8 Forced Motion of Shear Buildings

In the preceding chapter, we have shown that the free motion of a dynamic system may be expressed in terms of the normal modes of vibration. Our present interest is to demonstrate that the forced motion of such a system may also be expressed in terms of the normal modes of vibration and that the total response may be obtained as the superposition of the solution of independent modal equations. In other words, our aim in this chapter is to show that the normal modes may be used to transform the system of coupled differential equations into a set of uncoupled differential equations in which each equation contains only one dependent variable. Thus the modal superposition method reduced the problem of finding the response of a multi-degree-of-freedom system to the determination of the response of single-degree-of-freedom systems.

8.1 Modal Superposition Method

In Chap. 6, we have shown that any free motion of a multi-degree-of-freedom system may be expressed in terms of normal modes of vibration. It will now be demonstrated that the forced motion of such a system may also be expressed in terms of the normal modes of vibration. We return to the equations of motion, Eq. (3.40), which for the particular case of a two-degree-of-freedom shear building are

$$\begin{aligned} m\ddot{u}_1 + (k_1 + k_2) - k_2 u_2 &= F_1(t) \\ m_2\ddot{u}_2 - k_2 u_1 + k_2 u_2 &= F_2(t) \end{aligned} \tag{8.1}$$

We seek to transform this coupled system of differential equations into a system of independent or uncoupled equations in which each equation contains only one unknown function of time. It is first necessary to express the solution in terms of the normal modes multiplied by some factors determining the contributions of each mode. In the case of free motion, these factors were sinusoidal functions of time; in the present case, for forced motion they are general functions of time which we designate as $q_i(t)$. Hence, the solution of Eq. (8.1) is assumed to be of the form

$$\begin{aligned} u_1(t) &= a_{11}q_1(t) + a_{12}q_2(t) \\ u_2(t) &= a_{21}q_1(t) + a_{22}q_2(t) \end{aligned} \tag{8.2}$$

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Upon substitution into Eq. (8.1), we obtain

$$\begin{aligned} &m_1 a_{11}\ddot{q}_1 + (k_1+k_2)a_{11}q_1 - k_2 a_{21}q_1 + m_1 a_{12}\ddot{q}_2 + (k_1+k_2)a_{12}q_2 - k_2 a_{22}q_2 = F_1(t) \\ &m_2 a_{21}\ddot{q}_1 - k_2 a_{11}q_1 + k_2 a_{21}q_1 + m_2 a_{22}\ddot{q}_2 - k_2 a_{12}q_2 + k_2 a_{22}q_2 = F_2(t) \end{aligned} \tag{8.3}$$

To determine the appropriate functions $q_1(t)$ and $q_2(t)$ that will uncouple Eq. (8.3), it is necessary to make use of the orthogonality relations to separate the modes. The orthogonality relations are used by multiplying the first of Eq. (8.3) by a_{11} and the second by a_{21}. Addition of these equations after multiplication and simplification by using Eqs. (6.3) and (6.5) yields

$$\left(m_1 a_{11}^2 + m_2 a_{21}^2\right)\ddot{z}_1 + \omega_1^2\left(m_1 a_{11}^2 + m_2 a_{21}^2\right)z_1 = a_{11}F_1(t) + a_{21}F_2(t) \tag{8.4a}$$

Similarly, multiplying the first of Eq. (8.3) by a_{12} and the second by a_{22}, we obtain

$$\left(m_1 a_{12}^2 + m_2 a_{22}^2\right)\ddot{z}_2 + \omega_2^2\left(m_1 a_{12}^2 + m_2 a_{22}^2\right)z_2 = a_{12}F_1(t) + a_{22}F_2(t) \tag{8.4b}$$

The results obtained in Eq. (8.4) permit a simple physical interpretation. The force that is effective in exciting a mode is equal to the work done by the external force displaced by the modal shape in question. From the mathematical point of view, what we have accomplished is to separate or uncouple, by a change of variables, the original system of differential equations. Consequently, each of these equations, Eqs. (8.4a) and (8.4b), corresponds to a single-degree-of-freedom system which may be written as

$$\begin{aligned} M_1\ddot{q}_1 + K_1 q_1 &= P_1(t) \\ M_2\ddot{q}_2 + K_2 q_2 &= P_2(t) \end{aligned} \tag{8.5}$$

where $M_1 = m_1 a_{11}^2 + m_2 a_{21}^2$ and $M_2 = m_2 a_{12}^2 + m_2 a_{22}^2$ are the modal masses; $K_1 = \omega_1^2 M_1$ and $K_2 = \omega_2^2 M_2$, the modal spring constants; $P_1(t) = a_{11}F_1(t) + a_{21}F_2(t)$ and $P_2(t) = a_{12}F_1(t) + a_{22}F_2(t)$ the modal forces. Alternatively, if we make use of the previous normalization in Eq. (7.15) or (7.16), these equation may be written simply as

$$\begin{aligned} \ddot{q}_1 + \omega_1^2 q_1 &= P_1(t) \\ \ddot{q}_2 + \omega_2^2 q_2 &= P_2(t) \end{aligned} \tag{8.6}$$

where P_1 and P_2 are now given by

$$\begin{aligned} P_1 &= \phi_{11}F_1(t) + \phi_{21}F_2(t) \\ P_2 &= \phi_{12}F_1(t) + \phi_{22}F_2(t) \end{aligned} \tag{8.7}$$

The solution for the uncoupled differential equations, Eq. (8.5) or Eq. (8.6), may now be found by any of the methods presented in the previous chapters for the solution of a single degree of freedom system. In particular, Duhamel's integral provides a general solution for these equations regardless of the functions describing the forces acting on the structure. Also, maximum values of the response for each modal equation may readily be obtained using available response spectra. However, the superposition of modal maximum responses presents a problem. The fact is that these modal maximum values will in general not occur simultaneously as the transformation of coordinates, Eq. (8.2), requires. To obviate the difficulty, it is necessary to use an approximate method. An upper limit for the maximum response may be obtained by adding the absolute values of the maximum modal contributions, that is, by substituting q_1 and q_2 in Eq. (8.2) for the maximum modal responses, $q_{1\max}$ and $q_{2\max}$, and adding the absolute values of the terms in these equations, so that

$$\begin{aligned} u_{1\max} &= |\phi_{11} q_{1\max}| + |\phi_{12} q_{2\max}| \\ u_{2\max} &= |\phi_{21} q_{1\max}| + |\phi_{22} q_{2\max}| \end{aligned} \tag{8.8}$$

The results obtained by this method will generally overestimate the maximum response. Another method, which is widely accepted and which generally gives a reasonable estimate of the maximum response from these spectral values, is the Square Root of the Sums of Squares of the modal contributions (SRSS method). Thus the maximum displacements may be approximated by

$$u_{1\max} = \sqrt{(\phi_{11} q_{1\max})^2 + (\phi_{12} q_{2\max})^2}$$

and

$$u_{2\max} = \sqrt{(\phi_{21} q_{1\max})^2 + (\phi_{22} q_{2\max})^2} \tag{8.9}$$

The results obtained by application of the SRSS method (Square Root of the Sum of the Squares of modal contributions) may substantially underestimate or overestimate the total response when two or more modes are closely spaced. In this case, another method known as the Complete Quadratic Combination (CQC) for combining modal responses to obtain the total response is recommended. The discussion of such a method is presented in this chapter in Sect. 8.6

The transformation from a system of two coupled differential equations, Eq. (8.1), to a set of two uncoupled differential equations, Eq. (8.6), may be extended to a general and system of N degrees of freedom. For such a system, it is particularly convenient to use matrix notation. With such notation, the equation of motion for a linear system of N degrees of freedom is given by Eq. (7.3) as

$$[M]\{\ddot{u}\} + [K]\{u\} = \{F(t)\} \tag{8.10}$$

where $[M]$ and $[K]$ are respectively the mass and the stiffness matrices of the system, $\{F(t)\}$ the vector of external forces, and $\{u\}$ the vector of unknown displacements at the nodal coordinates.

Introducing into Eq. (8.10) the linear transformation of coordinates

$$\{u\} = [\Phi]\{q\} \tag{8.11}$$

in which $[\Phi]$ is the modal matrix of the system, yields

$$[M][\Phi]\{\ddot{q}\} + [K][\Phi]\{q\} = [F(t)\} \tag{8.12}$$

The pre-multiplication of Eq. (8.12) by the transpose of the ith modal vector, $\{\phi\}_i^T$, results in

$$\{\phi\}_i^T\,[M][\Phi]\{\ddot{q}\} + \{\phi\}_i^T[K][\Phi]\{q\} = \{\phi\}_i^T\{F(t)\} \tag{8.13}$$

The orthogonality conditions between normalized modes, Eqs. (7.17) and (7.19), imply that

$$\{\phi\}_i^T\,[M][\Phi] = 1 \tag{8.14}$$

and

$$\{\phi\}_i^T\ [K][\Phi] = \omega_i^2 \tag{8.15}$$

Consequently, Eq. (8.13) may be written as

$$\ddot{q}_i + \omega_i^2 q_i = P_i(t) \quad i = 1, 2, 3, \ldots, N \tag{8.16}$$

where the modal force $P_i(t)$ is given by

$$P_i(t) = \phi_{1i}F_1(t) + \phi_{2i}F_2(t) + \ldots + \phi_{Ni}F_N(t) \tag{8.17}$$

Equation (8.16) constitutes a set of N uncoupled or independent equations of motion in terms of the modal coordinates q_i. These uncoupled equations, as may be observed, may readily be written after the natural frequencies ω_i and the modal vectors, $\{\phi\}_i$ have been determined in the solution of the corresponding eigenproblem as presented in Chap. 7.

Illustrative Example 8.1

The two-story frame of Illustrative Example 7.1 is acted upon at the floor levels by horizontal triangular impulsive forces as shown in Fig. 8.1. For this frame, determine the maximum floor displacements and the maximum shear forces in the columns.

Solution:

The results obtained in Illustrative Examples 7.1 and 7.2 for the free vibration of this frame gave the following values for the natural frequencies and normalized modes:

$$\omega_1 = 11.83\,\text{rad/sec}, \qquad \omega_2 = 32.89\,\text{rad/sec}$$
$$\phi_{11} = 0.06437, \qquad \phi_{12} = 0.0567$$
$$\phi_{21} = 0.08130, \qquad \phi_{22} = -0.0924$$

The forces acting on the frame which are shown in Fig. 8.1b may be expressed by

$$F_1(t) = 10{,}000(1 - t/t_d)\,\text{lb}$$
$$F_2(t) = 20{,}000(1 - t/t_d)\,\text{lb} \quad \text{for } t \le 0.1\,\text{sec}$$

in which $t_d = 0.1$ sec and

$$F_1(t) = F_2(t) = 0, \qquad \text{for } t > 0.1\,\text{sec}$$

The substitution of these values into the uncoupled equations of motion, Eqs. (8.6) and (8.7) gives

$$\ddot{q}_1 + 140q_1 = 2270f(t)$$
$$\ddot{q}_2 + 1082.41q_2 = -1281f(t)$$

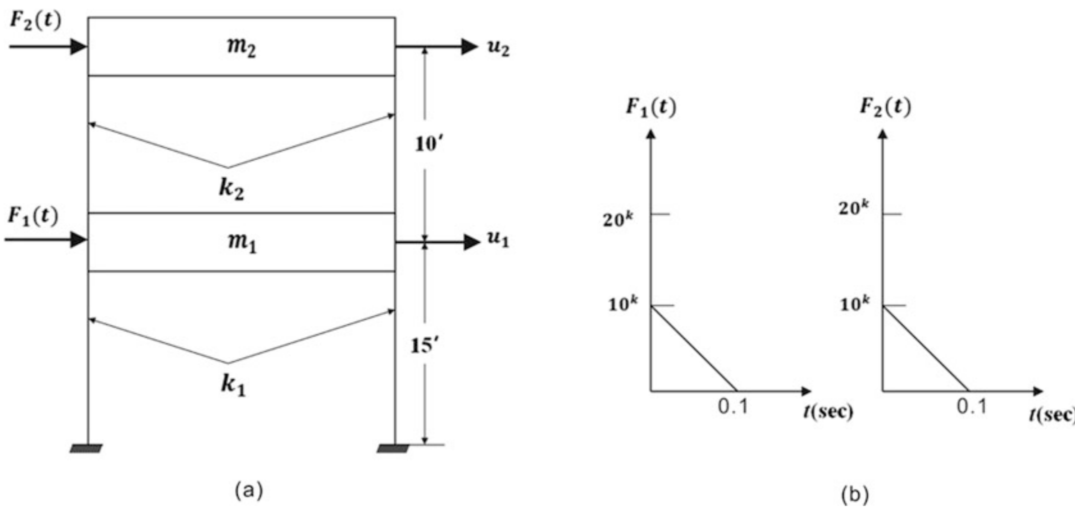

Fig. 8.1 Shear building with impulsive loadings. (**a**) Two-story shear building. (**b**) Impulsive loadings

in which $f(t) = 1 - t/t_d$ for $t \leq 0.1$ and $f(t) = 0$ for $t > 0.1$. The maximum values for q_1 and q_2 are then obtained from available spectral charts such as the one shown in Fig. 4.5. For this example,

$$\frac{t_d}{T_1} = \frac{0.1}{0.532} = 0.188$$

and

$$\frac{t_d}{T_2} = \frac{0.1}{0.191} = 0.524$$

in which the modal natural periods are calculated as

$$T_1 = \frac{2\pi}{\omega_1} = 0.532 \text{ sec} \quad \text{and} \quad T_2 = \frac{2\pi}{\omega_2} = 0.191 \text{ sec}$$

From Fig. 4.5, we obtain:

$$(DLF)_{1\max} = \frac{q_{1\max}}{q_{1st}} = 0.590$$

$$(DLF)_{2\max} = \frac{q_{2\max}}{q_{2st}} = 1.22$$

where the static deflections, q_{1st}, and q_{2st}, are calculated as

$$q_{1st} = \frac{F_{01}}{\omega_1^2} = \frac{2270}{140} = 16.3, \quad q_{2st} = \frac{F_{02}}{\omega_2^2} = \frac{-1281}{1082.41} = -1.18$$

Then the maximum values of the modal response are:

$$q_{1\max} = 0.590 \times 16.3 = 9.62, \quad q_{2\max} = 1.22 \times 1.18 = 1.44$$

As indicated above these maximum modal values do not occur simultaneously and therefore cannot simply be superimposed to obtain the maximum response of the system. However, an upper limit for the absolute maximum displacement may be calculated with Eq. (8.8) as

$$u_{1\max} = |0.06437 \times 9.62| + |0.0567 \times 1.44| = 0.70 \text{ in}$$

$$u_{2\max} = |0.08130 \times 9.62| + |-0.0924 \times 1.44| = 0.92 \text{ in}$$

A second acceptable estimate of the maximum response is obtained by taking the square root of the sum of the squared modal contributions as indicated by Eq. (8.9). For this example, we have

$$\begin{aligned} u_{1\max} &= \sqrt{(0.06437 \times 9.62)^2 + (0.0567 \times 1.44)^2} = 0.62 \text{ in} \\ u_{2\max} &= \sqrt{(0.08130 \times 9.62)^2 + (-0.0924 \times 1.44)^2} = 0.79 \text{ in} \end{aligned} \quad \text{(a)}$$

The maximum shear force $V_{\max}$ in the columns is given by

$$V_{\max} = k\Delta_u \quad (8.18)$$

in which k is the stiffness of the story and Δ_u the difference between the displacements at the two ends of the column. Since the maximum displacements calculated as in Eq. (a) may have positive or negative values, the relative displacement Δ_u cannot be determined as the difference of the absolute displacements of the two ends of the column. The maximum positive value for Δ_u could be estimated as the sum of the absolute maximum displacements at the ends of the columns. However, this procedure will in most cases greatly overestimate the actual forces in the columns. The recommended procedure is to calculate first the shear force in the columns for each mode, separately, and then combine these modal forces by a suitable method, such as the square root of the sum of the squares of modal contributions. This procedure is based on the fact that modal displacements are known with their correct relative sign and not as absolute values.

The maximum shear force V_{ij} at story i corresponding to mode j is given by

$$V_{ij} = q_{j\max}\left(\phi_{ij} - \phi_{i-1j}\right)k_i \quad (8.19)$$

where $q_{j\max}$ is the maximum modal response, $(\phi_{ij} - \phi_{i-1j})$ the relative modal displacement of story i (with $\phi_{0j} = 0$), and k_i the stiffness of the story. For this example we have for the first story

$$k_1 = \frac{12EI_1}{L_1^3} = \frac{12 \times 30 \times 10^6 \times 248.6}{(15 \times 12)^3} = 15{,}345 \text{ lb/in}$$

$$V_{11} = 9.62 \times 0.06437 \times 15{,}345 = 9502 \text{ lb}$$

$$V_{12} = 1.44 \times 0.0567 \times 15{,}345 = 1253 \text{ lb}$$

$$V_{1\max} = \sqrt{9502^2 + 1253^2} = 9584 \text{ lb}$$

and for a column in the second story

$$k_2 = \frac{12EI_2}{L_2^3} = \frac{12 \times 30 \times 10^6 \times 106.3}{(10 \times 12)^3} = 22,146 \text{ lb}$$

$$V_{21} = 9.62 \times (0.08130 - 0.06437) \times 22,146 = 3607 \text{ lb}$$

$$V_{22} = 1.44 \times (-0.0924 - 0.0567) \times 22,146 = -4755\ lb$$

$$V_{2\max} = \sqrt{3607^2 + 4755^2} = 5968 \text{ lb}$$

8.2 Response of a Shear Building to Base Motion

The response of a shear building to the base or foundation motion is conveniently obtained in terms of floor displacements relative to the base motion. For the two-story shear building of Fig. 8.2a, which is modeled as shown in Fig.8.2b, the equations of motion obtained by equating to zero the sum of forces in the free body diagrams of Fig. 8.2c are the following:

$$\begin{aligned} m_1\ddot{u}_1 + k_1(u_1 - u_s) - k_2(u_2 - u_1) &= 0 \\ m_2\ddot{u}_2 + k_2(u_2 - u_1) &= 0 \end{aligned} \tag{8.20}$$

where $u_s = u_s(t)$ is the displacement imposed at the foundation of the structure. Expressing the floor displacements relative to the base motion, we have

$$\begin{aligned} u_{r1} &= u_1 - u_s \\ u_{r2} &= u_2 - u_s \end{aligned} \tag{8.21}$$

The differentiation yields

$$\begin{aligned} \ddot{u}_1 &= \ddot{u}_{r1} + \ddot{u}_s \\ \ddot{u}_2 &= \ddot{u}_{r2} + \ddot{u}_s \end{aligned} \tag{8.22}$$

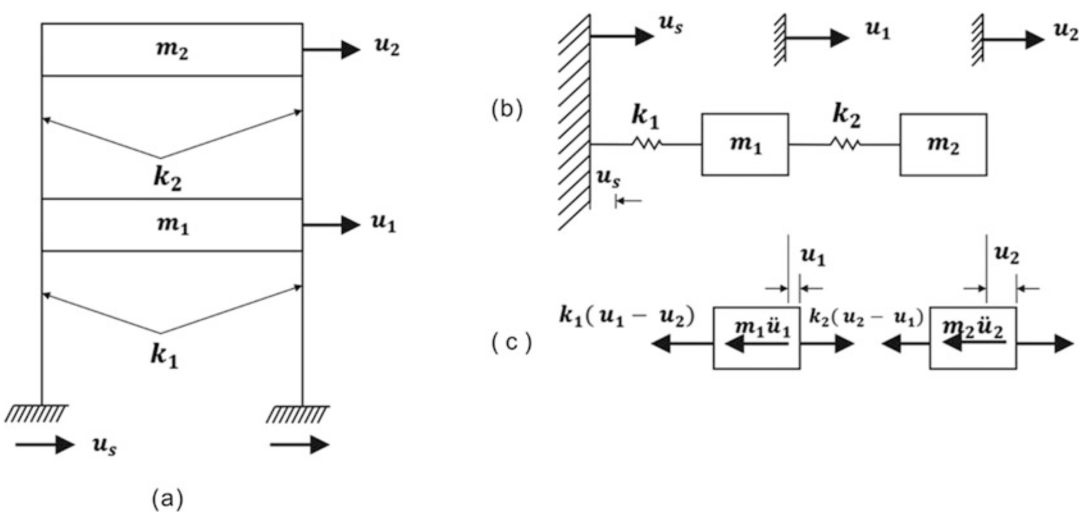

Fig. 8.2 Shear building with base motion. (**a**) Two-story shear building. (**b**) Mathematical model. (**c**) Free-body diagrams

Substitution of Eqs. (8.21) and (8.22) into Eq. (8.20) results in

$$\begin{aligned} m_1\ddot{u}_{r1} + (k_1 + k_2)u_{r1} - k_2 u_{r2} &= -m_1\ddot{u}_s \\ m_2\ddot{u}_{r2} - k_2 u_{r1} + k_2 u_{r2} &= m_2\ddot{u}_s \end{aligned} \tag{8.23}$$

We note that the right-hand sides of Eq. (8.23) are proportional to the same function of time, $\ddot{u}_s(t)$. This fact leads to a somewhat simpler solution compared to the solution of Eq. (8.6), which may contain different functions of time in each equation. For the base motion of the shear building, Eq. (8.4) may be written as

$$\begin{aligned} \ddot{q}_1 + \omega_1^2 q_1 &= -\frac{m_1 a_{11} + m_2 a_{211}}{m_1 a_{11}^2 + m_2 a_{21}^2}\ddot{u}_s(t) \\ \ddot{q}_2 + \omega_2^2 q_2 &= -\frac{m_1 a_{12} + m_2 a_{22}}{m_1 a_{12}^2 + m_2 a_{22}^2}\ddot{u}_s(t) \end{aligned} \tag{8.24}$$

or

$$\begin{aligned} \ddot{q}_1 + \omega_1^2 q_2 &= \Gamma_1 \ddot{u}_s(t) \\ \ddot{q}_2 + \omega_2^2 q_2 &= \Gamma_2 \ddot{u}_s(t) \end{aligned} \tag{8.25}$$

where Γ_1 and Γ_2 are called participation factors and are given by

$$\begin{aligned} \Gamma_1 &= -\frac{m_1 a_{11} + m_2 a_{21}}{m_1 a_{11}^2 + m_2 a_{21}^2} \\ \Gamma_2 &= -\frac{m_1 a_{12} + m_2 a_{22}}{m_1 a_{12}^2 + m_2 a_{22}^2} \end{aligned} \tag{8.26}$$

The relation between the modal displacements q_1, q_2 and the relative displacement u_{r1}, u_{r2} is given from Eq. (8.2) as

$$\begin{aligned} u_{r1} &= a_{11}q_1 + a_{12}q_2 \\ u_{r2} &= a_{21}q_1 + a_{22}q_2 \end{aligned} \tag{8.27}$$

In practice, it is convenient to introduce a change of variables in Eq. (8.25) such that the second members of these equations equal $\ddot{u}_s(t)$. The required change of variables to accomplish this simplification is

$$\begin{aligned} q_1 &= \Gamma_1 g_1 \\ q_2 &= \Gamma_2 g_2 \end{aligned} \tag{8.28}$$

which when introduced into Eq. (8.25) gives

$$\begin{aligned} \ddot{g}_1 + \omega_1^2 g_1 &= \ddot{u}_s(t) \\ \ddot{g}_2 + \omega_2^2 g_2 &= \ddot{u}_s(t) \end{aligned} \tag{8.29}$$

Finally, solving for $g_1(t)$ and $g_2(t)$ in the uncoupled Eq. (8.29) and substituting the solution into Eqs. (8.27) and (8.28) give the response as

$$u_{r1}(t) = \Gamma_1 a_{11} g_1(t) + \Gamma_2 a_{12} g_2(t)$$
$$u_{r2}(t) = \Gamma_1 a_{21} g_1(t) + \Gamma_2 a_{22} g_2(t) \tag{8.30}$$

When the maximum modal response $g_{1\max}$ and $g_{2\max}$ are obtained from spectral charts, we may estimate the maximum values $u_{r1\max}$ and $u_{r2\max}$ by the SRSS combination method as

$$u_{r1\max} = \sqrt{(\Gamma_1 a_{11} g_{1\max})^2 + (\Gamma_2 a_{12} g_{2\max})^2}$$
$$u_{r2\max} = \sqrt{(\Gamma_1 a_{21} g_{1\max})^2 + (\Gamma_2 a_{22} g_{2\max})^2} \tag{8.31}$$

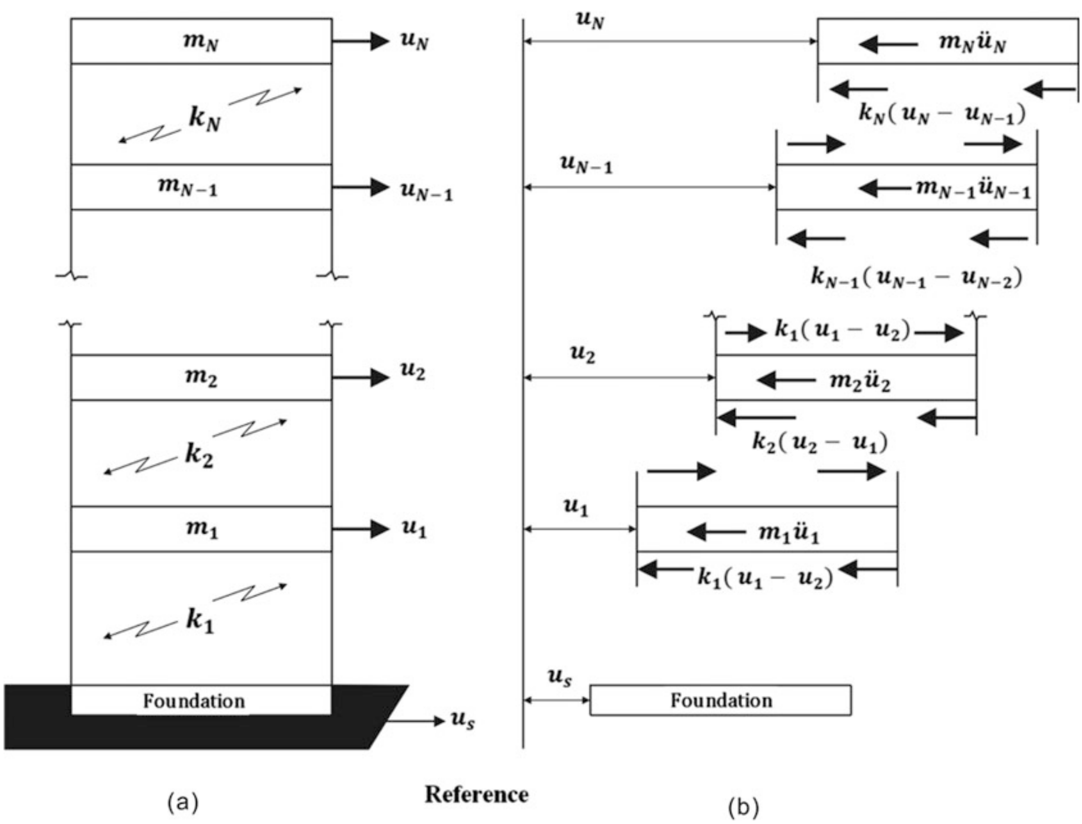

Fig. 8.3 Multistory shear building excited at the foundation (**a**) Structural Model. (**b**) Free body diagrams

The equations of motion for an N-story shear building (Fig. 8.3a) subjected to excitation motion at its base are obtained by equating to zero the sum of forces shown in the free body diagrams of Fig. 8.3b, namely

$$\begin{aligned}
&m_1\ddot{u}_1 + k_1(u_1 - u_s) - k_2(u_2 - u_1) = 0\\
&m_2\ddot{u}_2 + k_2(u_2 - u_1) - k_3(u_3 - u_2) = 0\\
&\qquad \cdots\\
&m_{N-1}\ddot{u}_{N-1} + k_{N-1}(u_{N-1} - u_{N-2}) - k_N(u_N - u_{N-1}) = 0\\
&m_N\ddot{u}_N + k_N(u_N - u_{N-1}) = 0
\end{aligned} \tag{8.32}$$

Introducing into Eq. (8.32)

$$u_{ri} = u_i - u_s \;\; (i = 1, 2, \ldots, N) \tag{8.33}$$

results in

$$\begin{aligned}
&m_1\ddot{u}_{r1} + k_1u_{r1} - k_2(u_{r2} - u_{r1}) = -m_1\ddot{u}_s\\
&m_2\ddot{u}_{r2} + k_2u_{r2} - k_3(u_{r3} - u_{r2}) = -m_2\ddot{u}_s\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&m_{N-1}\ddot{u}_{r\,N-1} + k_{N-1}(u_{rN-1} - u_{rN-2}) - k_N(u_{rN} - u_{rN-1}) = -m_{N-1}\ddot{u}_s\\
&m_N\ddot{u}_{rN} + k_N(u_{rN} - u_{rN-1}) = -m_N\ddot{u}_s
\end{aligned} \tag{8.34}$$

where $\ddot{u}_s = \ddot{u}_s(t)$ is the acceleration function exciting the base of the structure.

Equations (8.34) may be conveniently be written in matrix notation as

$$[M]\{\ddot{u}_r\} + [K]\{u_r\} = -[M]\{1\}\ddot{u}_s(t) \tag{8.35}$$

in which $[M]$, the mass matrix, is a symmetric matrix, $\{1\}$ is a vector with all its elements equal to 1, $\ddot{u}_s = \ddot{u}_s(t)$ is the applied acceleration at the foundation of the building, and $\{u_r\}$ and $\{\ddot{u}_r\}$ are, respectively, the displacement and acceleration vectors relative to the motion of the foundation.

As has been demonstrated, the system of differential equations (8.35) can be uncoupled through the transformation given by Eq. (8.11) as

$$\{u_r\} = [\Phi]\{q\} \tag{8.36}$$

where $[\Phi]$ is the modal matrix obtained in the solution of corresponding eigenproblem $[[K] - \omega^2[M]$ $\{\phi\} = \{0\}$.

The substitution of Eq. (8.36) into Eq. (8.35) followed by premultiplication by the transpose of the ith eigenvector, $\{\phi\}_i^T$ (the ith modal shape), results in

$$\{\phi\}_i^T[M][\Phi]\{\ddot{q}\} + \{\phi\}_i^T[K][\Phi]\{q\} = -\{\phi\}_i^T[M]\{1\}\ddot{u}_s(t) \tag{8.37}$$

which upon introduction of orthogonality property of the normalized eigenvectors [Eqs. (8.14) and (8.15)] results in the modal equations

$$\ddot{z}_i + \omega_1^2 z_i = \Gamma\ddot{u}_s(t) \qquad (i = 1, 2, \ldots, N) \tag{8.38}$$

Where the modal participation factor Γ_i is given in general by

$$\Gamma_i = \frac{\sum_{j=1}^{N} m_j \phi_{ji}}{\sum_{j=1}^{N} m_j \phi_{ji}^2} \tag{8.39}$$

and for normalized eigenvectors by

$$\Gamma_i = \sum_{j=1}^{N} m_j \phi_{ji} \quad (i = 1, 2, \ldots, N) \tag{8.40}$$

The maximum response in terms of maximum values for displacements ($u_{r\ i\ \max}$) or for acceleration ($\ddot{u}_{i\ \max}$) at the modal coordinates calculated by the SRSS method, is then given, respectively, by

$$u_{ri\max} = \sqrt{\sum_{j=1}^{N} \left(\Gamma_j \phi_{ij} S_{Dj}\right)^2} \tag{8.41}$$

and

$$\ddot{u}_{i\max} = \sqrt{\sum_{j=1}^{N} \left(\Gamma_j \phi_{ij} S_{Aj}\right)^2} \tag{8.42}$$

where S_{Dj} and S_{Aj} are, respectively, the spectral displacement and spectral acceleration for the jth mode.

The participation factors Γ_j indicated in Eqs. (8.39) and (8.40) are the coefficients of the excitation function $\ddot{u}_s(t)$ in Eq. (8.38). As presented in Chap. 5, response spectral charts are prepared as the solution of Eq. (8.38) (with $\Gamma_i = 1$). Therefore, the spectral values obtained from these charts S_{Dj} or S_{Aj} should be multiplied as indicated in Eqs. (8.41) and (8.42) by the participation factor T_j, which was omitted in the calculation of spectral values.

Illustrative Example 8.2

Determine the response of the frame of Illustrative Example 8.1 shown in Fig. 8.2 when it is subjected to a suddenly applied constant acceleration $\ddot{u}_s = 0.28$ g at its base.

Solution:

The natural frequencies and corresponding normal modes from calculations in Examples 7.1 and 7.2 are

$$\omega_1 = 11.83 \text{ rad/sec}, \qquad \omega_2 = 32.89 \text{ rad/sec}$$

$$\phi_{11} = 0.06437, \qquad \phi_{12} = 0.0567$$

$$\phi_{21} = 0.08130, \qquad \phi_{22} = -0.0924$$

The acceleration acting at the base of this structure is

$$\ddot{u}_s = 0.28 \times 386 = 108.47 \text{ in/sec}^2$$

The participation factors are calculated from Eq. (8.39) with the denominators set equal to unity since the modes are normalized. These factors are then

$$\begin{aligned} \Gamma_1 &= -(136 \times 0.06437 + 66 \times 0.08130) = -14.120 \\ \Gamma_2 &= -(136 \times 0.0567 - 66 \times 0.0924) = -1.613 \end{aligned} \tag{a}$$

The modal equations (8.29) are

$$\begin{aligned} \ddot{g}_1 + 140 g_1 &= 108.47 \\ \ddot{g}_2 + 1082 g_2 &= 108.47 \end{aligned} \tag{b}$$

and their solution, assuming zero initial conditions for velocity and displacement, is given by Eq. (4.5) as

$$\begin{aligned} g_1(t) &= \frac{108.47}{140}(1 - \cos 11.83t) \\ g_2(t) &= \frac{108.47}{1082}(1 - \cos 32.89t) \end{aligned} \tag{c}$$

The response in terms of the relative motion of the stories at the floor levels with respect to the displacement of the base is given as a function of time by Eqs. (8.27) and (8.28), as

$$\begin{aligned} u_{r1}(t) &= -14.120 \times 0.06437 \times 0.775(1 - \cos\ 11.83t) - 1.613 \times 0.0567 \times 0.100(1 - \cos\ 32.89t) \\ u_{r2}(t) &= -14.120 \times 0.08130 \times 0.775(1 - \cos\ 11.83t) + 1.613 \times 0.0924 \times 0.100(1 - \cos\ 32.89t) \end{aligned}$$

or, upon simplification, as

$$\begin{aligned} u_{r1} &= -0.7135 + 0.704 \cos\ 11.83t + 0.009 \cos\ 32.89t \\ u_{r2} &= -0.874 + 0.900 \cos\ 11.83t - 0.015 \cos\ 32.89t \end{aligned} \tag{d}$$

In this example, due to the simple excitation function (a constant acceleration), it was possible to obtain a closed solution of the problem as a function of time. For a complex excitation function such as the one produced by an actual earthquake, it would be necessary to resort to numerical integration to obtain the response or to use response spectra if available. The maximum modal response is obtained for the present example when the cosine functions in Eq. (c) are set equal to -1. In this case the maximum modal response is then

$$\begin{aligned} g_{1\max} &= 1.55 \\ g_{2\max} &= 0.20 \end{aligned} \tag{e}$$

and the maximum response, calculated from the approximate formulas (8.31), is

$$\begin{aligned} u_{r1\max} &= 1.409 \text{ in} \\ u_{r2\max} &= 1.800 \text{ in} \end{aligned} \tag{f}$$

The possible maximum values for the response calculated from Eq. (d) by setting the cosines function to their maximum value results in

$$
\begin{aligned}
u_{1\max} &= 1.426\,in \\
u_{2\max} &= 1.789\,in
\end{aligned} \tag{g}
$$

which for this particular example certainly compares very well with the approximate results obtained in Eq. (f) above.

8.3 Response by Modal Superposition Using MATLAB

MATLAB calculates the response of a linear system by superposition of the solutions of the modal equations. Before one can use this program, it is necessary to solve an eigenproblem to determine the natural frequencies and modal shapes of the structure. The program determines the response of the structure excited either by time-dependent forces applied at nodal coordinates or a time-dependent acceleration at the support of the structure.

Illustrative Example 8.3

$$\{u\} = [\Phi]\{q\}$$

$$\ddot{q}_i + \omega_i^2 q_i = \{\phi\}_i^T\{F\} = P_i$$

$$\Gamma_i' = \frac{\{\phi\}_i^T\{F\}}{\{\phi\}_i^T[M]\{\phi\}_i} = \{\phi\}_i^T\{F\}\ (i = 1, 2, \ldots N)$$

$m_1 = 136$ #-sec^2/in., $m_2 = 66$ #-sec^2/in., $k_1 = 30{,}700$ #/in., $k_2 = 44{,}300$ #/in.
$\underline{F_2} = \underline{5000\ \#,\ F_1} = \underline{0}$
Use computer MATLAB to determine natural frequencies and normalized mode shapes, and the participation factors.

Solution:
The natural frequencies and the modal matrix for this structure as calculated in Example 10.1 are

$$\omega_1 = 11.83\ \text{rad/sec}$$
$$\omega_2 = 32.89\ \text{rad/sec}$$

and

$$[\Phi] = \begin{bmatrix} 0.0644 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix}$$

The mass matrix is

$$[M] = \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix}$$

```
clc
clear all
close all

%%%%-GIVEN VALUES-%%%%

%%%Define Mass Matrix
M = [136 0; 0 66]

%%%Define Stiffness Matrix
K = [30700+44300 -44300;-44300 44300]

%%%Define Force Matrix
F =[0; 5000];
%____________________________________________________________________
% Solve the eignevalue problem and normalized eigenvectors
%____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(K, M)

[omegas,k] = sort(sqrt(diag(D)));

%%%Natural frequencies
omegas =sqrt(D)

%%%{a}1 before changing the unity in the first DOF.
a1 = a(:,1);

%%%{a}2 before changing the unity in the first DOF.
a2 = a(:,2);

%%%Change the {a} wrt the unity in the first DOF.
a11 = 1;
a21 = a1(2,1)./a1(1,1);
a12 = 1;
a22 = a2(2,1)./a2(1,1);

a =[];

%%%Calculate the {a}
a(:,1) = [a11, a21];                 %[a11,a21]
a(:,2) = [a12, a22];                 %[a12,a22]

%%%aMa = {a}'*[M]*(a}
aMa = a'*M*a;                        %Eq.7.14

%%%Normalization factor
norm_1 = sqrt(aMa(1,1));
norm_2 = sqrt(aMa(2,2));

%%%Normalized eigenvectors
nom_phi(:,1) = 1./norm_1.*a(:,1);   %Eq.7.16 for the first mode
nom_phi(:,2)= 1./norm_2.*a(:,2);    %Eq.7.16 for the 2nd mode
nom_phi


%fMf = {f}'*[M}*{f}
fMf=nom_phi'*M*nom_phi

%Omega: [Omega]
Omega = D'*fMf;
%P
P = nom_phi'*F

%q_st
q_st = inv(Omega)*P
```

8.4 Harmonic Force Excitation

When the excitation, that is, the external forces or base motion, is harmonic (sine or cosine function), the analysis is quite simple and the response can readily be found without the use of modal analysis. Let us consider the two-story shear building as shown in Fig. 8.4 subjected to a single harmonic force $F = F_0 \sin \overline{\omega} t$ which is applied at the level of the second floor. In this case, Eq. (8.1) with $F_1\,(t) = 0$ and $F_2 = F_0 \sin \overline{\omega} t$ become

$$\begin{aligned} m_1 \ddot{u}_1 + (k_1 + k_2) u_1 - k_2 u_2 &= 0 \\ m_2 \ddot{u}_2 - k_2 u_1 + k_2 u_2 &= F_0 \sin \overline{\omega} t \end{aligned} \tag{8.43}$$

For the steady-state response we seek a solution of the form

$$\begin{aligned} u_1 &= U_1 \sin \overline{\omega} t \\ u_2 &= U_2 \sin \overline{\omega} t \end{aligned} \tag{8.44}$$

After substitution of Eq. (8.44) into Eq. (8.43) and cancellation of the common factor $\sin \overline{\omega} t$, we obtain

$$\begin{aligned} \left(k_1 + k_2 - m_1 \overline{\omega}^2\right) U_1 - k_2 U_2 &= 0 \\ -k_2 U_1 + \left(k_2 - m_2 \overline{\omega}^2\right) U_2 &= F_0 \end{aligned} \tag{8.45}$$

which is a system of two equation in two unknowns, U_1, and U_2. This system always has a unique solution except in the case when the determinant formed by the coefficients of the unknowns is equal to zero. The reader should remember that in this case the forced frequency $\overline{\omega}$ would equal one of the natural frequencies, since this determinant when equated to zero is precisely the condition used for determining the natural frequencies. In other words, unless the structure is forced to vibrate at one of the resonant frequencies, the algebraic system of Eq. (8.43) has a unique solution for U_1, and U_2.

Illustrative Example 8.4

Determine the steady-state response of the two-story shear building of Illustrative Example 7.1 when a force $F_2(\mathrm{t}) = 10{,}000 \sin 20t$ is applied to the second story as shown in Fig. 8.4.

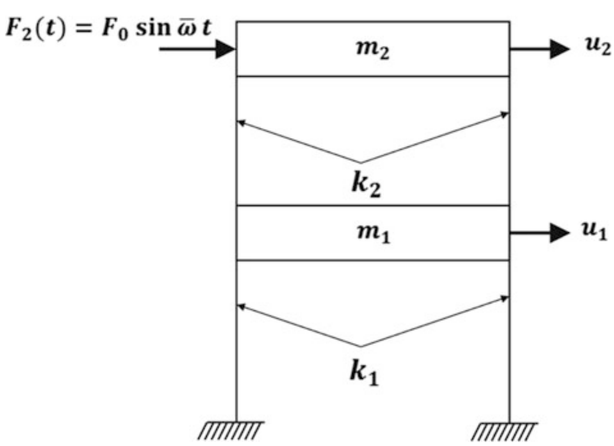

Fig. 8.4 Shear building with harmonic load

Solution:
The natural frequencies for this frame were determined in Illustrative Example 7.1 to be

$$\omega_1 = 11.83 \text{ rad/sec}$$
$$\omega_2 = 32.89 \text{ rad/sec}$$

Since the forcing frequency is 20 rad/sec, the system is not at resonance. The steady-state response is then given by solving Eq. (8.45) for U_1 and U_2. substituting numerical values in this system or equations, we have

$$\left(75,000 - 136 \times 20^2\right)U_1 - 44,300U_2 = 0$$
$$-44,300U_1 + \left(44,300 - 66 \times 20^2\right)U_2 = 10,000$$

Solving these equations simultaneously results in

$$U_1 = -0.28 \text{ in}, \qquad U_2 = -0.13 \text{ in}$$

Therefore, according to Eq. (8.44), the steady-state response is

$$\begin{aligned} u_1 &= -0.28 \sin 20t \text{ in} \\ u_2 &= -0.13 \sin 20t \text{ in} \end{aligned} \qquad \text{(Ans)}$$

Damping may be considered in the analysis by simply including damping elements in the model as it is shown in Fig. 8.5 for a two-story shear building. The equations of motion which are obtained by equating to zero the sum of the forces in the free body diagram shown in Fig. 8.5c are

$$\begin{aligned} m_1\ddot{u}_1 + (c_1 + c_2)\dot{u}_1 + (k_1 + k_2)u_1 - c_2\dot{u}_2 - k_2u_2 &= F_1(t) \\ m_2\ddot{u}_2 - c_2\dot{u}_1 - k_2u_1 + c_2\dot{u}_2 + k_2u_2 &= F_2(t) \end{aligned} \qquad (8.46)$$

Now, considering the general case of applied forces of the form given by

$$F(t) = F_c \cos \overline{\omega}t + F_s \sin \overline{\omega}t \qquad (8.47)$$

we, conveniently, express such force in complex form as

$$F(t) = (F_c - iF_s)e^{i\omega t} \qquad (8.48)$$

with the tacit understanding that only the real part of the Eq. (8.48) is the applied force. We show that the real part of the complex force in Eq. (8.48) is precisely the force in Eq. (8.47). Using Euler's formula $e^{i\overline{\omega}t} = \cos \overline{\omega}t + i \sin \overline{\omega}t$, we obtain

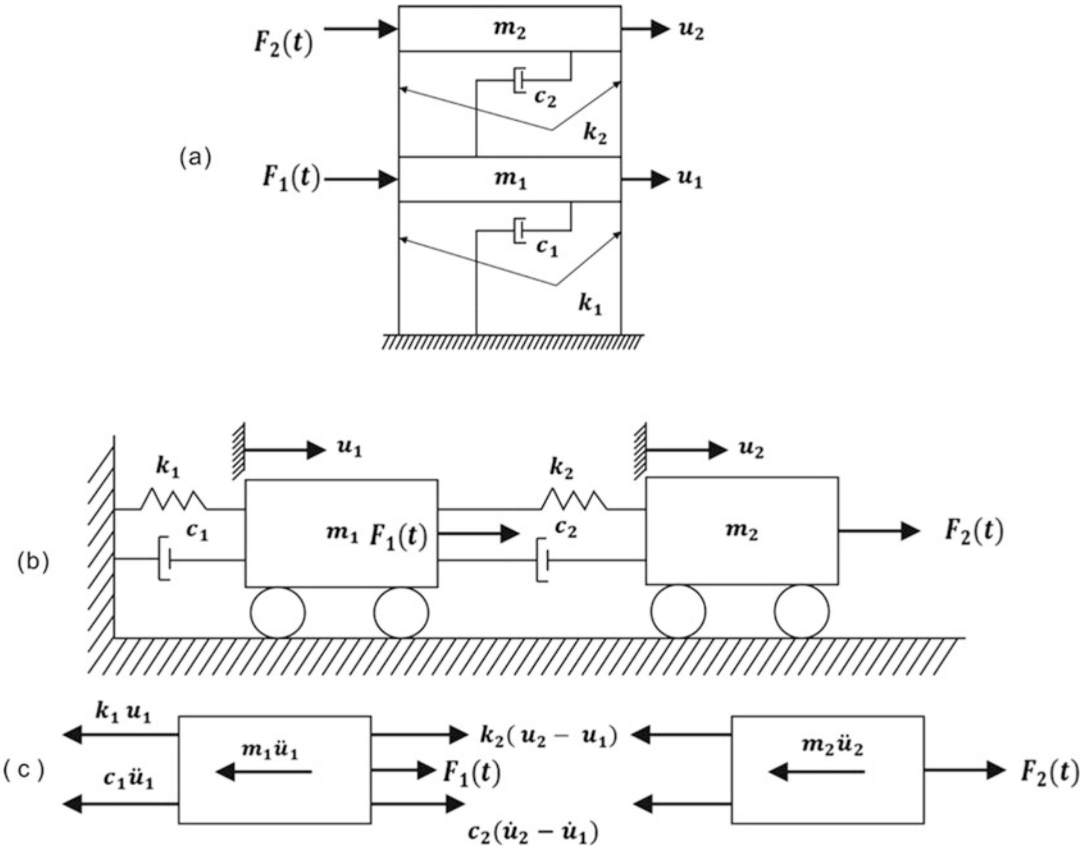

Fig. 8.5 (**a**) Damped shear building with harmonic load. (**b**) Multi-degree mass-spring model. (**c**) Free body diagrams

$$\begin{aligned} \text{Real}\left\{(F_c - iF_s)e^{i\overline{\omega}t}\right\} &= \text{Real}\left\{(F_c - i\text{F}_s)\left(\cos\overline{\omega}t + i\sin\overline{\omega}t\right)\right\} \\ &= F_c\cos\overline{\omega}t + F_s\sin\overline{\omega}t \end{aligned} \tag{8.49}$$

which is equal to the expression in Eq. (8.47).

Assuming that the forces $F_1(t)$ and $F_2(t)$ in Eq. (8.46) are in the form given by Eq. (8.47), we substitute Eq. (8.48) into Eq. (8.46) to obtain

$$\begin{aligned} m_1\ddot{u}_1 + (c_1 + c_2)\dot{u}_1 + (k_1 + k_2)u_1 - c_2\dot{u}_2 - k_2u_2 &= (F_{c1} - iF_{s1})e^{i\overline{\omega}t} \\ m_2\ddot{u}_2 - c_2\dot{u}_1 - k_2u_1 + c_2\dot{u}_2 + k_2u_2 &= (F_{c2} - iF_{s2})e^{i\overline{\omega}t} \end{aligned} \tag{8.50}$$

The solution of the complex system of Eq. (8.50) will, in general, be of the form of

$$\begin{aligned} \text{u}_1(\text{t}) &= (\text{U}_{c1} + \text{iU}_{s1}).e^{i\overline{\omega}t} \\ \text{u}_2(\text{t}) &= (\text{U}_{c2} + \text{iU}_{s2})\,e^{i\overline{\omega}t} \end{aligned} \tag{8.51}$$

The substitution of Eq. (8.51) together with the first and second derivatives of u_1 and u_2 into Eq. (8.50) results in the following system of complex algebraic equations:

$$\begin{aligned}&\left\{\left(k_1+k_2-m\overline{\omega}^2\right)+\mathrm{i}\overline{\omega}\left(c_1+c_2\right)\right\}\left(U_{ci}+iU_{si}\right)-\left\{\left(k_2+\mathrm{i}\overline{\omega}c_2\right)\left(U_{c2}+iU_{s2}\right)=F_{c1}-F_{s1}\right.\\&-\left(k_2+i\overline{\omega}c_2\right)\left(U_{ci}+\mathrm{i}U_{si}\right)+\left\{\left(k_2-m_2\overline{\omega}^2\right)+\mathrm{i}\overline{\omega}c_2\right\}\left(U_{c2}+\mathrm{i}U_{s2}\right)=F_{c2}-F_{s2}\end{aligned} \tag{8.52}$$

As already stated, the response is then found by solving the complex system of equations (11.54) and retaining only the real part of the solution. Hence, analogously to Eq. (8.49),

$$\begin{aligned}u_1(t)&=U_{c1}\cos\overline{\omega}t-U_{s1}\sin\overline{\omega}t\\u_2(\mathrm{t})&=U_{c2}\cos\overline{\omega}t-U_{s2}\sin\overline{\omega}t\end{aligned} \tag{8.53}$$

in which U_{c1}, U_{s2}, U_{c2}, U_{s2}, is the solution of the complex equations (8.52). The necessary calculations are better explained through the use of a numerical example.

Illustrative Example 8.5

Determine the steady state response for the two-story shear building of Illustrative Example 8.4 in which damping is considered in the analysis (Fig. 8.5). Assume for this example that the damping constants c_1 and c_2 are, respectively, proportional to the magnitude of spring constants k_1 and k_2 in which the factor of proportionality, $a_1 = 0.01$.

Solution:
The damping constants are calculated as

$$\begin{aligned}c_1&=a_1k_1=307\ \text{lb. sec}/\text{in}\\c_2&=a_1k_2=443\ \text{lb. sec}/\text{in}\end{aligned} \tag{a}$$

The substitution of numerical values for this example into Eq. (8.54) results in the following system of equations:

$$\begin{aligned}&(20,600+15,000i)U_1-(44,300+8860i)U_2=0\\&-(44,300+8860i)U_1+(17,900+8860i)U_2=-10,000i\end{aligned} \tag{b}$$

The solution of this system of equations is

$$\begin{aligned}\mathrm{U}_1&=0.000681+\mathrm{i}\,0.26865\\\mathrm{U}_2&=-0.06390+\mathrm{i}\,0.13777\end{aligned}$$

The, it follows from Eq. (8.55),

$$\begin{aligned}u_1(t)&=0.0006814\cos 20t-0.26865\sin 20t\\u_2(t)&=-0.0639\cos 20t-0.137775\sin 20t\end{aligned} \tag{8.54}$$

which may also be written as

$$\begin{aligned}u_1&=0.2686\sin\left(20t+3.144\right)\ \text{in}\\u_2&=0.1516\sin\left(20t+3.571\right)\ \text{in}\end{aligned} \tag{Ans}$$

When the results are compared with those obtained for the undamped structure in Illustrative Example 8.4, we note only a small change in the amplitude of motion. This is always the case for systems lightly damped and subjected to harmonic excitation of a frequency that is not close to one of the natural frequencies of the system. For this example, the forced frequency $\overline{\omega}=20$ rad/sec is relatively far from the natural frequencies, $\omega_1 = 11.83$ rad/sec or $\omega_2 = 32.89$ rad/sec which were calculated in Illustrative Example 5.1.

8.5 Harmonic Response: MATLAB Program

MATLAB calculates the response to harmonic excitations of a structural system having the stiffness and mass matrices.

Damping in the system is assumed to be proportional to the stiffness and/or mass coefficients, that is, the damping matrix is calculated as

$$[C] = a_0[M] + a_1[K] \tag{8.55}$$

in which a_0 and a_1 are constants specified in the input data. The program calculates the steady-state response for structures subjected to harmonic forces applied at the nodal coordinates or a harmonic acceleration applied at the base of the structure.

Illustrative Example 8.6

Obtain the response of the damped two-degree-of-freedom shear building of Illustrative Example 8.5 using MATLAB.

Solution:

```
clc
clear all
close all

%______________________________________________________________________
% Inputs:
%   M, K
%   F = forcing function
%   t = Time period
%   u0 = initial displacement
%   v0 = initial velocity
%______________________________________________________________________

%%%%-GIVEN VALUES-%%%%

%%%Time for Response
t = 0:0.01:10;

%%%Define Mass Matrix
M = [136 0; 0 66]

%%%Define Stiffness Matrix
k1=30700;
k2=44300;

K = [k1+k2 -k2;
    -k2 k2];

%%%Determine #s of DOFs
[n,n]= size(M);

%%%Define Force Matrix
F = zeros(n,1); F(2)=10000;

nstep = size(t');

%_____________________________________________________________________
% Initial conditions
%_____________________________________________________________________

u0 = zeros(n,1); u0(1) =0;
v0 = zeros(n,1); v0(1) =0;
[n,n]= size(M);
```

```
%_____________________________________________________________________
% Solve the eignevalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(K, M)

%%%Natural Frequencies
[omegas,k] = sort(sqrt(diag(D)));

%%%Eigenvectors
a = a(:,k)

%%%Natural Periods
T = 2*pi./omegas;

%%%aMa = {a}'*[M]*(a}
aMa = diag(a'*M*a)

%%%Normalized modal matrix
nom_phi = (a)*inv(sqrt(diag(aMa)))

%%%Normalized force, P = nom_F
P = nom_phi'*F;
q0 = nom_phi'*M*u0
dq0 = nom_phi'*M*v0

%_____________________________________________________________________
% Damping matrix using the proportional damping matrix
% [C] = a0[M]+a1[K] (Eq. 8.55)
% zetas = damping ratios
%_____________________________________________________________________
a0 = 0;
a1 = 0.01;
nom_C = nom_phi'*(a0*M+a1*K)*nom_phi;
zetas = diag((1/2)*nom_C*inv(diag(omegas)));

save ('temp1.mat', 'omegas', 'P' ,'zetas');
q = [];
r = [];

%_____________________________________________________________________
% Solve uncoupled equations of motions
%_____________________________________________________________________

%%%Iteration for uncoupled equations of motion (Eq. 8.6)
fori=1:n
q0_i = q0(i,:);
dq0_i = dq0(i,:);

load temp1.mat
omega = omegas(i,:);
P = P(i,:);
m = M(i,i);
zeta = zetas(i,:);

save ('temp2.mat', 'omega', 'P', 'm', 'zeta');

[t,q] = ode45(@MDOFP, t, [q0_i dq0_i]',[]);

r(:,i) = q(:,1);
save ('temp3.mat', 'r')

end

load ('temp3.mat', 'r');

%%%Response using Modal Superposition Method
yim = nom_phi*[r'];                    %Eq.8.11

%%%Response
figure
```

```
subplot(2,1,1);
plot(t, yim(1,:))                    %Response @ 1DOF
title('(a) u_1')
xlabel ('Time (sec)');
ylabel ('u_1(in.)');
grid on

subplot(2,1,2);
plot(t, yim(2,:))                    %Response @ 2DOF
title('(b) u_2')
xlabel ('Time (sec)');
ylabel ('u_2(in.)');
grid on

%%%Maximum response
umax_1=max(abs(yim(1,:)))            %u_max @ 1DOF
umax_2=max(abs(yim(2,:)))            %u_max @ 2DOF
```

This function of MATLAB is similar to the function defined in Chap. 3 (MATLAB function file: SDPF.m). This function is used to solve the partial differential equations for solving uncoupled equations of motions. The function of force is defined here except for participation factors which is found in the main MATLAB program.

```
function  q = MDOFP(t, q)
load ('temp2.mat', 'omega', 'P', 'm' ,'zeta')

%_______________________________________
%illustrative Example 8.4
%_______________________________________
P = P*sin(20*t);

%zeta =0 ;
q = [q(2); -omega*omega*q(1)-2*zeta*omega*q(2)+P];
```

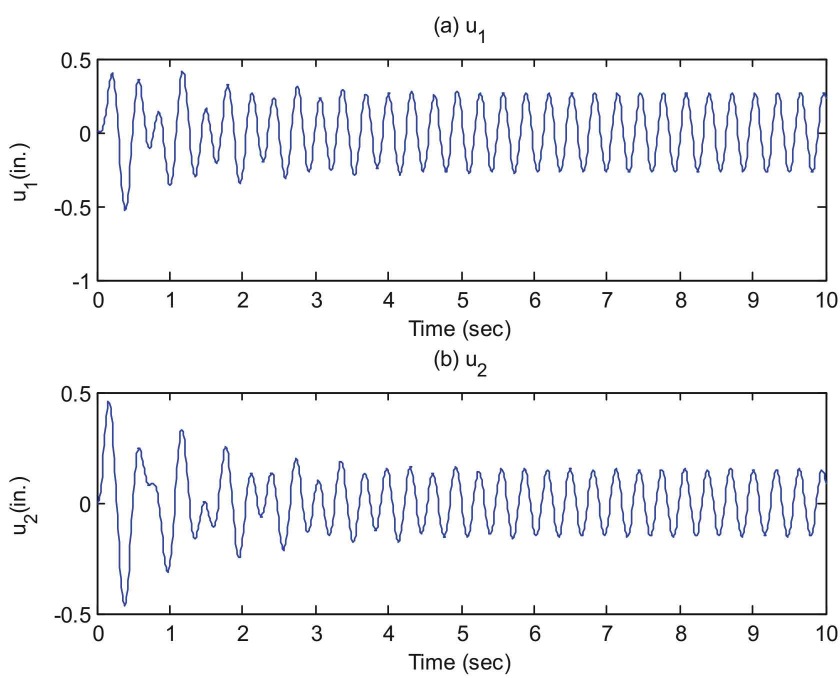

Fig. 8.6 (**a**) Response of 1st degree-of-freedom. (**b**) Response of 2nd degree-of-freedom

The results given by the MATLAB, as expected, are the same as the values calculated in Example 8.6. The steady-state maximum displacement is the same (Fig. 8.6).

8.6 Combining Maximum Values of Modal Response

The square root of the sum of squared contributions (SRSS), to estimate the total response from calculated maximum modal values, may be expressed, in general, from Eq. (11.41) or Eq. (11.42), as

$$R = \sqrt{\sum_{i=1}^{N} R_i^2} \tag{8.56}$$

where R is the estimated response (force, displacement, etc.) at a specified coordinate and R_I is the corresponding maximum response of the ith mode at that coordinate.

Application of the SRSS method for combining modal response generally provides an acceptable estimation of the total maximum response. However, when some of the modes are closely spaced, the use of the SRSS method may result in grossly underestimating or overestimating the maximum response. In particular, large errors have been found in the analysis of three-dimensional structures in which torsional effects are significant. The term "closely spaced" referring to modes, may be arbitrarily define the case when the difference between two natural frequencies is within 10% of the smallest of the two frequencies.

A formulation known as the Complete Quadratic Combinations (CQC), which is based on the theory of random vibrations, has been proposed by Kiureghian (1980) and by Wilson, et al. (1981). The CQC method, which may be considered as an extension of the SRSS method, is given by the following equation:

$$R = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} R_j \rho_{\text{ij}} R_j} \tag{8.57}$$

in which the cross-modal coefficient ρ_{ij}, may be approximated by

$$\rho_{ij} = \frac{8\left(\xi_i \xi_j\right)^{1/2}\left(\xi_i + r\xi_j\right) r^{3/2}}{\left(1 - r^2\right)^2 + 4\xi_i \xi_j r\left(1 - r^2\right) + 4\left(\xi_i^2 + \xi_j^2\right) r^2} \tag{8.58}$$

where r $= \omega_j/\omega_i$ is the ratio of the natural frequencies or order i and j and ξ_i and ξ_j the corresponding damping ratios for modes i and j. For constant modal damping ξ, Eq. (11.58) reduces to

$$\rho_{ij} = \frac{8\xi^2(1 - r)r^{3/2}}{\left(1 - r^2\right)^2 + 4\xi^2 r(1 + r)^2})\tag{8.59}$$

It is important to note that, for i = j, Eq. (11.58) or Eq. (11.59) yields $\rho_{\text{ij}} = 0$ for any value of the damping ratio, including.$\xi = 0$ Thus, for an undamped structure, the CQC method (Eq. 11.57) is identical to the SRSS method (Eq. 11.56).

8.7 Summary

For the solution of linear equations of motion, we may employ either the modal superposition method of dynamic analysis or a step-by-step numerical integration procedure. The modal superposition method is restricted to the analysis of structures governed by linear systems of equations whereas the step-by-step methods of numerical integration are equally applicable to systems with linear or nonlinear behavior. We have deferred the presentation of the multi-degree-of-freedom systems.

In the present chapter, we have introduced the modal superposition method in obtaining the responses of the shear building subjected to either force excitation or to base motion and have demonstrated that the use of normal modes of free vibration for transforming the coordinates leads to a set of uncoupled differential equations. The solution of these equations may then be obtained by any of the methods presented in Part 1 for the single-degree-of-freedom system.

When use is made of response spectra to determine maximum values for modal responses, these values are usually combined by the square root of the sum of squares (SRSS) method. However, the SRSS method could seriously overestimate or underestimate the total response when some of the natural frequencies are closely spaced. A more precise method of combining maximum values of the modal response is the Complete Quadratic Combination (CQC). This method has been strongly recommended in lieu of the SRSS method.

In the particular case of harmonic excitation, the response may be obtained in closed form by simply solving a system of algebraic equations in which the unknowns are the amplitudes of the response at the various coordinates.

8.8 Problems

Problem 8.1

Determine the response as a function of time for the two-story shear building of Problem 7.1 when a constant force of 500 lb is suddenly applied at the level of the second floor as shown in Fig. P8.1. Bays are 15 ft. apart.

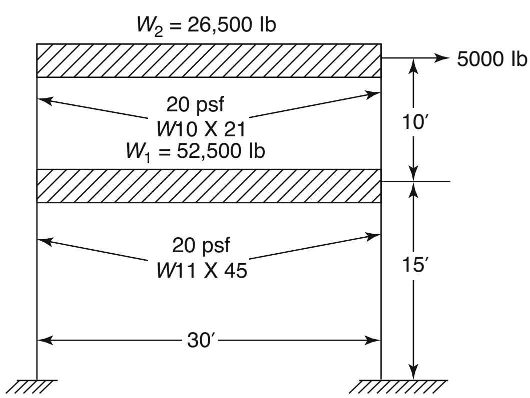

Fig. P8.1

Problem 8.2

Repeat Problem 8.1 if the excitation is applied to the base of the structure in the form of a suddenly applies acceleration of magnitude 0.5 g.

Problem 8.3
Determine the maximum displacement a the floor levels of the three-story shear building (Fig P8.3a) subjected to impulsive triangular loads as shown in Fig. P8.3b. The total stiffness of the columns of each story is $k = 1500$ lb/in and the mass at each floor is $m = 0.386$ lb. sec^2/in.

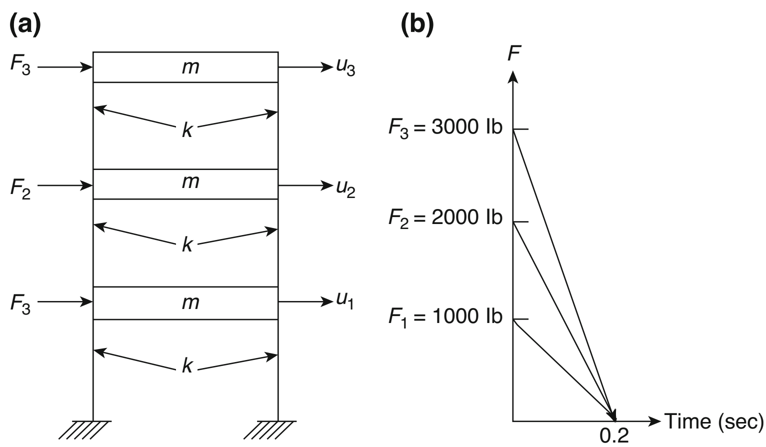

Fig. P8.3

Problem 8.4
Determine the maximum shear force in the columns of the second story of Problem 8.3. (Hint: Calculate modal shear forces and combine contributions using method of square root sum of squares.)

Problem 8.5
Use SAP 2000 to obtain the time history response of the three-story building in Fig. P8.5a subjected to the support acceleration plotted in Fig. P8.5b. Determine the response for a total time of 1.0 sec using time step) t = 0.05 sec, and modal damping coefficient of 10% for all the modes ($E = 30 \times 10^6$ psi).

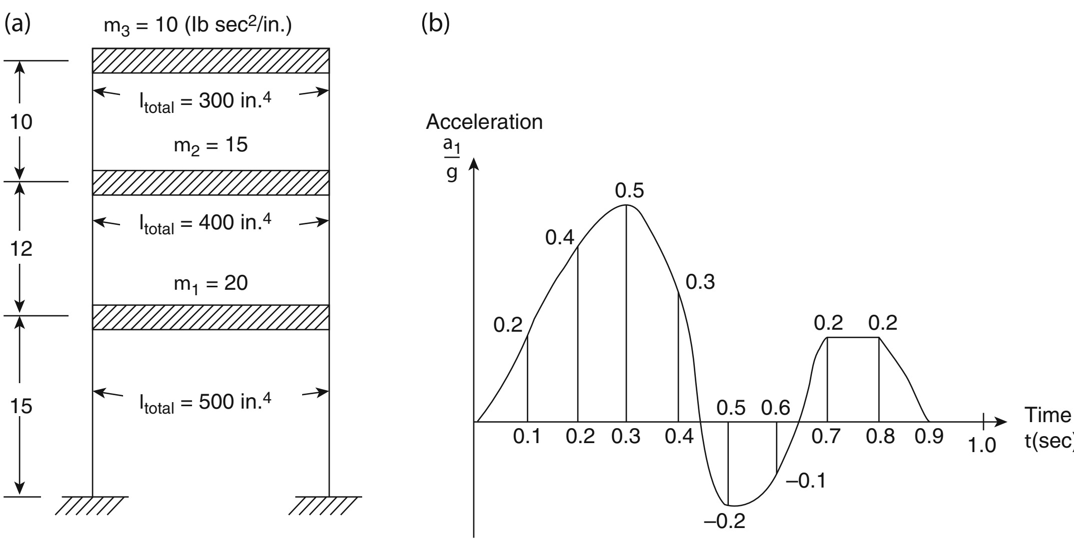

Fig. P8.5

Problem 8.6
Find the steady-state response of the shear building shown in Fig. P8.6 subjected to the harmonic forces indicated in the figure. Neglect damping.

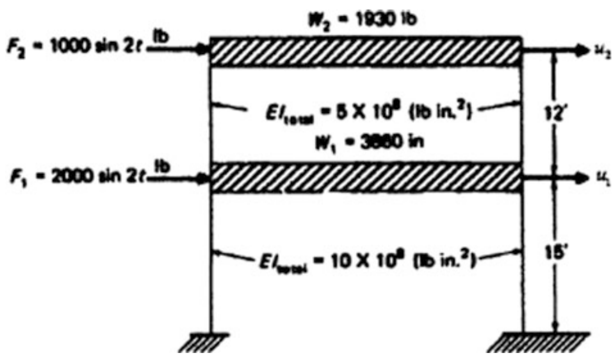

Fig. P8.6

Problem 8.7
Solve Problem 8.6 assuming damping coefficients proportional to the story stiffness, $c_i = 0.05\,K_i$.

Problem 8.8
For the structure (shear building) shown in Fig. P8.8 determine the steady-state motion for the following load systems (loads in pounds):

$$(a)\;\; F_1(\mathrm{t}) = 1000 \sin t, \quad F_2(\mathrm{t}) = 2000 \sin t, F_3(\mathrm{t}) = 1500 \sin t$$

$$(b)\; F_1(\mathrm{t}) = 2000 \cos t, \quad F_2(\mathrm{t}) = 3000 \cos t, F_3(\mathrm{t}) = 4000 \cos t$$

Also load the structure simultaneously with load systems (a) and (b) and verify the superposition of results.

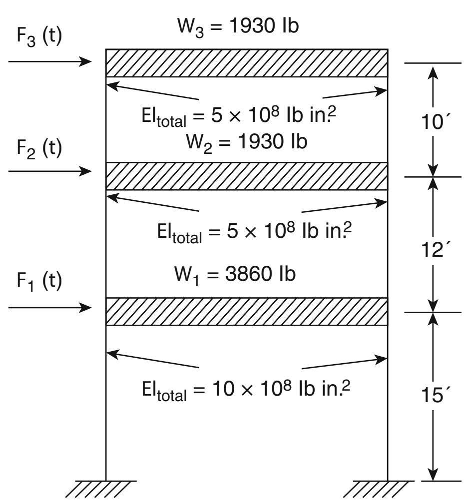

Fig. P8.8

Problem 8.9

For the structure modeled as a four-story shear building shown in Fig. P8.9 determine the steady-state response when it is subjected to a force F = 10,000 sin 20 t (lb) applied at the top floor of the building. The modulus of elasticity is $E = 2.0 \times 10^6$ psi. Assume damping in the system is proportional to the stiffness coefficient ($c_o = 0.01$)

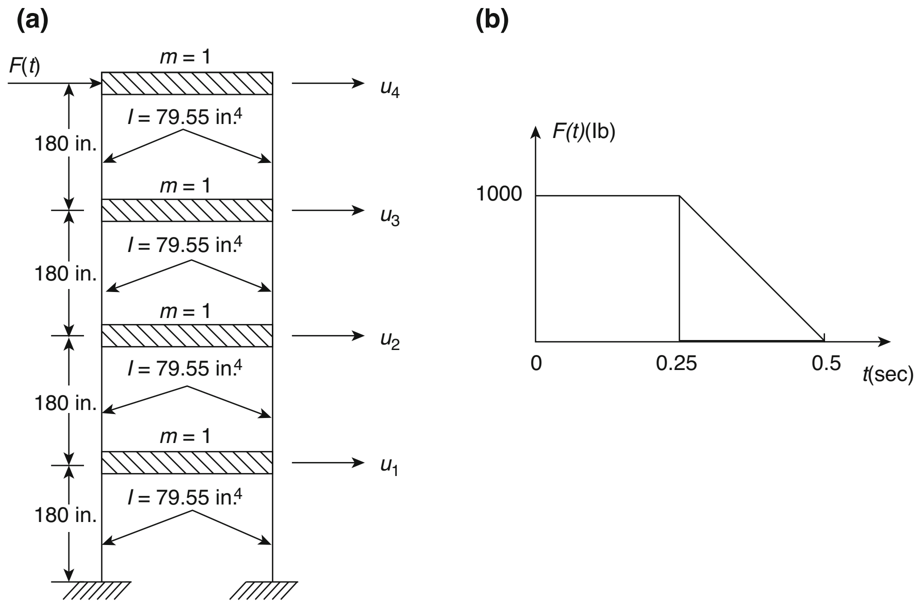

Fig. P8.9