21 Generalized Coordinates and Rayleigh's Method

In the preceding chapters we concentrated our efforts in obtaining the response to dynamic loads of structures modeled by the simple oscillator, that is, structures that may be analyzed as a damped or undamped spring-mass system. Our plan in the present chapter is to discuss the conditions under which a structural system consisting of multiple interconnected rigid bodies or having distributed mass and elasticity can still be modeled as a one-degree of-freedom system. We begin by presenting an alternative method to the direct application of Newton's Law of Motion, the principle of virtual work.

21.1 Principle of Virtual Work

An alternative approach to the direct method employed thus far for the formulation of the equations of motion is the use of the principle of virtual work. This principle is particularly useful for relatively complex structural systems which contain many interconnected parts. The principle of virtual work was I originally stated for a system in equilibrium. Nevertheless, the principle can readily be applied to dynamic systems by the simple recourse to D'Alembert's Principle, which establishes dynamic equilibrium by the inclusion of the inertial forces in the system.

The principle of virtual work may be stated as follows: For a system that is in equilibrium, the work done by all the forces during an assumed displacement (virtual displacement) that is compatible with the system constraints is equal to zero. In general the equations of motion are obtained by introducing virtual displacements corresponding to each degree of freedom and equating the resulting work done to zero.

To illustrate the application of the principle of virtual work to obtain the equation of motion for a single-degree-of-freedom system, let us consider the damped oscillator shown in Fig. 21.1a and its corresponding free body diagram in Fig. 21.1b. Since the inertial force has been included among the external forces, the system is in "equilibrium" (dynamic equilibrium). Consequently, the principle of virtual work is applicable. If a virtual displacement δu is assumed to have taken place, the total work done by the forces shown in Fig. 21.1b is equal to zero, that is,

$$m\ddot{u}\delta u + c\dot{u}\,\delta u + ku\delta u - F(t)\delta u = 0$$

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or

$$\{m\ddot{u} + c\dot{u} + ku - F(t)\}\delta u = 0 \tag{21.1}$$

Since δu is arbitrarily selected as not equal to zero, the other factor in Eq. (21.1) must equal zero. Hence,

$$m\ddot{u} + c\dot{u} + ku - F(t) = 0 \tag{21.2}$$

Thus we obtained in Eq. (21.2) the differential equation for the motion of the damped oscillator.

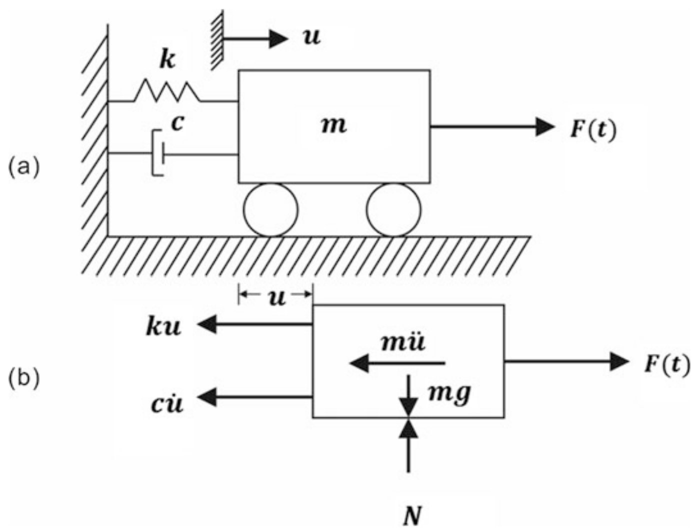

Fig. 21.1 Damped simple oscillator undergoing virtual displacement δu

21.2 Generalized Single-Degree-of-Freedom System–Rigid Body

Most frequently the configuration of a dynamic system is specified by coordinates indicating the linear or angular positions of elements of the system. However, coordinates do not necessarily have to correspond directly to displacements; they may in general be any independent quantities that are sufficient in number to specify the position of all parts of the system. These coordinates are usually called generalized coordinates and their number is equal to the number of degrees of freedom of the system.

The example of the rigid-body system shown in Fig. 21.2 consists of a rigid bar with distributed mass supporting a circular plate at one end. The bar is supported by springs and dampers in addition to a single frictionless support. Dynamic excitation is provided by a transverse load $F(x, t)$ varying linearly on the portion AB of the bar. Our purpose is to obtain the differential equation of motion and to identify the corresponding expressions for the parameters of the simple oscillator representing this system.

Since the bar is rigid, the system in Fig. 21.2 has only one degree of freedom, and, therefore, its dynamic response can be expressed with one equation of motion. The generalized coordinate could be selected as the vertical displacement of any point such as A, B, or C along the bar, or may be taken as the angular position of the bar. This last coordinate designated θ (t) is selected as the generalized coordinate of the system. The corresponding free body diagram showing all the forces including the inertial forces and the inertial moments is shown in Fig. 21.3. In evaluating the displacements of the different forces, it is assumed that the displacements of the system are small and, therefore, vertical displacements are simply equal to the product of the distance to support D multiplied by the angular displacement $\theta = \theta(t)$.

The displacements resulting at the points of application of the forces in Fig. 21.3 due to a virtual displacement $\delta\theta$ are indicated in this figure. By the principle of virtual work, the total work done by the forces during this virtual displacement is equal to zero. Hence

$$\delta\theta\left[I_0\ddot{\theta} + I_1\ddot{\theta} + 4L^3\bar{m}\ddot{\theta} + mL^2\ddot{\theta} + cL^2\dot{\theta} \ + 4kL^2\theta - \frac{7}{6}L^2 f(t)\right] = 0$$

or, since $\delta\theta$ *is* arbitrarily set not equal to zero, it follows that

$$\left(I_0 + I_1 + 4L^3\bar{m} + mL^2\right)\ddot{\theta} + cL^2\dot{\theta} \ + 4kL^2\dot{\theta} \ - \frac{7}{6}\,L^2 f(t) = 0, \tag{21.3}$$

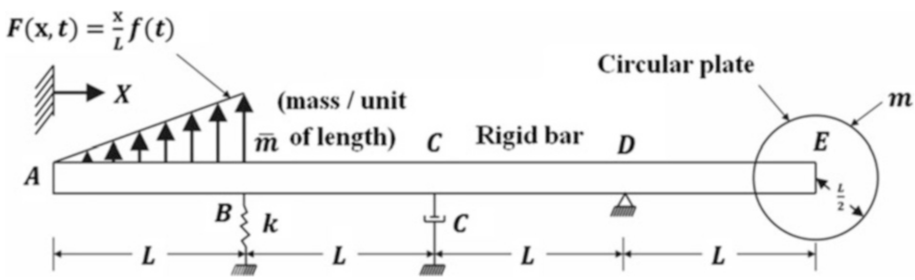

Fig. 21.2 Example of single-degree-of-freedom rigid system

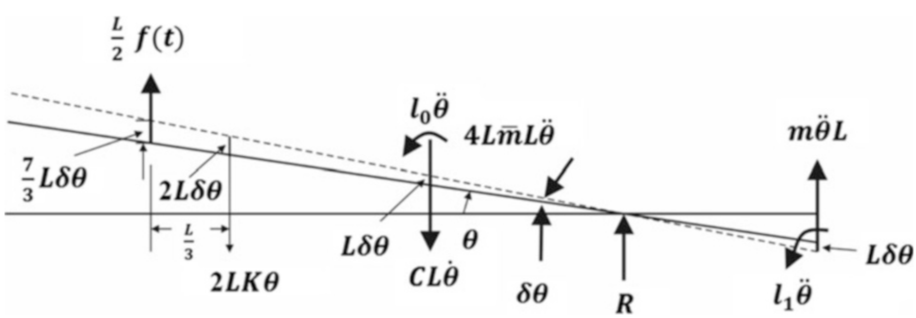

Fig. 21.3 Displacements and resultant forces for system in Fig. 21.2

where

$$I_0 = \frac{1}{12}(4\bar{m}L)(4L)^2 = \text{mass moment of inertia of the rod}$$

$$I_0 = \frac{1}{2}m\left(\frac{L}{2}\right)^2 = \text{mass moment of inertia of the circular plate}$$

The differential Eq. (21.3) governing the motion of this system may conveniently be written as

$$I^*\ddot{\theta} + C^*\dot{\theta} + K^*\theta = F^*(t) \tag{21.4}$$

where I^*, C^*, K^*, and $F^*(t)$ are, respectively, the generalized inertia, generalized damping, generalized stiffness, and generalized force for this system. These quantities are given in Eq. (21.3) by the factors corresponding to the acceleration, velocity, displacement, and force terms, namely,

$$I^* = I_0 + I_1 + 4\bar{m}L^3 + mL^2$$
$$C^* = cL^2$$
$$K^* = 4kL^2$$
$$F^*(t) = \frac{7}{6}L^2 f(t)$$

Illustrative Example 21.1

For the system shown in Fig. 21.4. determine the generalized physical properties M^*. C^*. K^* and generalized loading $F^*(t)$. Let $U(t)$ at the point A_2 in Fig. 1.4 be the generalized coordinate of the system.

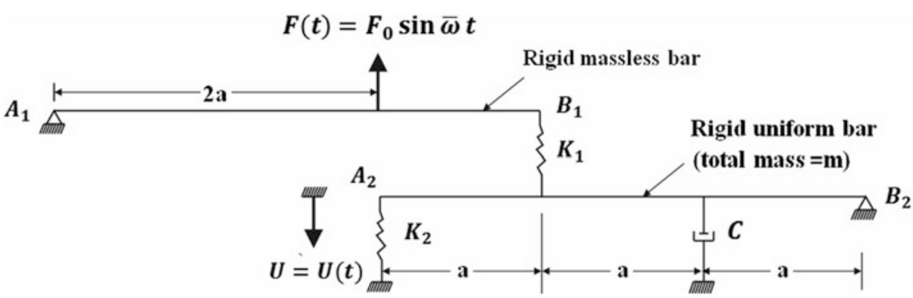

Fig. 21.4 System for Illustrative Example 21.1

Solution:

The free body diagram for the system is depicted in Fig. 21.5 which shows all the forces on the two bars of the system including the inertial force and the inertial moment. The generalized coordinate is $U(t)$ and the displacement of my point in the system should be expressed in terms of this coordinate; nevertheless, for convenience, we select also the auxiliary coordinate $U_1(t)$ as indicated in Fig. 21.5.

The summation of the moments about point A_1 of all the forces acting on bar A_1 -B_1. and the summation of moments about B_2 of the forces on bar A_2 -B_2. Give the following equations:

$$k_1\left(\frac{2}{3}U - U_1\right)3a = 2aF_0 \sin \overline{\omega} t \tag{21.5}$$

$$\frac{I_0}{3a}\ddot{U} + \frac{3}{4}ma\ddot{U} + \frac{a}{3}c\dot{U} + k_1\left(\frac{2}{3}U - U_1\right)2a + 3ak_2U = 0 \tag{21.6}$$

Substituting U_1 from Eq. (21.5) into Eq. (21.6), we obtain the differential equation for the motion of the system in terms of the generalized coordinate $U(t)$, namely

$$M^*\ddot{U}(t) + C^*\dot{U}(t) + K^*U(t) = F^*(t)$$

where the generalized quantities are given by

$$M^* = \frac{I_0}{3a^2} + \frac{3m}{4}$$
$$C^* = \frac{c}{3}$$
$$K^* = 3k_2$$

and

$$F^*(t) = -\frac{4}{3}F_0 \sin \varpi t$$

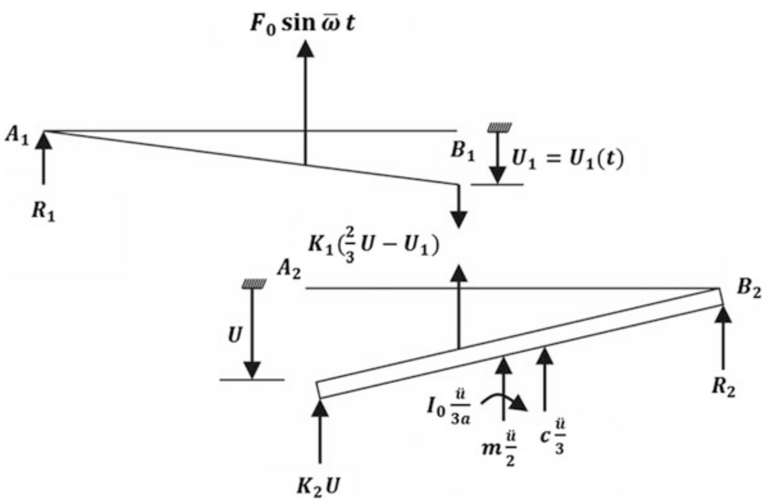

Fig. 21.5 Displacements and resultant forces for Example 21.1

21.3 Generalized Single-Degree-of-Freedom System–Distributed Elasticity

The example presented in the preceding section had only one degree of freedom in spite of the complexity of the various parts of the system because the two bars were interconnected through a spring and one of the bars was massless so that only one coordinate sufficed to completely specify the motion. If the bars were not rigid, but could deform in flexure, the system would have an infinite number of degrees of freedom. However, a single-degree-of-freedom analysis could still be made, provided that only a single shape could be developed during motion, that is, provided that the knowledge of the displacement of a single point in the system determines the displacement of the entire system.

As an illustration of this method for approximating the analysis of a system with an infinite number of degrees of freedom with a single degree of freedom, consider the cantilever beam shown in Fig. 21.6. In this illustration, the physical properties of the beam are the flexural stiffness $EI(x)$ and its mass per unit of length $m(x)$. It is assumed that the beam is subjected to an arbitrary distributed forcing function $p(x, t)$ and to an axial compressive force N.

In order to approximate the motion of this system with a single coordinate, it is necessary to assume that the beam deflects during its motion in a prescribed shape. Let $\phi\ (x)$ be the function describing this shape and, as a generalized coordinate, $U(t)$ the function describing the displacement of the motion corresponding to the free end of the beam. Therefore, the displacement at any point x along the beam is

$$u(x,\ \ t) = \phi(x)U(t) \tag{21.7}$$

where $\phi\ (L) = 1$.

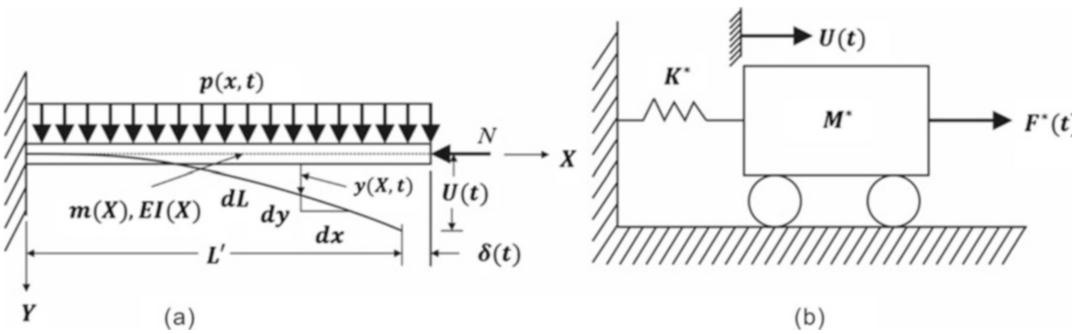

Fig. 21.6 Single-degree-of-freedom continuous system

The equivalent one-degree-of-freedom system (Fig. 21.6b) may be defined simply as the system for which the kinetic energy, potential energy (strain energy), and work done by the external forces have at all times the same values in the two systems.

The kinetic energy T of the beam in Fig. 21.6 vibrating in the pattern indicated by Eq. (21.7) is

$$T = \int_0^L \frac{1}{2} m(x) \{\phi(x)U(t)\}^2 dx \tag{21.8}$$

Equating this expression for the kinetic energy of the continuous system to the kinetic energy of the equivalent single-degree-of-freedom system $\frac{1}{2}M^* U^2(t)$ and solving the resulting equation for the generalized mass, we obtain

$$M^* = \int_0^L m(x)\ \ \phi^2(x) dx \tag{21.9}$$

The flexural strain energy V of a prismatic beam may be determined as the work done by the bending moment $M(x)$ undergoing an angular displacement $d\theta$. This angular displacement is obtained from the well-known formula for the flexural curvature of a beam, namely

$$\frac{d^2u}{dx^2} = \frac{d\theta}{dx} = \frac{M(x)}{EI} \tag{21.10}$$

or

$$d\theta = \frac{M(x)}{EI}dx \tag{21.11}$$

since $du/dx = \theta$, where θ, being assumed small, is taken as the slope of the elastic curve. Consequently, the strain energy is given by

$$V = \int_0^L \frac{1}{2}M(x)d\theta \tag{21.12}$$

The factor $\frac{1}{2}$ is required for the correct evaluation of the work done by the flexural moment increasing from zero to its final value $M(x)$ [average value $M(x)/2$]. Now, utilizing Eqs. (21.10 and 21.11) in Eq. (21.12), we obtain

$$V = \int_0^L \frac{1}{2}EI(x)\left(\frac{d^2u}{dx^2}\right)^2 dx \tag{21.13}$$

Finally, equating the potential energy, Eq. (21.13), for the continuous system to the potential energy of the equivalent system and using Eq. (21.7) results in

$$\frac{1}{2}K^*U(t)^2 = \int_0^L \frac{1}{2}EI(x)\{\phi''(x)U(t)\}^2 dx$$

or

$$K^* = \int_0^L EI(x)\{\phi''(x)\}^2 dx \tag{21.14}$$

where

$$\phi''(x) = \frac{d^2u}{dx^2}$$

The generalized force $F^*(t)$ may be found from the virtual displacement $\delta U(t)$ of the generalized coordinate $U(t)$ upon equating the work performed by the external forces in the structure to the work done by the generalized force: in the equivalent single-degree-of-freedom system. The work of the distributed external force $p(x, t)$ due to this virtual displacement is given by

$$W = \int_0^L p(x,t)\delta u dx$$

Substituting $\delta u = \phi(x)\, \delta U$ from Eq. (21.7) gives

$$W = \int_0^L p(x,t)\phi(x)\delta U dx \tag{21.15}$$

The work of the generalized force $F^{*}(t)$ in the equivalent system corresponding to the virtual displacement δU of the generalized coordinate is

$$W^{*} = F^{*}(t)\delta U \tag{21.16}$$

Equating Eq. (21.15) with Eq. (21.16) and canceling the factor $\delta\, u$, which is taken to be different from zero, we obtain the generalized force as

$$\mathrm{F}^{*}(\mathrm{t}) = \int_0^L p(x,t)\phi(x)dx \tag{21.17}$$

Similarly, to determine the generalized damping coefficient, assume a virtual displacement and equate the work of the damping forces in the physical system with the work of the damping force in the equivalent single-degree of-freedom system. Hence

$$C^{*}\dot{U}\,\delta U = \int_0^L c(x)u\;\delta u dx$$

where $c(x)$ is the distributed damping coefficient per unit length along the beam. Substituting $\delta u = \phi(x)\delta u$ and $\dot{u} = \phi(x)\dot{U}$ from Eq. (21.7) and canceling the common factors, we obtain

$$C^{*} = \int_0^L c(x)[\phi(x)]^2 dx \tag{21.18}$$

which is the expression for the generalized damping coefficient.

To calculate the potential energy of the axial force N which is unchanged during the vibration of the beam and consequently is a conservative force, it is necessary to evaluate the horizontal component of the motion $\delta(t)$ of the free end of the beam as indicated in Fig. 21.6. For this purpose, we consider a differential element of length dL along the beam as shown in Fig. 21.6a. The length of this element may be expressed as

$$dL = \left(dx^2 + du^2\right)^{1/2}$$

or

$$dL = \left(1 + (du/dx)^2\right)^{½} dx \tag{21.19}$$

Now, integrating over the horizontal projection of the length of beam (L') and expanding in series the binomial expression, we obtain

$$\begin{aligned} L &= \int_0^{L'} \left(1 + \left(\tfrac{du}{dx}\right)^2\right)^{1/2} dx \\ &= \int_0^{L'} \left\{1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2 - \frac{1}{8}\left(\frac{du}{dx}\right)^4 + \ldots\right\} dx \end{aligned}$$

Retaining only the first two terms of the series results in

$$L = L' + \int_0^L \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx \tag{21.20}$$

or

$$\delta(t) = L - L' = \int_0^L \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx \tag{21.21}$$

The reader should realize that Eqs. (21.20 and 21.21) involve approximations since the series was truncated and the upper limit of the integral in the final expression was conveniently set equal to the initial length of the beam L instead of to its horizontal component L'.

Now we define a new stiffness coefficient to be called the generalized geometric stiffness K^*_G as the stiffness of the equivalent system required to store the same potential energy as the potential energy stored by the normal force N, that is,

$$\frac{1}{2}K^*{}_G U(t)^2 = N\delta(t)$$

Substituting $\delta(t)$ from Eq. (21.21) and the derivative du/dx from Eq. (21.7), we have

$$\frac{1}{2}K^*{}_G U(t)^2 = \frac{1}{2}N\int_0^L \left\{U(t)\frac{d\phi}{dx}\right\}^2 dx$$

or

$$K^*{}_G = N\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \tag{21.22}$$

Equations (21.9, 21.14, 21.17, 21.18 and 21.22) give, respectively, the generalized expression for the mass, stiffness, force, damping, and geometric stiffness for a beam with distributed properties and load, modeling it as a simple oscillator.

For the case of an axial compressive force, the potential energy in the beam decreases with a loss of stiffness in the beam. The opposite is true for a tensile axial force, which results in an increase of the flexural stiffness of the beam. Customarily, the geometric stiffness is determined for a compressive axial force. Consequently, the combined generalized stiffness K^*_G is then given by

$$K^*{}_c = K^* - K^*{}_G \tag{21.23}$$

Finally, the differential equation for the equivalent system may be written as

$$M^*\ddot{U}(t) + C^*U(t) + K^*{}_C U(t) = F^*(t) \tag{21.24}$$

The critical buckling load N_{cr} is defined as the axial compressive load that reduces the combined stiffness to zero, that is,

$$K^*{}_c = K^* - K^*{}_G = 0$$

The substitutian of K^* and K^*_G from Eqs. (21.14 and 21.22) gives

$$\int_0^L EI\left(\frac{d^2\phi}{dx^2}\right)^2 dx - N_{cr}\int\left(\frac{d\phi}{dx}\right)^2 dx = 0$$

and solving for the critical buckling load, we obtain

$$N_{cr} = \frac{\int_0^L EI\left(d^2\phi/dx^2\right)^2 dx}{\int_0^L (d\phi/dx)^2 dx} \tag{21.25}$$

Once the generalized stiffness K* and generalized mass M* have been determined, the system can be analyzed by any of the methods presented in the preceding chapters for single-degree-of-freedom systems. In particular, the square of the natural frequency, ω^2, is given from Eqs. (21.9 and 21.14) by

$$\omega^2 = \frac{K^*}{M^*} = \frac{\int_0^L EI(x)\phi''(x)^2 dx}{\int_0^L m(x)\phi^2(x) dx} \tag{21.26}$$

The displacement $U(t)$ of the generalized single-degree-of-freedom system is found as the solution of the differential equation

$$M^* U(1) + C^* U(1) + K^* U(1) = F^*(1) \tag{21.27}$$

and the displacement $u(x, t)$ at location x and time t is then calculated by Eq. (21.7).

21.4 Shear Forces and Bending Moments

The internal forces-shear forces and bending moments-associated with the displacements $u(x, t)$ may be obtained by loading the structure with the equivalent forces that would produce the dynamic displacements calculated. From elementary beam theory, the rate of loading $p(x, t)$ per unit of length along abeam is related to the lateral displacement $u = u(x, t)$ by the differential equation

$$p(x,t) = \frac{\partial}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] \tag{21.28}$$

or substituting $u(x, t) = \phi(x)\cdot U(t)$ from Eq. (21.7) by

$$p(x,t) = \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 \Phi}{\partial x^2}\right]U(t) \tag{21.29}$$

The equivalent force $p(x, i)$, calculated using Eq. (21.29), depending on derivatives of the shape function, will in general, give internal forces that are less accurate than the displacements, because the derivatives of approximate shape functions tends to increase errors in the shape function.

A better approach to estimate the equivalent forces is to compute the inertial forces that resulted in the calculated displacements. These inertial forces are given by

$$p(x,t) = m(x)\ddot{u}(x,t) \tag{21.30}$$

or substituting $\ddot{U}(x, t)$ from Eq. (21.7) by

$$p(x,t) = m(x)\phi(x)\ddot{U}(t) \tag{21.31}$$

At any time t the equivalent load $p(x, t)$ given by Eq. (21.31) is applied to the structure as a static load and the internal forces (shear forces and bending moments), as well as the stresses resulting from these internal forces, are determined.

To provide an example of the determination of the equivalent one degree of freedom for a system with distributed mass and stiffness, consider the water tower in Fig. 21.7 to have uniformly distributed mass iii and stiffness EI along its length with a concentrated mass $M = \bar{m}L$ at the top. The tower is subjected to an earthquake ground motion excitation of acceleration $a_g(t)$ and to an axial compressive load due to the weight of its distributed mass and concentrated mass at the top. Neglect damping in the system. Assume that during the motion the shape of the tower is given by

$$\phi(x) = 1 - \cos\frac{\pi x}{2L} \tag{21.32}$$

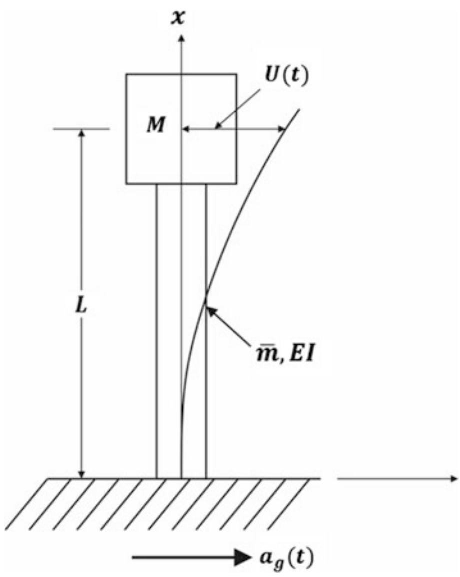

Fig. 21.7 Water tower with distributed properties for Illustrative Example 21.2

Selecting the lateral displacement $u(t)$ at the top of the tower as the generalized coordinate as shown in Fig. 21.7, we obtain for the displacement at any point

$$U(x,t) = U(t)\phi(x) = U(t)\left(1 - \cos\frac{\pi x}{2L}\right) \tag{21.33}$$

The generalized mass and the generalized stiffness of the tower are computed, respectively, from Eqs. (21.9 and 21.14) as

$$M^* = \bar{m}L + \bar{m}\int_0^L \left(1 - \cos\frac{\pi x}{2L}\right)^2 dx$$

$$M^* = \frac{\bar{m}L}{2\pi}(5\pi - 8) \tag{21.34}$$

and

$$K^* = \int_0^L EI\left(\frac{\pi}{2L}\right)^4 \cos^2\frac{\pi x}{2L}dx$$

$$K^* = \frac{\pi^4 EI}{32L^3} \tag{21.35}$$

The axial force is due to the weight of the tower above a particular section, including the concentrated weight at the top, and may be expressed as

$$N(x) = \bar{m}\ Lg\left(2 - \frac{x}{L}\right) \tag{21.36}$$

where g is the gravitational acceleration. Since the normal force in this case is a function of x, it is necessary in using Eq. (21.22) to include $N(x)$ under the integral sign. The geometric stiffness coefficient K^*_G is then given by

$$K^*_G = \int_0^L \bar{m}Lg\left(2 - \frac{x}{L}\right)\left(\frac{\pi}{2L}\right)^2 \sin^2\frac{\pi x}{2L}dx$$

which upon integration yields

$$K^*_G = \frac{\bar{m}g}{16}\left(3\pi^2 - 4\right) \tag{21.37}$$

Consequently, the combined stiffness from Eqs. (21.35 and 21.37) is

$$K^*_c = K^* - K^*_G = \frac{\pi^4 EI}{32L^3} - \frac{\bar{m}g}{16}\left(3\pi^2 - 4\right) \tag{21.38}$$

By setting $K^*_c = 0$, we obtain

$$\frac{\pi^4 EI}{32L^3} - \frac{\bar{m}g}{16}\left(3\pi^2 - 4\right) = 0$$

which gives the critical load

$$\left(\bar{m}g\right)_{cr} = \frac{\pi^4 EI}{2(3\pi^3 - 4)L^3} \tag{21.39}$$

The equation of motion in terms of the relative motion $u = u(t) - u_g(t)$ is given by Eq. (3.50) for the undamped system as

$$M^*\ddot{u}_r + K^*_c u_r = F^*_{eff}(t) \tag{21.40}$$

where M^* is given by Eq. (21.34), K^*_c by Eq. (21.38), and the effective force by Eq. (21.17) for the effective distributed force and by $-\ \bar{m}\ La_g(t)$ for the effective concentrated force at the top of the tower. Hence

$$F^*_{eff}(t) = \int_0^L p_{eff}(x,t)\phi(x)dx - \bar{m}La_g(t)$$

where $p_{eff}(x,\ \ t) = -\bar{m}\ \ a_g(t)$ is the effective distributed force. Then

$$F^*_{eff}(t) = \int_0^L -\bar{m}a_g(t)\phi(x)dx - \bar{m}La_g(t)$$

Substitution of $\phi(x)$ from Eq. (21.32) into the last equation yields upon integration

$$F^*_{eff}(t) = -\frac{2\bar{m}a_g(t)L}{\pi}(\pi - 1) \tag{21.41}$$

Illustrative Example 21.2

As a numerical example of calculating the response of a system with distributed properties, consider the water tower shown in Fig. 21.7 excited by a sinusoidal ground acceleration $a_g(t) = 20\sin 6.36\, t$ (in/sec^2).

Model the structure by assuming the shape given by Eq. (21.32) and determine the response.

Solution:

The numerical values for this example are:

$$\bar{m} = 0.1 \ \text{Kip} \cdot \sec^2/\text{in per unit of length}$$
$$EI = 1.2 \ \ 10^{13} \ \text{Kip} \cdot \text{in}^2$$
$$L = 100 \ \text{ft} \ = \ 1200 \ \text{in}$$
$$\varpi = 6.36 \ \text{rad/sec}$$

From Eq. (21.34) the generalized mass is

$$M^* = \frac{0.1 \times 1200}{2\pi}(5\pi - 8) = 147.21 \ \text{Kip} \cdot \sec^2/\text{in}$$

and from Eq. (21.38) the generalized combined stiffness is

$$K^*_c = \frac{\pi^4 1.2 10^{13}}{32 \times (1200)^3} - \frac{0.1 \times 386}{16}\left(3\pi^2 - 4\right) = 21{,}077 \ \text{Kip/in}$$

The natural frequency is

$$\omega = \sqrt{K^*_c/M^*} = 11.96 \ \text{rad/sec}$$

and the frequency ratio

$$r = \frac{\varpi}{\omega} = \frac{6.36}{11.96} = 0.532$$

From Eq. (21.41) the effective force is

$$F^*_{eff} = -\frac{2(0.1)(1200)(\pi - 1)}{\pi} 20.0 \sin 6.36t$$

or

$$F^*_{eff} = -3272 \sin 6.36t(\text{Kip})$$

Hence the steady-state response (neglecting damping) in terms of relative motion is given from Eq. (3.9) as

$$u_r = \frac{F^*_{eff}/k^*}{1-r^2}\sin\varpi t$$

$$= -\frac{3272/21,077}{1-(0.532)^2}\sin 6.36t$$

$$= -0.217\sin 6.36t \ \text{ in} \qquad \text{(Ans)}$$

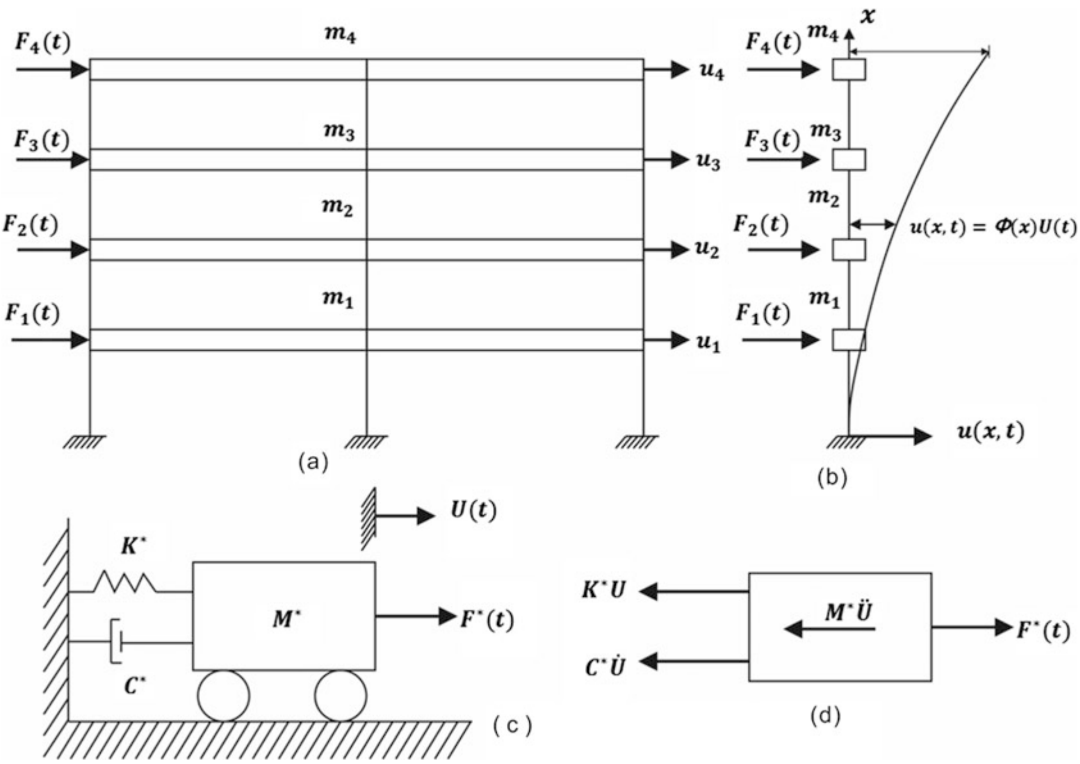

Fig. 21.8 (**a**) Multistory building subjected to lateral forces $F_i(t)$. (**b**) Column modeling the building showing assumed lateral displacement function $u(x,t) = \phi(x)U(t)$. (**c**) Equivalent single-degree-of-freedom system. (**d**) Free body diagram

21.5 Generalized Equation of Motion for a Multistory Building

Consider a multistory building such as the model for the four-story building shown in Fig. 21.8a subjected to lateral dynamic forces $F_i(t)$ at the different levels of the building. The mass of the building is assumed to be concentrated at the various levels (floors and roof) which are assumed to be rigid in their own planes; thus only horizontal displacements are possible in such a building.

To model this structure as a single degree of freedom (Fig. 21.8c), the lateral displacement shape u (x, t) is defined in terms of a single generalized coordinate $U(t)$ as

$$u(x,t) = \phi(x)U(t) \tag{21.42}$$

The generalized coordinate $f(t)$ in Eq. (21.42) is selected as the lateral displacement at the top level of the building which requires that shape function be assigned a unit value at that level; that is, $\phi(H) = 1.0$ where H is the height of the building.

The equation of motion for the generalized single-degree-of-freedom system in Fig. 21.8c is obtained by equating to zero the sum of forces in the corresponding free-body diagram (Fig. 21.8d), that is

$$M^*\ddot{U} + C^*\dot{U} + K^*U = F^*(t) \tag{21.43}$$

where M^*, C^*, K^*, and $F^*(t)$ are, respectively, the generalized mass, generalized damping, generalized stiffness, and generalized force, and are given for a discrete system, modeled in Fig. 21.8b, by

$$M^* = \sum_{i=1}^{N} m_i \Phi_i^2 \tag{21.44a}$$

$$C^* = \sum_{i=1}^{N} c_i \Delta\Phi_i^2 \tag{21.44b}$$

$$K^* = \sum_{i=1}^{N} k_i \Delta\Phi_i^2 \tag{21.44c}$$

$$F^*(t) = \sum_{i=1}^{N} F_i(t)\Phi_i \tag{21.44d}$$

where the upper index N in the summations is equal to the number of stories or levels in the building.

The various expressions for the equivalent parameters in Eq. (21.44) are obtained by equating the kinetic energy, potential energy, and the virtual work done by the damping forces and by external forces in the actual structure with the corresponding expressions for the generalized single-degree-of-freedom system.

In Eq. (21.44), the relative displacement $\Delta\phi_i$ between two consecutive levels of the building is given by

$$\Delta\phi_i = \phi_i - \phi_{i-1} \tag{21.45}$$

with $\phi_0 = 0$ at the ground level.

As shown in Fig. 21.8b, m_i and $F_i(t)$ are, respectively, the mass and the external force at level i of the building, while k_i and c_i are the stiffness and damping coefficients corresponding to the ith story.

It is convenient to express the generalized damping coefficient C^* of Eq. (21.44b) in terms of generalized damping ratio ξ^*; thus by Eq. (2.7)

$$C^* = \sum_{i=1}^{N} c_i \Delta\Phi_i^2 = \xi^* C^*_{cr} = 2\xi^* \; M^*\omega \tag{21.46}$$

in which w is the natural frequency calculated for the generalized system as

$$\omega = \sqrt{\frac{K^*}{M^*}} \tag{21.47}$$

The substitution of the generalized damping coefficient, ξ^*, from Eq. (21.46) and of $K^* = \omega^2 M^*$ from Eq. (21.47) into the differential equation of motion, Eq. (21.43), results in

$$\ddot{U} + 2\xi^* \omega \dot{U} + \omega^2 U = \frac{F^*(t)}{M^*} \tag{21.48}$$

When the excitation of the building is due to an acceleration function, $a(t) = U_0(t)$, acting at the base of the building, the displacements $u_i(t)$ are conveniently measured relative to the motion of the base $u_0(t)$, that is,

$$u_{ri}(t) = u_i(t) - u_0(t) \tag{21.49}$$

Also, in this case, the effective forces $F_{eff,i}$ at the various levels of the building are given from Eq. (4.39) as

$$F_{eff,i} = -m_i \ddot{u}_0(t) \tag{21.50}$$

The differential equation for the generalized single-degree-of-freedom system excited at its base is then written as

$$M^*{}_{\ddot{U}^*_r}(t) + C^*{}_{\dot{U}_r}(t) + K^* U_r(t) = F^*{}_{eff}(t) \tag{21.51}$$

where the generalized coordinate for the relative displacement, $U_r(t)$ is

$$U_r(t) = U(t) - U_{r0}(t) \tag{21.52}$$

The generalized effective force $F^*{}_{eff}(t)$ is calculated by Eqs. (21.44d and 21.50) as

$$F^*{}_{eff} = -\ddot{U}_0(t) \sum_{i=1}^{N} m_i \Phi_i \tag{21.53}$$

The generalized equation of motion may then be expressed as

$$\ddot{U}_r(t) + 2\xi^* \omega \dot{U}_r(t) + \omega^2 U_r(t) = \Gamma^* \ddot{U}_{r0}(t) \tag{21.54}$$

in which the coefficient Γ^* is the generalized participation factor given by

$$\Gamma^* = -\frac{\sum_{i=1}^{N} m_i \Phi_i}{\sum_{i=1}^{N} m_i \Phi_i^2} \tag{21.55}$$

21.6 Shape Function

The use of generalized coordinates transforms a multidegree-of-freedom system into an equivalent single-degree-of-freedom system. The shape function describing the deformed structure could be any arbitrary function that satisfies the boundary conditions. However, in practical applications, the success of this approach will depend on how close the assumed shape function approximates the actual displacements of the dynamic system. For structural buildings, selection of the shape function is most appropriate by considering the aspect ratio of the structure, which is defined as the ratio of the building height to the dimension of the base. The recommended shape functions for high-rise, mid-rise, and low-rise buildings are summarized in Fig. 21.9. Most seismic building codes use the straight-line shape which is shown for the mid-rise building. The displacements in the structure are calculated using Eq. (21.7) after the dynamic response is obtained in terms of the generalized coordinate.

Illustrative Example 21.3

A four-story reinforced concrete framed building has the dimensions shown in Fig. 6.10. The sizes of the exterior columns (nine each on lines *A* and *C*) are 12 in × 20 in. and the interior columns (nine on line *B*) are 12 in × 24 in for the bottom two stories, and, respectively, 12 in × 16 in and 12 in × 20 in for the highest two stories. The height between floors is 12 ft. The dead load per unit area of the floor (floor slab, beam, half the weight of columns above and below the floor, partition walls, etc.) is estimated to be 140 psf. The design live load is taken as 25% of an assumed live load of 125 psf. Determine the generalized mass, generalized stiffness, generalized damping (for damping ratio $\xi^* = 0.1$), and the fundamental period for lateral vibration perpendicular to the long axis of the building. Assume the following shape functions: (a) $\phi(x) = x/H$ and (b) $\phi(x)$ sin $(\pi x/2H)$ where H is the height of the building (Fig. 21.10).

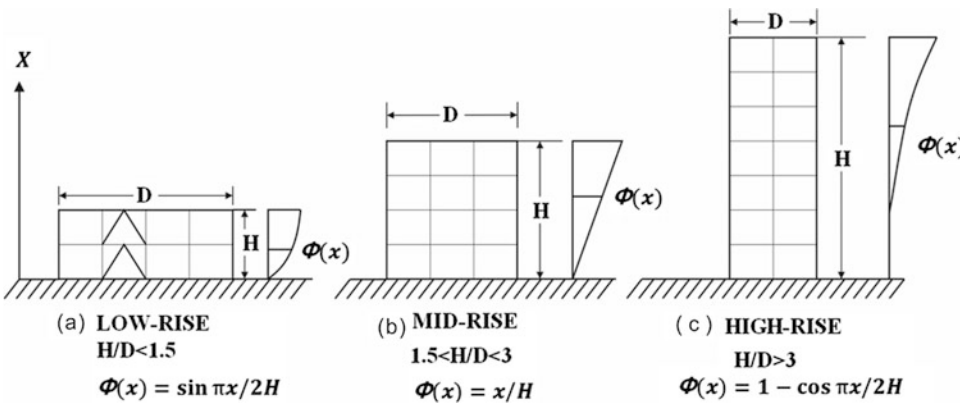

Fig. 21.9 Possible shape functions based on aspect ratio (Naeim 1989, p. 100)

Solution:

1. Effective weight at various floors:

No live load needs to be considered on the roof. Hence, the effective weight at all floors, except at the roof, will be 140 + 0.25 × 125 = 171.25 psf, and the effective weight for the roof will be 140 psf. The plan area is 48 ft. × 96 ft. =4608 ft^2. Hence, the weights of various levels are:

$W_1 = W_2 = W_3 = 4608 \times 0.17125 = 789.1$ Kips.
$W_4 = 4608 \times 0.140 = 645.1$ Kips.

The total seismic design weight of the building is then
$W = 789.1 \times 3 + 645.1 = 3012.4$ Kips

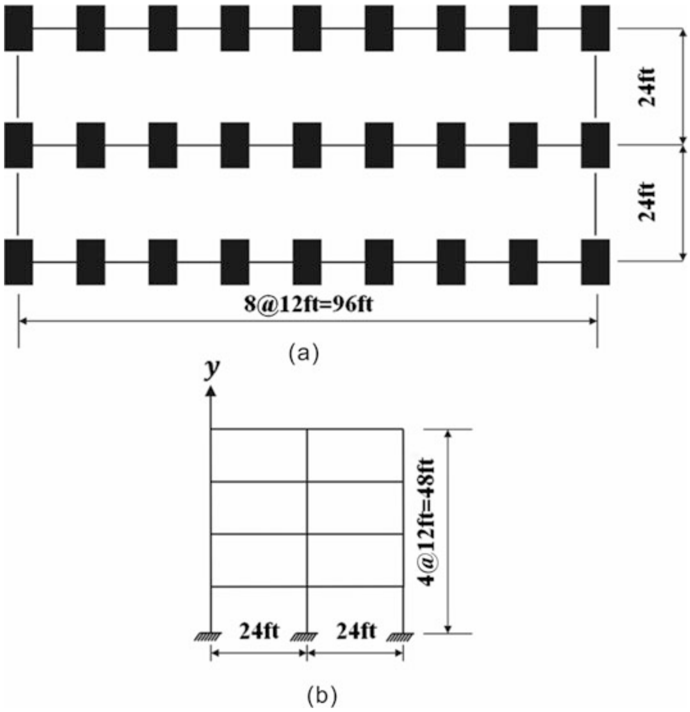

Fig. 21.10 Plan and elevation for a four-story building for Example 21.3; (**a**) Plan. (**b**) Elevation

2. Story lateral stiffness:

It will be assumed that horizontal beam-and-floor diaphragms are rigid compared to the columns of the building in order to simplify the hand calculation. In this case, the stiffness between two consecutive levels is given by

$$k = \frac{12EI}{L^3}$$

where

$L = 12$ ft. (distance between two floors)

$E = 3 \times 10^3$ ksi (modulus of elasticity of concrete)

$I = \frac{1}{12} 12 \times 20^3 = 8000$ in^4 (moment of inertia for the concrete section for columns 12 in $\times$ 20 in)

Therefore, for these columns,

$$k = \frac{12 \times 3 \times 10^3 \times 8000}{144^3} = 96.450 \text{ Kip/in}$$

Similarly, for columns 12 in $\times$ 24 in,

$$I = 13{,}824 \text{ in}^4 \text{ and } k = 166.667 \text{ Kip/in}$$

The total stiffness for the first and second stories is then

$$K_1 = K_2 = 18 \times 96.45 + 9 \times 166.67 = 3236 \text{ Kip/in}$$

Similarly, for the columns 12 in × 16 in. of the top and third stories,

$$I = \frac{1}{12} 12 \times 16^3 = 4096 \ \text{in}^4, \ \ k = \frac{12 \times 3 \times 10^3 \times 4096}{144^3} = 49.4 \ \text{Kip/in}$$

and for the columns 12 in × 20 in,

$$I = \frac{1}{12} 12 \times 20^3 = 8000 \ \text{in}^4, \ \ k = \frac{12 \times 3 \times 10^3 \times 8000}{144^3} = 96.5 \ \text{Kip/in}$$

Hence, total stiffness for the third or fourth stories is

$$K_3 = K_4 = 18 \times 49.4 + 9 \times 96.5 = 1757.7 \ \text{Kip/in}$$

3. *Generalized mass and stiffness:*
   (a) Assuming $\phi\ (x) = x/H$,

Table 21.1 shows the necessary calculations to obtain using Eqs. (21.44a and 21.44c) the generalized mass M^* and the generalized stiffness K^* for this example assuming $\phi(x) = x/H$.

The natural frequency is then calculated from Eq. (21.47) as

$$\omega = \sqrt{\frac{K^*}{M^*}} = \sqrt{\frac{625.250}{3.559}} = 13.25 rad/\sec, \ \ \text{then} \ \ T_a = \frac{2\pi}{\omega} = 0.47 \sec$$

and the generalized critical damping C^*_{cr} and the absolute generalized damping C^*, respectivaly, by Eqs. (2.6 and 2.19) as

$$C*_{cr} = 2\sqrt{K^* M^*} = 2\sqrt{(625.250)(3.559)} = 94.345 (lb \cdot \sec/in)$$

and

$$C^* = \xi^* C^*_{cr} = (0.1)(94.345) = 9.43 (lb \cdot \sec/in)$$

Table 21.1 Calculation of M^* and K^* Assuming $\phi\ (x) = x/H$

Level	k_i(Kip/in)	m_i(Kip · sec^2/in)	ϕ_i	$\Delta\phi_i$	$m_i\phi_i^2$(Kip sec^2/in)	$k_i\Delta\phi_i^2$ (Kip/in)
4		1.671	1.000		1.671	
	1758			0.250		109.875
3		2.044	0.750		1.150	
	1758			0.250		109.875
2		2.044	0.500		0.610	
	3236			0.250		202.250
1		2.044	0.250		0.128	
	3236			0.250		202.250
					$M^* = 3.559$	$K^* = 625.250$

(b) Assuming $\phi\ (x) = \sin(\pi\ x/2H)$.

Table 21.2 shows the necessary calculations to obtain using Eqs. (21.44a and 21.44c) the generalized mass M^* and the generalized stiffness K^* for this example assuming $\phi\ (x) = \sin(\pi\ x/2H)$.

Table 21.2 Calculation of M^* and K* Assuming ϕ (x) = sin (π $x/2H$)

Level	k_i(Kip/in)	m_i(Kip·sec^2/in)	ϕ_i	$\Delta\phi_i$	$m_i\phi_i^2$(Kip·sec^2/in)	$k_i\Delta\phi_i^2$(Kip/in)
4		1.671	1.000		1.671	
	1758			0.076		10.154
3		2.040	0.924		1.745	
	1758			0.217		82.782
		2.040	0.707		1.022	
	3236			0.324		339.702
1		2.040	0.383		0.300	
	3236			0.383		476.686
					M^* = 4.738	K^* = 909.324

The natural frequency is then calculated from Eq. (21.47) as

$$\omega = \sqrt{\frac{K^*}{M^*}} = \sqrt{\frac{909.324}{4.738}} = 13.85 rad/\sec \quad \text{then} \quad T_b = \frac{2\pi}{\omega} = 0.45 \sec$$

and the generalized critical damping C^*_{cr} and the absolute generalized damping C^*, respectively, by Eqs. (21.48 and 21.49) as

$$C^*_{cr} = 2\sqrt{K^*M^*} = 2\sqrt{(909.324)(4.738)} = 131.276(lb \cdot \sec/in)$$

and

$$C^* = \xi^* C^*_{cr} = (0.1)(131.276) = 13.13(lb \cdot \sec/in)$$

For this example, either of the two assumed shape functions results in essentially the same value for the fundamental period. However, assuming ϕ (x) = x/H is a slightly better approximation[1] to the true deflected shape than is ϕ (x) = sin $(\pi x/2H)$ because $T_a > T_b$.

21.7 Rayleigh's Method

In the preceding sections of this chapter the differential equation for a vibrating system was obtained by application of the principle of virtual work as an alternative method of considering the dynamic equilibrium of the system. However, the differential equation of motion for an undamped system in free vibration may also be obtained with the application of the *Principle of Conservation of Energy*. This principle may be stated as follows: If no external forces are acting on the system and there is no dissipation of energy due to damping, then the total energy of the system must remain constant during motion and consequently its derivative with respect to time must be equal to zero.

To illustrate the application of the Principle of Conservation of Energy in obtaining the differential equation of motion, consider the spring-mass system shown in Fig. 21.11. The total energy in this

[1] For an assumed displacement shape closer to the actual displacement shape, the structure will vibrate closer to free condition with less imposed constrains, thus with the stiffness reduced and a longer period.

case consists of the sum of the kinetic energy of the mass and the potential energy of the spring. In this case the kinetic energy T is given by.

$$T = \frac{1}{2} my\dot{u}^2 \tag{21.56}$$

where $\dot{u}$ is the instantaneous velocity of the mass.

The force in the spring, when displaced y units from the equilibrium position, is ku and the work done by this force on the mass for an additional displacement du is $-ku\ du$. This work is negative because the force ku acting on the mass is opposite to the incremental displacement du given in the positive direction of coordinate u. However, by definition, the potential energy is the value of this work but with opposite sign. It follows then that the total potential energy (PE) in the spring for a final displacement u will be

$$(PE) = \int_0^y kudu = \frac{1}{2} ku^2 \tag{21.57}$$

Adding Eqs. (21.56 and 21.57), and setting this sum equal to a constant, will give

$$\frac{1}{2} my\dot{u}^2 + \frac{1}{2} ku^2 = C_0 \tag{21.58}$$

Differentiation with respect to time yields

$$my\dot{u}\,\ddot{u} + ku\dot{u} = 0$$

Since $\dot{y}$ cannot be zero for all values of t, it follows that

$$m\ddot{u} + ku = 0 \tag{21.59}$$

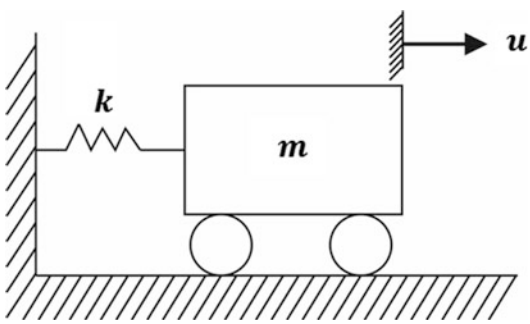

Fig. 21.11 Spring-mass system in free vibration

This equation is identical with Eq. (1.11) of Chap. 1 obtained by application of Newton's Law of Motion. Used in this manner, the energy method has no particular advantage over the equilibrium method. However, in many practical problems it is only the natural frequency that is desired. Consider again the simple oscillator of Fig. 21.11, and assume that the motion is harmonic. This assumption leads to the equation of motion of the form

$$u = C \sin(\omega t + \alpha) \tag{21.60}$$

and velocity

$$\dot{u} = \omega C \cos(\omega t + \alpha) \tag{21.61}$$

where C is the maximum displacement and ωC the maximum velocity. Then, at the neutral position ($u = 0$), there will be no force in the spring and the potential energy is zero. Consequently, the entire energy is then kinetic energy and

$$T_{\max} = \frac{1}{2} m(\omega C)^2 \tag{21.62}$$

At the maximum displacement the velocity of the mass is zero and all the energy is then potential energy, thus

$$(PE)_{\max} = \frac{1}{2} k C^2 \tag{21.63}$$

The energy in the system changes gradually over one-quarter of the cycle from purely kinetic energy, as given by Eq. (21.62), to purely potential energy, as given by Eq. (21.63). If no energy has been added or lost during the quarter cycle, the two expressions for this energy must be equal. Thus

$$\frac{1}{2} m\omega^2 C^2 = \frac{1}{2} k C^2 \tag{21.64}$$

Canceling common factors and solving Eq. (21.64) will give

$$\omega = \sqrt{\frac{k}{m}} \tag{21.65}$$

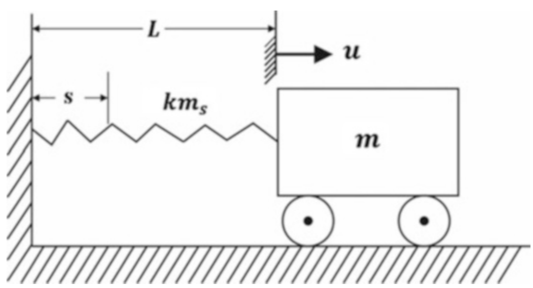

Fig. 21.12 Spring-mass system with heavy spring

which is the natural frequency for the simple oscillator obtained previously from the differential equation of motion. This method, in which the natural frequency is obtained by equating maximum kinetic energy with maximum potential energy, is known as Rayleigh's Method.

Illustrative Example 21.4

In the previous calculations on the spring-mass system, the mass of the spring was assumed to be so small that its effect on the natural frequency could be neglected. A better approximation to the true value of the natural frequency may be obtained using Rayleigh's Method. The distributed mass of the spring could easily be considered in the calculation by simply assuming that the deflection of the spring along its length is linear. In this case, consider in Fig. 21.12 the spring-mass system for which the spring has a length L and a total mass m_s. Use Rayleigh's Method to determine the fraction of the spring mass that should be added to the vibrating mass.

Solution:
The displacement of an arbitrary section of the spring at a distance s from the support will now be assumed to be $u_r = su/L$. Assuming that the motion of the mass m is harmonic and given by Eq. (21.60), we obtain

$$u_r = \frac{s}{L}u = \frac{s}{L}C\sin(\omega t + \alpha) \tag{21.66}$$

The potential energy of the uniformly stretched spring is given by Eq. (21.57) and its maximum value is

$$(PE)_{\max} = \frac{1}{2}kC^2 \tag{21.67}$$

A differential element of the spring of length ds has mass equal to $m_s ds/L$ and maximum velocity $\dot{u}_{r\max} = \omega u_{r\max} = \omega s\ \ C/L$. Consequently the total kinetic energy in the system at its maximum value is

$$T_{\max} = \int_0^L \frac{1}{2}\frac{m_s}{L}ds\left(\omega\frac{s}{L}C\right)^2 + \frac{1}{2}m\omega^2C^2 \tag{21.68}$$

After integrating Eq. (21.68) and equating it with Eq. (21.67), we obtain

$$\frac{1}{2}kC^2 = \frac{1}{2}\omega^2C^2\left(m + \frac{m_s}{3}\right) \tag{21.69}$$

Solving for the natural frequency yields

$$\omega = \sqrt{\frac{k}{m + \frac{m_s}{3}}} \tag{21.70}$$

or in cycles per second (cps),

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m + \frac{m_s}{3}}} \tag{21.71}$$

The application of Rayleigh's Method shows that a better value for the natural frequency may be obtained by adding one-third of the mass of the spring to that of the main vibrating mass.

Rayleigh's Method may also be used to determine the natural frequency of a continuous system provided that the deformed shape of the structure is described as a generalized coordinate. The deformed shape of continuous structures and also of discrete structures of multiple degrees of freedom could in general be assumed arbitrarily. However, in practical applications, the success of the method depends on how close the assumed deformed shape will come to match the actual shape of the structure during vibration. Once the deformed shape has been specified, the maximum kinetic energy and the maximum potential energy may be determined by application of pertinent equations such as Eqs. (21.8 and 21.13). However, if the deformed shape has been defined as the shape resulting from statically applied forces, it would be simpler to calculate the work done by the external forces, instead of directly determining the potential energy. Consequently, in this case, the maximum kinetic energy is equated to the work of the forces applied statically. The following examples illustrate the application of Rayleigh's Method to systems with distributed properties.

Illustrative Example 21.5

Determine the natural frequency of vibration of a cantilever beam with a concentrated mass at its end when the distributed mass of the beam is taken into account. The beam has a total mass m_b and length L. The flexural rigidity of the beam is EI and the concentrated mass at its end is m, as shown in Fig. 21.13.

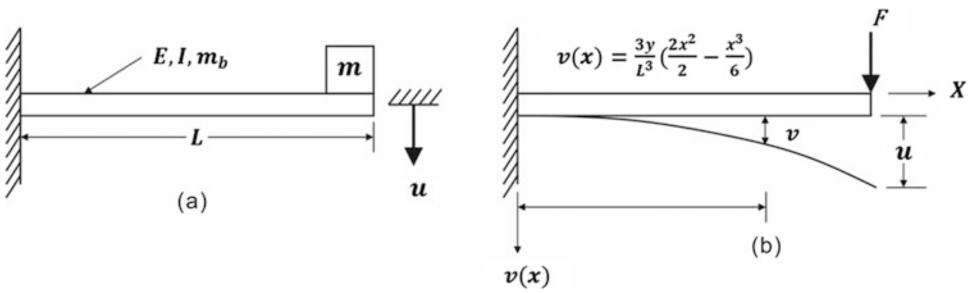

Fig. 21.13 (**a**) Cantilever beam of uniform mass with a mass concentrated at its tip. (**b**) Assumed deflection curve

Solution:

It will be assumed that the shape of deflection curve of the beam is that of the beam acted upon by a concentrated force F applied at the free end as shown in Fig. 21.13b. For this static load the deflection at a distance x from the support is

$$v(x) = \frac{3u}{L}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \tag{21.72}$$

where u = deflection at the free end of the beam. Upon substitution into Eq. (21.72) of $v(x) = C \sin(\omega t + \alpha)$, which is the harmonic deflection of the free end, we obtain

$$v(x) = \frac{3x^2L - x^3}{2L} C \sin(\omega t + \alpha) \tag{21.73}$$

The potential energy (PE) is equated to the work done by the force F as it gradually increases from zero to the final value F. This work is equal to $\frac{1}{2}Fv$, and its maximum value $(PE)_{\max}$ which is equal to the maximum potential energy is then

$$(PE)_{\max} = \frac{1}{2}FC = \frac{3EI}{2L^3}C^2 \tag{21.74}$$

since the force F is related to the maximum deflection by the formula from elementary strength of materials,

$$(PE)_{\max} = u = C = \frac{FL^3}{3EI} \tag{21.75}$$

The kinetic energy due to the distributed mass of the beam is given by

$$T = \int_0^L \frac{1}{2}\left(\frac{m_b}{L}\right)\dot{u}^2{}_r dx \tag{21.76}$$

and using Eq. (21.73) the maximum value for total kinetic energy will then be

$$T_{\max} = \frac{m_b}{2L}\int_0^L \left(\frac{3x^2L - x^3}{2L^3}\omega C\right)^2 dx + \frac{m}{2}\omega^2 C^2 \tag{21.77}$$

After integrating Eq. (21.77) and equating it with Eq. (21.74), we obtain

$$\frac{3EI}{2L^3}C^2 = \frac{1}{2}\omega^2 C^2\left(m + \frac{33}{140}m_b\right) \tag{21.78}$$

and the natural frequency becomes

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3EI}{L^3\left(m + \frac{33}{140}m_b\right)}} \tag{21.79}$$

It is seen, then, that by concentrating a mass equal to (33/140) m_b at the end of the beam, a more accurate value for the natural frequency of the cantilever beam is obtained compared to the result obtained by simply neglecting its distributed mass. In practice the fraction 33/140 is rounded to 1/4, thus approximating the natural frequency of a cantilever beam by

$$f = \frac{1}{2\pi}\sqrt{\frac{3EI}{L^3\left(m + \frac{33}{140}m_b\right)}} \tag{21.80}$$

The approximation given by either Eq. (21.79) or Eq. (21.80) is a good one even for the case in which $m = 0$. For this case the error given by these formulas is about 1.5% compared to the exact solution which will be presented in Chap. 21.

Illustrative Example 21.6

Consider in Fig. 21.14 the case of a simple beam carrying several concentrated masses. Neglect the mass of the beam and determine an expression for the natural frequency by application of Rayleigh's Method.

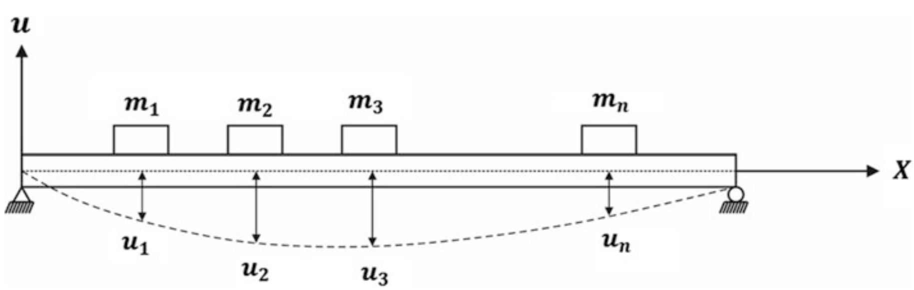

Fig. 21.14 Simple beam carrying concentrated masses

Solution:

In the application of Rayleigh's Method, it is necessary to choose a suitable curve to represent the deformed shape that the beam will have during vibration. A choice of a shape that gives consistently good results is the curve produced by forces proportional to the magnitude of the masses acting on the structure. For the simple beam, these forces could be assumed to be the weights $W_1 = m_1\ g$, $W_2 = m_2\ g$, …, $W_N = m_N g$ due to gravitational action on the concentrated masses. The static deflections under these weights may then be designated by $U_1, U_2, \ldots, U_N$. The potential energy is then equal to work done during the loading of the beam, thus,

$$(PE)_{\max} = \frac{1}{2}W_1u_1 + \frac{1}{2}W_2u_2 + \quad \ldots \quad + \frac{1}{2}W_Nu_N \tag{21.81}$$

For harmonic motion in free vibration, the maximum velocities under the weights would be ωu_1, $\omega u_2, \ldots, \omega u_N$, and therefore the maximum kinetic energy would be

$$T_{\max} = \frac{1}{2}\frac{W_1}{g}(\omega u_1)^2 + \frac{1}{2}\frac{W_2}{g}(\omega u_2)^2 + \ldots + \frac{W_N}{g}(\omega u_N)^2 \tag{21.82}$$

When the maximum potential energy, Eq. (21.81), is equated with the maximum kinetic energy, Eq. (21.82), the natural frequency is found to be

$$\omega = \sqrt{\frac{g(W_1u_1 + W_2u_2 + \ldots + W_Nu_N)}{W_1u_1^2 + W_2u_2^2 + \ldots + .W_Nu_N^2}}$$

or

$$\omega = \sqrt{\frac{g\sum_{i=1}^{N} W_iu_i}{\sum_{i=1}^{N} W_iu_i^2}} \tag{21.83}$$

Where u_i is the deflection at coordinate i and W_i the weight at this coordinate.

This method is directly applicable to any beam, but in applying the method, it must be remembered that these are not gravity forces at all but substituted forces for the inertial forces. For example, in the case of a simple beam with overhang (Fig. 21.15) the force at the free end should be proportional to m_3 ($F_3 = m_3\,g$) but directed upward in order to obtain the proper shape for the deformed beam.

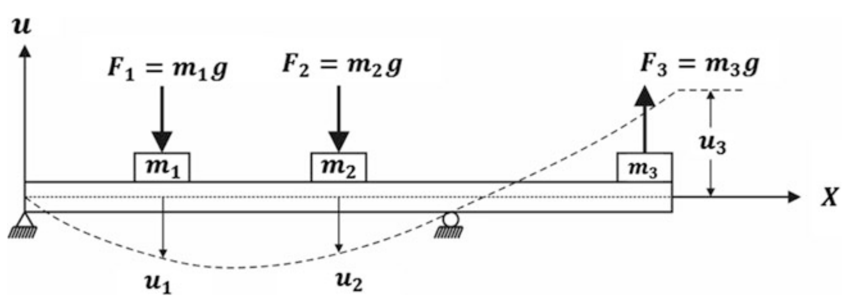

Fig. 21.15 Overhanging massless beam carrying concentrated masses

In the application of Rayleigh's Method, the forces producing the deflected shape do not necessarily have to be produced by gravitational forces. The only requirement is that these forces produce the expected deflection shape for the fundamental mode. For example, if the deflected shape for the beam shown in Fig. 21.14 is produced by forces designated by $f_1, f_2 \ldots f_N$ instead of the gravitation forces $W_1, W_2, \ldots, W_N$, we will obtain, as in Eq. (21.81), the maximum potential energy

$$(PE)_{\max} = \frac{1}{2}f_1u_1 + \frac{1}{2}f_2u_2 + \ldots + \frac{1}{2}f_Nu_N \tag{21.84}$$

which equated to the maximum kinetic energy, Eq. (21.82), will result in the following formula for the fundamental frequency:

$$\omega = \sqrt{\frac{g\sum_{i=1}^{N} f_iu_i}{\sum_{i=1}^{N} W_iu_i^2}} \tag{21.85}$$

Consequently, the fundamental period could be calculated as

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\sum_{i=1}^{N} W_iu_i^2}{g\sum_{i=1}^{N} f_iu_i}} \tag{21.86}$$

21.8 Improved Rayleigh's Method

The concept of applying inertial forces as static loads in determining the deformed shape for Rayleigh's Method may be used in developing an improved scheme for the method. In the application of the improved Rayleigh's Method, one would start from an assumed deformation curve followed by the calculation of the maximum values for the kinetic energy and for the potential energy of the system. An approximate value for natural frequency is calculated by equating maximum kinetic energy with the maximum potential energy. Then an improved value for the natural frequency may be obtained by loading the structure with the inertial loads associated with the assumed deflection. This load results in a new deformed shape which is used in calculating the maximum potential energy. The method is better explained with the aid of numerical examples.

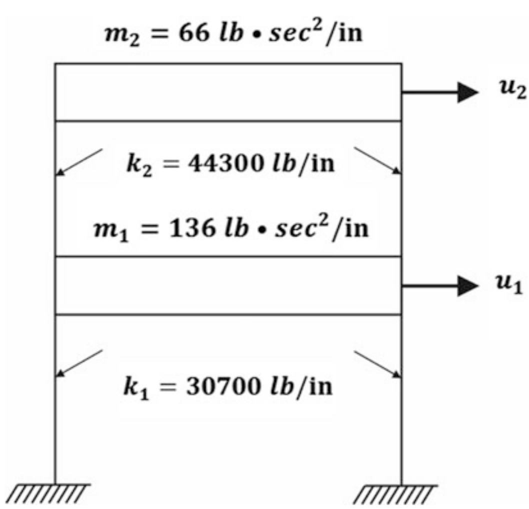

Fig. 21.16 Two-story frame for Example 21.7

Illustrative Example 21.7

By Rayleigh's Method, determine the natural frequency (lower or fundamental frequency) of the two-story frame shown in Fig. 21.16. Assume that the horizontal members are very rigid compared to the columns of the frame. This assumption reduces the system to only two degrees of freedom, indicated by coordinates u_1 and u_2 in the figure. The mass of the structure, which is lumped at the floor levels, has values $m_1 = 136$ lb·sec^2/in and $m_2 = 66$ lb·sec^2/in. The total stiffness of the first story is $k_1 = 30{,}700$ lb/in and of the second story $k_2 = 44{,}300$ lb/in, as indicated in Fig. 21.16.

Solution:

This structure may be modeled by the two mass systems shown in Fig. 21.17. In applying Rayleigh's Method, let us assume a deformed shape for which $u_1 = 1$ and $u_2 = 2$. The maximum potential energy is then

$$\begin{aligned}(PE)_{\max} &= \frac{1}{2}K_1 u_1^2 + \frac{1}{2}K_2(u_2 - u_1)^2 \\ &= \frac{1}{2}(30,700)(1)^2 + \frac{1}{2}(44,300)(1)^2 \\ &= 37,500 \ \text{lb.in}\end{aligned} \tag{a}$$

and the maximum kinetic energy

$$\begin{aligned}T_{\max} &= \frac{1}{2}m_1(\omega u_1)^2 + \frac{1}{2}m_2(\omega u_2)^2 \\ &= \frac{1}{2}(136)\omega^2 + \frac{1}{2}(66)(2\omega)^2 \\ &= 200\omega^2\end{aligned} \tag{b}$$

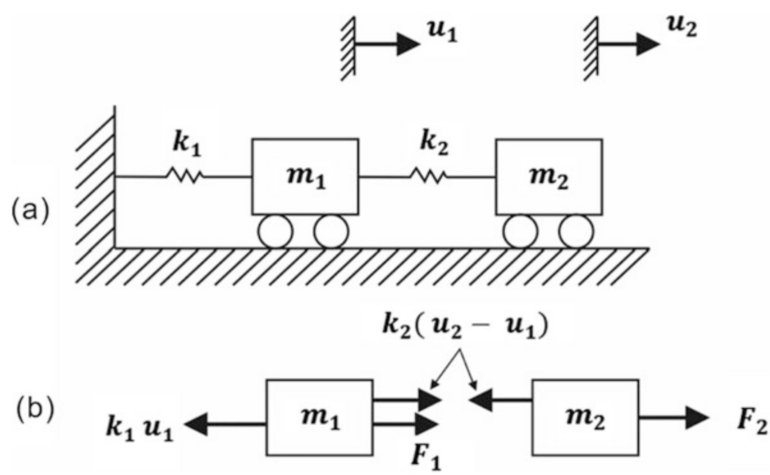

Fig. 21.17 Mathematical model for structure of Example 21.7

Equating maximum potential energy with maximum kinetic energy and solving for the natural frequency gives

$$\omega = 13.69 \ \text{rad/sec}$$

or

$$f = \frac{\omega}{2\pi} = 2.18 \ \text{cps}$$

The natural frequency calculated as $f = 2.18$ cps is only an approximation to the exact value, since the deformed shape was assumed for the purpose of applying Rayleigh's Method. To improve this calculated value of the natural frequency, let us load the mathematical model in Fig. 21.17a with the inertial load calculated as

$$F_1 = m_1\omega^2 u_1 = (136)(13.69)^2(1) = 25,489$$
$$F_2 = m_2\omega^2 u_2 = (66)(13.69)^2(2) = 24,739$$

The equilibrium equations obtained by equating to zero the sum of the forces in the free body diagram shown in Fig. 21.17b gives

$$30,700 \ \ u_1 - 44,300(u_2 - u_1) = 25,489$$

$$44,300(u_2 - u_1) = 24,739$$

and solving

$$u_1 = 1.64$$

$$u_2 = 2.19$$

or in the ratio

$$u_1 = 1.00$$

$$u_2 = 1.34 \tag{c}$$

Introducing these improved values for the displacements u_1 and u_2 into Eqs. (a) and (b) to recalculate the maximum potential energy and maximum kinetic energy results in

$$(PE)_{\max} = 25 \tag{d}$$

$$T_{\max} = 160.03\omega^2 \tag{e}$$

and upon equating $(PE)_{\max}$ and $T_{\max}$, we obtain

$$\omega = 12.57 \ \ \text{rad/sec}$$

or

$$f = 2.00 \ \ \text{cps}$$

This last calculated value for the natural frequency $f = 2.00$ cps could be further improved by applying a new inertial load in the system based on the last value of the natural 'frequency and repeating anew cycle of calculations. Table 21.3 shows results obtained for four cycles.

The exact natural frequency and deformed shape, which are calculated for this system in Chap. 7, Example 7.1, as a two-degrees-of-freedom system, checks with the values obtained in the last cycle of the calculations s shown in Table 21.3.

Table 21.3 Improved Rayleigh's Method Applied to Example 21.7

Cycle	Deformed Shape	Inertial Load		Natural Frequency
		F_1	F_2	
1	1:2.00			2.18 cps
2	1:1.34	25,489	24,739	2.00
3	1:1.32	21,489	18,725	1.88
4	1:1.27	19,091	12,230	1.88

21.9 Shear Walls

Horizontal forces in buildings, such as those produced by earthquake motion or wind action, are often resisted by structural elements called shear walls. These structural elements are generally designed as reinforced concrete walls fixed at the foundation. A single cantilever shear wall can be expected to behave as an ordinary flexural member if its length-to-depth ratio (L/D) is greater than about 2. For short shear walls ($L/D < 2$), the shear strength assumes preeminence and both flexural and shear deformations should be considered in the analysis.

When the floor system of a multistory building is rigid, the structure's weights or masses at each floor may be treated as concentrated loads, as shown in Fig. 21.18 for a three-story building. The response of the structure is then a function of these masses and of the stiffness of the shear wall. In practice a mathematical model is developed in which the mass as well as the stiffness of the structure are combined at each floor level. The fundamental frequency (lowest natural frequency) for such a structure can then be obtained using Rayleigh's Method, as shown in the following illustrative example.

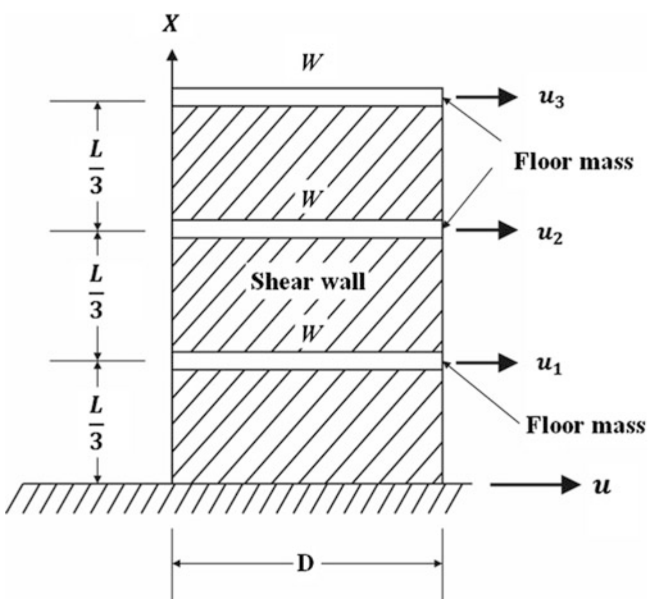

Fig. 21.18 Mathematical model for shear wall and rigid floors

Illustrative Example 21.8

Determine, using Rayleigh's Method, the natural period of the three-story building shown in Fig. 21.18. All the floors have equal weight W. Assume the mass of the wall negligible compared to the floor masses and consider only flexural deformations ($L/D > 2$).

Solution:

The natural frequency can be calculated using Eq. (21.83), which is repeated here for convenience:

$$\omega = \sqrt{\frac{g\sum_{i=1}^{N} W_i u_i}{\sum_{i=1}^{N} W_i u_i^2}}$$

The deformed shape equation is assumed as the deflection curve produced on a cantilever beam supporting three concentrated weights W, as shown in Fig. 21.19. The static deflections u_1, u_2, and u_3 calculated by using basic knowledge of strength of materials are

$$u_1 = \frac{15}{162}\frac{WL^3}{EI} = 0.0926\frac{WL^3}{EI}$$
$$u_2 = \frac{15}{162}\frac{WL^3}{EI} = 0.3025\frac{WL^3}{EI}$$
$$u_3 = \frac{92}{162}\frac{WL^3}{EI} = 0.5679\frac{WL^3}{EI}$$

The natural frequency by Eq. (a) is then calculated as

$$\omega = \sqrt{\frac{386(0.0926 + 0.3025 + 0.5679)}{(0.0926^2 + 0.3025^2 + 0.5679^2)} \cdot \frac{EI}{WL^3}} = 29.66\sqrt{\frac{EI}{WL^3}} \text{ rad/ sec}$$

or

$$f = \frac{\omega}{2\pi W} = 4.72\sqrt{\frac{EI}{WL^3}}\text{cps}$$

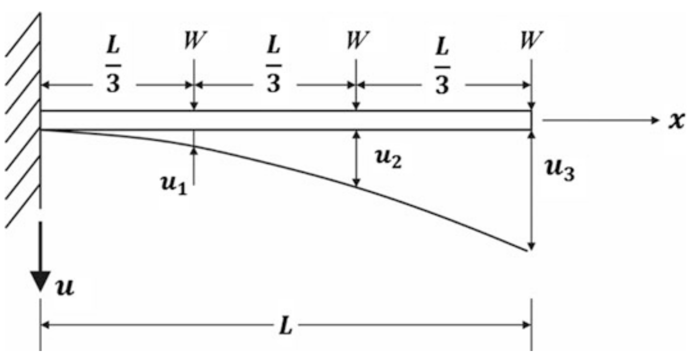

Fig. 21.19 Assumed deflection curve for Example 21.8

Illustrative Example 21.9

For the mathematical model of a three-story building shown in Fig. 21.18, determine the total deflections at the floor levels considering both flexural and shear deformations.

Solution:
The lateral deflection Δu_s, considering only shear deformation for a beam segment of length Δx, is given by

$$\Delta u_s = \frac{V\Delta x}{\alpha AG} \tag{a}$$

where
$V =$ shear force
$A =$ cross-sectional area
$\alpha =$ shear constant ($\alpha = 1.2$ for rectangular sections)
$G =$ shear modulus of elasticity
At the first story, $v = 3\ W$. Therefore, by Eq. (a) the shear deflection at the first story is

$$u_{s1} = \frac{3W(L/3)}{\alpha AG} = \frac{WL}{\alpha AG} \tag{b}$$

At the second floor the shear deflection is equal to the first floor deflection plus the relative deflection between floors, that is

$$u_{s2} = u_{s1} + \frac{2W(L/3)}{\alpha AG} = \frac{5WL}{3\alpha AG} \tag{c}$$

since the shear force of the second story is $V = 2\ W$, and at the third floor

$$u_{s3} = u_{s2} + \frac{W(L/3)}{\alpha AG} = \frac{6WL}{3\alpha AG} \tag{d}$$

The total deflection is then obtained by adding the flexural deflection determined in Example 21.8 to the above shear deflections. Hence,

$$\begin{aligned} u_1 &= \frac{15}{162}\frac{WL^3}{EI} + \frac{WL}{\alpha AG} \\ u_2 &= \frac{49}{162}\frac{WL^3}{EI} + \frac{5WL}{3\alpha AG} \\ u_3 &= \frac{92}{162}\frac{WL^3}{EI} + \frac{6WL}{3\alpha AG} \end{aligned} \tag{e}$$

We can see better the relative importance of the shear contribution to the total deflection by factoring the first terms in Eq. (e). Considering a rectangular wall for which $A = D \times t$, $E/G = 2.5$, $I = tD^3/12$, $\alpha = 1.2$ ($t =$ thickness of the wall), we obtain

$$\begin{aligned} u_1 &= \frac{15}{162}\frac{WL^3}{EI}\left[1 + 1.875\left(\frac{D}{L}\right)^2\right] \\ u_2 &= \frac{49}{162}\frac{WL^3}{EI}\left[1 + 0.957\left(\frac{D}{L}\right)^2\right] \\ u_3 &= \frac{92}{162}\frac{WL^3}{EI}\left[1 + 0.611\left(\frac{D}{L}\right)^2\right] \end{aligned} \tag{f}$$

Table 21.4 Calculation of the Natural Frequency for the Shear Wall Modeled s Shown in Fig. 21.18

D/L[a]	u_1[b](in)	u_2[b](in)	u_3[b](in)	ω[c](rad/sec)	f[c](cps)
0.00	0.09259	0.30247	0.56790	29.66	4.72
0.50	0.13600	0.37483	0.65465	27.67	4.40
1.00	0.26620	0.59193	0.91489	23.30	3.71
1.50	0.48322	0.95376	1.34862	19.05	3.03
2.00	0.78704	1.46032	1.95585	15.71	2.50
2.50	1.17765	2.11161	2.73658	13.21	2.10
3.00	1.65509	2.90764	3.69079	11.33	1.80

[a]$D/L = 0$ is equivalent, to neglect shear deformations
[b]Factor of WL^3/EI
[c]Factor of $\sqrt{EI/WL^3}$

The next illustrative example presents a table showing the relative importance that shear deformation has in calculating the natural frequency for a series of values of the ratio D/L.

Illustrative Example 21.10
For the structure modeled as shown in Fig. 21.18, study the relative importance of shear deformation in calculating the natural frequency. Solution: In this study we will consider, for the wall, a range of values from 0 to 3.0 for the ratio D/L (depth-to-length ratio). The deflections u_1, u_2, u_3 at the floor levels are given by Example 21.9 and the natural frequency by Eq. (21.83). The necessary calculations are conveniently shown in Table 21.4. It may be seen from the last column of Table 21.4, that for this example the natural frequency neglecting shear deformation ($D/L = 0$) is $f = 4.72\sqrt{EI/WL^3}$ cps. For short walls ($D/L > 0.5$) the effect of shear deformation becomes increasingly important.

21.10 Summary

The concept of generalized coordinate presented in this chapter permits the analysis of multiple interconnected rigid or elastic bodies with distributed properties as single-degree-of-freedom systems. The analysis as one-degree-of freedom systems can be made provided that by the specification of a single coordinate (the generalized coordinate) the configuration of the whole system is determined. Such a system may then be modeled as the simple oscillator with its various parameters of mass, stiffness, damping, and load, calculated to be dynamically equivalent to the actual system to be analyzed. The solution of this model provides the response in terms of the generalized coordinate.

The principle of virtual work which is applicable to systems in static or dynamic equilibrium is a powerful method for obtaining the equations of motion as an alternative to the direct application of Newton's law. The principle of virtual work states that for a system in equilibrium the summation of the work done by all its forces during any displacement compatible with the constraints of the system is equal to zero.

Rayleigh's Method for determining the natural frequency of a vibrating system is based on the principle of conservation of energy. In practice, it is applied by equating the maximum potential energy with the maximum kinetic energy of the system. To use Rayleigh's Method for the determining of the natural frequency of a discrete or a continuous system, it is necessary to assume a deformed shape. Often, this shape is selected as the one produced by gravitational loads acting in the direction of the expected displacements. This approach leads to the following formula for calculating the natural frequency:

$$\omega = \sqrt{\frac{g \sum_i W_i u_i}{\sum_i W_i u_i^2}} \qquad (21.83)\text{ repeated}$$

where u_i is the deflection at coordinate i and W_i concentrated weight at this coordinate. Shear walls are structural walls designed to resist lateral forces in buildings. For short walls ($L/D \leq 2$) shear deformations are important and should be considered in the analysis in addition to the flexural deformations.

21.11 Problems

Problem 21.1

For the system shown in Fig. P21.1 determine the generalized mass M^*, damping C^*, stiffness K^*, and the generalized load $F^*(t)$. Select $U(t)$ as the generalized coordinate.

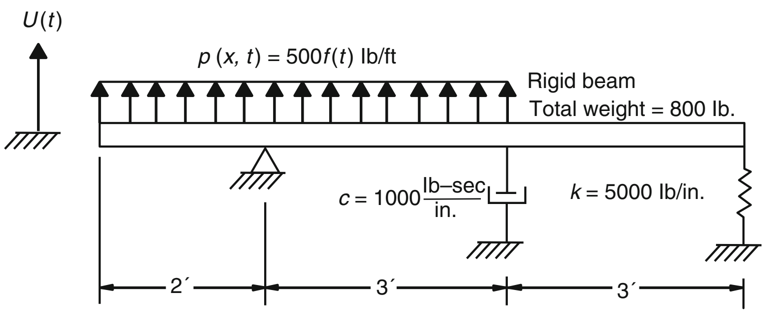

Fig. P21.1

Problem 21.2

Determine the generalized quantities M^*, C^*, K^*, and $F^*(t)$ for the structure shown in Fig. P21.2. Select $U(t)$ as the generalized coordinate.

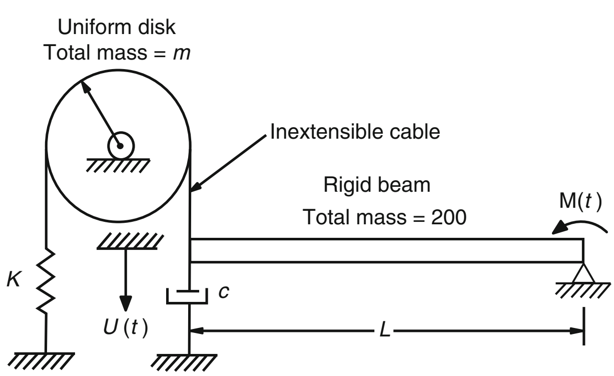

Fig. P21.2

Problem 21.3

Determine the generalized quantities M^*, C^*, K^*, and $F^*(t)$ for the structure shown in Fig. P21.3. Select $\theta(t)$ as the generalized coordinate.

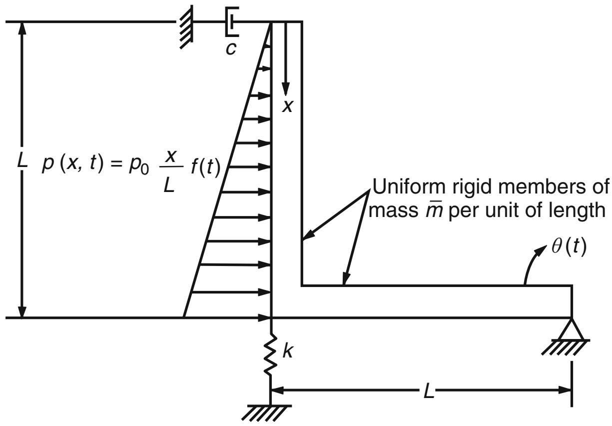

Fig. P21.3

Problem 21.4

For the elastic cantilever beam shown in Fig. P21.4, determine the generalized quantities M^*, K^*, and $F^*(t)$. Neglect damping. Assume that the deflected shape is given by $\phi(x) = 1 - \cos(\pi x/2L)$ and select $U(t)$ as the generalized coordinate as shown in Fig. P21.4. The beam is excited by a concentrated force $F(t) = F_0 f(t)$ at midspan.

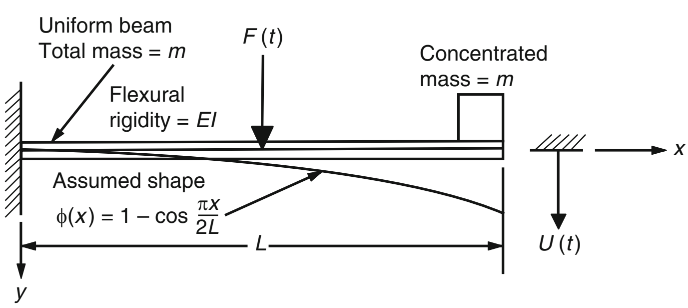

Fig. P21.4

Problem 21.5

Determine the generalized geometric stiffness K^*_G for the system in Fig. P21.4 if an axial tensile force N is applied at the free end of the beam along the x direction. What is the combined generalized stiffness K^*_c?

Problem 21.6

A concrete conical post of diameter d at the base and height L is shown in Fig. P21.6. It is assumed that the wind produces a dynamic pressure $p_0(t)$ per unit of projected area along a vertical plane. Determine the generalized quantities M^*, K^*, and $F^*(t)$ (Take modulus of elasticity $E_c = 3 \times 10^6$ psi; specific weight $\gamma = 150$ lb./ft^3 for concrete.)

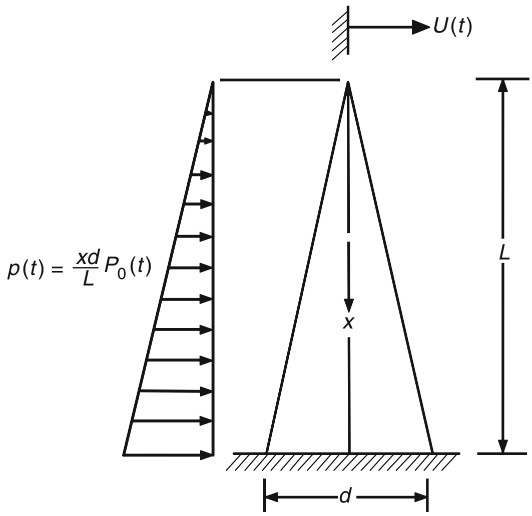

Fig. P21.6

Problem 21.7

A simply supported beam of total uniformly distributed mass m_b, flexural rigidity EI, and length L, carries a concentrated mass m at its center. Assume the deflection curve to be the deflection curve due to a concentrated force at the center of the beam and determine the natural frequency using Rayleigh's Method.

Problem 21.8

Determine the natural frequency of a simply supported beam with overhang which has a total uniformly distributed mass m_b, flexural rigidity EI, and dimensions shown in Fig. P21.8. Assume that during vibration the beam deflected curve is of the shape produced by a concentrated force applied at the free end of the beam.

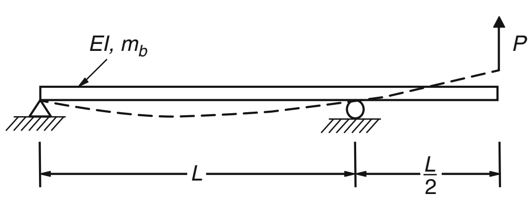

Fig. P21.8

Problem 21.9

Determine the natural frequency of the simply supported beam shown in Fig. P21.9 using Rayleigh's Method. Assume the deflection curve given by $\phi\ (x) = U\sin\ \pi x/L$. The total mass of the beam is $m_b = 10$ lb. sec^2/in, flexural rigidity $EI = 10^8$ lb.·in^2, and length $L = 100$ in. The beam carries a concentrated mass at the center $m = 5$ lb.·sec^2/in.

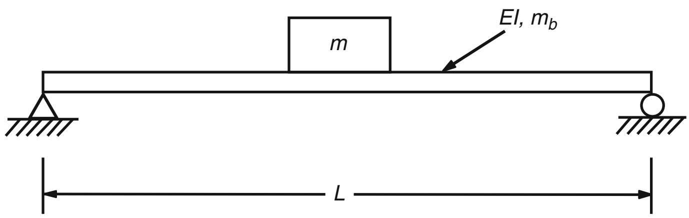

Fig. P21.9

Problem 21.10

A two-story building is modeled as the frame shown in Fig. P21.10. Use Rayleigh's Method to determine the natural frequency of vibration for the case in which only flexural deformation needs to be considered. Neglect the mass of the columns and assume rigid floors. (Hint: Use Eq. (21.83)).

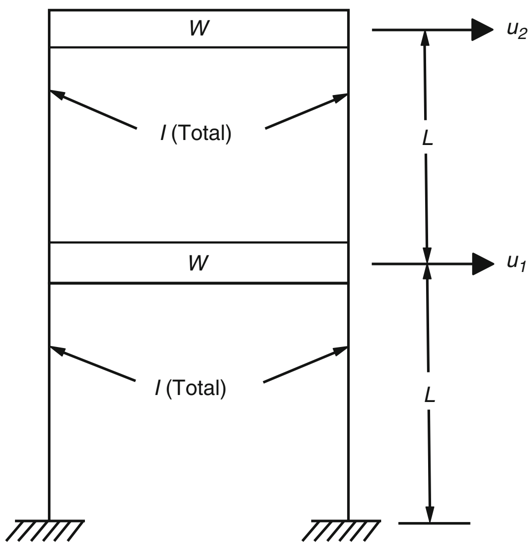

Fig. P21.10

Problem 21.11

Solve Problem 21.10 for the case in which the columns are short and only shear deformation needs to be considered. (The lateral force V for a fixed column of length L, cross-sectional area A, is approximately given by $V = AG\Delta/L$, where G is the shear modulus of elasticity and Δ the lateral deflection.)

Problem 21.12

Calculate the natural frequency of the shear wall carrying concentrated masses at the floor levels of a three-story building as shown in Fig. P21.12. Assume that the deflection shape of the shear wall is that resulting from a concentrated lateral force applied at its tip. Take flexural rigidity, $EI = 3.0 \times 10^{11}$ lb.·in^2; length $L = 36$ ft.; concentrated masses, $m = 100$ lb.·sec^2/in and mass per unit of length along the wall, $\bar{m} = 10\ \text{lb} \cdot \text{sec}^2/\text{in}^2$.

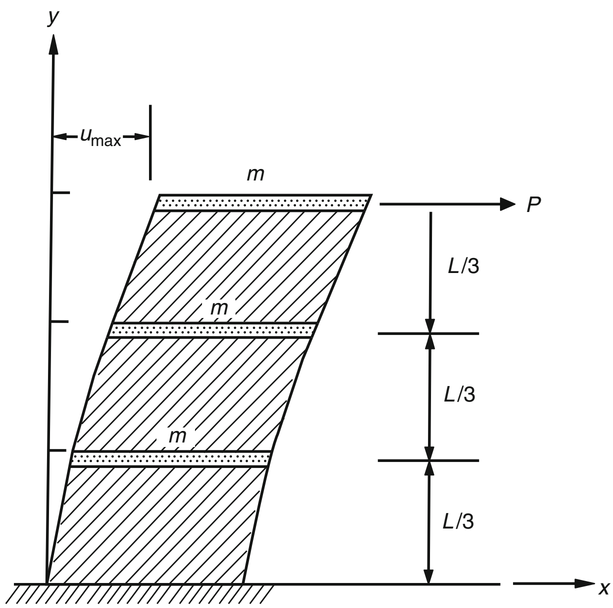

Fig. P21.12

Problem 21.13

Solve Problem 21.12 on the assumption that the deflection shape of the shear wall is that resulting from a lateral uniform load applied along its length.