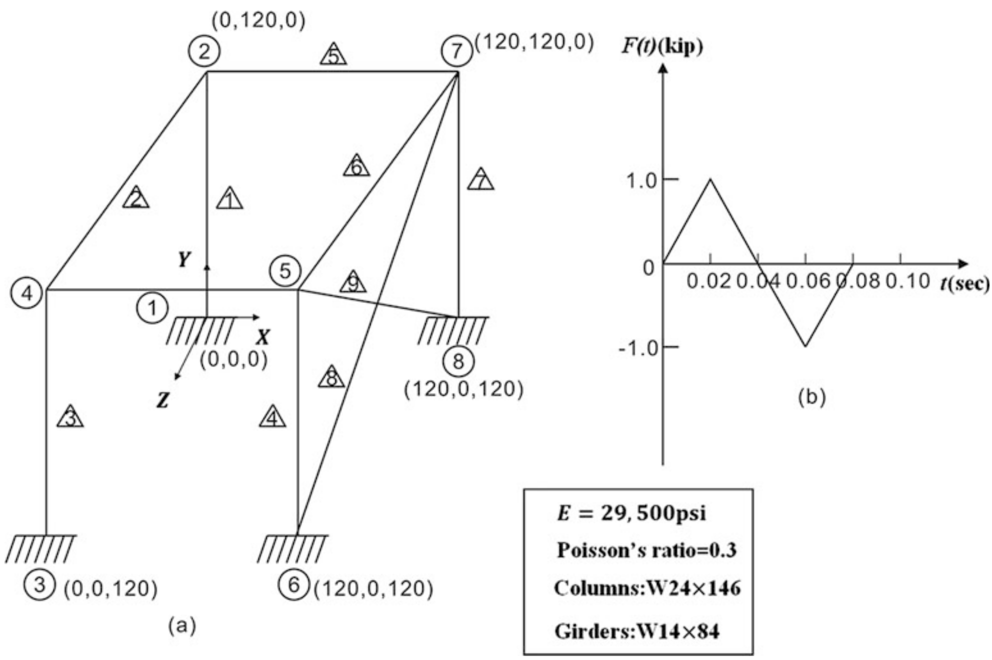

Fig. 13.5 Space frame for Problem 13.1