

The static analysis of trusses whose members are pin-connected reduces to the problem of determining the bar forces due to a set of loads applied at the joints. When the same trusses are subjected to the action of dynamics forces, the simple situation of only axial stresses in the members is no longer present. The inertial forces developed along the members of the truss will, in general, produce flexural bending in addition to axial forces. The bending moments at the ends of the truss members will still remain zero in the absence of external joint moments. The dynamic stiffness method for the analysis of trusses is developed as in the case of framed structures by establishing the basic relations between external forces, elastic forces, damping forces, inertial forces and the resulting displacements, velocities, and accelerations at the nodal coordinates, that is, by determining the stiffness, damping, and mass matrices for a member of the truss. The assemblage of system stiffness, damping, and mass matrices of the truss as well as the solution for the displacements at the nodal coordinates follows along the standard method presented in the preceding chapters for framed structures.

14.1 Stiffness and Mass Matrices for the Plane Truss

A member of a plane truss has two nodal coordinates at each joint, that is, a total of four nodal coordinates (Fig. 14.1). For small deflections, it may be assumed that the force-displacement relationship for the nodal coordinates along the axis of the member (coordinates 1 and 3 in Fig. 14.1) are independent of the transverse displacements along nodal coordinates 2 and 4. This assumption is equivalent to stating that a displacement along nodal coordinates 1 or 3 does not produce forces along nodal coordinates 2 or 4 and vice versa.

The stiffness and mass coefficients corresponding to the axial nodal coordinates were derived in Chap. 11 and are given, in general, by Eq. (11.17) for the stiffness coefficients and by Eq. (11.23) for consistent mass coefficients. Applying these equations to a uniform beam element, we obtain, using the notation of Fig. 14.1, the following coefficients:

$$k_{11} = k_{33} = \frac{AE}{L}, \quad k_{13} = k_{31} = -\frac{AE}{L} \quad (14.1)$$

$$m_{11} = m_{33} = \frac{\bar{m}L}{3}, \quad m_{13} = m_{31} = \frac{\bar{m}L}{6} \quad (14.2)$$

in which \bar{m} is the mass per unit length, A is the cross-sectional area, and L is the length of the element.

The stiffness coefficients, for pin-ended elements, corresponding to the nodal coordinates 2 and 4 are all equal to zero, since a force is not required to produce displacements at these coordinates. Therefore, arranging the coefficients given by Eq. (14.1), we obtain the stiffness equation for a uniform member of a truss as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \quad (14.3)$$

or in condensed notation

$$\{P\} = [K]\{\delta\} \quad (14.4)$$

in which $[K]$ is the element stiffness matrix.

The consistent mass matrix is obtained, as previously demonstrated, using expressions for static displacement functions in the application of the principle of virtual work. The displacement functions corresponding to a unit deflection at nodal coordinates 2 and 4 indicated in Fig. 14.2 are given by

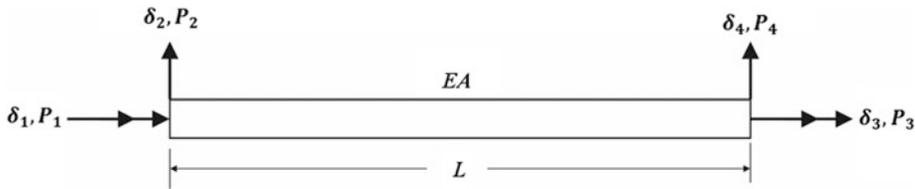


Fig. 14.1 Member of a plane truss showing nodal displacements and forces

$$u_2 = 1 - \frac{x}{L} \quad (14.5)$$

and

$$u_4 = \frac{x}{L} \quad (14.6)$$

The consistent mass coefficients are given by the general expression, Eq. (11.23), which is repeated here for convenience, namely,

$$m_{ij} = \int_0^L \bar{m}(x) u_i(x) u_j(x) dx \quad (14.7)$$

For a uniform member of mass \bar{m} per unit length, the substitution of Eqs. (14.5) and (14.6) into Eq. (14.7) yields

$$\begin{aligned} m_{22} &= m_{44} = \frac{\bar{m}L}{3} \\ m_{24} &= m_{42} = \frac{\bar{m}L}{6} \end{aligned} \quad (14.8)$$

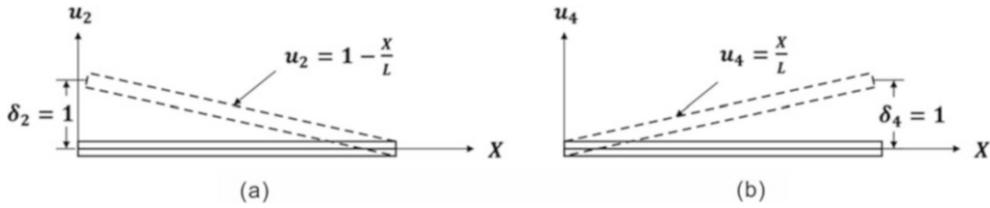


Fig. 14.2 Displacement functions. (a) For a unit displacement $\delta_2 = 1$. (b) For a unit displacement $\delta_4 = 1$

Finally, the combination of the mass coefficients from Eqs. (14.2) and (14.8) forms the consistent mass matrix relating forces to accelerations at the nodal coordinates for a uniform member of a plane truss, namely,

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{\bar{m}E}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \end{Bmatrix} \quad (14.9)$$

or in concise notation

$$\{P\} = [M]\{\ddot{\delta}\} \quad (14.10)$$

14.2 Transformation of Coordinates

The stiffness matrix, Eq. (14.3), and the mass matrix, Eq. (14.9), were derived in reference to nodal coordinates associated with the local or element system of coordinates. As discussed before in the chapters on framed structures, it is necessary to transform these matrices to a common system of reference, the global coordinate system. The transformation of displacements and forces at the nodal coordinates is accomplished, as was demonstrated in Chap. 11, performing a rotation of coordinates. Deleting the angular coordinates in Eq. (11.31) and relabeling the remaining coordinates result in the following transformation for the nodal forces:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \\ \bar{P}_4 \end{Bmatrix} \quad (14.11)$$

or in condensed notation,

$$\{P\} = [T]\{\bar{P}\} \quad (14.12)$$

in which $\{P\}$ and $\{\bar{P}\}$ are the nodal forces in reference to local and global coordinates, respectively, and $[T]$ the transformation matrix defined in Eq. (14.11). The same transformation matrix $[T]$ also serves to transform the nodal displacement vector $\{\bar{\delta}\}$ in the global coordinate system to the nodal displacement vector $\{\delta\}$ in local coordinates:

$$\{\delta\} = [T]\{\bar{\delta}\} \quad (14.13)$$

The substitution of Eqs. (14.12) and (14.13) into the stiffness Eq. (14.4) gives

$$[T][\bar{P}] = [K][T]\{\bar{\delta}\}$$

Since $[T]$ is an orthogonal matrix ($[T]^{-1} = [T]^T$), it follows that

$$\{\bar{P}\}[T]^T[K][T]\{\bar{\delta}\}$$

or

$$\{\bar{P}\} = [\bar{K}]\{\bar{\delta}\} \quad (14.14)$$

in which

$$[\bar{K}] = [T]^T[K][T] \quad (14.15)$$

is the element stiffness matrix in the global coordinate system. Analogously, substituting Eqs. (14.12) and (14.13) into Eq. (14.10) results in

$$\{\bar{P}\} = [T]^T[M][T]\{\bar{\delta}\} \quad (14.16)$$

or

$$\{\bar{P}\} = [\bar{M}]\{\bar{\delta}\} \quad (14.17)$$

$$[\bar{M}] = [T]^T[M][T] \quad (14.18)$$

in which $[\bar{M}]$ is the element mass matrix referred to the global system of coordinates. However, there is no need to use Eq. (14.18) to calculate matrix $[\bar{M}]$. This matrix is equal to the mass matrix $[M]$ in reference to local axes of coordinates. To verify this fact, we substitute into Eq. (14.18) matrices $[M]$ and $[T]$, respectively, from Eqs. (14.9) and (14.11) to obtain

$$\begin{aligned} [\bar{M}] &= \frac{\bar{m}L}{6} \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & c & s & c \end{bmatrix} \\ &= \frac{\bar{m}L}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} = [M] \end{aligned} \quad (14.19)$$

in which we use the notation $c = \cos \theta$, $s = \sin \theta$ and the fact that $\cos^2 \theta + \sin^2 \theta = 1$.

A similar relationship is also obtained for the element damping matrix, namely,

$$[\bar{C}] = [T]^T[C][T] \quad (14.20)$$

in which $[\bar{C}]$ and $[C]$ are the damping matrices referred, respectively, to the global and the local systems of coordinates.

Illustrative Example 14.1

The plane truss shown in Fig. 14.3 which has only three members is used to illustrate the application of the stiffness method for trusses. For this truss determine the system stiffness and the system consistent mass matrices.

Solution:

The stiffness matrix, Eq. (14.3), the mass matrix, Eq. (14.9), and the transformation matrix, Eq. (14.11), are applied to the three members of this truss. For member 1, $\theta = 90^\circ$,

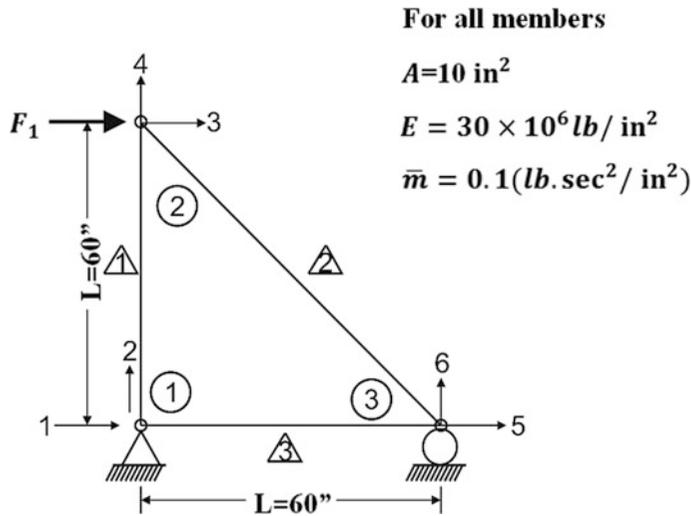


Fig. 14.3 Plane truss for Illustrative Example 14.1

$$[K_1] = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [\bar{M}_1] = [M_1] = \frac{\bar{m}E}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

and

$$[T_1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Then from Eqs. (14.15)

$$[\bar{K}_1] = [T_1]^T [K_1] [T_1] = \frac{AE}{L} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

For member 2, $\theta = 135^\circ$,

$$[K_2] = \frac{AE}{\sqrt{2}L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, [\bar{M}_2] = [M_2] = \frac{\bar{m}\sqrt{2}L}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

and

$$[T_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

Then from Eqs. (14.15)

$$[\bar{K}_2] = [T_2]^T [K_2] [T_2] = \frac{AE}{2\sqrt{2}L} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

For member 3, $\theta = 0^\circ$,

$$[\bar{K}_3] = [K_3] = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\bar{M}_3] = [M_3] = \frac{\bar{m}L}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

substituting the proper numerical values for this example; $L = 60$ in, $A = 10$ in², $\bar{m} = 0.1$ lb. • sec²/in², $E = 30 \times 10^6$ lb./in², and following the rules of the direct method of assembling the system stiffness and mass matrix from the above element matrices, we obtain

$$[K_s] = 10^6 \begin{bmatrix} 1.768 & -1.768 & -1.768 \\ -1.768 & 6.768 & 1.768 \\ -1.768 & 1.768 & 6.768 \end{bmatrix}$$

$$[M_s] = \begin{bmatrix} 4.828 & 0 & 1.414 \\ 0 & 4.828 & 0 \\ 1.414 & 0 & 4.828 \end{bmatrix}$$

where $[K_s]$ and $[M_s]$ are, respectively, the system stiffness and mass matrices for the truss shown in Fig. 14.3.

Illustrative Example 14.2

Determine the natural frequencies and normal modes for the truss of Example 14.1.

Solution:

The differential equations of motion for this system are

$$[M_S]\{\ddot{u}\} + [K_S]\{u\} = 0 \quad (\text{a})$$

Substituting $\{u\} = \{a\} \sin \omega t$, we obtain

$$([K_S] - \omega^2[M_S])\{a\} = \{0\} \quad (\text{b})$$

For the nontrivial solution, we require

$$|[K_S] - \omega^2[M_S]| = 0 \quad (\text{c})$$

Substituting from Example 14.1 $[K_S]$ and $[M_S]$ and expanding the above determinant give a cubic equation in $\lambda = \omega^2 \bar{m}L^2/6AE$ which has the following roots

$$\lambda_1 = 0.00344 \text{ or } \omega_1 = 415 \text{ rad/sec}$$

$$\lambda_2 = 0.0214 \text{ or } \omega_2 = 1034 \text{ rad/sec}$$

$$\lambda_3 = 0.0466 \text{ or } \omega_3 = 1526 \text{ rad/sec}$$

Substituting in turn ω_1 , ω_2 and ω_3 into Eq. (b), setting $a_1 = 1$, and solving for a_2 and a_3 give the modal vectors.

$$\{a_1\} = \begin{Bmatrix} 1.000 \\ 0.216 \\ 0.274 \end{Bmatrix}, \{a_2\} = \begin{Bmatrix} 1.000 \\ 5.488 \\ -4.000 \end{Bmatrix}, \{a_3\} = \begin{Bmatrix} 1.000 \\ -1.000 \\ -1.524 \end{Bmatrix}$$

which may be normalized using the factors

$$\sqrt{\{a_1\}^T [M_S] \{a_1\}} = 2.489, \sqrt{\{a_2\}^T [M_S] \{a_2\}} = 14.695$$

$$\sqrt{\{a_3\}^T [M_S] \{a_3\}} = 4.066$$

This normalization results in

$$\{\phi_1\} = \begin{Bmatrix} 0.402 \\ 0.087 \\ 0.110 \end{Bmatrix}, \{\phi_2\} = \begin{Bmatrix} 0.068 \\ 0.373 \\ -0.272 \end{Bmatrix}, \{\phi_3\} = \begin{Bmatrix} 0.246 \\ -0.246 \\ -0.375 \end{Bmatrix}$$

These normalized eigenvectors form the modal matrix:

$$[\Phi] = \begin{bmatrix} 0.402 & 0.068 & 0.246 \\ 0.087 & 0.373 & -0.246 \\ 0.110 & -0.272 & -0.375 \end{bmatrix}$$

The MATLAB program is presented to demonstrate Illustrative Examples 14.1 and 14.2.

```

clc
close all
clear all

%
% Determine System Matrices/Determine Force
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%GIVEN VALUES-%%%

E=30*10^6; A=10; %E (psi), A (in.^2)
P = 5000; %P (lbs)

%%%Create frame model (ith row of nodes is ith node)
nodes = 60*[0, 0; 0, 1; 1, 0]; % Unit: in.
%%%Element number (ith row = ith element with two nodes)
conn = [1,2; 2,3; 1,3];
%%%Dofs for ith element (ith row)
lmm = [1, 2, 3, 4; 3, 4, 5, 6; 1, 2, 5, 6];
m_bar = 0.1; %Distributed mass (lb-sec^2/in/in)

dof = 2*length(nodes); %Total No.dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1:3
    lm=lmm(i,:);
    con=conn(i,:);
    k=PlaneTrussElement(E, A, nodes(con,:));
    K(lm, lm) = K(lm, lm) + k;
end

%%%Generate mass matrix for each element and assemble them.
for i=1:3
    lm=lmm(i,:);
    con=conn(i,:);
    m=PlaneTrussMass(m_bar, nodes(con,:));
    M(lm, lm) = M(lm, lm) + m;
end

K
M

%%%Define the load vector
F = zeros(dof,1); F(3) = 5000; %Applied force at specific dofs

%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F, [1,2,6]);

Kf
Mf
Rf
%
% Solve the eigenvalue problem and normalized eigenvectors
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(Kf, Mf);

[omegas,ii] = sort(sqrt(diag(D))); %Natural Frequencies

omegas

a = a(:,ii) %Mode Shapes

T = 2*pi./omegas; %Natural Periods

save ('temp0.mat', 'Mf', 'Kf', 'Rf');

```

The function file of MATLAB is used to assemble the stiffness matrix of plane truss element for global stiffness matrix.

```
function k = PlaneTrussElement(E, A, coord)
% PlaneTrussElement(e, A, coord)
% Generates stiffness matrix for a plane truss element
% E = modulus of elasticity (psi)
% A = area of cross-section (in^2)
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
ls=(x2-x1)/L; ms=(y2-y1)/L;
T = [ls ms 0 0; -ms ls 0 0; 0 0 ls ms; 0 0 -ms ls];
k = E*A/L*T.*[1 0 -1 0;
              0 0 0 0;
              -1 0 1 0;
              0 0 0 0]*T;
```

The function file of MATLAB is used to assemble the mass matrix of each plane truss element for global mass matrix.

```
function m = PlaneTrussMass(m_bar, coord)
% PlaneTrussMass(m_bar, coord)
% Generates mass matrix for a plane truss element
% m = distributed mass (lb.sec^2/in/in)
% L = length (in.)
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
ls=(x2-x1)/L; ms=(y2-y1)/L;
T = [ls ms 0 0; -ms ls 0 0; 0 0 ls ms; 0 0 -ms ls];
m = m_bar*L/6*T.*[2 0 1 0;
                  0 2 0 1;
                  1 0 2 0;
                  0 1 0 2]*T;
```

Illustrative Example 14.3

Determine the response of the truss in Examples 14.1 and 14.2 when a constant force $F_1 = 5000$ lb. is suddenly applied at node 2 in x axis as shown in Fig. 14.3.

Solution:

The modal equations are given in general [Eq. (8.6)] by

$$\ddot{z}_n + \omega_n^2 z_n = P_n$$

in which the modal force.

$$P_n = \sum_i \phi_{in} F_i$$

Hence using the results that were calculated in Example 18.2 we obtain.

$$\ddot{z}_1 + (415)^2 z_1 = 2010$$

$$\ddot{z}_2 + (1034)^2 z_2 = 340$$

$$\ddot{z}_3 + (1526)^2 z_3 = 1230$$

The solution of the above equations for zero initial conditions ($z_n = 0, \dot{z}_n = 0$) is given by Eqs. (4.5) as.

$$z_1 = \frac{2010}{(415)^2} (1 - \cos 415t)$$

$$z_2 = \frac{340}{(1034)^2} (1 - \cos 1034t)$$

$$z_3 = \frac{1230}{(1526)^2} (1 - \cos 1526t)$$

The response at the nodal coordinates is then calculated from

$$\{u\} = [\Phi]\{z\}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 0.402 & 0.068 & 0.246 \\ 0.087 & 0.373 & -0.246 \\ 0.110 & -0.272 & -0.375 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix}$$

or

$$u_1 = 10^{-3} [4.843 - 4.692 \cos 415t - 0.022 \cos 1034t - 0.130 \cos 1526t]$$

$$u_2 = 10^{-3} [1.004 - 1.015 \cos 415t - 0.119 \cos 1034t + 0.130 \cos 1526t]$$

$$u_3 = 10^{-3} [0.999 - 1.284 \cos 415t + 0.087 \cos 1034t + 0.198 \cos 1526t]$$

The system matrices in Example 14.2 are used to plot the responses over 5 s at 0.01 s interval. The initial conditions are zeros for both displacement and velocity. The program prints out the maximum responses and figure below.

```

clc
clear all
close all

%
% Inputs:
% M, K
% F = forcing function
% t = Time period
% u0 = initial displacement
% v0 = initial velocity
%
t = 0:0.01:5;

load ('temp0.mat', 'Mf', 'Kf', 'Rf')

%%Deifne Mass Matrix

```

```

M = Mf
%%%Deifne Stiffness Matrix
K = Kf

[n,n]= size(M);

F = Rf;

nstep = size(t');

u0 = zeros(n,1); u0(1) =0;
v0 = zeros(n,1); v0(1) =0;
[n,n]= size(M);

%
% Solve the eignevalue problem and normalized the eigenvectors
%
[a, D] = eig(K, M) % Solve for eigenvalues (D) and eigenvectors (a)

[omegas,k] = sort(sqrt(diag(D))); % Natural Frequencies
a = a(:,k)

T = 2*pi./omegas; % Natural Periods

aMa = diag(a'*M*a); % aMa = {a}'*[M]*(a)

nom_phi = (a)*inv(sqrt(diag(aMa))); % Normalized modal matrix

orth_M = nom_phi'*M*nom_phi % Check the orthogonality condition for Mass Matrix
orth_K = nom_phi'*K*nom_phi % Check the orthogonality condition for Stiffness
Matrix
%
% Initial conditions
%
P = nom_phi'*F % Normalized force, P = nom_F
q0 = nom_phi'*M*u0;
dq0 = nom_phi'*M*v0;

%
% Damping matrix using the proportional damping matrix
% [C] = a[M]+b[K]
% zetas = damping ratios
%
a = 0;
b = 0;
nom_C = nom_phi'*(a*M+b*K)*nom_phi;
zetas = diag((1/2)*nom_C*inv(diag(omegas)));

save ('temp1.mat', 'omegas', 'P', 'zetas');
q = [];
r = [];
for i=1:n
q0_i = q0(i,:);
dq0_i = dq0(i,:);

load temp1.mat
omega = omegas(i,:);
P = P(i,:);
m = M(i,i);
zeta = zetas(i,:);
save ('temp2.mat', 'omega', 'P', 'm', 'zeta');

[t,q] = ode45(@MDOFP, t, [q0_i dq0_i]',[]);

r(:,i) = q(:,1);
save ('temp3.mat', 'r')

end

```

```

load ('temp3.mat', 'r');
yim = nom_phi*[r'];
save ('response.mat','yim');

figure
subplot(3,1,1);          % Node 2: x displ (in.).
plot(t, yim(1,:))
xlabel ('Time (sec)'); ylabel ('u_1(in.)'); grid on

subplot(3,1,2);          % Node 2: y displ (in.).
plot(t, yim(2,:))
xlabel ('Time (sec)'); ylabel ('u_2(in.)'); grid on

subplot(3,1,3);          % Node 3: x displ (in.).
plot(t, yim(3,:))
xlabel ('Time (sec)'); ylabel ('u_3(in.)'); grid on

umax_1=max(abs(yim(1,:)))
umax_2=max(abs(yim(2,:)))
umax_3=max(abs(yim(3,:)))

```

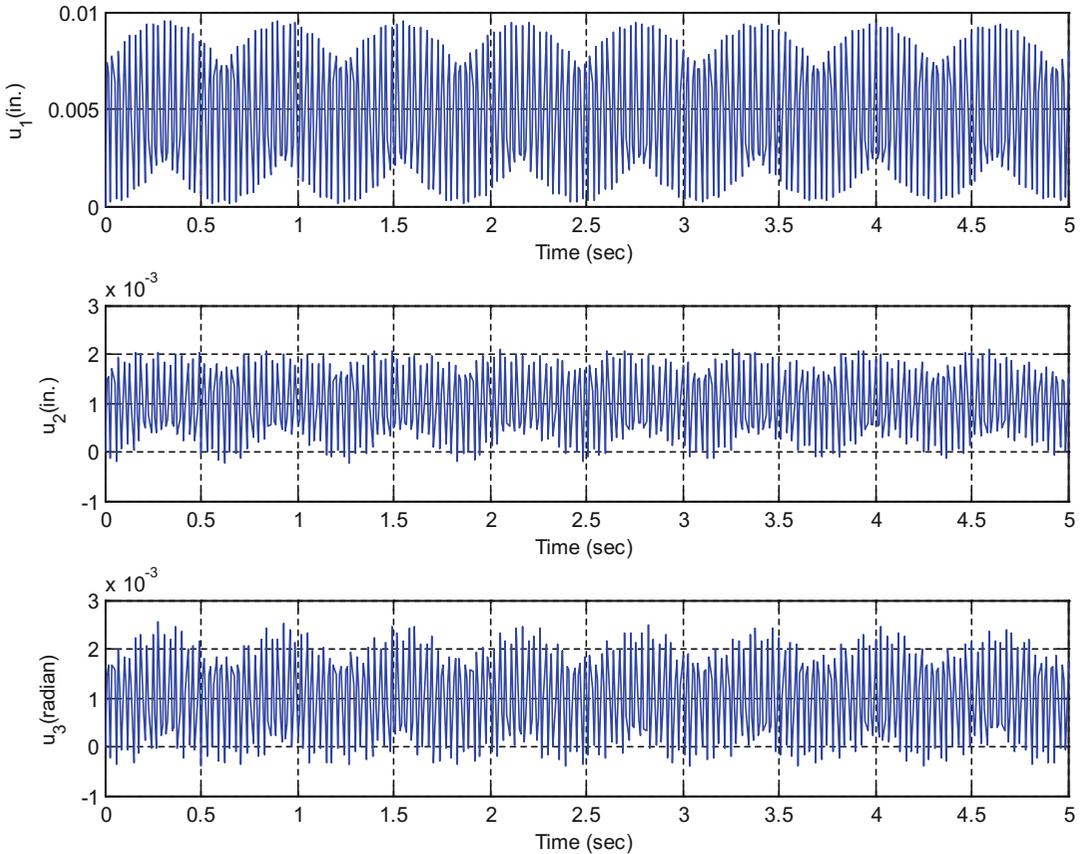


Fig. 14.4 Response of Illustrative Example 14.3

The maximum displacements at the nodal coordinates were estimated using MATLAB.

$$u_{1\max} = 0.0096 \text{ in} \quad u_{2\max} = 0.0021 \text{ in} \quad u_{3\max} = 0.0025 \text{ in}$$

14.3 Stiffness and Mass Matrices for Space Trusses

The stiffness matrix and the mass matrix for a space truss can be obtained as an extension of the corresponding matrices for the plane truss. Figure 14.4 shows the nodal coordinates in the local system (unbarred) and in the global system (barred) for a member of a space truss. The local x axis is directed along the longitudinal axes of the member while the y and z axes are set to agree with the principal directions of the cross section of the member. The following matrices may then be written for a uniform member of a space truss as an extension of the stiffness, Eq. (14.3), and the mass, Eq. (14.9), matrices for a member of a plane truss.

Stiffness matrix:

$$[K] = \frac{AE}{L} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (14.21)$$

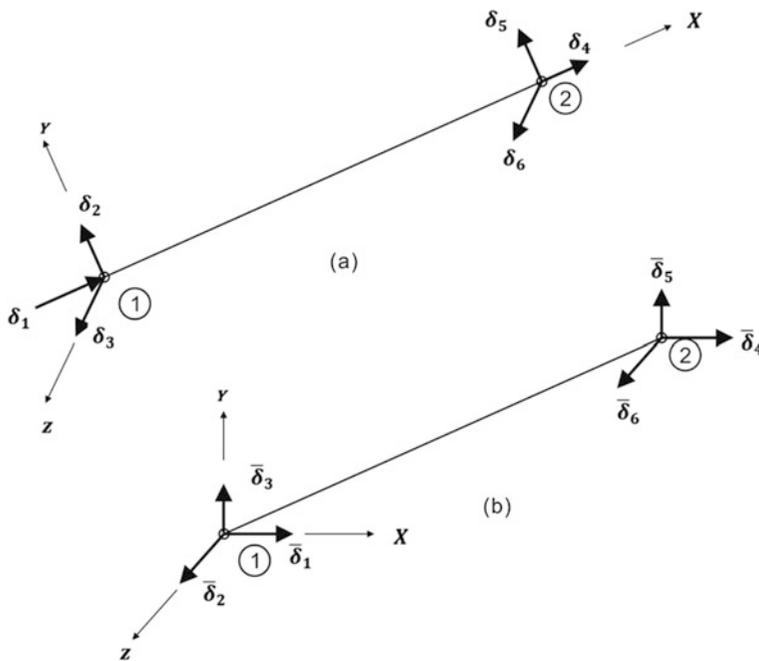


Fig. 14.5 Member of a space truss showing nodal coordinates. (a) In the local system (unbarred). (b) In the global system (barred)

Consistent mass matrix:

$$[M_c] = \frac{\bar{m}L}{6} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \quad (14.22)$$

Lumped mass matrix:

$$[M_L] = \frac{\bar{m}L}{2} [1 \ 1 \ 1 \ 1 \ 1 \ 1] \quad (14.23)$$

The transformation matrix $[T_1]$ corresponding to three nodal coordinates at a joint is given by Eq. (13.9). It is repeated here for convenience.

$$[T_1] = \begin{bmatrix} \cos xX & \cos xY & \cos xZ \\ \cos yX & \cos yY & \cos yZ \\ \cos zX & \cos zY & \cos zZ \end{bmatrix} \quad (14.24)$$

The direction cosines in the first row of Eq. (14.24), $c_1 = \cos xX$, $c_2 = \cos xY$, and $c_3 = \cos xZ$, may be calculated from the coordinates of the two points $P_1 (X_1, Y_1, Z_1)$ and $P_2 (X_2, Y_2, Z_2)$ at the ends of the truss element, that is

$$c_1 = \frac{X_1 - X_2}{L}, c_2 = \frac{Y_2 - Y_1}{L} \text{ and } c_3 = \frac{Z_2 - Z_1}{L} \quad (14.25)$$

with

$$L = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \quad (14.26)$$

The transformation matrix for the nodal coordinates at the two ends of a truss member is then given by

$$[T] = \begin{bmatrix} [T_1] & [0] \\ [0] & [T_1] \end{bmatrix} \quad (14.27)$$

in which $[T_1]$ is given by Eq. (14.24). The following transformations are then required to obtain the member stiffness matrix $[\bar{K}]$ and the member mass matrix $[\bar{M}]$ in reference to the global system of coordinates:

$$[\bar{K}] = [T]^T [K] [T] \quad (14.28)$$

and

$$[\bar{M}] = [T]^T [M] [T] \quad (14.29)$$

In the case of an element of a space truss, it is only necessary to calculate the direction cosines of the centroidal axis of the element which are given by Eq. (14.25). The other direction cosines in

Eq. (14.24) do not appear in the final expression for the element stiffness matrix $[\bar{K}]$ as may be verified by substituting Eqs. (14.24) and (14.27) into Eq. (14.28) and proceeding to multiply the matrices indicated in this last equation. The final result of this operation may be written as follows.

$$[K] = \frac{AE}{L} \begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 & -c_1^2 & -c_1c_2 & -c_1c_3 \\ c_2c_1 & c_2^2 & c_2c_3 & -c_2c_1 & -c_2^2 & -c_2c_3 \\ c_3c_1 & c_3c_2 & c_3^2 & -c_3c_1 & -c_3c_2 & -c_3^2 \\ -c_1^2 & -c_1c_2 & -c_1c_3 & c_1^2 & c_1c_2 & c_1c_3 \\ -c_2c_1 & -c_2^2 & -c_2c_3 & c_2c_1 & c_2^2 & c_2c_3 \\ -c_3c_1 & -c_3c_2 & -c_3^2 & c_3c_1 & c_3c_2 & c_3^2 \end{bmatrix} \quad (14.30)$$

Consequently, the determination of the stiffness matrix for an element of a space truss, in reference to the global system of coordinates by Eq. (14.30), requires the evaluation by Eq. (14.25) of only the direction cosines of the local axis x along the element.

Also, analogously, to Eq. (14.19) for an element of a plane truss, the mass matrix $[\bar{M}]$ for an element of a space truss in reference to global coordinates is equal to the mass matrix of the element $[M]$ in local coordinates. Thus, there is no need to perform the operations indicated in Eq. (14.29). The substitution into this equation of $[M]$ from Eq. (14.22) and $[T]$ from Eqs. (14.24) and (14.27) results, after performing the multiplications established by Eq. (14.29), in the same matrix $[M]$, that is,

$$[\bar{M}] = [M] \quad (14.31)$$

14.4 Equation of Motion for Space Trusses

The dynamic equilibrium conditions at the nodes of the space truss result in the differential equations of motion which in matrix notation may be written as follows:

$$[\bar{M}]\{\ddot{u}\} + [\bar{C}]\{\dot{u}\} + [\bar{K}]\{u\} = \{F(t)\} \quad (14.32)$$

in which $\{u\}$, $\{\dot{u}\}$, and $\{\ddot{u}\}$ are, respectively, the displacement, velocity, and acceleration vectors at the nodal coordinates, $\{F(t)\}$ is the vector of external nodal forces, and $[\bar{M}]$, $[\bar{C}]$, and $[\bar{K}]$ are the system mass, damping, and stiffness matrices.

In the stiffness method of analysis, the system matrices in Eq. (14.32) are obtained by appropriate superposition of the corresponding member matrices using the direct method as we have shown previously for the framed structures. As was discussed in the preceding chapters, the practical way of evaluating damping is to prescribe damping ratios relative to the critical damping for each mode. Consequently, when Eq. (14.32) is solved using the modal superposition method, the specified modal damping ratios are introduced directly into the modal equations. In this case, there is no need for explicitly obtaining the system damping matrix $[C]$. However, this matrix is required when the solution of Eq. (14.32) is sought by other methods of solution, such as the step-by-step integration method. In this case, the system damping matrix $[C]$ can be obtained from the specified modal damping ratios by any of the methods discussed in Chap. 22.

14.5 Modeling Structures as Space Trusses Using MATLAB

The MATLAB calculates the stiffness and the mass matrices for the space truss. MATLAB can determine natural frequencies or calculate the response of the structure subjected to external excitation.

Illustrative Example 14.4

Determine the stiffness and mass matrices for the space truss shown in Fig. 14.5. The mass per unit length of any member is $\bar{M} = 0.1$ (lb · sec²/in²). Also determine the first three natural frequencies of the truss. Plot the response of the structure when $F_1 = 5000$ lb is suddenly applied at node 3 in z axis (Fig. 14.6).

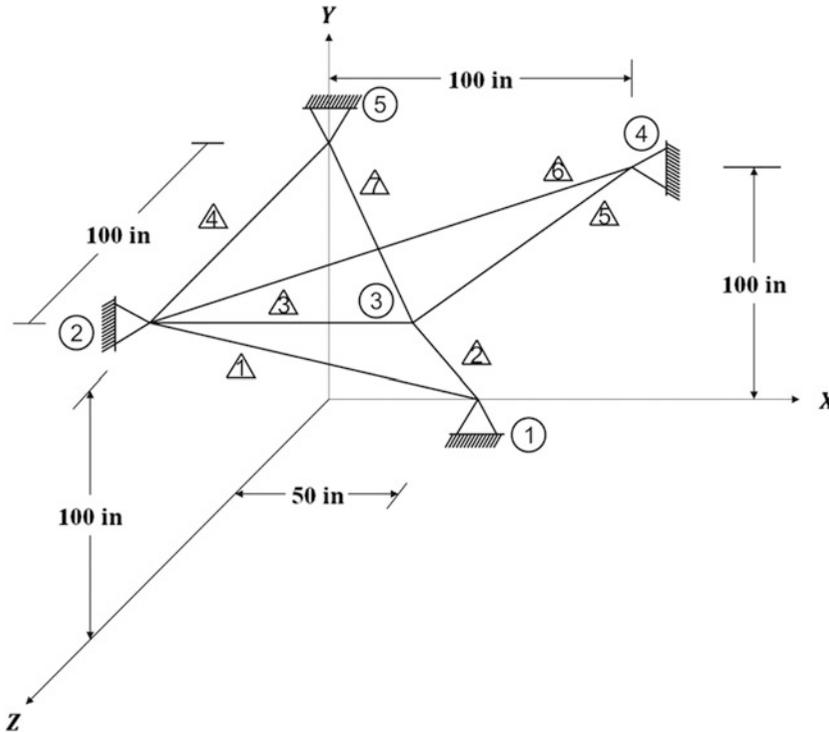


Fig. 14.6 Space truss for Example 14.4

Solution:

The force is applied at node 3 in z axis (which is corresponding to 9th degree-of-freedom).

```

clc
close all
clear all

%
% Determine System Matrices/Determine Force
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%GIVEN VALUES-%%%
E=30*10^6; A=10; %E (psi), A (in^2)
P = 5000; %P (lbs)

%%Create frame model (ith row of nodes is ith node)
nodes = [50, 0, 0;
         0, 100, 100;
         100,100, 100;
         100,100, 0;
         0, 100, 0]; % Unit: in.

%%Element number (ith row = ith element with two nodes)
conn = [2,1;
        3,1;
        2,3;
        2,5;
        3,4;
        2,4;
        5,3];

%%Dofs for ith element (ith row)
lmm = [4, 5, 6, 1, 2, 3;
       7, 8, 9, 1, 2, 3;
       4, 5, 6, 7, 8, 9;
       4, 5, 6, 13,14,15;
       7, 8, 9, 10,11,12;
       4, 5, 6, 10,11,12;
       13,14,15, 7, 8, 9];

m_bar = 0.1; %Distributed mass (lb-sec^2/in/in)

dof = 3*length(nodes); %Total No.dofs

K= zeros(dof);
M= zeros(dof);

%%Generate equations for each element and assemble them.
for i=1:7
    lm=lmm(i,:);
    con=conn(i,:);
    k=SpaceTrussElement(E, A, nodes(con,:));
    K(lm, lm) = K(lm, lm) + k;
end

%%Generate mass matrix for each element and assemble them.
for i=1:7
    lm=lmm(i,:);
    con=conn(i,:);
    m=SpaceTrussMass(m_bar, nodes(con,:));
    M(lm, lm) = M(lm, lm) + m;
end

```

```

K
M

%%Define the load vector
F = zeros(dof,1); F(9) = 5000;           %Applied force at specific dofs

%%System Matrices
[Kf, Mf, Rf] = System(K, M, F, [1:6,10:dof])

Kf
Mf
Rf
%
% Solve the eigenvalue problem and normalized eigenvectors
%
%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(Kf, Mf);

[omegas,ii] = sort(sqrt(diag(D))); %Natural Frequencies

omegas

a = a(:,ii) %Mode Shapes

T = 2*pi./omegas; %Natural Periods

save ('temp0.mat', 'Mf', 'Kf', 'Rf');

```

The function file of MATLAB is used to assemble the stiffness matrix of space truss element for global stiffness matrix.

```

function k = SpaceTrussElement(e, A, coord)
% k = SpaceTrussElement(e, A, coord)
% Generates stiffness matrix of a space truss element
% E = modulus of elasticity (psi)
% A = Area of cross-section (in^2)
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2); z1=coord(1,3);
x2=coord(2,1); y2=coord(2,2); z2=coord(2,3);
L=sqrt((x2-x1)^2+(y2-y1)^2+(z2-z1)^2);
C1=(x2-x1)/L; C2=(y2-y1)/L; C3=(z2-z1)/L;

k = e*A/L*[C1^2, C1*C2, C1*C3, -(C1^2), -(C1*C2), -(C1*C3);
           C2*C1, C2^2, C2*C3, -(C2*C1), -(C2^2), -(C2*C3);
           C3*C1, C3*C2, C3^2, -(C3*C1), -(C3*C2), -(C3^2);
           -(C1^2), -(C1*C2), -(C1*C3), C1^2, C1*C2, C1*C3;
           -(C2*C1), -(C2^2), -(C2*C3), C2*C1, C2^2, C2*C3;
           -(C3*C1), -(C3*C2), -(C3^2), C3*C1, C3*C2, C3^2];

```

The function file of MATLAB is used to assemble the mass matrix of each space truss element for global mass matrix.

```

function m = SpaceTrussMass(m_bar, coord)
% SpaceTrussElement(e, A, coord)
% Generates mass matrix for a space truss element
% m = distributed mass (lb.sec^2/in/in)
% L = length (in.)
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2); z1=coord(1,3);
x2=coord(2,1); y2=coord(2,2); z2=coord(2,3);
L=sqrt((x2-x1)^2+(y2-y1)^2+(z2-z1)^2);

m = m_bar*L/6*[2 0 0 1 0 0;
               0 2 0 0 1 0;
               0 0 2 0 0 1;
               1 0 0 2 0 0;
               0 1 0 0 2 0;
               0 0 1 0 0 2];

```

The same MATLAB file can be used in Example 14.4 to plot the response of the space truss structures (Fig. 14.7).

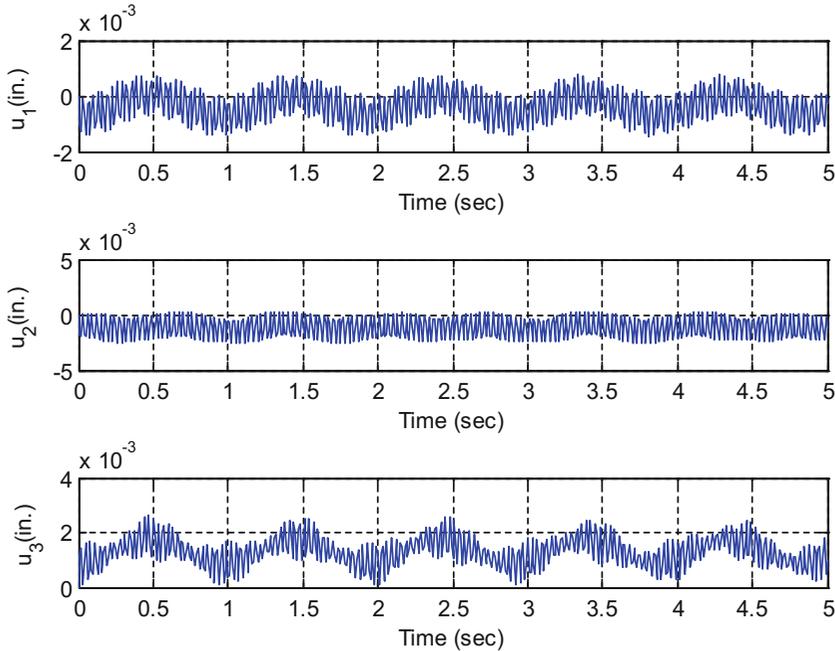


Fig. 14.7 Response of Illustrative Example 14.4

The maximum displacements at the nodal coordinates were estimated using MATLAB.

$$u_{1\max} = 0.0014 \text{ in.} \quad u_{2\max} = 0.0026 \text{ in.} \quad u_{3\max} = 0.0026 \text{ in.}$$

14.6 Summary

The dynamic analysis of trusses by the stiffness matrix method was presented in this chapter. As in the case of framed structures, discussed in the preceding chapters, the stiffness and mass matrices for a member of a truss were developed. The system matrices for a truss are assembled as explained for framed structures by the appropriate superposition of the coefficients in the matrices of the elements.

14.7 Problems

Problem 14.1

For the plane truss shown in Fig. P14.1 determine the system stiffness and mass matrices corresponding to the two nodal coordinates indicated in the figure.

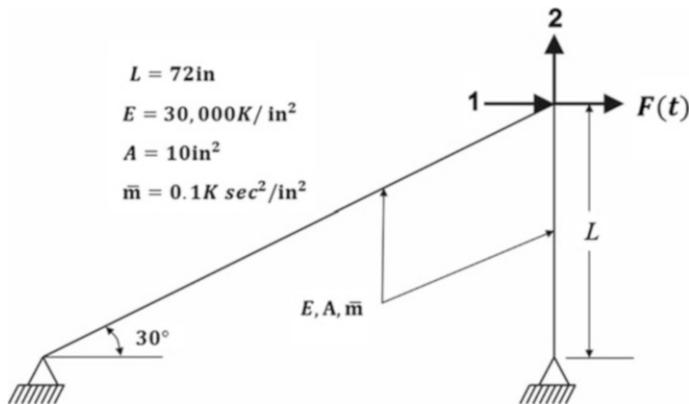


Fig. P14.1

Problem 14.2

Determine the natural frequencies and corresponding normal modes for the truss shown in Fig. P14.1.

Problem 14.3

Determine the response of the truss shown in Fig. P14.1 when subjected to a force $F(t) = 10 \text{ Kip}$ suddenly applied for 2 s at nodal coordinate 1. Use the results of Problem 14.2 to obtain the modal equations. Neglect damping in the system.

Problem 14.4

Solve Problem 14.3 assuming 10% damping in all the modes.

Determine the maximum response of the truss shown in Fig. P14.1 when subjected to a rectangular pulse of magnitude $F_0 = 10 \text{ Kip}$ and duration $t_d = 0.1 \text{ sec}$. Use the appropriate response spectrum to determine the maximum modal response (Fig. 4.4). Neglect damping in the system.

Problem 14.5

Determine the dynamic response of the frame shown in Fig. P14.1 when subjected to a harmonic force $F(t) = 10 \sin 10 t$ (Kips) along nodal coordinate 1. Neglect damping in the system.

Problem 14.6

Repeat Problem 14.6 assuming that the damping in the system is proportional to the stiffness, $[C] = a_0 [K]$ where $a_0 = 0.1$.

Problem 14.7

Determine the response of the truss shown in Fig. P14.8 when acted upon by the forces $F_1(t) = 10t$ and $F_2(t) = 5t^2$ during 1 sec. Neglect damping.

Problem 14.8

Determine the response of the truss shown in Fig. P14.8 when acted upon by the forces $F_1(t) = 10t$ and $F_2(t) = 5t^2$ during 1 s. Neglect damping.

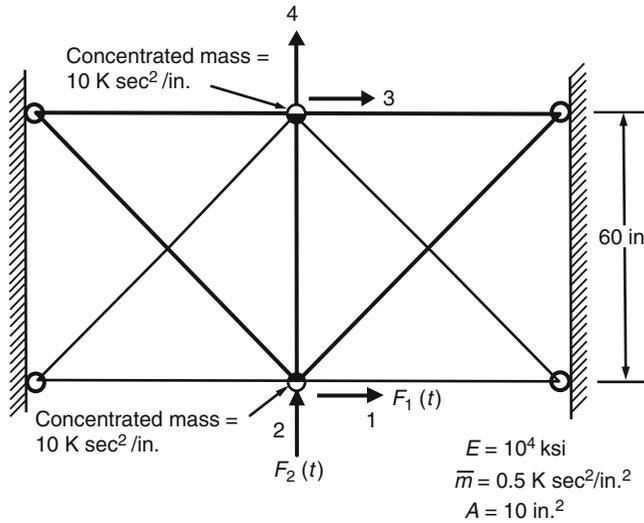


Fig. P14.8

Problem 14.9

Solve problem 14.8 assuming 10% modal damping in all the modes.