7 Free Vibration of a Shear Building

In Part I we analyzed and obtained the dynamic response for structures modeled as a single-degree-of-freedom system. Only if the structure can assume a unique shape during its motion will the single-degree model provide the exact dynamic response. Otherwise, when the structure takes more than one possible shape during motion, the solution obtained from a single-degree model will be at best, only an approximation to the true dynamic behavior.

Structures cannot always be described by a single-degree-of-freedom model and, in general, have to be represented by multiple-degree models. In fact, structures are continuous systems and as such possess an infinite number of degrees of freedom. There are analytical methods to describe the dynamic behavior of continuous structures that have uniform material properties and regular geometry. These methods of analysis, though interesting in revealing information for the discrete modeling of structures, are rather complex and are applicable only to relatively simple actual structures. They require considerable mathematical analysis, including the solution of partial differential equations which will be presented in Part IV. For the present, we shall consider one of the most instructive and practical types of structure which involve many degrees of freedom, the multistory shear building.

7.1 Stiffness Equations for the Shear Building

A shear building may be defined as a structure in which there is no rotation of a horizontal section at the level of the floors. In this respect, the deflected building will have many of the features of a cantilever beam that is deflected by shear forces only, hence the name shear building. To accomplish such deflection in a building, we must assume that: (1) the total mass of the structure is concentrated at the levels of the floors; (2) the slabs or girders on the floors are infinitely rigid as compared to the columns; and (3) the deformation of the structure is independent of the axial forces present in the columns. These assumptions transform the problem from a structure with an infinite number of degrees of freedom (due to the distributed mass) to a structure that has only as many degrees as it has lumped masses at the floor levels. A three-story structure modeled as a shear building (Fig. 7.1a) will have three degrees of freedom, that is, the three horizontal displacements at the floor levels. The second assumption introduces the requirement that the joints between girders and columns are fixed against rotation. The third assumption leads to the condition that the rigid girders will remain horizontal during motion.

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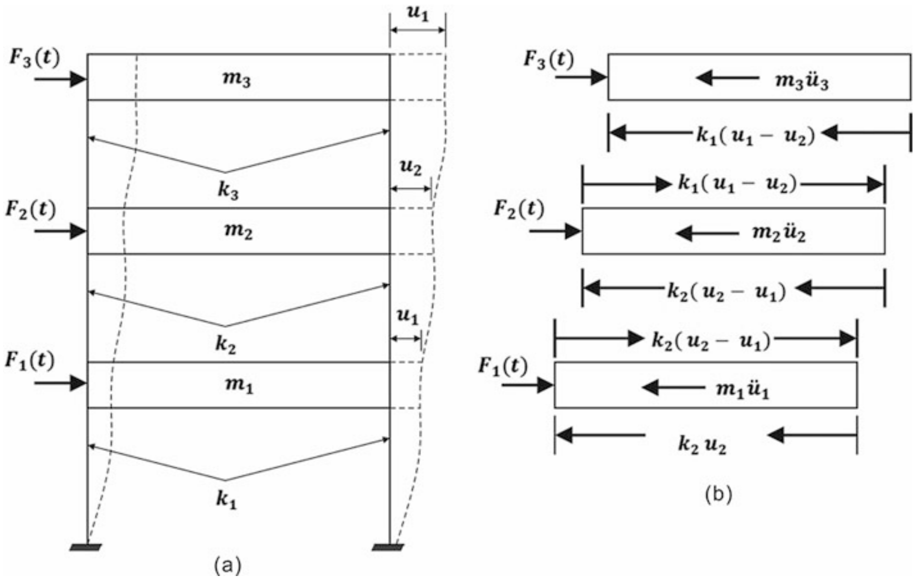


Fig. 7.1 (**a**) Single-bay model representation of a shear building. (**b**) Free body diagram

It should be noted that the building may have any number of bays and that it is only a matter of convenience that we represent the shear building solely in terms of a single bay. Actually, we can further idealize the shear building as a single column (Fig. 7.2a), having concentrated masses at the floor levels with the understanding that only horizontal displacements of these masses are possible. Another alternative is to adopt a multimass-spring system shown in Fig. 7.3a to represent the shear building. In any of the three representations depicted in these figures, the stiffness coefficient, or spring constant k_i, shown between any two consecutive masses is the force required to produce a relative unit displacement of the two adjacent floor levels

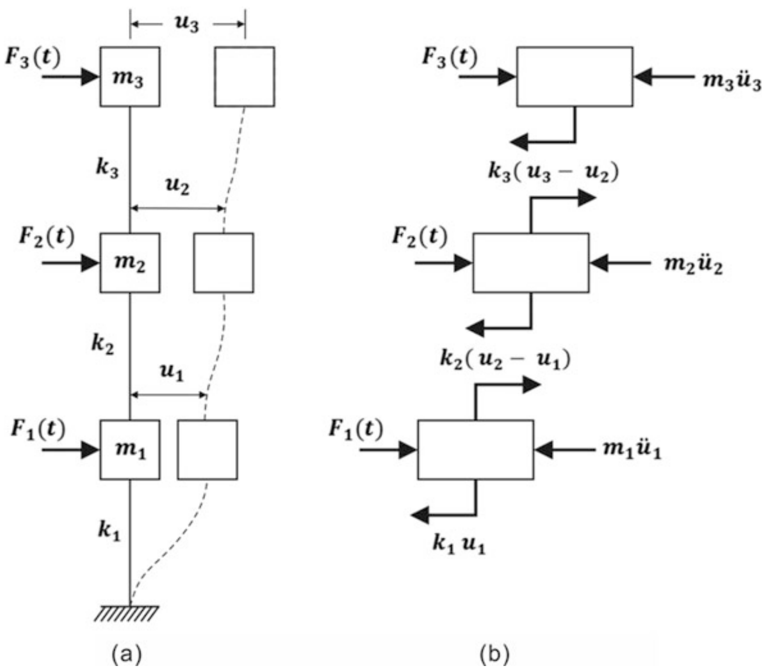


Fig. 7.2 (**a**) Single-column model representation of a shear building. (**b**) Free body diagram

For a uniform column with the two ends fixed against rotation, the stiffness or spring constant, k, is given by

$$k = \frac{12EI}{L^3} \tag{7.1a}$$

and for a column with one end fixed and the other pinned by

$$k = \frac{3EI}{L^3} \tag{7.1b}$$

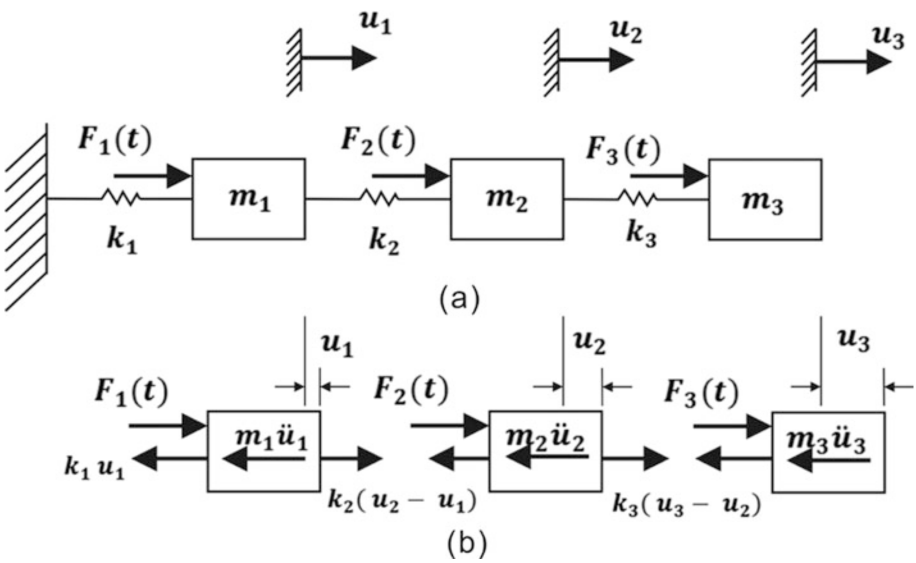


Fig. 7.3 (a) Multimass-spring model representation of a shear. (b) Free body diagram

where E is the material modulus of elasticity, I the cross-sectional moment of inertia, and L the length of the column.

It should be clear that all of the three representations shown in Figs. 7.1, 7.2 and 7.3 for the shear building are equivalent. Consequently, the following equations of motion for the thee-story shear building are obtained from any of the corresponding free body diagrams shown in these figures by equating to zero the sum of the forces acting on each mass. Hence

$$\begin{aligned} m_1\ddot{u}_1 + k_1u_1 - k_2(u_2 - u_1) - F_1(t) = 0 \\ m_2\ddot{u}_2 + k_2(u_2 - u_1) - k_3(u_3 - u_2) - F_2(t) = 0 \\ m_3\ddot{u}_3 + k_3(u_3 - u_2) - F_3(t) = 0 \end{aligned} \tag{7.2}$$

This system of equations constitutes the stiffness formulation of the equations of motion for a three-story shear building. It may conveniently be written in matrix notation as

$$[M]\{\ddot{u}\} + [K]\{u\} = \{F\} \tag{7.3}$$

where $[M]$ and $[K]$ are the mass and stiffness matrices given, respectively, by

$$[M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{7.4}$$

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \tag{7.5}$$

and $\{u\}$, $\{\ddot{u}\}$ and $\{F\}$ are, respectively, the displacement, acceleration and force vectors given by

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \qquad \{\ddot{u}\} = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}, \qquad \{F\} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{Bmatrix} \tag{7.6}$$

It should be noted that the mass matrix, Eq. (7.4), corresponding to the shear building is a diagonal matrix (the non-zero elements are only in the main diagonal). The elements of the stiffness matrix, Eq. (7.5), are designated *stiffness coefficients*. In general, the stiffness coefficient, k_{ij}, is defined as the force at coordinate i when a unit displacement is given at j, all other coordinates being fixed. For example, the coefficient in the second row and second column of Eq. (7.5), $k_{22} = k_2 + k_3$, is the force required at the second floor when a unit displacement is given to this floor.

7.2 Natural Frequencies and Normal Modes

The problem of free vibration requires that the force vector $\{F\}$ be equal to zero in Eq. (7.3). Namely,

$$[M]\{\ddot{u}\} + [K]\{u\} = 0 \tag{7.7}$$

For free vibrations of the undamped structure, we seek solutions of Eq. (7.7) in the form

$$u_i = a_i \sin(\omega t - \alpha), \qquad i = 1, 2, \ldots, n$$

or in vector notation

$$\{u\} = \{a\} \sin(\omega t - \alpha) \tag{7.8}$$

where a_i is the amplitude of motion of the ith coordinate and n is the number of degrees of freedom. The substitution of Eq. (7.8) into Eq. (7.7) gives

$$-\omega^2 [M]\{a\} \sin(\omega t - \alpha) + [K]\{a\} \sin(\omega t - \alpha) = 0$$

or factoring out sin(ωt–α) and rearranging terms

$$\left[[K] - \omega^2 [M]\right]\{a\} = \{0\} \tag{7.9}$$

which for the general case, is set for n homogenous (right-hand side equal to zero) algebraic system of linear equations with n unknown displacements a_i and an unknown parameter ω^2. The formulation of Eq. (7.9) is an important mathematical problem known as an eigenproblem. Its nontrivial solution, that is, the solution for which not all $a_i = 0$, requires that the determinant of the matrix factor of $\{a\}$ be equal to zero; in this case

$$\left|[K] - \omega^2[M]\right| = 0 \tag{7.10}$$

In general, the expansion of the determinant in Eq. (7.10) results in a polynomial equation of degree n in ω^2 which should be satisfied for n values of ω^2. This polynomial is known as the characteristic equation of the system. For each of these values of ω^2 satisfying the characteristic Eq. (7.10) we can solve Eq. (7.9) for $a_1, a_2, \ldots, a_n$ in terms of an arbitrary constant. The necessary calculations are better explained through a numerical example.

Illustrative Example 7.1

The structure to be analyzed is the two-story steel rigid frame shown in Fig. 7.4. The weights of the floors and walls are indicated in the figure and are assumed to include the structural weight as well. The building consists of a series of frames spaced 15 ft. apart. It is further assumed that the structural properties are uniform along the length of the building and, therefore, the analysis to be made of an

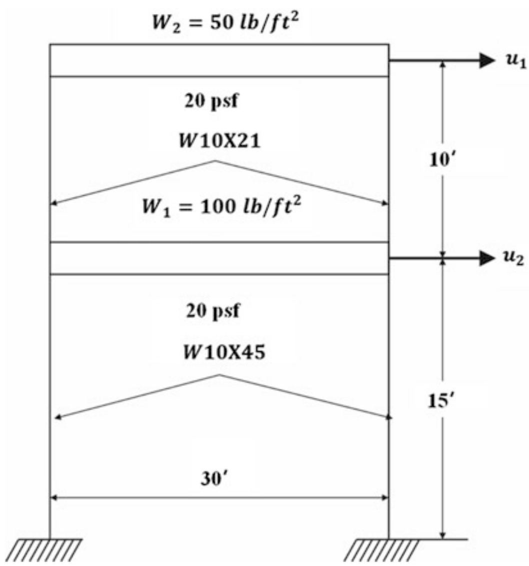


Fig. 7.4 Two-story shear building for Illustrative Example 7.1

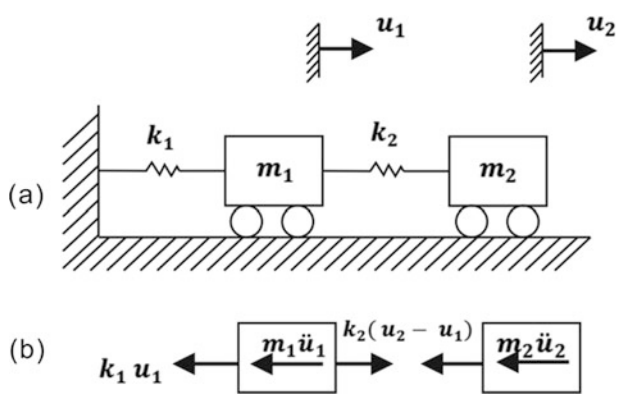


Fig. 7.5 Multimass-spring model for a two-story shear building of Illustrative Example 7.1. (**a**) Model, (**b**) Free body diagram

interior frame yields the response of the entire building. Determine (a) the natural frequencies and corresponding modal shapes, (b) the equations of motion with initial conditions for displacements u_{01}, u_{02}, and for velocities $\dot{u}_{01}$, and $\dot{u}_{02}$, respectively, for the first and second stories of the building.

Solution:

(a) Natural Frequencies and Modal Shapes
The building is modeled as a shear building and, under the assumptions stated, the entire building may be represented by the spring –mass system shown in Fig. 7.5. The concentrated weights, which are each taken as the total floor weight plus that of the tributary walls, are computed as follows:

$$W_1 = 100 \times 30 \times 15 + 20 \times 12.5 \times 15 \times 2 = 52,500 \text{ lb}$$
$$m_1 = 136 \text{ lb. sec}^2/\text{in}$$
$$W_2 = 50 \times 30 \times 15 + 20 \times 5 \times 15 \times 2 = 25,500 \text{ lb}$$
$$m_2 = 66 \text{ lb. sec}^2/\text{in}$$

Since the girders are assumed to be rigid and fixed at the two ends, the stiffness (spring constant) of each story is given by Eq. (7.1a) as

$$k = \frac{12E(2I)}{L^3}$$

and the individual values for the steel column sections indicated are thus

$$k_1 = \frac{12 \times 30 \times 10^6 \times 248 \times 2}{(15 \times 12)^3} = 30,700 \text{ lb/in}$$
$$k_2 = \frac{12 \times 30 \times 10^6 \times 118 \times 2}{(10 \times 12)^3} = 44,300 \text{ lb/in}$$

The equations of motion for the system, which are obtained by considering in Fig. 7.5b the dynamic equilibrium of each mass in free vibration, are

$$m_1\ddot{u}_1 + k_1u_1 - k_2(u_2 - u_1) = 0$$
$$m_2\ddot{u}_2 + k_2(u_2 - u_1) = 0$$

In the usual manner, these equations of motion are solved for free vibration by substituting

$$\begin{aligned} u_1 &= a_1 \sin(\omega t - \alpha) \\ u_2 &= a_2 \sin(\omega t - \alpha) \end{aligned} \tag{a}$$

for the displacements and

$$\ddot{u}_1 = -a_1\omega^2 \sin(\omega t - \alpha)$$
$$\ddot{u}_2 = -a_2\omega^2 \sin(\omega t - \alpha)$$

for the accelerations. In matrix notation, we obtain

$$\begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{b}$$

For a nontrivial solution, we require that the determinant of the coefficients be equal to zero, that is,

$$\begin{vmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{vmatrix} = 0 \tag{c}$$

The expansion of this determinant gives a quadratic equation in ω^2, namely

$$m_1 m_2 \omega^4 - [(k_1 - k_2)m_2 + m_1 k_2]\omega^2 + k_1 k_2 = 0 \tag{d}$$

or by introducing the numerical values for this example, we obtain

$$8976\omega^4 - 10,974,800\omega^2 + 1.36 \times 10^6 = 0 \tag{e}$$

The roots of this quadratic are

$$\omega_1^2 = 140$$
$$\omega_2^2 = 1082$$

Therefore, the natural frequencies of the structure are

$$\omega_1 = 11.83 \ \text{rad/sec}$$
$$\omega_2 = 32.89 \ \text{rad/sec}$$

or in cycles per second (cps)

$$f_1 = \omega_1/2\pi = 1.88 \ \text{cps}$$
$$f_2 = \omega_2/2\pi = 5.24 \ \text{cps}$$

and the corresponding natural periods:

$$T_1 = \frac{1}{f_1} = 0.532 \quad \text{sec}$$
$$T_2 = \frac{1}{f_2} = 0.191 \quad \text{sec}$$

To solve Eq. (b) for the amplitudes a_1 and a_2, we note that by equating the determinant to zero in Eq. (c), the number of independent equations is one less. Thus in the present case, the system of two equations is reduced to one independent equation. Considering the first equation in Eq. (b) and substituting the first natural frequency, $\omega_1 = 11.8$ rad/sec, we obtain

$$55,960a_{11} - 44,300a_{21} = 0 \tag{f}$$

We have introduced a second sub-index in a_1 and a_2 to indicate that the value ω_1 has been used in this equation. Since in the present case there are two unknowns and only one equation, we can solve Eq. (f) only for the relative value of a_{21} to a_{11}. This relative value is known as the normal mode or modal shape corresponding to the first frequency. For this example, Eq. (f) gives

$$\frac{a_{21}}{a_{11}} = 1.263$$

It is customary to describe the normal modes by assigning a unit value to one of the amplitudes; thus, for the first mode we set a_{11} equal to unity so that

$$a_{11} = 1.000$$
$$a_{21} = 1.263 \tag{g}$$

Similarly, substituting the second natural frequency, $\omega_2 = 32.9$ rad/sec into Eq. (b) we obtain the second normal mode as

$$a_{12} = 1.000$$
$$a_{22} = -1.629 \tag{h}$$

It should be noted that although we obtained only ratios, the amplitudes of motion could, of course, be found from initial conditions.

We have now arrived at two possible simple harmonic motions of the structure which can take place in such a way that all the masses move in phase in the same frequency, either ω_1 or ω_2. Such a motion of an undamped system is called a normal or natural mode of vibration. The shapes for these modes (a_{21}/a_{11} and a_{22}/a_{12}) for this example are called normal mode shapes of simply modal shapes for the corresponding natural frequencies ω_1 and ω_2. These two modes for this example are depicted in Fig. 7.6.

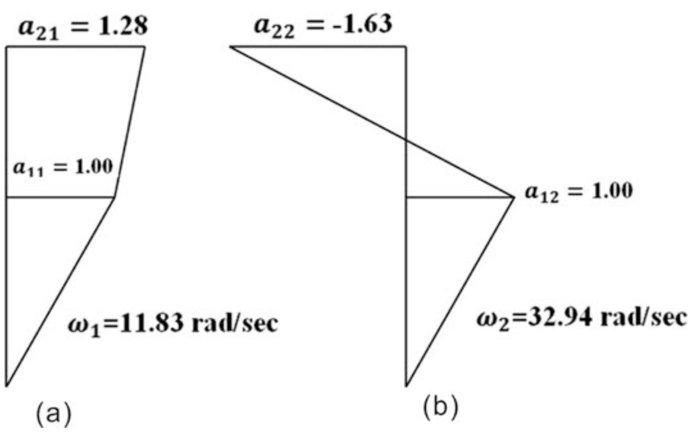


Fig. 7.6 Normal modes for Illustrative Example 7.1 (**a**) First mode (**b**) Second mode

We often use the designation first mode or fundamental mode to refer to the mode associated with the lowest frequency. The other modes are sometimes called harmonics or higher harmonics. It is evident that the modes of vibration, each having its own frequency, behave essentially as single-degree-of-freedom systems.

(b) Equations of Motion

The total motion of the system, that is, the total solution of the equations of motion, Eq. (7.7), is given by the superposition of the modal harmonic vibrations which in terms of arbitrary constants of integration may be written as

$$u_1(t) = C_1^{'} a_{11} \sin(\omega_1 t - \alpha_1) + C_2^{'} a_{12} \sin(\omega_2 t - \alpha_2)$$
$$u_1(t) = C_1^{'} a_{21} \sin(\omega_1 t - \alpha_1) + C_2^{'} a_{22} \sin(\omega_2 t - \alpha_2) \tag{i}$$

Here $C_1^{'}$ and $C_2^{'}$ as well as α_1 and α_2 are four constants of integration to be determined from four initial conditions which are the initial displacement and velocity for each mass in the system. For a two-degree-of-freedom system, these initial conditions are

$$\begin{aligned} u_1(0) &= u_{01}, \qquad \dot{u}_1(0) = \dot{u}_{01} \\ u_2(0) &= u_{02}, \qquad \dot{u}_2(0) = \dot{u}_{02} \end{aligned} \tag{j}$$

For computational purposes, it is convenient to eliminate the phase angles [α_1 and α_2 in Eq. (i)] in favor of other constants. Expanding the trigonometric functions in Eq. (i) and renaming the constants, we obtain

$$\begin{aligned} u_1(t) &= C_1 a_{11} \sin \omega_1 t + C_2 a_{11} \cos \omega_1 t + C_3 a_{12} \sin \omega_2 t + C_4 a_{12} \cos \omega_2 t \\ u_2(t) &= C_1 a_{21} \sin \omega_1 t + C_2 a_{21} \cos \omega_1 t + C_3 a_{22} \sin \omega_2 t + C_4 a_{22} \cos \omega_2 t \end{aligned} \tag{k}$$

in which C_1, C_2, C_3 and C_4 are the new renamed constants of integration. From the first two initial conditions in Eq. (j), we obtain the following two equations:

$$\begin{aligned} u_{01} &= C_2 a_{11} + C_4 a_{12} \\ u_{02} &= C_2 a_{21} + C_4 a_{22} \end{aligned} \tag{l}$$

Since the modes are independent, these equations can always be solved for C_2 and C_4. Similarly, by expressing in Eq. (k) the velocities at time equal to zero, we find

$$\begin{aligned} \dot{u}_{01} &= \omega_1 C_1 a_{11} + \omega_2 C_3 a_{12} \\ \dot{u}_{02} &= \omega_1 C_1 a_{21} + \omega_2 C_3 a_{22} \end{aligned} \tag{m}$$

The solution of these two sets of Eqs. (l) and (m), allows us to express the motion of the system in terms of the two modal vibrations, each proceeding at its own frequency, completely independent of the other, the amplitudes and phases being determined by the initial conditions.

7.3 Orthogonality Property of the Normal Modes

We shall now introduce an important property of the normal modes, the orthogonality property. This property constitutes the basis of the most important method for solving dynamic problems, the Modal Superposition Method of multi-degree-of-freedom systems. We begin by rewriting the equations of motion in free vibration, Eq. (7.7) as

$$[K]\{a\} = \omega^2 [M]\{a\} \tag{7.11}$$

For the two-degree-of-freedom system, we obtain from Eq. (b) of Illustrative Example 7.1

$$\begin{aligned} (k_1 + k_2)a_1 - k_2 a_2 &= \omega^2 m_1 a_1 \\ -k_2 a_1 + k_2 a_2 &= \omega^2 m_2 a_2 \end{aligned} \tag{7.12}$$

These equations are exactly the same as Eq. (b) of Illustrative Example 7.1 but written in this form they may be given a static interpretation as the equilibrium equations for the system acted on by forces of magnitude $\omega^2 m_1 a_1$ and $\omega^2 m_2 a_2$ applied to masses m_1 and m_2, respectively. The modal shapes may then be considered as the static deflections resulting from the forces on the right-hand side of Eq. (7.12) for any of the two modes. This interpretation, as a static problem, allows us to use the results of the general static theory of linear structures. In particular, we may use of Betti's theorem which states: For a structure acted upon by two systems of loads and corresponding displacements, the work done by the first system of loads moving through the displacements of the second system is equal to the work done by this second system of loads undergoing the displacements produced by the first load system. The two systems of loading and corresponding displacements which we shall consider are as follows:

System I:

Forces	$\omega_1^2 m_1 a_{11}, \quad \omega_1^2 m_2 a_{21}$
Displacements	$a_{11}, \quad a_{21}$

System II:

Forces	$\omega_2^2 m_1 a_{12} \quad \omega_2^2 m_2 a_{22}$
Displacements	$a_{12}, \quad a_{22}$

The application of Betti's theorem for these two systems yields

$$\omega_1^2 m_1 a_{11} a_{12} + \omega_1^2 m_2 a_{21} a_{22} = \omega_2^2 m_1 a_{12} a_{11} + \omega_2^2 m_2 a_{22} a_{21}$$

or

$$\left(\omega_1^2 - \omega_2^2\right)(m_1 a_{11} a_{12} + m_2 a_{21} a_{22}) = 0$$

If the natural frequencies are different ($\omega_1 \neq \omega_2$), it follows that

$$m_1 a_{11} a_{12} + m_2 a_{21} a_{22} = 0$$

which is the so-called orthogonality relationship between modal shapes of a two degree-of-freedom system. For an n-degree-of-freedom system in which the mass matrix is diagonal, the orthogonality condition between any two modes i and j may be expressed as

$$\sum_{k=1}^{n} m_k a_{ki} a_{kj} = 0, \qquad \text{for } i \neq j \tag{7.13}$$

and in general for any n-degree-of-freedom system as

$$\{a\}_i^T [M] \{a\}_j = 0 \qquad \text{for } i \neq j \tag{7.14}$$

in which $\{a_i\}$ and $\{a_j\}$ are any two modal vectors and $[M]$ is the mass matrix of the system.

As mentioned before, the amplitudes of vibration in a normal mode are only relative values which may be scaled or normalized to some extent as a matter of choice. The following is an especially convenient normalization for a general system:

$$\phi_{ij} = \frac{a_{ij}}{\sqrt{\{a\}_i^T\,[M]\{a\}_j}} \tag{7.15}$$

which, for a system having a diagonal mass matrix, may be written as

$$\phi_{ij} = \frac{a_{ij}}{\sqrt{\sum_{k=1}^{n} m_k a_{kj}^2}} \tag{7.16}$$

in which ϕ_{ij} is the normalized i component of the j modal vector. For normalized eigenvectors, the orthogonality condition is given by

$$\begin{aligned} \{\phi\}_i^T[M]\{\phi\}_j &= 0 \quad \text{for } i \neq j \\ &= 1 \quad \text{for } i = j \end{aligned} \tag{7.17}$$

Another orthogonality condition is obtained by writing Eq. (7.9) for the normalized j mode as

$$[K]\{\phi\}_j = \omega_j^2[M]\{\phi\}_j \tag{7.18}$$

Then pre-multiplying Eq. (7.18) by $\{\phi\}_i^T$ we obtain, in view of Eq. (7.17), the following orthogonality condition between eigenvectors:

$$\begin{aligned} \{\phi\}_i^T[K]\{\phi\}_j &= 0 \quad \text{for } i \neq j \\ &= \omega_{\mathrm{j}}^2 \quad \text{for } i = j \end{aligned} \tag{7.19}$$

Illustrative Example 7.2

For the two-story shear building of Illustrative Example 7.1 determine (a) the normalized modal shapes of vibration, and (b) verify the orthogonality condition between the modes.

Solution:

The substitution of Eqs. (g) and (h) from Illustrative Example 7.1 together with the values of the masses from Illustrative Example 7.1 into the normalization factor required in Eq. (7.16) gives

$$\sqrt{(136)(1.00)^2 + (66)(1.263)^2} = \sqrt{241.31}$$

$$\sqrt{(136)(1.00)^2 + (66)(-1.629)^2} = \sqrt{311.08}$$

Consequently, the normalized modes are

$$\phi_{11} = \frac{1.00}{\sqrt{241.31}} = 0.06437, \qquad \phi_{12} = \frac{1.00}{\sqrt{311.08}} = 0.0567$$

$$\phi_{21} = \frac{1.263}{\sqrt{241.31}} = 0.0813, \qquad \phi_{22} = \frac{-1.629}{\sqrt{311.08}} = -0.0924$$

The normal modes may be conveniently arranged in the columns of a matrix known as the modal matrix of the system. For the general case of n degrees of freedom, the modal matrix is written as

$$[\Phi] = \begin{bmatrix} \phi_{11} & \phi_{12}\cdots & \phi_{1n} \\ \phi_{21} & \phi_{22}\cdots & \phi_{2n} \\ \phi_{n1} & \phi_{n2}\cdots & \phi_{nn} \end{bmatrix} \tag{7.20}$$

The orthogonality condition may then be expressed in general as

$$[\Phi]^T[M][\Phi] = [I] \tag{7.21}$$

where $[\Phi]^T$ is the matrix transpose of $[\Phi]$, and $[M]$ the mass matrix of the system. For this example of two degrees of freedom, the modal matrix is

$$[\Phi] = \begin{bmatrix} 0.06437 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix} \tag{a}$$

To check the orthogonality condition, we simply substitute the normal modes from Eq. (a) into Eq. (7.21) and obtain

$$\begin{bmatrix} 0.06437 & 0.0813 \\ 0.0567 & -0.0924 \end{bmatrix} \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix} \begin{bmatrix} 0.06437 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We have seen that to determine the natural frequencies and normal modes of vibration of a structural system, we have to solve an eigenvalue problem. The direct method of solution based on the expansion of the determinant and the solution of the resulting characteristic equation is limited in practice to systems having only a few degrees of freedom. For a system of many degrees of freedom, the algebraic and numerical work required for the solution of an eigenproblem becomes so immense as to make the direct method impossible. However, there are many numerical methods available for the calculation of eigenvalues and eigenvectors of an eigenproblem. The discussion of these methods belongs in a mathematical text on numerical methods rather than in a text such as this on structural dynamics. One of the most popular methods for the numerical solution of an eigenproblem is the Jacobi Method, which is an iterative method to calculate the eigenvalues and eigenvectors of the system. The basic Jacobi solution method has been developed for the solution of standard eigenproblems (i.e., $[M]$ being the identity matrix). The method was proposed over a century ago and has been used extensively. This method can be applied to all symmetric matrices $[K]$ with no restriction on the eigenvalues. It is possible to transform the generalized eigenproblem, $[[K] - \omega^2[M]] = \{\Phi\} = \{0\}$ into the standard form and still maintain the symmetry required for the Jacobi Method. However, this transformation can be dispensed with by using a generalized Jacobi solution method (Bathe, K. J. 1982) which operates directly on $[K]$ and $[M]$.

Examples 7.1 and 7.2 can be solved using MATLAB program. The function of MATLAB are used to solve eigenproblem using built-in function, eig(K, M). The natural frequencies and normal modes are estimated using the following MATLAB codes. The outcomes are the natural frequencies (natural periods) and normalized modal matrix. This MATLAB code adopts the framework proposed by Anderson and Naeim (2012).

```
clear all
close all

%%%%-GIVEN VALUES-%%%%

%%%Define Mass Matrix
M = [136 0; 0 66]

%%%Define Stiffness Matrix
K = [30700+44300 -44300;-44300 44300]

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(K, M)

[omegas,k] = sort(sqrt(diag(D)));

%%%Natural frequencies
omegas =sqrt(D)

%%%Natural periods
T =2*pi./omegas;

T1 = 2*pi./omegas(1,1);
T2 = 2*pi./omegas(2,2);

%%%{a}1before changing the unity in the first DOF.
a1 = a(:,1);

%%%{a}2before changing the unity in the first DOF.
a2 = a(:,2);

%%%Change the {a} w.r.t. the unity in the first DOF.
a11 = 1;
a21 = a1(2,1)./a1(1,1);
a12 = 1;
a22 = a2(2,1)./a2(1,1);

a =[];

%%%Calculate the {a}
a(:,1) = [a11, a21]                   %[a11,a21]
a(:,2) = [a12, a22]                   %[a12,a22]

%%%aMa = {a}'*[M]*(a}
aMa = a'*M*a;                         %Eq.7.14

%%%Normalization factor
norm_1 = sqrt(aMa(1,1));
norm_2 = sqrt(aMa(2,2));

%%%Normalized eigenvectors
nom_phi(:,1) = 1./norm_1.*a(:,1);    %Eq.7.16 for the first mode
nom_phi(:,2)= 1./norm_2.*a(:,2);     %Eq.7.16 for the 2nd mode
nom_phi
%Check the orthogonality condition for Mass Matrix
orth_M = nom_phi'*M*nom_phi;         %Eq.7.17

%Check the orthogonality condition for Stiffness Matrix
orth_K = nom_phi'*K*nom_phi;         %Eq.7.19
```

7.4 Rayleigh's Quotient

Several iterative methods for the solution of an eigenproblem make use of the Rayleigh's quotient. The Rayleigh's quotient may be obtained by pre-multiplying Eq. (7.18) by the transpose of the modal vector $\{\phi\}_j^T$. Hence,

$$\{\phi\}_j^T[K]\{\phi\}_j = \omega_j^2\{\phi\}_j^T[M]\{\phi\}_j$$

The property of the mass matrix $[M]$ being positive definite[1] renders the product $\{\phi\}_j^T[M]\{\phi\}_j \neq 0$, thus, it is permissible to solve for ω_j^2:

$$\omega^2 = \frac{\{\phi\}^T[K]\{\phi\}}{\{\phi\}^T[M]\{\phi\}} \tag{7.22}$$

in which for convenience the sub-index j has been omitted.

The ratio given by Eq. (7.22) is known as the Rayleigh's quotient. This quotient has the following properties: (1) It provides the eigenvalue ω_j^2 when the corresponding eigenvector $\{\phi\}_j$ is introduced in Eq. (7.22). (2) When a vector $\{\phi\}$ different from an eigenvector is used, then Eq. (7.22) provides a value ω^2 that lies between the smallest eigenvalue, ω_1^2 and the largest eigenvalue ω_N^2. (3) Finally, if a vector $\{\phi\}$ that is an approximation to eigenvector $\{\phi\}_j$, correct to d decimals is used, then the value of ω^2 obtained from Eq. (7.22) is accurate to $2d$ number of decimals as an approximation to ω_j^2.

Illustrative Example 7.3

Use Rayleigh's quotient to calculate an approximate value for the first eigenvalue of the structure in Illustrative Example 7.1 beginning with the approximate eigenvector for the first mode $\{\phi\}^T = \{1.00\ 1.50\}$, then iterate using Eqs. (7.12) and (7.22) to converge to the eigenvalue and eigenvector for the first mode.

Solution:

The substitution of the given vector $\{\phi\}^T = \{1.00\ 1.50\}$ and the matrices $[K]$ and $[M]$ into the numerator of Eq. (7.22) results in

$$\{1.00\ \ 1.50\}\begin{bmatrix} 75,000 & -44,300 \\ -44,300 & 44,300 \end{bmatrix}\begin{Bmatrix} 1.00 \\ 1.50 \end{Bmatrix} = 42,025$$

and also into the denominator

$$\{\ 1.00 \quad 1.50\ \}\begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix}\begin{Bmatrix} 1.00 \\ 1.50 \end{Bmatrix} = 284.5$$

which substituted into Eq. (7.22) yields

$$\omega^2 = \frac{42,025}{284.5} = 147.9$$

[1] Matrix $[A]$ is defined as positive definite if it satisfies the condition that for any arbitrary nonzero vector $\{v\}$, the product $\{v)^T\ [A]\{v\} > 0$.

The use of the calculated value $\omega^2 = 147.9$ together with $a_1 = 1.00$ into the first Eq. (7.12) yields

$$a_1 = 1.00 \quad \text{and} \quad a_2 = 1.24$$

A second iteration of Eq. (7.22) with $\{\Phi\}^T = \{1.00 \quad 1.24\}$ yields

$$\omega^2 = 140.02$$

This value of ω^2 is virtually equal to the solution $\omega_1 = 140.02$ obtained for the first mode in Illustrative Example 7.1

Another popular iterative method to solve an eigenproblem, that is, for structural dynamics, to calculate natural frequencies and modal shapes, is the Subspace Iteration Method.

7.5 Summary

The motion of an undamped dynamic system in free vibration is governed by a homogenous system of differential equations which in matrix notation is

$$[M]\{\ddot{u}\} + [K]\{u\} = 0$$

The process of solving this system of equations leads to a homogenous system of linear algebraic equations of the form

$$\left[[K] - \omega^2[M]\right]\{a\} = \{0\}$$

which mathematically is known as an eigenproblem.

For a nontrivial solution of this problem, it is required that the determinant of the coefficients of the unknown $\{a\}$ be equal to zero, that is,

$$\left|[K] - \omega^2[M]\right| = 0$$

The roots of this equation provide the natural frequencies ω_i, (i = 1, 2, ... n). It is then possible to solve for the unknowns $\{a\}_i$, in terms of relative values. The vectors $\{a\}_i$ corresponding to the roots ω_i^2 are the modal shapes (eigenvectors) of the dynamic system. The arrangement in matrix format of the modal shapes constitutes the modal matrix $[\Phi]$ of the system. It is particularly convenient to normalize the eigenvectors to satisfy the following condition:

$$\{\phi\}_i^T[M]\{\phi\}_i = [I] \quad i = 1, 2, \ldots, n$$

where the normalized modal vectors $\{\phi\}_i$ are obtained by dividing the components of the vector $\{a\}_{\text{ij}}$ by $\sqrt{\{a\}_i^T[M]\{a\}_j}$.

The normalized modal vectors satisfy the following important conditions of orthogonality:

$$\begin{aligned} \{\phi\}_i^T[M]\{\phi\}_j &= 0 \quad \text{for } i \neq j \\ \{\phi\}_i[M]\{\phi\}_j &= 1 \quad \text{for } i = j \end{aligned}$$

and

$$\begin{aligned} \{\phi\}_i^T[K]\{\phi\}_j &= 0 \quad \text{for } i \neq j \\ \{\phi\}_i[K]\{\phi\}_j &= \omega_i^2 \quad \text{for } i = j \end{aligned}$$

The above relations are equivalent to

$$[\Phi]^T[M][\Phi] = [I]$$

and

$$[\Phi]^T[K][\Phi] = [\Omega]$$

in which $[\Phi]$ is the modal matrix of the system and $[\Omega]$ is a diagonal matrix containing the eigenvalues ω_i^2 in the main diagonal.

For a dynamic system with only a few degrees of freedom, the natural frequencies and modal shapes may be determined by expanding the determinant and calculating the roots of the resulting characteristic equation. However, for a system with a large number of degrees of freedom, this direct method of solution becomes impractical. It is then necessary to resort to other numerical methods which usually require an iteration process.

7.6 Problems

Problem 7.1 Determine the natural frequencies and normal modes for the two-story shear building shown in Fig. P7.1.

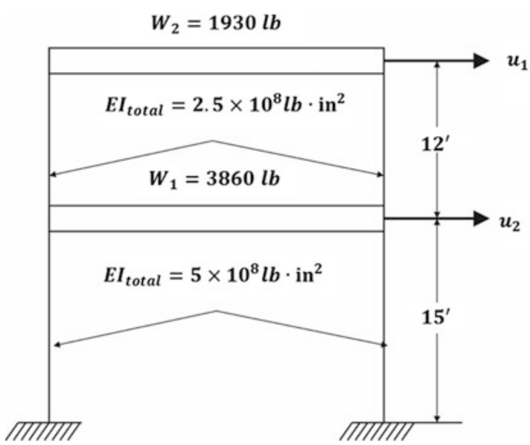


Fig. P7.1

Problem 7.2
A certain structure has been modeled as a three-degree-of-freedom system having the numerical values indicated in Fig. P7.2. Determine the natural frequencies of the structure and the corresponding normal modes. Check your answer using MATLAB.

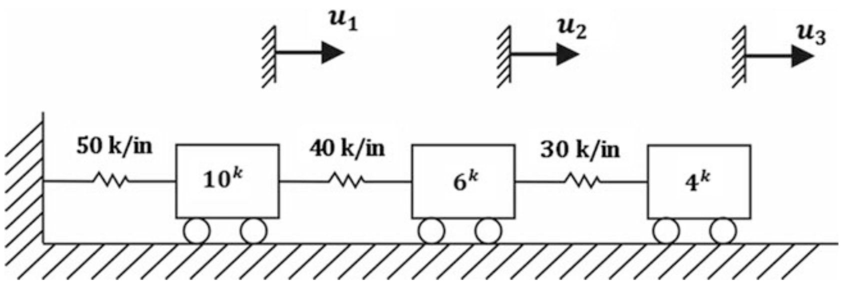


Fig. P7.2

Problem 7.3

Assume a shear building model for the frame shown in Fig. P7.3 and determine the natural frequencies and normal modes.

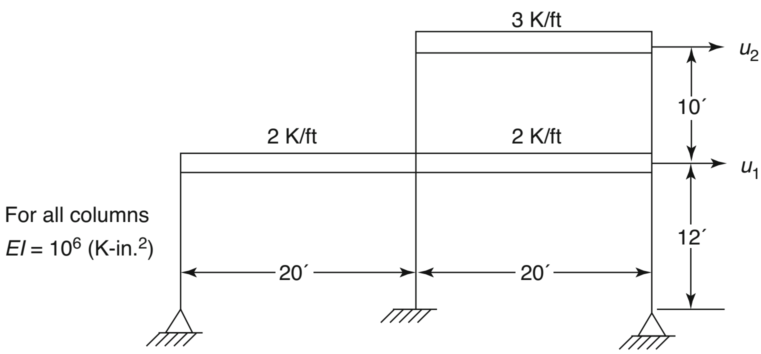


Fig. P7.3

Problem 7.4

Assume a shear building model for the frame shown in Fig. P7.4 and determine the natural frequencies and normal modes.

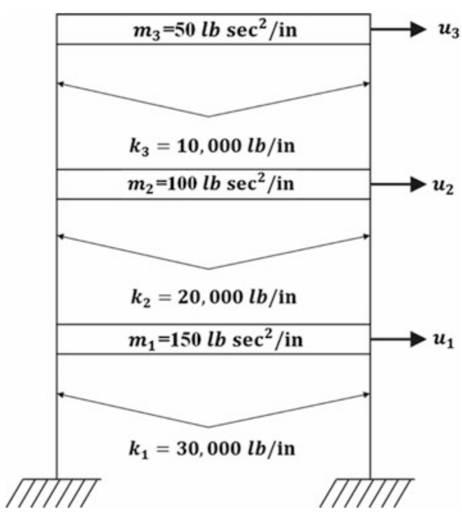


Fig. P7.4

Problem 7.5
Consider the uniform shear building in which the mass of each floor is m and the stiffness of each story is k. Determine the general form of the system of differential equations for a uniform shear building of N stories.

Problem 7.6
Find the natural frequencies and modal shapes for the three-degree-of-freedom shear building in Fig. P7.6.

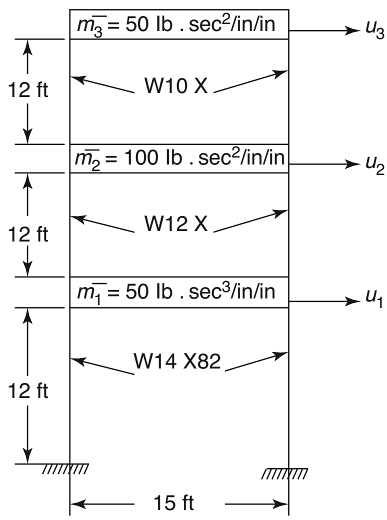
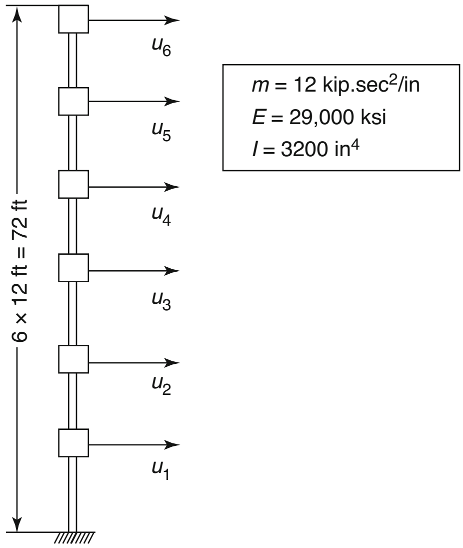


Fig. P7.6

Problem 7.7
Use the results of Problem 7.6 to write the expressions for the free vibration displacements u_1, u_2, and u_3 of the shear building in Fig. P7.6 in terms of constants of integration.

Problem 7.8
Use MATLAB to determine the natural frequencies for the six-story uniform shear building modeled as a column shown in Fig. P7.8.



u_6
u_5
u_4
u_3
u_2
u_1
6 × 12 ft = 72 ft
m = 12 kip.sec^2/in
E = 29,000 ksi
I = 3200 in^4

Fig. P7.8