

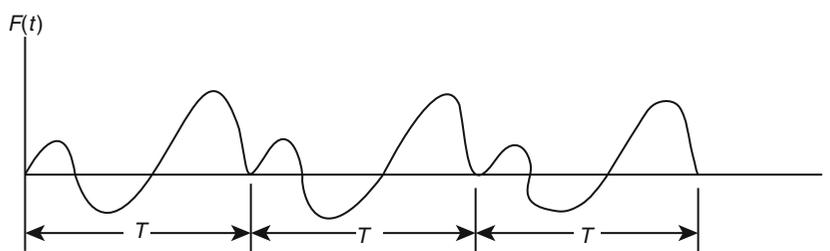
# Fourier Analysis and Response in the Frequency Domain

# 19

This chapter presents the application of Fourier series to determine: (1) the response of a system to periodic forces, and (2) the response of a system to nonperiodic forces in the frequency domain as an alternate approach to the usual analysis in the time domain. In either case, the calculations require the evaluation of integrals that, except for some relatively simple loading functions, employ numerical methods for their computation. Thus, in general, to make practical use of the Fourier method, it is necessary to replace the integrations with finite sums.

## 19.1 Fourier Analysis

The subject of Fourier series and Fourier analysis has extensive ramifications in its application to many fields of science and mathematics. We begin by considering a single-degree-of-freedom system under the action of a periodic loading, that is, a forcing function that repeats itself at equal intervals of time,  $T$  (the period of the function). Fourier has shown that a periodic function may be expressed as the summation of an infinite number of sine and cosine terms. Such a sum is known as a Fourier series.



**Fig. 19.1** Arbitrary periodic function

For a periodic function, such as the one shown in Fig. 19.1, the Fourier series may be written as

$$F(t) = a_0 + a_1 \cos \varpi t + a_2 \cos 2\varpi t + a_3 \cos 3\varpi t + a_4 \cos 4\varpi t + \dots a_n \cos n\varpi t + \dots \\ + b_1 \sin \varpi t + b_2 \sin 2\varpi t + b_3 \sin 3\varpi t + \dots b_n \sin n\varpi t + \dots \quad (19.1)$$

or

$$F(t) = +a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\varpi t + b_n \sin n\varpi t\} \quad (19.2)$$

where  $\varpi = 2\pi/T$  is the frequency and  $T$  the period of the function. The evaluation of the coefficients  $a_0$ ,  $a_n$ , and  $b_n$  for a given function  $F(t)$  is determined from the following expressions:

$$a_0 = \frac{1}{T} \int_{t_1}^{t_1+T} F(t) dt \\ a_n = \frac{2}{T} \int_{t_1}^{t_1+T} F(t) \cos n\varpi t dt \\ b_n = \frac{2}{T} \int_{t_1}^{t_1+T} F(t) \sin n\varpi t dt \quad (19.3)$$

where  $t_1$  in the limits of the integrals may be any value of time, but is usually equal to either  $-T/2$  or zero. The constant  $a_0$  equals the average of the periodic function  $F(t)$ .

## 19.2 Response to a Loading Represented by Fourier Series

The response of a single-degree-of-freedom system to a periodic force represented by its Fourier series is found as the superposition of the response to each component of the series. When the transient is omitted, the response of an undamped system to any term of the series is given by Eq. (3.9) as

$$u_n(t) = \frac{b_n/k}{1 - r_n^2} \sin n\varpi t \quad (19.4)$$

where  $r_n = n\varpi/\omega$  and  $\omega = \sqrt{k/m}$

Similarly, the response to any cosine term is

$$u_n(t) = \frac{a_n/k}{1 - r_n^2} \cos n\varpi t \quad (19.5)$$

The total response of an undamped, single-degree-of-freedom system may then be expressed as the superposition of the responses to all the force terms of the series, including the response  $a_0/k$  (steady-state response) to the constant force  $a_0$ . Hence we have

$$u(t) = \frac{a_0}{k} + \sum \frac{1}{1 - r_n^2} \left( \frac{a_n}{k} \cos n\varpi t + \frac{b_n}{k} \sin n\varpi t \right) \quad (19.6)$$

When the damping in the system is considered, the steady-state response for the general sine term of the series is given from Eq. (3.20) as

$$u_n(t) = \frac{b_n/k \sin(n\omega t - \theta)}{\sqrt{(1 - r_n^2)^2 + (2r_n\xi)^2}} \tag{19.7}$$

or

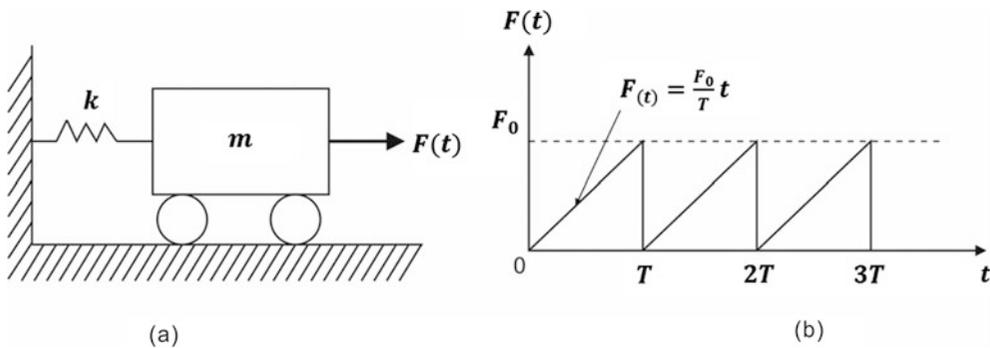
$$u_n(t) = \frac{b_n}{k} \cdot \frac{\sin n\omega t \cos \theta - \cos n\omega t \sin \theta}{\sqrt{(1 - r_n^2)^2 + (2r_n\xi)^2}}$$

The substitution of  $\sin\theta$  and  $\cos\theta$  from Eq. (3.21) gives

$$u_s(t) = \frac{b_n}{k} \frac{(1 - r_n^2) \sin n\omega t + 2r_n\xi \cos n\omega t}{(1 - r_n^2)^2 + (2r_n\xi)^2} \tag{19.8}$$

Similarly, for a cosine term of the series, we obtain

$$u_s(t) = \frac{a_n}{k} \frac{(1 - r_n^2) \sin n\omega t + 2r_n\xi \sin n\omega t}{(1 - r_n^2)^2 + (2r_n\xi)^2} \tag{19.9}$$



**Fig. 19.2** Undamped oscillator acted upon by a periodic force

Finally, the total response is then given by the superposition of the terms expressed by Eqs. (19.8) and (19.9) in addition to the response to the constant term of the series. Therefore, the total response of a damped single-degree-of freedom system may be expressed as

$$u(t) = \frac{a_0}{k} + \frac{1}{k} \sum_{n=1}^{\infty} \left\{ \frac{a_n 2r_n \xi + b_n (1 - r_n^2)}{(1 - r_n^2)^2 + (2r_n \xi)^2} \sin n\omega t + \frac{a_n (1 - r_n^2) - b_n 2r_n \xi}{(1 - r_n^2)^2 + (2r_n \xi)^2} \cos n\omega t \right\} \tag{19.10}$$

**Illustrative Example 19.1**

As an application of the use of Fourier series in determining the response of a system to a periodic loading, consider the undamped simple oscillator in Fig. 19.2a which is acted upon by the periodic force shown in Fig. 19.2b.

Solution:

The first step is to determine the Fourier series expansion of  $F(t)$ . The corresponding coefficients are determined from Eqs. (19.3) as follows:

$$a_0 = \frac{1}{T} \int_0^T \frac{F_0}{T} t \, dt = \frac{F_0}{2}$$

$$a_n = \frac{2}{T} \int_0^T \frac{F_0}{T} t \cos n\omega t \, dt = 0$$

$$b_n = \frac{2}{T} \int_0^T \frac{F_0}{T} t \sin n\omega t \, dt = -\frac{F_0}{n\pi}$$

The response of the undamped system is then given from Eq. (19.6) as

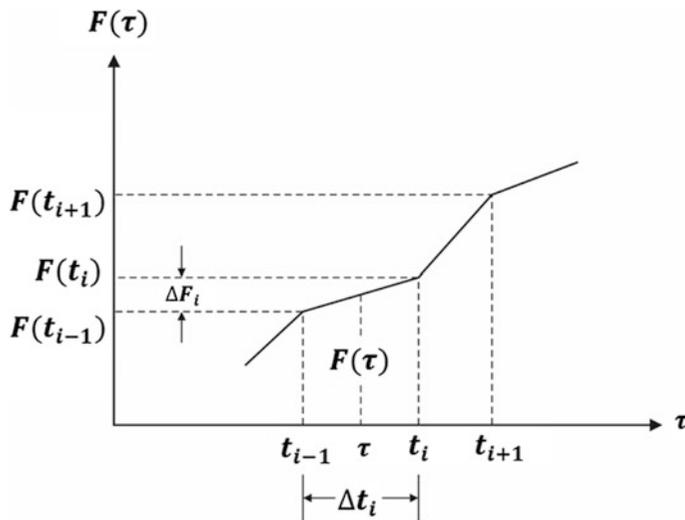
$$U(t) = \frac{F_0}{2k} - \sum_{n=1}^{\infty} \frac{F_0 \sin n\omega t}{n\pi k(1-r_n^2)}$$

or in expanded form as

$$U(t) = \frac{F_0}{2k} - \frac{F_0 \sin \omega t}{\pi k(1-r_1^2)} - \frac{F_0 \sin 2\omega t}{\pi k(1-4r_1^2)} - \frac{F_0 \sin 3\omega t}{\pi k(1-9r_1^2)} - \dots$$

where

$$r_1 = \omega/\omega_n, \omega = \sqrt{k/m}, \text{ and } \omega = 2\pi/T$$



**Fig. 19.3** Piecewise linear forcing function

### 19.3 Fourier Coefficients for Piecewise Linear Functions

Proceeding as before in the evaluation of Duhamel’s integral, we can represent the forcing function by piecewise linear function as shown in Fig. 19.3. The calculation of Fourier coefficients, Eq. (19.3), is then obtained as a summation of the integrals evaluated for each linear segment of the forcing function, that is, as

$$a_0 = \frac{1}{T} \sum_{i=1}^N \int_{t_{i-1}}^{t_i} F(t) dt \quad (19.11)$$

$$a_n = \frac{2}{T} \sum_{i=1}^N \int_{t_{i-1}}^{t_i} F(t) \cos n\omega t dt \quad (19.12)$$

$$b_n = \frac{2}{T} \sum_{i=1}^N \int_{t_{i-1}}^{t_i} F(t) \sin n\omega t dt \quad (19.13)$$

where  $N$  is the number of segments of the piecewise forcing function. The forcing function in any interval  $t_{i-1} \leq t \leq t_i$  is expressed by Eq. (4.20) as

$$F(t) = F(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} (t - t_{i-1}) \quad (19.14)$$

in which  $\Delta F_i = F(t_i) - F(t_{i-1})$  and  $\Delta t_i = t_i - t_{i-1}$ . The integrals required in the expressions of  $a_n$  and  $b_n$  have been evaluated in Eqs. (4.21) and (4.22) and designated as  $A(t_i)$  and  $B(t_i)$  in the recurrent expressions (4.18) and (4.19). The use of Eqs. (4.18) through (4.22) to evaluate the coefficients  $a_n$  and  $b_n$  yields

$$a_n = \frac{2}{T} \sum_{i=1}^N \left\{ \frac{1}{n\omega} \left( F(t_{i-1}) - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) (\sin n\omega t_i - \sin n\omega t_{i-1}) \right. \\ \left. + \frac{\Delta F_i}{n^2 \omega^2 \Delta t_i} ((\cos n\omega t_i - \cos n\omega t_{i-1}) + n\omega (t_i \sin n\omega t_i - t_{i-1} \sin n\omega t_{i-1})) \right\} \quad (19.15)$$

$$b_n = \frac{2}{T} \sum_{i=1}^N \left\{ \frac{1}{n\omega} \left( F(t_{i-1}) - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) (\cos n\omega t_{i-1} - \cos n\omega t_i) \right. \\ \left. + \frac{\Delta F_i}{n^2 \omega^2 \Delta t_i} ((\sin n\omega t_i - \sin n\omega t_{i-1}) - n\omega (t_i \cos n\omega t_i - t_{i-1} \cos n\omega t_{i-1})) \right\} \quad (19.16)$$

The integral appearing in the coefficient  $a_0$  of Eq. (19.3) is readily evaluated after substituting  $F(t)$  from Eq. (19.14) into Eq. (19.11). This evaluation yields

$$a_0 = \frac{1}{T} \sum_{i=1}^N \{ \Delta t_i (F_i + F_{i-1}) / 2 \} \quad (19.17)$$

## 19.4 Exponential Form of Fourier Series

The Fourier series expression given by Eq. (19.2) may also be written in exponential form by substituting the trigonometric functions using Euler's relationships:

$$\sin n\omega t = \frac{e^{in\omega t} - e^{-in\omega t}}{2i}$$

$$\cos n \varpi t = \frac{e^{in\varpi t} + e^{-in\varpi t}}{2} \quad (19.18)$$

The result of this substitution may be written as

$$F(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\varpi t} \quad (19.19)$$

where

$$C_n = \frac{1}{T} \int_0^T F(t) e^{-in\varpi t} dt \quad (19.20)$$

The interval of integration in Eq. (19.20) has been selected from zero to  $T$  for the periodic function. It should be noted that the exponential form for the Fourier series in Eq. (19.19) has the advantage of simplicity when compared to the equivalent trigonometric series, Eq. (19.2). The exponential form of the Fourier series can be used as before to determine the dynamic response of structural systems. However, a more efficient method is available for the determination of the coefficients  $C_n$  as well as for the calculation of the response for the single degree of freedom excited by the force expanded as in Eq. (19.19). This method, which is based on Fourier analysis for the discrete case, is presented in the next sections.

## 19.5 Discrete Fourier Analysis

When the periodic function  $F(t)$  is supplied only at  $N$  equally spaced time intervals ( $\Delta t = T/N$ )  $t_0, t_1, t_2, \dots, t_{N-1}$ , where  $t_j = j \Delta t$ , the integrals in Eq. (19.3) may be replaced approximately by the summations

$$a_n = \frac{1}{T} \sum_{j=0}^{N-1} F(t_j) \cos n \varpi t_j \Delta t \quad (19.21)$$

$$b_n = \frac{1}{T} \sum_{j=0}^{N-1} F(t_j) \sin n \varpi t_j \Delta t, \quad n = 0, 1, 2, \dots$$

where  $\varpi = 2\pi/T$ . The above definitions for the Fourier coefficients have been slightly altered by omitting the factor 2 in the expressions for  $a_n$  and  $b_n$ . In this case Eq. (19.2) is then written as

$$F(t_i) = 2 \sum_{n=1}^{\infty} \{a_n \cos n \varpi t + b_n \sin n \varpi t\} \quad (19.22)$$

If we use complex notation, Eq. (19.21) can be combined into a single form by defining

$$C_n = a_n - i b_n \quad (19.23)$$

and using Euler's relationship

$$e^{-in\varpi t_j} = \cos n \varpi t_j - i \sin n \varpi t_j \quad (19.24)$$

to obtain after substituting Eq. (19.21) into Eq. (19.23)

$$C_n = \frac{1}{T} \sum_{j=0}^{N-1} F(t_j) e^{-in \varpi t_j \Delta t} \quad (19.25)$$

Substituting  $t_j = j \Delta t$ ,  $T = N \Delta t$ , and  $\varpi = 2\pi/T$  into Eq. (19.25), we obtain

$$C_n = \frac{1}{N} \sum_{j=0}^{N-1} F(t_j) e^{-2\pi i(nj/N)}, \quad n = 0, 1, 2, \dots \quad (19.26)$$

Equation (19.26) may be considered as an approximate formula for calculating the complex Fourier coefficients in Eq. (19.20). The discrete coefficients given by Eq. (19.26) do not provide sufficient information to obtain a continuous function for  $F(t)$ ; however, it is a most important fact that it does allow to obtain all the discrete values of the series  $\{F(t_j)\}$  exactly (Newland 1984). This fact leads to the formal definition of the discrete Fourier transform of the series  $\{F(t_j)\}$ ,  $j = 0, 1, 2, \dots, N-1$ , given by

$$C_n = \frac{1}{N} \sum_{j=0}^{N-1} F(t_j) e^{-2\pi i(nj/N)}, \quad n = 0, 1, 2, \dots, (N-1) \quad (19.27)$$

and its inverse discrete Fourier transform by

$$F(t_j) = \sum_{n=0}^{N-1} C_n e^{2\pi i(nj/N)}, \quad j = 0, 1, 2, \dots, (N-1) \quad (19.28)$$

The range of the summation in Eq. (19.28) has been limited from 0 to  $(N-1)$  in order to maintain the symmetry of transform pair Eqs. (19.27) and (19.28). It is important to realize that in the calculation of the summation indicated in Eq. (19.28), the frequencies increase with increasing index  $n$  up to  $n = N/2$ . It will be shown very shortly that, for  $n > N/2$ , the corresponding frequencies are equal to the negative of frequencies of order  $N-n$ . This fact restricts the harmonic components that may be represented in the series to a maximum of  $N/2$ . The frequency corresponding to this maximum order  $\omega_{N/2} = (N/2)\varpi$  is known as the Nyquist frequency or sometimes as the *folding frequency*. Moreover, if there are harmonic components above  $\omega_{N/2}$  in the original function, these higher components will introduce distortions in the lower harmonic components of the series. This phenomenon is called *aliasing* (Newland 1984, p. 118). In view of this fact, it is recommended that the number of intervals or sampled points  $N$  should be at least twice the highest harmonic component present in the function.

The Nyquist frequency  $\omega_u$  is given in radians per second by

$$\omega_u = \frac{2\pi N/2}{T} = \frac{2\pi N/2}{N\Delta t} = \frac{\pi}{\Delta t} \left( \frac{rad}{sec} \right) \quad (19.29)$$

and in cycles per second by

$$f_u = \frac{\omega_u}{2\pi} = \frac{1}{2\Delta t} (cps) \quad (19.30)$$

As a matter of interest, Example 19.4 is presented later in this chapter to illustrate the importance of choosing the number of sampling points  $N$  for the excitation function sufficiently large to avoid spurious results due to aliasing.

Having represented an arbitrary discrete function by a finite sum, we may then also obtain as a discrete function the response of a simple oscillator excited by the harmonic components of the

loading function. Again, only the steady-state response will be considered. The introduction of the unit exponential forcing function  $E_n = e^{i\omega_n t}$  into the equation of motion, Eq. (3.13), leads to

$$m\ddot{u} + c\dot{u} + ku = e^{i\omega_n t} \quad (19.31)$$

which has a steady-state solution of the form

$$u(t) = H(\omega_n)e^{i\omega_n t} \quad (19.32)$$

When Eq. (19.32) is introduced into Eq. (19.31), it is found that the function  $H(\omega_n)$ , which will be designated as the *complex frequency response* function, takes the form

$$H(\omega_n) = \frac{1}{k - m\omega_n^2 + ic\omega_n} \quad (19.33)$$

Upon introducing the frequency ratio

$$r_n = \frac{\omega_n}{\omega}$$

and the damping ratio

$$\xi = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}}$$

Eq. (19.33) becomes

$$H(\omega_n) = \frac{1}{k(1 + r_n^2 + 2ir_n\xi)}$$

Therefore, the response  $y_n(t_j)$  at time  $t_j = j \Delta t$  to a harmonic force component of amplitude  $C_n$  indicated in Eq. (19.28) is given by

$$u_n(t_j) = \frac{C_n e^{2\pi i(nj/N)}}{k(1 - r_n^2 + 2ir_n\xi)} \quad (19.34)$$

and the total response due to the  $N$  harmonic force components by

$$u_n(t_j) = \sum_{n=0}^{N-1} \frac{C_n e^{2\pi i(nj/N)}}{k(1 - r_n^2 + 2ir_n\xi)} \quad (19.35)$$

where  $C_n$  is expressed in discrete form by Eq. (19.27). In the determination of the response  $y(t_j)$  using Eq. (19.35), it is necessary to bear in mind that in Eq. (19.28) the force component of the frequency of order  $n$  is equal to the negative of the component of the frequency of order  $N-n$ . This fact may be verified by substituting  $-(N-n)$  for  $n$  in the exponential factor of Eq. (19.28). In this case we obtain,

$$e^{-2\pi i[(N-n)j/N]} = e^{-2\pi ij} e^{2\pi i(nj/N)} = e^{2\pi i(nj/N)} \quad (19.36)$$

since  $e^{-2\pi ij} = \cos 2\pi j - \sin 2\pi j = 1$  for all integer values of  $j$ . Equation (19.36) together with Eq. (19.28) shows that harmonic components of the force corresponding to frequencies of orders  $n$  and  $-(N-n)$  have the same value. As a consequence of this fact,  $r_n = \omega_n/\omega$ , where  $\omega = \sqrt{k/m}$  should be evaluated (selecting  $N$  as an even number) as

$$\omega_n = n\varpi \quad \text{for} \quad n \leq N/2$$

and

$$\omega_n = -(N - n)\varpi \quad \text{for} \quad n > N/2$$

where the frequency corresponding to  $n = N/2$ , as already mentioned, is the highest frequency that can be considered in the discrete Fourier series.

The evaluation of the sums necessary to determine the response using the discrete Fourier transform is greatly simplified by the fact that the exponential functions involved are harmonic and extend over a range of  $N^2$  as demonstrated in the next section.

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## 19.6 Fast Fourier Transform

A numerical technique is available that is efficient for computer determination of the response in the frequency domain. This method is known as the fast *Fourier transform* (FFT) (Cooley et al. 1965). The corresponding computer program is reproduced as a subroutine of computer Program 4. The response in frequency domain of a single-degree-of-freedom system to a general force is given by Eq. (19.35) and the coefficients required are computed from Eq. (19.27). It can be seen that either Eq. (19.35) or Eq. (19.27) may be represented, except for sign in the exponent by the exponential function as

$$A(j) = \sum_{n=0}^{N-1} A^{(0)}(n) W_N^{jn} \quad (19.37)$$

where

$$W_n = e^{2\pi i/N} \quad (19.38)$$

The evaluation of the sum in Eq. (19.37) will be most efficient if the number of time increments  $N$  into which the period  $T$  is divided is a power of 2, that is,

$$N = 2^M \quad (19.39)$$

where  $M$  is an integer. In this case, the integers  $j$  and  $n$  can be expressed in binary form. For the purpose of illustration, we will consider a very simple case where the load period is divided into only eight time increments, that is,  $N = 8$ ,  $M = 3$ . In this case, the indices in Eqs. (19.27) and (19.35) will have the binary representation

$$\begin{aligned} j &= j_0 + 2j_1 + 4j_2 \\ n &= n_0 + 2n_1 + 4n_2 \end{aligned} \quad (19.40)$$

and Eq. (19.37) may be written as

$$A(j) = \sum_{n_2=0}^1 \sum_{n_1=0}^1 \sum_{n_0=0}^1 A^{(0)}(n) W_8^{(j_0+2j_1+4j_2)(n_0+2n_1+4n_2)} \quad (19.41)$$

The exponential factor can be written as

$$W_8^{jn} = W_8^{8(j_1n_2+2j_2n+j_2n_1)} W_8^{4n_2j_0} W_8^{2n_1(2j_1+j_0)} W_8^{n_0(4j_2+2j_1+j_0)}$$

We note that the first factor on the right-hand side is unity since from Eq. (19.38)

$$W_8^{8I} = e^{2\pi i(8/8)I} = \cos 2\pi i + \sin 2\pi i = 1$$

where  $I = j_1n_2 + 2j_2n_2 + j_2n_1$  is an integer. Therefore, only the remaining three factors need to be considered in the summations. These summations may be performed conveniently in sequence by introducing a new notation to indicate the successive steps in the summation process. Thus the first step can be indicated by

$$A^{(1)}(j_0, n_1, n_0) = \sum_{n_2=0}^1 A^{(0)}(n_2, n_1, n_0) W_8^{4n_2j_0}$$

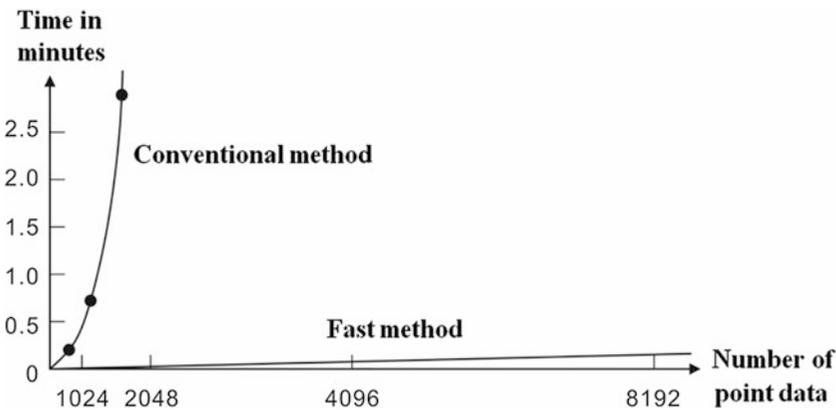
where  $A^{(0)}(n_2, n_1, n_0) = A^{(0)}(n)$  in Eq. (19.37). Similarly, the second step is

$$A^{(2)}(j_0, j_1, n_0) = \sum_{n_1=0}^1 A^{(1)}(j_2, n_1, n_0) W_8^{2n_1(2j_1+j_0)}$$

and the third step (final step for  $M = 3$ ) is

$$A^{(3)}(j_0, j_1, j_2) = \sum_{n_0=0}^1 A^{(2)}(j_0, j_1, n_0) W_8^{n_0(4j_2+2j_1+j_0)}$$

The final result  $A^{(3)}(j_0, j_1, j_2)$  is equal to  $A(j)$  in Eq. (19.37) or (19.41). This process, indicated for  $N = 8$ , can readily be extended to any integer  $N = 2M$ . The method is particularly efficient because the results of one step are immediately used in the next step, thus reducing storage requirements and also because the exponential takes the value of unity in the first factor of the summation. The reduction in computational time that results from this formulation is significant when the time interval is divided into a large number of increments. The comparative times required for computing the Fourier series by a conventional program and by the fast Fourier transform algorithm are illustrated in Fig. 19.4. It is seen here how, for large values of  $N$ , one can rapidly consume so much computer time as to make the conventional method unfeasible.



**Fig. 19.4** Time required for Fourier transform using conventional and fast method. (From Cooley, J. W., Lewis, P. A. W., and Welch, P. D. (1969), IEEE Trans. Education, E-12 (1))

### 19.7 Response in the Frequency Domain Using MATLAB

The MATLAB program presented in this chapter calculates the response in the frequency domain for a damped single-degree-of-freedom system. The excitation is input as a discrete function of time. The program output prints the displacement history of the steady-state motion of the response. The main body of this program performs the tasks of calculating, using the FFT algorithm, the coefficients  $C_n$  in Eq. (19.27), and the function  $F(t_j)$  in Eq. (19.28), and the response  $u(t_j)$  in Eq. (19.35).

#### Illustrative Example 19.2

Determine the response of the tower shown in Fig. 19.5a subjected to the impulsive load of duration 0.64 sec as shown in Fig. 19.5b.

Assume damping equal to 10% of the critical damping.

Solution:

Problem Data:

Mass:	$m = 38,600/386 = 100 \text{ (lb.sec}^2/\text{in.)}$
Spring constant:	$k = 100,000 \text{ (lb/in.)}$
Damping coefficient:	$c = 2 \xi \sqrt{km} = 632 \text{ (lb. sec /in.)}$

Select  $M$  such that  $2^M = 8$ ;  $M = 3$

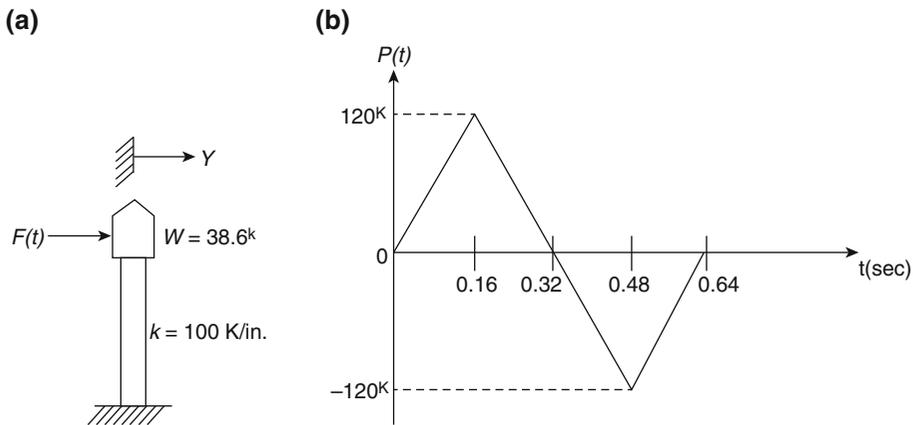


Fig. 19.5 Idealized structure and loading for Illustrative Example 19.2

Excitation function:

Time (sec)	Force (lb)
0.00	0
0.16	120,000
0.48	120,000
0.64	0

The MATLAB file to calculate total response is presented below (Fig. 19.6).

```

close all
clear all
clc

%%%GIVEN VALUES-%%%
m=100;           %Mass (lb.sec^2/in.)
k =100000;      %Stiffness (lb/in.)
xi =0.1;        %Damping ratio
omega = sqrt(k/m); %Natural frequency
c=2*m*omega*xi; %Damping coefficient. (lb.sec/in.)

T = 0.64;       %Time period, T(sec)
omega_bar = 2*pi/T; %Excitation frequency (rad/sec)
M= 3;          %Select M, M=3
N = 2^M;       %The number of time increments N

t=0:0.08:0.64; %Time ranging from 0 to 0.64 sec with deltat = 0.08 sec
Dt = t(2)-t(1); %Deltat = 0.08
tt= length(t); %Total number of calculation

for i= 1:tt-1
%%%Define the function of N harmonic force
    if t(i)<=0.16
        F(i) = 120000*t(i)/0.16;
    elseif t(i) <=0.48
        F(i) =-750000*(t(i)-0.16)+120000;
    else
        F(i)=min(0, 750000*(t(i)-0.64)) ;
    end

    %Define the discrete Fourier transform of the series
    Cn=fft(F/N); %Eq.19.27

    %Calculate frequency ratio, r_n
    if i<=N/2
        omega_n(i) = (i-1)*omega_bar;
    else
        omega_n(i) = -(N-(i-1))*omega_bar;
    end

    rn(i)=omega_n(i)/omega; %Frequency ratio, r_n

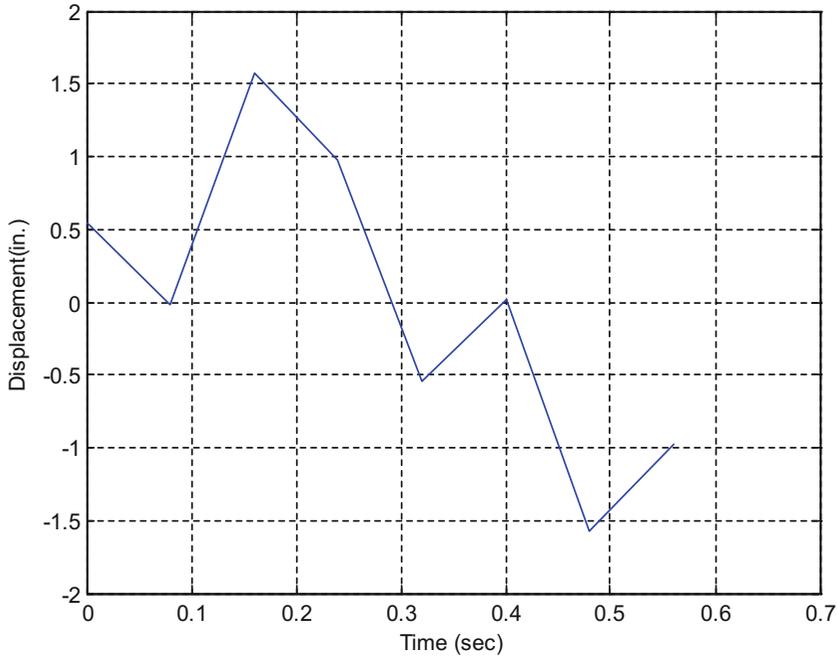
end

%%%Total response due to the N harmonic force
uu=Cn./(k*(1-rn.^2+2*xi.*sqrt(-1)*rn)); %Eq.19.35

%%%FFT Discrete Fourier transform (Built-in-MATLAB function)
u=fft(uu')

%%%Reponse
figure(1)
t = 0:Dt:(0.64-Dt);
plot(t',real(u))
xlabel ('Time (sec)'); ylabel ('Displacement(in.)'); grid on

```



**Fig. 19.6** Response of Illustrative Example 19.2

**Illustrative Example 19.3**

Determine the response of the simple oscillator shown in Fig. 19.7a when subjected to the forcing function depicted in Fig. 19.7b. Use  $M = 4$  for the exponent in  $N = 2M$ . Assume 15% of the critical damping.

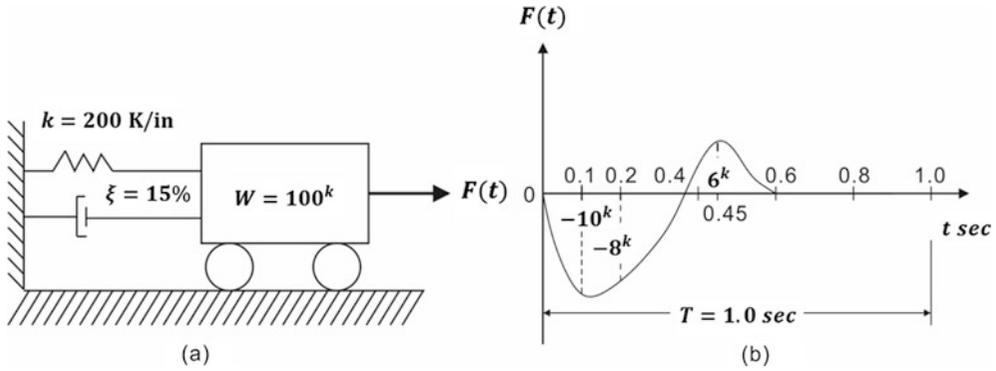
Solution:

Problem Data:

Mass:	$m = 100/386 = 0.259$ (Kip. sec <sup>2</sup> /in)
Spring constant:	$k = 200$ Kip/in.
Damping coefficient:	$c = 2\xi\sqrt{km}$
	$c = 2 \times 0.15\sqrt{200 \times 0.259} = 2.159$ (Kip. sec /in)
Exponent of $N = 2^M$	$M = 4$
Gravitational index	$G = 0$ (force on the mass)

Excitation function:

Time (sec)	Force (Kip)
0.00	0
0.10	-10
0.20	-8
0.40	0
0.45	6
0.60	0
1.00	0



**Fig. 19.7** Simple oscillator and loading for Illustrative Example 19.3

The MATLAB file to calculate total response is presented below (Fig. 19.8).

```

close all
clear all
clc

%%%-GIVEN VALUES-%%
m=0.259;           %Mass (lb.sec^2/in.)
k =200;           %Stiffness (lb/in.)
xi =0.15;         %Damping ratio
omega = sqrt(k/m); %Natural frequency
c=2*m*omega*xi;   %Damping coefficient. (lb.sec/in.)

T = 1;           %Time period, T(sec)
omega_bar = 2*pi/T; %Excitation frequency (rad/sec)
M = 4;           %Select M, M=4
N = 2^M;         %The number of time increments N

Dt = T/N;        %Deltat
t=0:Dt:T;        %Time ranging from 0 to T sec with deltat
tt= length(t);   %Total number of calculation
F = zeros(1,N);  %Setting up N harmonic force

for i= 1:tt-1
    %%Define the function of N harmonic force
    F(2)=-6.25; F(3)=-9.5; F(4)=-8.25; F(5)=-6;
    F(6)=-3.5; F(7)=-1; F(8)=4.5; F(9)=4.0;
    F(10)=1.5;

    %%Define the discrete Fourier transform of the series
    Cn=fft(F/N); %Eq.19.27

    %%Calculate frequency ratio, r_n
    if i<=N/2
        omega_n(i) = (i-1)*omega_bar;
    else
        omega_n(i) = -(N-(i-1))*omega_bar;
    end

    rn(i)=omega_n(i)/omega; %Frequency ratio, r_n
end
end

```

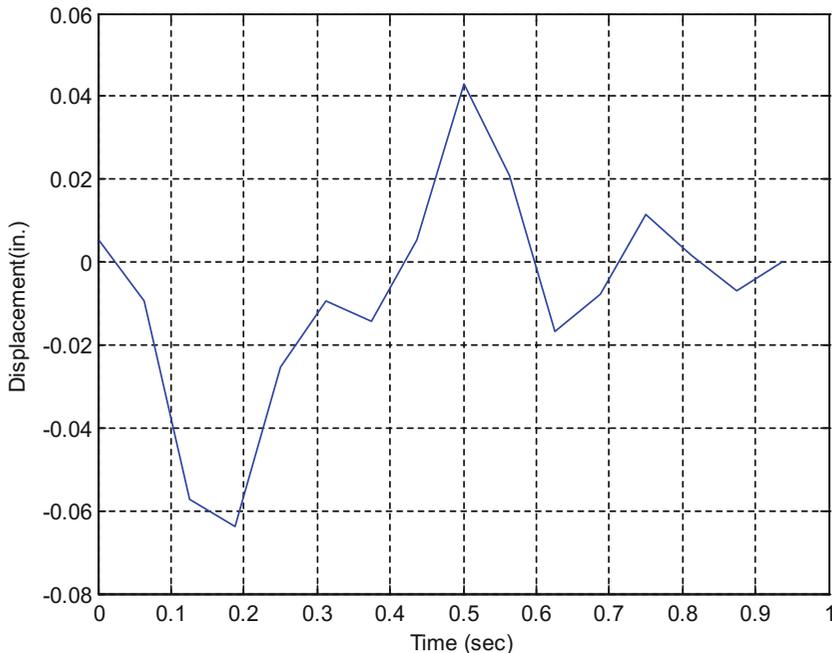
```

%%Total response due to the N harmonic force
uu=Cn./(k*(1-rn.^2+2*xi.*sqrt(-1)*rn))      %Eq.19.35

%%FFT Discrete Fourier transform (Built-in-MATLAB function)
u=fft(uu')

%%Response
figure(1)
t = 0:Dt:(1-Dt);
plot(t',real(u))
xlabel('Time (sec)'); ylabel('Displacement(in.)'); grid on

```



**Fig. 19.8** Response of Illustrative Example 19.3

#### Illustrative Example 19.4

Consider a single-degree-of-freedom undamped system in which  $k = 200$  lb/in,  $m = 100$  lb.sec<sup>2</sup>/in subjected to a force expressed as

$$P(t) = \sum_{n=1}^{16} 100 \cos 2\pi n t \quad (\text{a})$$

Determine the steady-state response of the system using MATLAB with  $M = 3, 4, 5,$  and  $6$  corresponding to  $N = 8, 16, 32,$  and  $64$  sampled points. Then discuss the results in relation to the limitations imposed by the Nyquist frequency.

**Solution:**

The fundamental frequency of the excitation function, Eq. (a), is  $\omega_1 = 2\pi$  and its period  $T = \omega_1 / 2\pi = 1$  sec. Since the highest component in Eq. (a) is of order  $\omega_{16} = 16\omega_1$ , to avoid aliasing, the number of sampled points should be at least twice that order, that is, the minimum number of sampled points should be  $N = 32$ .

The applied force is calculated in the MATLAB program. The results given by the MATLAB for this example are conveniently arranged in two tables: Table 19.1, giving the displacement response to

the excitation having all 16 harmonic components as prescribed for this problem; and Table 19.2, showing the displacement response to a reduced number of harmonic terms in the excitation function.

For this example, in which the exciting force is supplied in 16 harmonic components, the response given in Table 19.1 corresponding to  $N = 32$  or  $N = 64$  may be considered the exact solution. A comparison of the response shown for sample points  $N = 8$  or  $N = 16$  with the exact solution ( $N = 32$ ) dramatically demonstrates the risk of not choosing  $N$  sufficiently large enough so that none of the frequencies of the components in the exciting force exceed the Nyquist frequency. The response obtained for  $N = 8$  or  $N = 16$  gives spurious numerical results.

As the demonstration purpose, this MATLAB program is presented for the response for the  $N$  value of 8 and 16 harmonic force. To produce the Table 19.1, you need to update the line for the  $M = 3$  and the iteration from  $j = 1:16$ .

```
clear all
close all
clc

%%%-GIVEN VALUES-%%
m=100; %Mass (lb.sec^2/in.)
k=200; %Stiffness (lb/in.)
omega = sqrt(k/m); %Natural frequency
c=0; %Damping coefficient. (lb.sec/in.)
xi =c/(2*m*omega); %Damping ratio

T = 1.0; %Time period, T(sec)
omega_bar = 2*pi/T; %Excitation frequency (rad/sec)
M= 3; %Select M, M=3
N = 2^M; %The number of time increments N

Dt=T/N; %Deltat
t=0:Dt:1.0; %Time ranging from 0 to T sec with deltat
tt= length(t); %Total number of calculation

for i= 1:tt-1
    %%Define the function of 16 harmonic force
    for j=1:16
        F3(j,i)=100*cos((2*pi()*j)*t(i));
    end

    F3s= sum(F3);

    %%Calculate frequency ratio, r_n
    if i<=N/2
        omega_n(i) = (i-1)*omega_bar;
    else
        omega_n(i) = -(N-(i-1))*omega_bar;
    end

    rn(i)=omega_n(i)/omega; %Frequency ratio, r_n
end

%%Define the discrete Fourier transform of the series
Cn3 =fft(F3s/N);

%%Total response due to the N harmonic force
uu=Cn3./(k*(1-rn.^2+2*xi.*sqrt(-1)*rn)); %Eq.19.35

%%FFT Discrete Fourier transform (Built-in-MATLAB function)
u=fft(uu')

%%Reponse
figure(1)
t = 0:Dt:(1-Dt);
plot(t',real(u))
xlabel ('Time (sec)'); ylabel ('Displacement(in.)'); grid on
```

**Table 19.1** Displacement Response for Example 19.4 (Excitation Having 16 Harmonics)

Time(sec)	Number of Sampling Points for the Excitation			
	$N = 8$	$N = 16$	$N = 32$	$N = 64$
0	0.8531	0.4201	-0.0416	-0.0416
0.125	0.9357	0.4698	-0.0153	-0.0153
0.250	1.022	0.5107	0.0052	0.0052
0.375	1.071	0.5358	0.0178	0.0178
0.500	1.089	0.5443	0.0221	0.0221
0.625	1.071	0.5358	0.0178	0.0178
0.750	1.022	0.5107	0.0052	0.0052
0.875	0.9357	0.4698	-0.0153	-0.0153
1.000	0.8531	0.4201	-0.0416	-0.0416

Results in Table 19.2, which were obtained using  $N = 8$  sampled points, also verify that when the exciting force contains harmonic components higher than the Nyquist frequency which corresponds, in this case, to  $N_y = 4$ , the results are again spurious.

A final comment is in order. The example presented, having equal amplitude for all the components of the exciting force, serves to emphasize the importance of choosing the number of sampling points  $N$  sufficiently large to avoid aliasing. In practical situations normally the higher harmonics have much smaller amplitude than that of the fundamental or lower frequencies. Consequently, the distortion in the response might not be as dramatic as shown in Tables 19.1 and 19.2.

**Table 19.2** Displacement Response for Example 19.4 (Excitation Force Sampled at  $N = 8$  Points)

Time(sec)	Number of Sampling Points for the Excitation		
	$N = 16$	$N = 8$	$N = 4$
0	-0.0375	0.4246	0.8531
0.125	-0.0153	0.4679	0.9357
0.250	0.0048	0.5112	1.0220
0.375	0.0184	0.5353	1.0710
0.500	0.0215	0.5446	1.0890
0.625	0.0184	0.5353	1.0710
0.750	0.0048	0.5112	1.0220
0.875	-0.0153	0.4679	0.9357
1.000	-0.0375	0.4246	0.8531

## 19.8 Summary

In general, any periodic function may be expanded into a Fourier series, Eq. (19.1), whose terms are sine and cosine functions of successive multiples of the fundamental frequency. The coefficients of these functions may be calculated by integrating over a period the product of the periodic function multiplied by a sine or cosine function, Eq. (19.3). The response of the dynamic system is then obtained as the superposition of the response for each term of the Fourier series expansion of the excitation function. The extension of the Fourier series to non-periodic functions results in integrals

which are known as Fourier transforms. The discrete form of these transforms, Eqs. (19.27) and (19.28), permits their use in numerical applications. An extremely efficient algorithm known as the Fast Fourier Transform (FFT) can save as much as 99% of the computer time otherwise consumed in the evaluation of Fourier complex coefficients for the excitation function and for the response of a dynamic system.

## 19.9 Problems

### Problem 19.1

Determine the first three terms of the Fourier series expansion for the time varying force shown in Fig. P19.1.

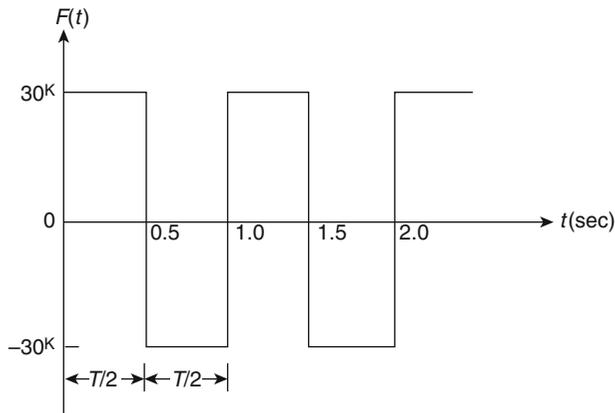


Fig. P19.1

### Problem 19.2

Determine the steady-state response for the damped spring-mass system shown in Fig. P19.2 that is acted upon by the forcing function of Problem 19.1.

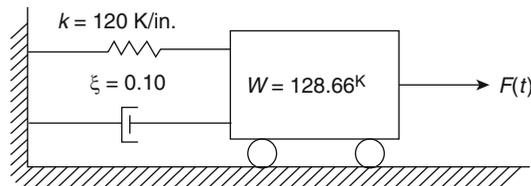


Fig. P19.2

### Problem 19.3

The spring-mass system of Fig. P19.2 is acted upon by the time-varying force shown in Fig. P19.3. Assume that the force is periodic of period  $T = 1$  sec. Determine the steady-state response of the system by applying Fourier series expansion of  $F(t)$ .

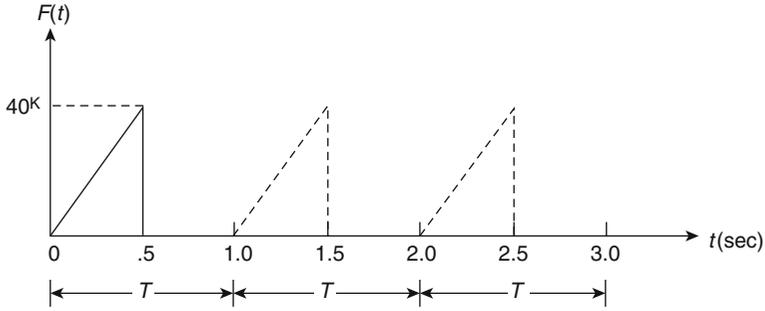


Fig. P19.3

**Problem 19.4**

The cantilever beam shown in Fig. P19.4a carries a concentrated weight at its free end and it is subjected to a periodic acceleration at its support which is the rectified sine function of period  $T = 0.4$  sec and amplitude  $\ddot{u}_0 = 180$  in/sec<sup>2</sup> as shown in Fig. P19.4b. Determine: (a) the Fourier series expansion of the forcing function and (b) the steady-state response considering only three terms of the series. Neglect damping in the system and assume the beam massless.

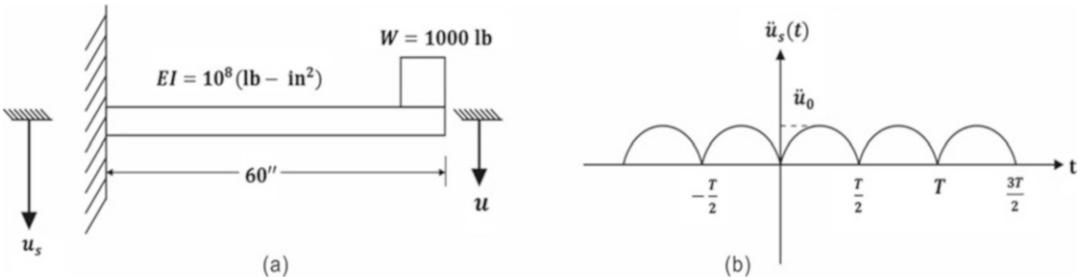


Fig. P19.4

**Problem 19.5**

Solve Problem 19.4 using Program 4. Take 16 Fourier terms. Input the values of the excitation functions at intervals of 0.025 sec.

**Problem 19.6**

Solve Problem 19.4 in the frequency domain using Program 4. Take the exponent of  $N = 2^M$ ,  $M = 4$ . Input the effective force,  $F_{eff} = -m\ddot{u}_s(t)$  calculated for every 0.025 sec.

**Problem 19.7**

Repeat Problem 19.6 assuming 20% of critical damping.

**Problem 19.8**

The forcing function shown in Fig. P19.8a is assumed to be periodic in the extended interval  $T = 1.4$  sec. Use Program 4 to determine the first eight Fourier coefficients and the steady-state response of a structure modeled by the undamped oscillator shown in Fig. P19.8b.

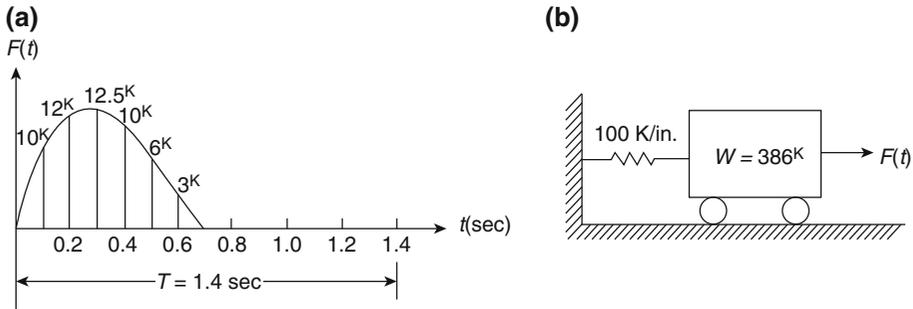


Fig. P19.8

**Problem 19.9**

Solve Problem 19.8 in the frequency domain using Program 4. Take  $M = 4$  for the exponent in  $N = 2M$ .

**Problem 19.10**

Use Program 4 to determine: (1) the Fourier series expansion of the forcing function shown in Fig. P19.10a and (2) the steady-state response calculated in the frequency domain for the spring-mass system shown in Fig. P19.10b. Assume 15% of the critical damping. Take  $M = 3$  for the exponent in  $N = 2^M$  and compare results with those obtained in the solution of Illustrative Example 19.3.

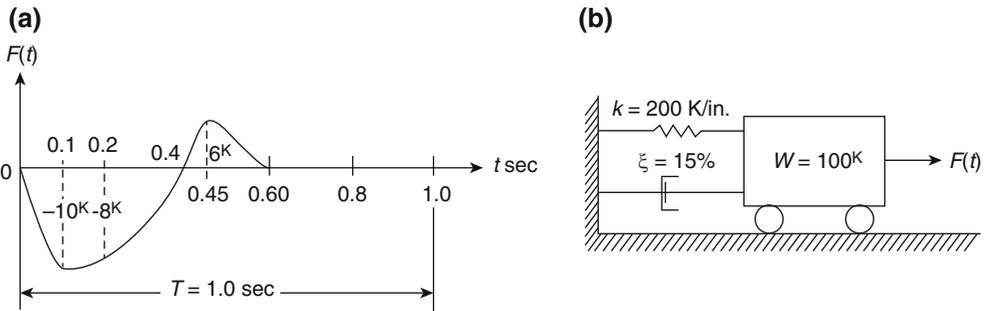


Fig. P19.10

**Problem 19.11**

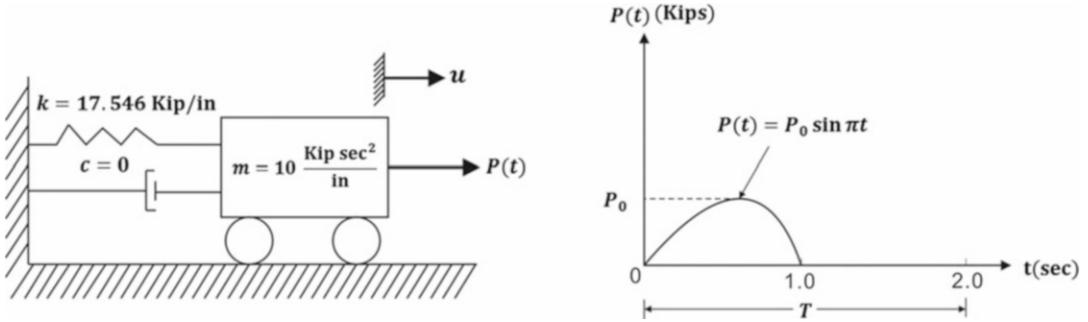
Solve Problem 19.10 in the frequency domain using Program 4. Take  $M = 5$  for the exponent in  $N = 2^M$ . Compare results with those in the solution of Example 19.3.

**Problem 19.12**

Consider the system shown in Fig. P19.12 and its loading with assumed period  $T = 2$  sec. Determine: (a) the first four terms of the Fourier series expansion for the forcing function in terms of  $P_0$ ; (b) the first four terms of the Fourier series expansion for the response.

**Problem 19.13**

Using Program 4, determine the response of the system and its load in Problem 19.12. Take 32 terms of Fourier series and input the force at intervals of 0.10 sec.



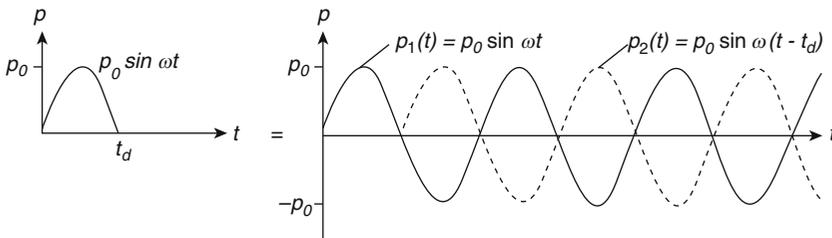
**Fig. P19.12**

**Problem 19.14**

Repeat Problem 19.13 assuming that the system has 20% of the critical damping.

**Problem 19.15**

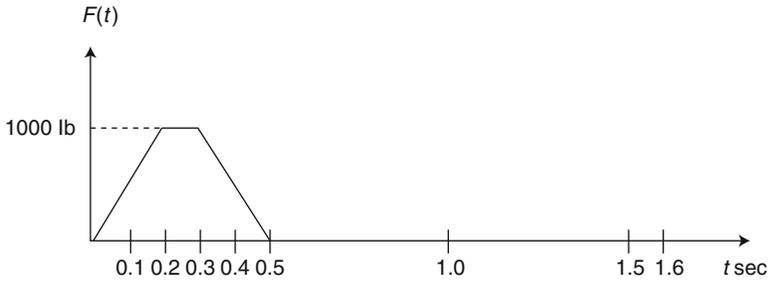
Obtain the close solution for the system in Problem 19.12 by considering the half-cycle sinusoidal excitation as the superposition of two sinusoidal functions:  $P_1 = P_0 \sin \pi t$  starting at  $t = 0$  and  $P_2 = P_0 \sin \pi(t-1)$  starting at  $t = 1$  sec as shown in Fig. P19.15.



**Fig. P19.15**

**Problem 19.16**

A single-degree-of-freedom system having a natural period of 0.8 sec and stiffness of 5000 lb fin subjected to an impulse of duration 0.5 sec which varies as shown in Fig. P19.16. Compute the response with an extension of 1.1 sec for which the value of force is zero. Use Program 4 to obtain: (a) the discrete transform of the forcing function and of the response, (b) the displacement response, and (c) the applied force calculated using the inverse discrete transform. Neglect damping and discretize the forcing function using a time step  $\Delta t = 0.1$  sec.



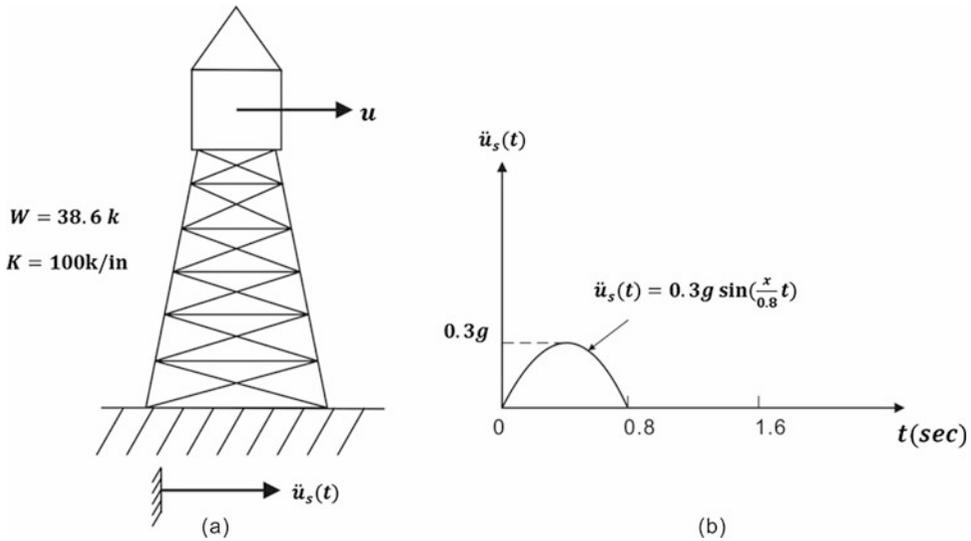
**Fig. P19.16**

**Problem 19.17**

Solve Problem 19.16 assuming a 10% of the critical damping in the system.

**Problem 19.18**

The water tower shown in Fig. P19.8a is subjected to impulsive acceleration of its base that varies as half the sine function shown in Fig. P19.18b. Use Program 4 to determine: (a) the discrete Fourier coefficients for the excitation and for the response, (b) the relative displacement of the tower with respect to the ground displacement, and (c) the excitation obtained by the inverse discrete transform. Use an extended excitation of total duration 1.6 sec and time step  $\Delta t = 0.1$  sec. Neglect damping.



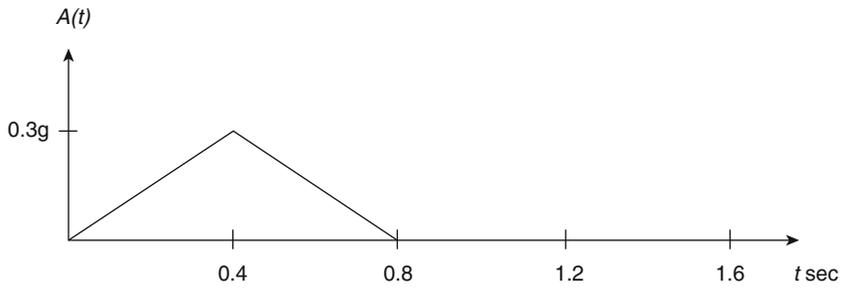
**Fig. P19.18**

**Problem 19.19**

Repeat Problem 19.18 assuming that the damping in the system is 5% of the critical.

**Problem 19.20**

Solve Problem 19.18 for an acceleration at the base of tower that varies as a symmetrical triangular load as shown in Fig. P19.20.



**Fig. P19.20**