6 Nonlinear Structural Response

In discussing the dynamic behavior of single-degree-of-freedom systems, we assumed that in the model representing the structure, the restoring force was proportional to the displacement. We also assumed the dissipation of energy through a viscous damping mechanism in which the damping force was proportional to the velocity. In addition, the mass in the model was always considered to be unchanging with time. As a consequence of these assumptions, the equation of motion for such a system resulted in a linear, second order ordinary differential equation with constant coefficients, namely,

$$m\ddot{u} + c\dot{u} + ku = F(t) \tag{6.1}$$

In the previous chapters it was illustrated that for particular forcing functions such as harmonic functions, it was relatively simple to solve this Eq. (6.1) and that a general solution always existed in terms of Duhamel's integral. Equation (6.1) thus represents the dynamic behavior of many structures modeled as a single-degree-of-freedom system. There are, however, physical situations for which this linear model does not adequately represent the dynamic characteristics of the structure. The analysis in such cases requires the introduction of a model in which the spring force or the damping force may not remain proportional, respectively, to the displacement or to the velocity. Consequently, the resulting equation of motion will no longer be linear and its mathematical solution, in general, will have a much greater complexity, often requiring a numerical procedure for its integration.

6.1 Nonlinear Single-Degree-of-Freedom Model

Figure 6.1a shows the model for a single-degree-of-freedom system and in Fig. 6.1b the corresponding free body diagram. The dynamic equilibrium in the system is established by equating to zero the sum of the inertial force $F_I(t)$, the damping force $F_D(t)$ the spring force $F_s(t)$, and the external force $F(t)$. Hence, at time t, the equilibrium of these forces is expressed as

$$F_I(t) + F_D(t) + F_s(t) = F(t) \tag{6.2}$$

Considering the case that in this equation, the mass is constant, $F_I(t) = m\ddot{u}$; that the damping force is proportional to the velocity with the damping coefficient also constant, $F_D(t) = c\dot{u}$; and the resisting force or spring force is a function of the displacement, $F_s(t) = F_s(u)$, we may then express Eq. (6.2) as

M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_6

$$m\ddot{u} + c\dot{u} + F_s(u) = F(t) \tag{6.3}$$

Hence, at time t_i, using the notation $u_i = u(t_i)$, $\dot{u}_i = \dot{u}(t_i)$ and $\ddot{u}_i = \ddot{u}(t_i)$,

$$m\ddot{u}_i + c\dot{u}_i + F_s(u_i) = F(t_i) \tag{6.4}$$

and at shoet time later, $t_{i+1} = t_i + \Delta t$

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + F_s(u_{i+1}) = F(t_{i+1}) \tag{6.5}$$

Subtracting Eq. (6.4) from Eq. (6.5) results in the difference equation of motion in terms of increments, namely

$$m\Delta\ddot{u}_i + c\Delta\dot{u}_i + \Delta F_s(u_i) = \Delta F_i \tag{6.6}$$

Furthermore, we assume that the incremental resisting or spring force is proportional to the incremental displacement, that is $\Delta F_s(u_i) = k_i \Delta u_i$,

$$m\Delta\ddot{u}_i + c\Delta\dot{u}_i + k_i\Delta u_i = \Delta F_i \tag{6.7}$$

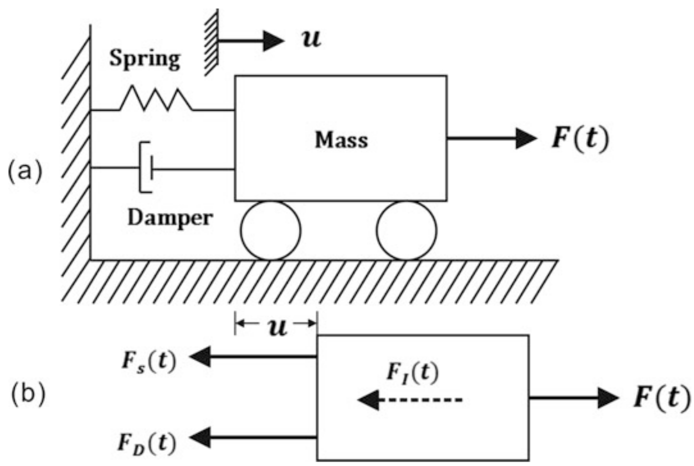

Fig. 6.1 (**a**) Model for a single-degree-of-freedom system. (**b**) Free body diagram showing the inertial force, the damping force, the spring force, and the external force

The coefficient k_i, is defined as the current evaluation for the resisting force per unit displacement (stiffness coefficient) which may be taken as the slope of the tangent of the force-displacement function, at the initiation of the time step Δt for the interval Δt or as the slope of the secant line as shown in Fig. 6.2 for the plot of the resisting force $F_s(u)$. The value of the coefficient k_i, is calculated at a displacement corresponding to time t, and assumed to remain constant during the increment of time Δt. Since, in general, this coefficient does not remain constant during that time increment, Eq. (6.7) is an approximate equation.

The incremental displacement Δu_i incremental velocity $\Delta\dot{u}_i$ and incremental acceleration $\Delta\ddot{u}$ are given by

$$\Delta u_i = u(t_i + \Delta t) - u(t_i) \tag{6.8}$$

$$\Delta\dot{u}_i = \dot{u}(t_i + \Delta t) - \dot{u}(t_i) \tag{6.9}$$

$$\Delta\ddot{u}_i = \ddot{u}(t_i + \Delta t) - \ddot{u}(t_i) \tag{6.10}$$

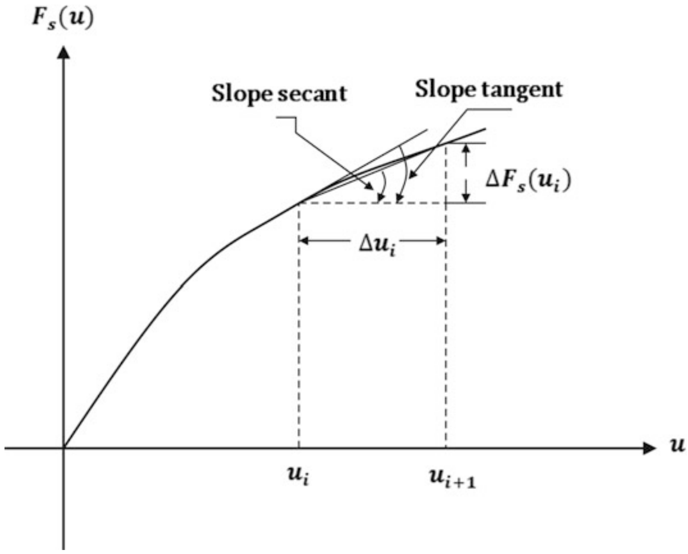

Fig. 6.2 Nonlinear stiffness function, spring force v.s. displacement

6.2 Integration of the Nonlinear Equation of Motion

Among the many methods available for the solution of the nonlinear equation of motion, probably one of the most effective is the step-by-step integration method. In this method, the response is evaluated at successive increments Δt of time, usually taken of equal lengths of time for computational convenience. At the beginning of each interval, the condition of dynamic equilibrium is established. Then, the response for a time increment Δt is evaluated approximately on the basis that the coefficients $k(u)$ and $c(\dot{u})$ remain constant during the interval Δt. The nonlinear characteristics of these coefficients are considered in the analysis by reevaluating these coefficients at the beginning of each time increment. The response is then obtained using the displacement and velocity calculated at the end of the time interval as the initial conditions for the next time step.

As we have said for each time interval, the stiffness coefficient $k(u)$ and the damping coefficient $c(\dot{u})$ are evaluated at the initiation of the interval but are assumed to remain constant until the next step; thus the nonlinear behavior of the system is approximated by a sequence of successively changing linear systems. It should also be obvious that the assumption of constant mass is unnecessary; it could just as well also be represented by a variable coefficient.

There are many procedures available for performing the step-by-step integration of Eq. (6.12). Two of the most popular methods are the constant acceleration method and the linear acceleration method. As the names of these methods imply, in the first method the acceleration is assumed to remain constant during the time interval Δt, while in the second method, the acceleration is assumed to vary linearly during the interval. As may be expected, the constant acceleration method is simpler but less accurate when compared with the linear acceleration method for the same value of the time increment. We shall present here in detail both methods.

6.3 Constant Acceleration Method

In the constant acceleration method, it is assumes that acceleration remains constant for the time step between times t_i and $t_{i+1} = t_i + \Delta t$ as shown in Fig. 6.3. The value of the constant acceleration during the interval Δt is taken as the average of the values of the acceleration $\ddot{u}_i$ at the initiation of the time step and $\ddot{u}_{i+1},$ the acceleration at the end of the time step. Thus, the acceleration $\ddot{u}(\mathrm{t})$ at any time t during the time interval Δt is given by

$$\ddot{u}(t) = \frac{1}{2}\left(\ddot{u}_i + \ddot{u}_{i+1}\right) \tag{6.11}$$

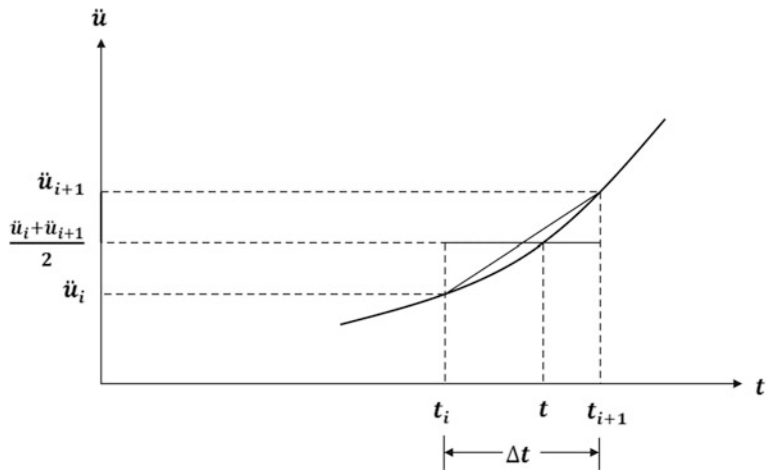

Fig. 6.3 Constant acceleration assumed during time interval Δt

Integrating this equation twice with respect to the time between the limits t_i and t results in

$$\dot{u}\,(t) = \dot{u}_{\,i} + \frac{1}{2}\left(\ddot{u}_i + \ddot{u}_{i+1}\right)(t - t_i) \tag{6.12}$$

and

$$u(t) = u_i + \dot{u}_{\,i}(t - t_i) + \frac{1}{4}\left(\ddot{u}_i + \ddot{u}_{i+1}\right)(t - t_i)^2 \tag{6.13}$$

The evaluation of Eqs. (6.12) and (6.13) at time $t_{i\,+\,1} = t_i + \Delta t$ gives

$$\Delta\dot{u}_{\,i} = \frac{\Delta t}{2}\left(\ddot{u}_i + \ddot{u}_{i+1}\right) \tag{6.14}$$

and

$$\Delta u_i = \dot{u}_{\,i}\Delta t + \frac{\Delta t^2}{2}\left(\ddot{u}_i + \ddot{u}_{i+1}\right) \tag{6.15}$$

where, Δu_i and, $\Delta\dot{u}_{\,i}$ are respectively the incremental displacement and incremental velocity defined by Eqs. (6.8) and (6.9).

To use the incremental displacement in the analysis, Eq. (6.15) is solved for $\ddot{u}_{i+1}$ and substituted into Eq. (6.14) to obtain:

$$\ddot{u}_{i+1} = \frac{4}{\Delta t^2}\Delta u_i - \frac{4}{\Delta t}\dot{u}_i - \ddot{u}_i \tag{6.16}$$

and

$$\Delta \dot{u}_i = \frac{2}{\Delta t}\Delta u_i - 2\dot{u}_i \tag{6.17}$$

Now subtracting $\ddot{u}_i$ on both sides of Eq. (6.16) results in

$$\Delta \ddot{u}_i = \frac{4}{\Delta t^2}\Delta u - \frac{4}{\Delta t}\dot{u}_i - 2\ddot{u}_i \tag{6.18}$$

The substitution into Eq. (6.7) of $\Delta \dot{u}_i$ and $\Delta \ddot{u}_i$, respectively, from Eqs. (6.17) and (6.18) gives

$$m\left(\frac{4}{\Delta t^2}\Delta u_i - \frac{4}{\Delta t}\dot{u}_i - 2\ddot{u}_i\right) + c\left(\frac{2}{\Delta t}\Delta u - 2\dot{u}_i\right) + k_i \Delta u_i = \Delta F_i \tag{6.19}$$

Equation (6.19) is then solved for the incremental displacement Δu_i to obtainin

$$\Delta u_i = \frac{\Delta \bar{F}_i}{\bar{k}_i} \tag{6.20}$$

in which the effective stiffness $\bar{k}_i$ is

$$\bar{k}_i = \frac{4m}{\Delta t^2} + \frac{2c}{\Delta t} + k_i \tag{6.21}$$

and the effective incremental force $\Delta \bar{F}_i$ *is*

$$\Delta \bar{F}_i = \Delta F_i + \frac{4m}{\Delta t}\dot{u}_i + 2m\Delta \ddot{u}_i + 2c\dot{u}_i \tag{6.22}$$

The displacement $u_{i+1} = u\,(t_i + \Delta t)$ at time $t_{i+1} = t_i + \Delta t$ is obtained from Eq. (6.8) after solving for incremental displacement Δu_i in Eq. (6.20). The incremental velocity is calculated by Eq. (6.17) and the velocity at time $t_{i+1} = t_i + \Delta t$ from Eq. (6.9) as

$$\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i \tag{6.23}$$

Finally, the acceleration $\ddot{u}_{i+1}$ at the end of the time step, $t_{i+1} = t_i + \Delta t$, is obtained directly from the differential equation of motion, Eq. (6.3), rather than using Eq. (6.16), Hence, from Eq. (6.3):

$$\ddot{u}_{i+1} = \frac{1}{m}\left[F(t_{i+1}) - c\dot{u}_{i+1} - F_s(u_{i+1})\right] \tag{6.24}$$

in which $F_s(u_{i+1})$ is the restoring force evaluated at time $t_{i+1} = t_i + \Delta t$

After the displacement, velocity, and acceleration have been determined at time $t_{i+1} = t_i + \Delta t$, the outlined procedure is repeated to calculate these quantities at the following time step $t_{i+2} = t_{i+1} + \Delta t$, and the process is continued to any desired final value of time.

6.4 Linear Acceleration Step-by-Step Method

In the linear acceleration method, it is assumed that the acceleration may be expressed by a linear function of time during the time interval Δt. Let t_i and $t_{i+1} = t_i + \Delta t$ be, respectively, the designation for the time at the beginning and at the end of the time interval Δt. In this type of analysis, the material properties of the system c_i and k_i may include any form of nonlinearity. Thus it is not necessary for the spring force to be only a function of displacement or for the damping force to be specified only as a function of velocity. The only restriction in the analysis is that we evaluate these coefficients at an instant of time t_i and then assume that they remain constant during the increment of time Δt. When the acceleration is assumed to be a linear function of time for the interval of time t_i or $t_{i+1} = t_i + \Delta t$ as depicted in Fig. 6.4, we may express the acceleration as

$$\ddot{u}(t) = \ddot{u}_i + \frac{\Delta \ddot{u}_i}{\Delta t}(t - t_i) \tag{6.25}$$

where $\Delta \ddot{u}_i$ is given by Eq. (6.10). Integrating Eq. (6.25) twice with respect to time between the limits t_i and t yields

$$\dot{u}(t) = \dot{u}_i + \ddot{u}_i(t - t_i) + \frac{1}{2}\frac{\Delta \ddot{u}_i}{\Delta t}(t - t_i)^2 \tag{6.26}$$

and

$$u(t) = u_i + \dot{u}_i(t - t_i) + \frac{1}{2}\ddot{u}_i(t - t_i)^2 + \frac{1}{6}\frac{\Delta \ddot{u}_i}{\Delta t}(t - t_i)^3 \tag{6.27}$$

The evaluation of Eqs. (6.26) and (6.27) at time $t = t_i + \Delta$t gives

$$\Delta \dot{u}_i = \ddot{u}_i \Delta t + \frac{1}{2}\Delta \ddot{u}_i \Delta t \tag{6.28}$$

and

$$\Delta u_i = \dot{u}_i \Delta t + \frac{1}{2}\ddot{u}_i \Delta t^2 + \frac{1}{6}\Delta \ddot{u}_i + \Delta t^3 \tag{6.29}$$

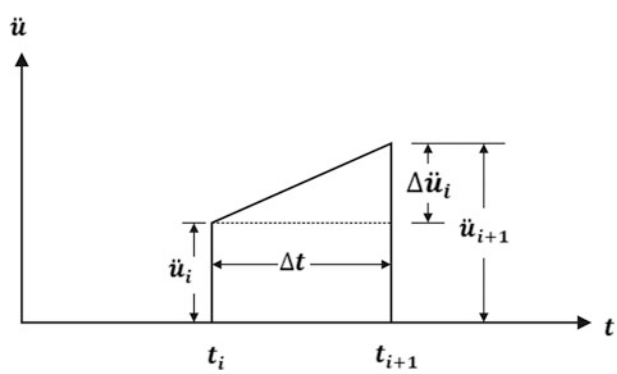

Fig. 6.4 Assumed linear variation of the acceleration during a time interval

where Δu_i and $\Delta \dot{u}_i$ are defined in Eqs. (6.8) and (6.9), respectively. Now, to use the incremental displacement Δu as the basic variable in the analysis, Eq. (6.29) is solved for the incremental acceleration $\Delta \ddot{u}_i$ and then substituted into Eq. (6.28) to obtain

$$\Delta \ddot{u}_i = \frac{6}{\Delta t^2}\Delta u_i - \frac{6}{\Delta t}\dot{u}_i - 3\ddot{u}_i \tag{6.30}$$

and

$$\Delta \dot{u}_i = \frac{3}{\Delta t}\Delta u_i - 3\dot{u}_i - \frac{\Delta t}{2}\ddot{u}_i \tag{6.31}$$

The substitution of Eqs. (6.30) and (6.31) into Eq. (6.7) leads to the following form of the equation of motion:

$$m\left\{\frac{6}{\Delta t^2}\Delta u_i - \frac{6}{\Delta t}\dot{u}_i - 3\ddot{u}_i\right\} + c_i\left\{\frac{3}{\Delta t}\Delta u_i - 3\dot{u}_i - \frac{\Delta t}{2}\ddot{u}_i\right\} + k_i\Delta u = \Delta F_i \tag{6.32}$$

Finally, transferring in Eq. (6.32) all the terms containing the unknown incremental displacement Δu_i to the left-hand side gives

$$\bar{k}_i \Delta u_i = \Delta \bar{F}_i \tag{6.33}$$

in which $\bar{k}_i$ is the effective spring constant, given by

$$\bar{k}_i = k_i + \frac{6m}{\Delta t^2} + \frac{3c_i}{\Delta t} \tag{6.34}$$

and $\Delta \bar{F}_i$ is the effective incremental force, expressed by

$$\Delta \bar{F}_i = \Delta F_i + m\left\{\frac{6}{\Delta t}\dot{u}_1 + 3\ddot{u}_i\right\} + c_i\left\{3\dot{u}_i + \frac{\Delta t}{2}\ddot{u}_i\right\} \tag{6.35}$$

It should be noted that Eq. (6.33) is equivalent to the static incremental equilibrium equation, and may be solved for the incremental displacement by simply dividing the effective incremental force $\Delta \bar{F}_i$ by the effective spring constant $\bar{k}_i$, that is,

$$\Delta u_i = \frac{\Delta \bar{F}_i}{\bar{k}_i} \tag{6.36}$$

To obtain the displacement $u_{i+1} = u\,(t_i + \Delta t)$ at time $t_{i+1} = t_i + \Delta t$, this value of Δu_i is substituted into Eq. (6.8) yielding

$$u_{i+1} = u_i + \Delta u_i \tag{6.37}$$

Then the incremental velocity $\Delta \dot{u}_i$ is obtained from Eq. (6.31) and the velocity at time $t_{i+1} = t_i\, \Delta + t$ from Eq. (6.9) as

$$\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i \tag{6.38}$$

Finally, the acceleration $\dot{u}_{i+1}$ at the end of the time step is obtained directly from the differential equation of motion, Eq. (6.2), where the equation is written for time $t_{i+1} = t_i + \Delta t$. Hence, after setting $F_1 = m\ddot{u}_{i+1}$ in Eq. (6.2), it follows that

$$\ddot{u}_{i+1} = \frac{1}{m}\{F(t_{i+1}) - F_D(t_{i+1}) - F_s(t_{i+1})\} \tag{6.39}$$

where that damping force F_D (t_{i+1}).and the spring force F_S (t_{i+1}) are now evaluated at time $t_{i+1} = t_i + \Delta t$.

After the displacement, velocity, and acceleration have been determined at time $t_{i+1} = t_i + \Delta t$, the outlined procedure is repeated to calculate these quantities at the following time step $t_{i+2} = t_{i+1} + \Delta t$, and the process is continued to any desired final value of time. The reader should, however, realize that this numerical procedure involves two significant approximations: (1) the acceleration is assumed to vary linearly during the time increment Δt; and (2) the damping and stiffness properties of the system are evaluated at the initiation of each time increment and assumed to remain constant during the time interval. In general, these two assumptions introduce errors that are small if the time step is short. However, these errors generally might tend to accumulate from step to step. This accumulation of errors should be avoided by imposing a total dynamic equilibrium condition at each step in the analysis. This is accomplished by expressing the acceleration at each step using the differential equation of motion in which the displacement and velocity as well as the stiffness and damping forces are evaluated at that time step.

There still remains the problem of the selection of the proper time increment Δt. As in any numerical method, the accuracy of the step-by-step integration method depends upon the magnitude of the time increment selected. The following factors should be considered in the selection of Δt: (1) the natural period of the structure; (2) the rate of variation of the loading function; and (3) the complexity of the stiffness and damping functions.

In general, it has been found that sufficiently accurate results can be obtained if the time interval is taken to be no longer than one-tenth of the natural period of the structure. The second consideration is that the interval should be small enough to represent properly the variation of the load with respect to time. The third point that should be considered is any abrupt variation in the rate of change of the stiffness or damping function. For example, in the usual assumption of elastoplastic materials, the stiffness suddenly changes from linear elastic to a yielding plastic phase. In this case, to obtain the best accuracy, it would be desirable to select smaller time steps in the neighborhood of such drastic changes.

6.5 The Newmark: β Method

The Newmark-β Method includes, in its formulation, several time-step methods used for the solution of linear or nonlinear equations. It uses a numerical parameter designated as β. The method, as originally proposed by Newmark (1959), contained in addition to β, a second parameter γ. Particular numerical values for these parameters leads to well-known methods for the solution of the differential equation of motion, the constant acceleration method, and the linear acceleration method.

The Newmark equation can be written in incremental quantities for a constant time step Δt, as

$$\Delta\dot{u}_i = \ddot{u}_i\Delta t + \gamma\Delta\ddot{u}_i\Delta t \tag{6.40}$$

and

$$\Delta u_i = \dot{u}_i\Delta t + \frac{1}{2}\ddot{u}_i\Delta t^2 + \beta\Delta\ddot{u}_i\Delta t^2 \tag{6.41}$$

in which the incremental displacement Δu_i and incremental velocity $\Delta\dot{u}_i$ are defined, respectively, by Eqs. (6.8) and (6.9).

It has been found that for values of γ different than 1/2, the method introduced a superfluous damping in this system. For this reason, this parameter is generally set as $\gamma = 1/2$. The solution of Eq. (6.41) for $\Delta \ddot{u}_i$ and its subsequent substitution into Eq. (6.40) after setting $\gamma = 1/2$ yield

$$\Delta \ddot{u}_i = \frac{1}{\beta \Delta t^2}\Delta u_i - \frac{1}{\beta \Delta t}\dot{u}_i - \frac{1}{2\beta}\ddot{u}_i \tag{6.42}$$

and

$$\Delta \dot{u}_i = \frac{1}{2\beta \Delta t}\Delta u_i - \frac{1}{2\beta}\dot{u}_i + \left(1 - \frac{1}{4\beta}\right)\Delta t \cdot \ddot{u}_i \tag{6.43}$$

Now, the substitution of Eqs. (6.42) and (6.43) into the incremental equation of motion, Eq. (6.7), results in an equation to calculate the incremental displacement Δu_i, namely,

$$\bar{k}_i \, \Delta u_i = \Delta \bar{F}_i \tag{6.44}$$

where the effective stiffness $\bar{k}_i$, and the effective incremental force $\Delta \bar{F}_i$ are given respectively by

$$\bar{k}_i = k_i + \frac{m}{\beta \Delta t^2} + \frac{c_i}{2\beta \Delta t} \tag{6.45}$$

and

$$\Delta \bar{F}_i = \Delta F_i + \frac{m}{\beta \Delta t}\dot{u}_i + \frac{c_i}{2\beta}\ddot{u}_i + \frac{m}{2\beta}\ddot{u}_i - c_i \Delta t\left(1 - \frac{1}{4\beta}\right)\ddot{u}_i \tag{6.46}$$

In these equations, c_i and k_i are respectively the damping and stiffness coefficients evaluated at the initial time t_i of the time step $\Delta t = t_{i+1} - t_i$

In the implementation of the Newmark Beta Method, a numerical value for the parameter β is selected. Newmark suggested a value in the range $1/6 \le \beta \le 1/2$. For $\beta = 1/4$, the method corresponds to the constant acceleration method and for $\beta = 1/6$ to the linear acceleration method.

Illustrative Example 6.1: Response by Newmark-β Method Using MATLAB

There are two methods to obtain solution depending on the selection of gamma and beta in the equation.

Using the linear variation method of Newmark-β approach, solve the Problem 4.2

Solution:

From Fig. 4.11, *we have the following data:*

Mass	$m = W/g = (38.6 \times 1000)/386 = 100$ (lb · sec2/in)
Spring constant	$k = 100 \times 1000 = 100{,}000$ (lb · /in)
Damping coefficient	$c = 2\xi\sqrt{km} = 1265$ (lb · sec/in)
Natural period	$T = 2\pi\sqrt{m/k} = 0.20$ sec
Select time step for integration	$\Delta t = T/10 = 0.02$ sec

```
close all
clear all
clc

%%%%-GIVEN VALUES-%%%%

t=0:0.02:0.1;
Dt = t(2)-t(1);                 %Time interval

m=38.6/386;                     %Mass
k =100;                         %Stiffness
xi =0.2;                        %Damping ratio
omega = sqrt(k/m);              %Natural frequency
c=2*m*omega*xi;                 %Damping coefficient

%%%%Linear acceleration method (Newmark beta method[Ch.6])

gamma =1/2;                     %Parameter gamma
beta = 1/6;                     %Parameter beta
tt= length(t);

%%%%Define the Forcing Function, F, and using Matlab function diff, an
%%%%array containing changes in F during each time step

for i= 1:tt
    if t(i)<=0.02
       F(i) = 120*t(i)/0.02;
    elseif t(i) <=0.04
       F(i) =120;
    else
       F(i)=max(0, -120*(t(i)-0.06)/0.02);
    end
end
DF1 =diff(F)                             %Delta F

%%%Initial calculation

u(1)=0;                                  %Initial condition; Displ.
v(1)=0;                                  %Initial condition; Velocity
a(1)=(F(1)-c*v(1)-k*u(1))/m;
kbar = k +gamma*c/(beta*Dt)+m/(beta*Dt*Dt);     %Eq.6.45
A = m/(beta*Dt)+gamma*c/beta;                   %A in DFbar = DF + A*v0+B*a0 (Eq. 6.46)
B = m/(2*beta)+Dt*c*((0.5*gamma/beta)-1);       %B in DFbar = DF + A*v0+B*a0 (Eq. 6.46)

%%%Setting up for initial value of Loop over

u0=u;
v0=v;
a0=a;
t =t(1);

%%%%Iteration for each time step using Newmark beta method

fori = 1:(tt-1)
DF=DF1(i);

[t,u,v,a] = Newmark( t, A, B, DF, Dt, kbar, u0, v0, a0, gamma, beta);

ti(:,i)=t(:,1)';
u_t1(i,:) = u(1,:)';
v_t1(:,i) = v(1,:)';
a_t1(:,i) = a(1,:)';

u0 = u;
v0 = v;
a0 = a;
t = t;
end

%%%Maximum values

umax = max(u_t1)
vmax = max(v_t1)
amax = max(a_t1)

%%%Plot response

plot (ti, u_t1);
xlabel ('t(sec)');
ylabel ('Displacement(in.)');
```

The function of Newmark-β method in MATLAB is presented below (Fig. 6.5 and 6.6).

```
function  [t,u,v,a] = Newmark(t, A, B, DF, Dt, kbar, u0, v0, a0, gamma, beta)

    DFbar = DF + A*v0+B*a0;                                              %Eq.6.46
    Du = DFbar/kbar;                                                     %Eq.6.44
    Dudot = gamma*Du/(beta*Dt)-gamma*v0/beta+ Dt*a0*(1-0.5*gamma/beta); %Eq.6.43
    Dudotdot = Du/(beta*Dt*Dt)-v0/(beta*Dt)-a0/(2*beta);                %Eq.6.42

    u=u0+Du;
    v=v0+Dudot;
    a=a0+Dudotdot;
    t=t+Dt;
```

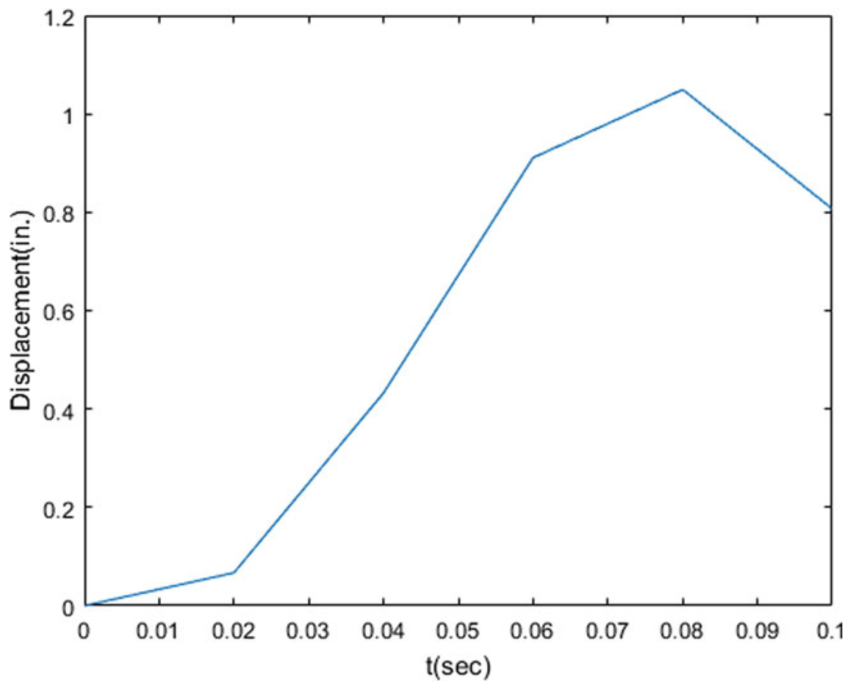

Fig. 6.5 Calculation of the response for Illustrative Example 6.1 ($\Delta t = 0.02$ sec)

The following table records the results provided by the MATLAB and their comparison with the Table 4.1 in Example 4.4:

Time (sec)	Direct integration (Table 4.1)	Δt = 0.02 sec		Δt = 0.01 sec		Δt = 0.005 sec	
		Displ.		Displ.		Displ.	
		u(t) (in.)	% error	u(t) (in.)	% error	u(t) (in.)	% error
0.000	0	0	0	0	0	0	0
0.020	0.074	0.067	−9.5	0.072	−2.7	0.073	−1.4
0.040	0.451	0.433	−4.0	0.446	−1.1	0.45	−0.2
0.060	0.926	0.911	−1.6	0.922	−0.4	0.925	−0.1
0.080	1.044	1.049	0.5	1.045	0.1	1.044	0.0
0.100	0.778	0.807	3.7	0.785	0.9	0.78	0.3

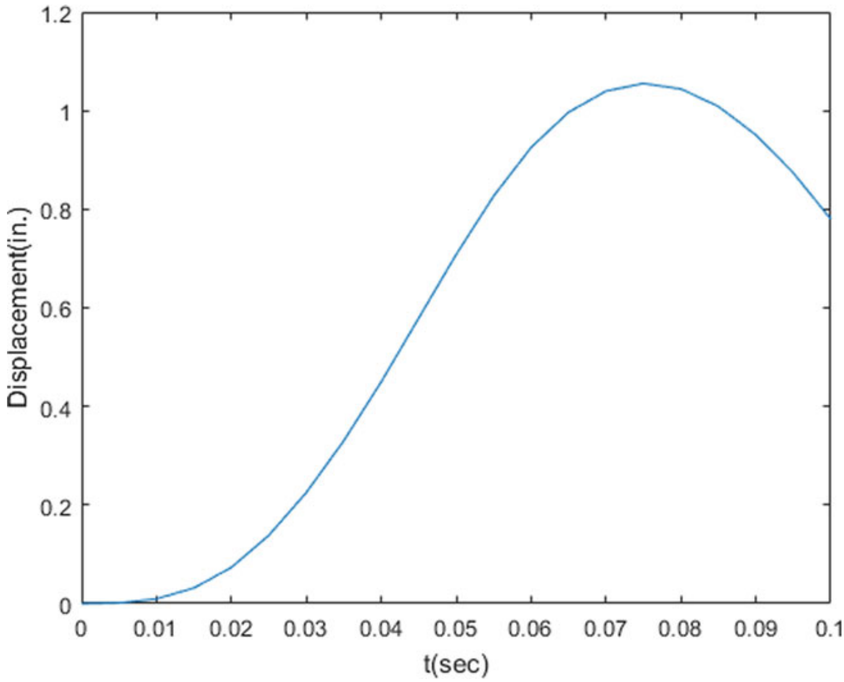

Fig. 6.6 Calculation of the response for Illustrative Example 6.1 ($\Delta t = 0.005$ sec)

Illustrative Example 6.2

(a) Determine the dynamic response of the tower shown in Fig. 6.7 *subjected to the sinusoidal force* F (t) = F_0 *sin* ϖ *t applied at its top for 0.30 sec. (b) Check results using the exact solution which in this case is available in closed form. Neglect damping.*

Solution:

(a) *The following data is obtained from* Fig. 6.7*:*

Mass	$m = w/g = (38.6 \times 1000)/386 = 100\ (lb \cdot sec2/in)$
Spring constant	$k = 100 \cdot 1000 = 100{,}000\ (lb/in)$
Natural frequency	$\omega = \sqrt{k/m} = 31.623\ rad/sec$
Natural period	$T = 2\pi/\omega = 0.20\ sec$
Select time step	$\Delta T = 0.01\ sec$

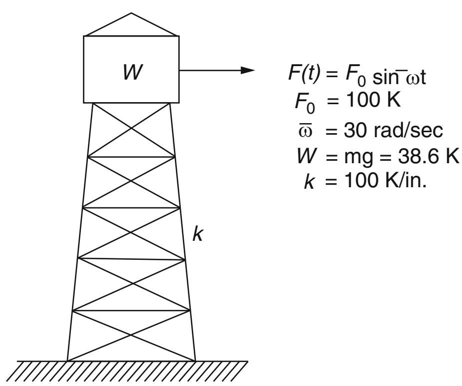

Fig. 6.7 Tower for Illustrative Example 6.2

(b) *The exact solution for the response of a simple oscillator to the sinusoidal force* $F_0 = \sin \varpi t$, *with zero initial displacement and velocity, from* Eq. (3.8) *is*

$$u(t) = \frac{F_0}{k - m\varpi^2}\left(\sin \varpi t - \frac{\varpi}{\omega}\sin \omega t\right)$$

where ω *is the natural frequency in rad/sec,* ϖ *the forced frequency also in rad/sec, and* F_0 *the amplitude of the sinusoidal force.*

Substituting corresponding numerical values for this example yields

$$u(t) = \frac{100,000}{100,000 - 100(30)^2}\left(\sin 30t - \frac{30}{31.623t}\sin 31.623t\right)$$
$$= 10(\sin 30t - 0.94868 \sin 31.623t)$$

The velocity and acceleration functions are then given by

$$\dot{u}(t) = 300 \cos 30t - 300 \cos 31.623t$$

and

$$\ddot{u}(t) = -9000 \sin 30t + 948.7 \sin 31.623t$$

The evaluation of the response at specific values of time results in the following table:

t (sec)	u(t) (in)	$\dot{u}$ (t) (in/sec)	ü(t) (in/sec^2)
0.1	1.6076	2.9379	−1466.51
0.2	−3.1865	−11.6917	2907.53
0.3	4.7420	26.0822	−4298.04

Illustrative Example 6.3

Solve Example 4.5 selecting the time step Δt *equal to 0.02 and 0.005. Then compare the displacements at time* t = *0.1, 0.2, and 0.3 sec with the response obtained in Example 4.5 using the exact solution of the differential equation.*

Solution:

In solving the problem using MATLAB, the results from Newmark's method and ODE 45 are compared. The following table presented the summary of results:

Time (sec)	Exact displ. (in.)	Newmark's method (Linear variation method)				ODE 45 method			
		Δt = 0.02 sec		Δt = 0.005 sec		Δt = 0.02 sec		Δt = 0.005 sec	
		Displ.		Displ.		Displ.		Displ.	
		(in.)	% error	(in.)	% error	(in.)	% error	(in.)	% error
0.1	1.6076	1.547	−3.8	1.604	−0.2	1.608	0.0	1.608	0.0
0.2	−3.1865	−3.055	−4.1	−3.178	−0.3	−3.187	0.0	−3.188	0.0
0.3	4.742	4.487	−5.4	4.696	−1.0	4.71	−0.7	4.71	−0.7

Results shown in the above table are sufficiently close to corresponding values given by the MATLAB in part (a) of this problem (Fig. 6.8).

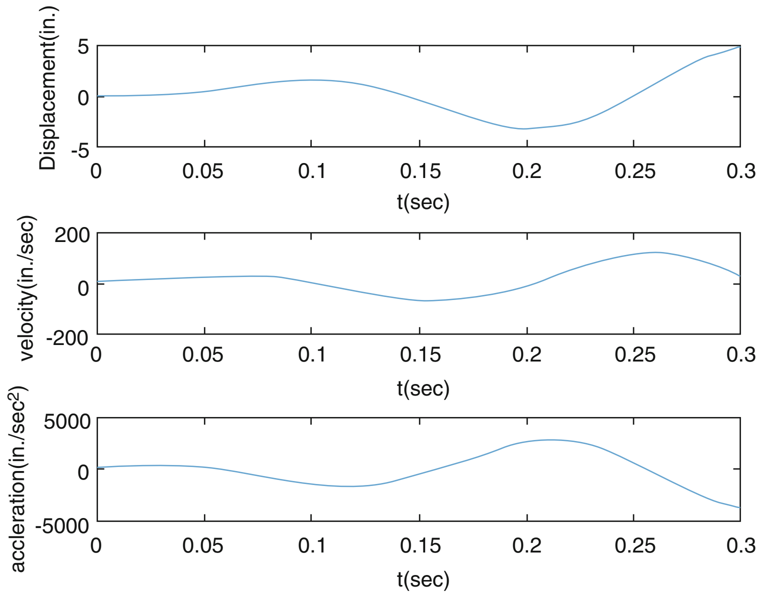

Fig. 6.8 Displacement, velocity, and acceleration for Illustrative Example 4.4 ($\Delta t = 0.02$ sec)

6.6 Elastoplastic Behavior

If any structure modeled as a single-degree-of-freedom system (spring-mass system) is allowed to yield plastically, then the restoring force exerted is likely to be of the form shown in Fig. 6.5a. There is a portion of the curve in which linear elastic behavior occurs, whereupon, for any further deformation, plastic yielding takes place. When the structure is unloaded, the behavior is again elastic until further reverse loading produces compressive plastic yielding. The structure may be subjected to cyclic loading and unloading in this manner. Energy is dissipated during each cycle by an amount that is proportional to the area under the curve (hysteresis loop) as indicated in Fig. 6.5a. This behavior is often simplified by assuming a definite yield point beyond which additional displacement takes place at a constant value for the restoring force without any further increase in the load. Such behavior is known as elastoplastic behavior; the corresponding force-displacement curve is shown in Fig. 6.5b.

For the structure modeled as a spring-mass system, expressions of the restoring force for a system with elastoplastic behavior are easily written (Fig. 6.9).

These expressions depend on the magnitude of the restoring force as well as upon whether the motion is such that the displacement is increasing $\left(\dot{u} > 0\right)$ or decreasing $\left(\dot{u} < 0\right)$. Referring to Fig. 6.5b in which a general elastoplastic cycle is represented, we assume that the initial conditions are zero ($u_0 = 0, \dot{u} = 0$) for the unloaded structure. Hence, initially, as the load is applied, the system behaves elastically along curve E_0. The displacement u_t at which plastic behavior in tension may be initiated, and the displacement u_c, at which plastic behavior in compression may be initiated, are calculated, respectively, from

$$u_t = R_t/k \tag{6.47}$$

and

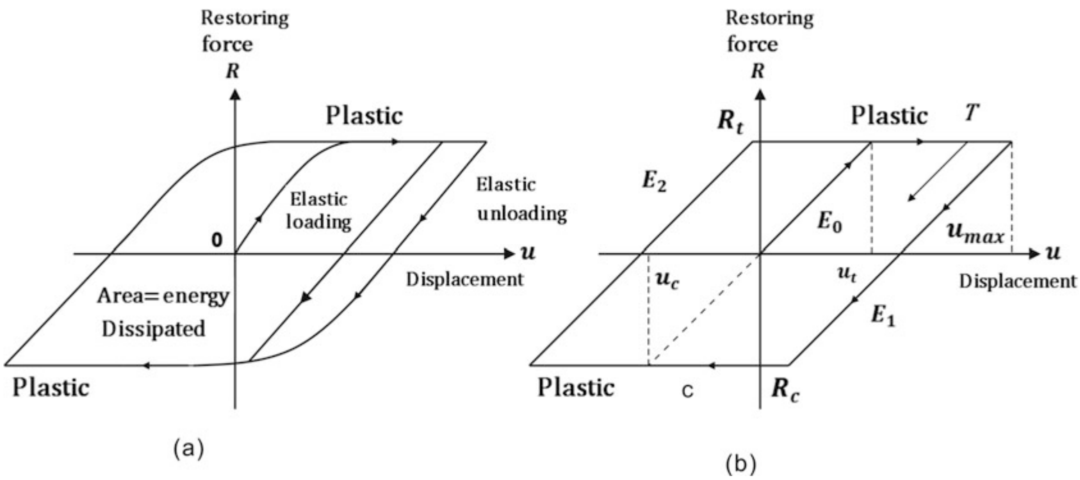

Fig. 6.9 Elastic-plastic structural models. (**a**) General plastic behavior. (**b**) Elastoplastic behavoir

$$u_c = R_c/k$$

where R_t and R_c are the respective values of the forces that produce yielding in tension and compression and k is the elastic stiffness of the structure. The system will remain on curve E_0 as long as the displacement u satisfies

$$u_c < u < u_t \tag{6.48}$$

If the displacement u increases to u_t the system begins to behave plastically in tension along curve T on Fig. 6.5b; it remains on curve T as long as the velocity $\dot{u} > 0$. When $\dot{u} < 0$, the system reverses to elastic behavior on a curve such as E_1 with new yielding points given by

$$u_t = u_{\max}$$

$$u_c = u_{\max} - (R_t - R_c)/k \tag{6.49}$$

in which u_{max} is the maximum displacement along curve T, which occurs when $\dot{u} = 0$.

Conversely, if u decreases to u_c the system begins a plastic behavior in compression along curve C and it remains on this curve as long as $\dot{u} < 0$. The system returns to an elastic behavior when the velocity again changes direction and $\dot{u} > 0$. In this case, the new yielding limits are given by

$$u_c = u_{\min}$$

$$u_t = u_{\min} + (R_t - R_c)/k \tag{6.50}$$

in which u_{min} is the minimum displacement along curve C, which occurs when $\dot{u} = 0$. The same condition given by Eq. (6.48) is valid for the system to remain operating along any elastic segment such as E_0, E_1, E_2,. . .as shown in Fig. 6.5b.

We are now interested in calculating the restoring force at each of the possible segments of the elastoplastic cycle. The restoring force on an elastic phase of the cycle (E_0. E_1. E_2,. . .) may be calculated as.

$$R = R_t - (u_t - u)k \tag{6.51}$$

on a plastic phase in tension as

$$R = R_t \tag{6.52}$$

and on the plastic compressive phase as

$$R = R_c \tag{6.53}$$

The algorithm for the step-by-step linear acceleration method of a single degree-of-freedom system assuming an elastoplastic behavior is outlined in the following section.

6.7 Algorithm for Step-by-Step Solution for Elastoplastic Single-Degree-of-Freedom System

Initialize and input data:

1. Input values for k, m, c, R_1, R_c, and a table giving the time t_i and magnitude of the excitation F_j.
2. Set $u_0 = 0$ and $\dot{u}_0 = 0$.
3. Calculate initial acceleration:

$$\ddot{u}_0 = \frac{F(t = 0)}{m} \tag{6.54}$$

4. Select time step Δt and calculate constants:

$$a_1 = 3/\Delta t, a_2 = 6/\Delta t, a_3 = \Delta t/2, a_4 = 6/\Delta t^2$$

5. Calculate initial yield points:

$$\begin{aligned} u_t &= R_t/k \\ u_c &= R_c/k \end{aligned} \tag{6.55}$$

For each time step:

1. Use the following code to establish the elastic or plastic state of the system:

$$\begin{aligned} &\text{KEY} = 0 \text{ (elastic behavior)} \\ &\text{KEY} = -1 \text{ (plastic behavior in compression)} \\ &\text{KEY} = 1 \text{ (plastic behavior in tension)} \end{aligned} \tag{6.56}$$

2. Calculate the displacement $\dot{u}$ and velocity u at the end of the time step and set the value of KEY according to the following conditions:

   (a) When the system is behaving elastically at the beginning of the time step and

$$u_c < u < u_t \qquad \text{KEY} = 0$$
$$u > u_t \qquad \text{KEY} = 1$$
$$u < u_c \qquad \text{KEY} = -1$$

(b) When the system is behaving plastically in tension at the beginning of the time step and

$$\dot{u} > 0 \text{ KEY} = 1$$
$$\dot{u} < 0 \text{ KEY} = 0$$

(c) When the system is behaving plastically in compression at the beginning of the time step and

$$\dot{u} < 0 \text{ KEY} = -1$$
$$\dot{u} > 0 \text{ KEY} = 0$$

3. Calculate the effective stiffness:

$$\bar{k}_i = k_p + a_4 m + a_1 c_i \tag{6.57}$$

where

$$\begin{aligned} k_p &= k \quad \text{forelasticbehavior}\,(\text{KEY} = 0) \\ k_p &= 0 \quad \text{forplasticbehavior}\,(\text{KEY} = 1\,\text{or} - 1) \end{aligned} \tag{6.58}$$

4. Calculate the incremental effective force:

$$\Delta\bar{F}_i = \Delta F_i + (a_2 m + 3c_i) + (3m + a_3 c_i)\ddot{u}_i \tag{6.59}$$

5. Solve for the incremental displacement:

$$\Delta u_i = \Delta\bar{F}_i/\bar{k}_i \tag{6.60}$$

6. Calculate the incremental velocity:

$$\Delta\dot{u}_i = a_1 \Delta u_i - 3\dot{u}_i - a_3 \ddot{u}_i \tag{6.61}$$

7. Calculate displacement and velocity at the end of time interval:

$$u_{i+1} = u_i + \Delta u_i \tag{6.62}$$

$$\dot{u}_{i+1} = \dot{u}_i + \Delta u_i \tag{6.63}$$

8. Calculate acceleration $\ddot{u}_{i+1}$ at the end of time interval using the dynamic equation of equilibrium:

$$\ddot{u}_{i+1} = \frac{1}{m}[F(t_{i+1}) - c_{i+1}\dot{u}_{i+1} - R] \tag{6.64}$$

in which

$$\begin{aligned} R &= R_t - (u_i - u_{i+1})k \quad \text{if KEY} = 0 \\ R &= R_t \qquad\qquad\qquad\quad \text{if KEY} = 1 \end{aligned} \tag{6.65}$$

or

$$R = R_c \qquad \text{if KEY} = -1$$

Illustrative Example 6.4

To illustrate the hand calculations in applying the step-by-step integration method described above, consider the single-degree-of-freedom system in Fig. 6.6 with elastoplastic behavior subjected to the loading history as shown. For this example, we assume that the damping coefficient remains constant ($\xi = 0.087$). Hence the only nonlinearities in the system arise from the changes in stiffness as yielding occurs (Fig. 6.10).

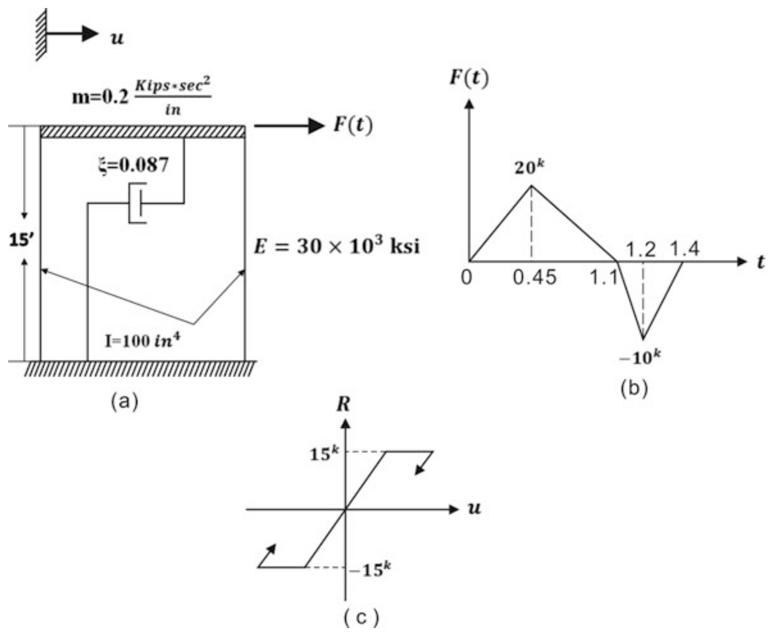

Fig. 6.10 Frame with elastoplastic behavior subjected to dynamic loading. (**a**) Frame. (**b**) Loading. (**c**) Elastoplastic behavior

Solution:

The stiffness of the system during elastic behavior is

$$k = \frac{12E(2I)}{L^3} = \frac{12x30x10^3x2x100}{(15x12)^3} = 12.35\ kip/in$$

and the damping coefficient

$$c = \xi c_{cr} = (0.087)(2)\sqrt{0.2 \times 12.35} = 0.274\ \text{kip} \cdot \text{sec}/\text{in}$$

Initial displacement and initial velocity are $u_0 = \dot{u}_0 = 0$.

The initial acceleration is

$$\ddot{u}_0 = \frac{F(0)}{k} = 0$$

Yield displacements are

$$u_t = \frac{R_t}{k} = \frac{15}{12.35} = 1.215 \text{ in}$$

and

$$u_c = -1.215 \text{ in}$$

The natural period is $T = 2\pi\sqrt{m/k} = 0.8$ (for the elastic system). For numerical convenience, we select $\boldsymbol{\Delta} t = 0.1$ sec. The effective stiffness from Eq. (6.57) is

$$\bar{k} = k_p + \frac{6}{0.1^2}0.2 + \frac{3}{0.1}0.274$$

or

$$\bar{k} = k_p + 128.22$$

where

$$k_p = k = 12.35 \qquad \text{(elasticbehavior)}$$
$$k_p = 0 \qquad \text{(plasticbehavior)}$$

The effective incremental loading from Eq. (6.59) is

$$\Delta\bar{F}_i = \Delta F_i + \left(\frac{6}{\Delta t}m + 3c\right)\dot{u}_i + \left(3m + \frac{\Delta t}{2}\right)\ddot{u}_i$$
$$\Delta\bar{F}_i = \Delta F_i + 12.822\dot{u}_i + 0.613\ddot{u}_i$$

The velocity increment given by Eq. (6.61) becomes

$$\Delta\dot{u}_i = 30\Delta u_i - 3\dot{u}_i - 0.05u_i$$

The necessary calculations may be conveniently arranged as illustrated in Table 6.1. In this example with elastoplastic behavior, the response changes abruptly as the yielding starts and stops. To obtain better accuracy, it would be desirable to subdivide the time step in the neighborhood of the change of state; however, an iterative procedure would be required to establish the length of the subintervals. This refinement has not been used in the present analysis or in the computer program described in the next section. The stiffness computed at the initiation of the time step has been assumed to remain constant during the entire time increment. The reader is again cautioned that a significant error may arise during phase transitions unless the time step is selected relatively small.

Table 6.1 Nonlinear response – linear acceleration step-by-step method for Illustrative Example 6.4

t (sec)	F (Kip)	u (in)	KEY	$\dot{u}$ (in/sec)	R (Kip)	$\ddot{u}$ (in/sec^2)	k_p Kip/in	$\bar{k}$ Kip/in	ΔF (Kip)	$\Delta \bar{F}$ (Kip)	Δu (in)	$\Delta \dot{u}$ (in/sec)
0	0		0	0	0	0	12.35	140.57	4.444	4.444	0.0316	0.9485
0.1	4.444	0.0316	0	0.9485	0.390	18.972	12.35	140.57	4.444	28.249	0.2010	2.2359
0.2	8.888	0.2326	0	3.1844	2.871	25.723	12.35	140.57	4.444	61.050	0.4343	2.193
0.3	13.333	0.6669	0	5.3760	8.233	18.134	12.35	140 57	4.444	84.510	0.6012	1.000
0.4	17.777	1.2681	1	6.3768	15.00	5.152	0	128.22	0.685	85.609	0.6677	0.6422
0.5	18.462	1.9358	1	7.0190	15.00	7.691	0	128.22	−3.077	91.641	0.7147	−0.0001
0.6	15.238	2.6505	1	7.0189	15.00	−7.693	0	128.22	−3.077	82.199	0.6409	−1.440
0.7	12.308	3.2916	1	5.5791	15.00	−2I.105	0	128.22	−3.077	55.506	0.4329	−2.695
0.8	9.231	3.7244	1	2.8840	15.00	−32.797	0	128.22	−3.077	13.773	0.1074	−3.789
0.9	6.154	3.8319	0	−0.9054	15.00	−42.990	12.35	140.57	−3.077	41.069	−0.2922	−3.899
1.0	3.077	3.5397	0	−4.8048	11.39	−34.998	12.35	140.57	−3.077	−86.162	−0.6130	−2.225
1.1	0	2.9268	0	−7.0295	3.825	−9.497	12.35	140.57	−10	−105.96	−0.7538	−1.051
1.2	−10	2.1729	0	−8.0806	−54.81	−11.525	12.35	140.57	5	−105.68	−0.7518	2.263
1.3	−5	1.4211	0	−5.8177	−14.76	56.784	12.35	140.57	5	−34.746	−0.2472	7.198
1.4	0	1.1739	−1	1.3860	−15.00	73.109	0	128.22	0	62.568	0.4880	6.842
1.5	0	1.6619	0	8.2227	−15.00	63.735	12.35	140 57	0	144.55	1.0283	2.995

6.8 Response for Elastoplastic Behavior Using MATLAB

The same as for the other programs presented in this book, MATLAB program can be used. After the user has selected one of these options, the program requests the name of the file and the necessary input data. The program continues by setting the initial values to the various constants and variables in the equations. Then by linear interpolation, values of the forcing function are computed at time increments equal to the selected time step Δt for the integration process. Results are presented in Table 6.1. Using values of force and time step, the displacement, velocity, and acceleration are computed at each time step. The nonlinear behavior of the restoring force is appropriately considered in the calculation by the variable KEY which is tested through a series of conditional statements in order to determine the correct expressions for the yield points and the magnitude of the restoring force in the system.

The output consists of a table giving the displacement, velocity, and acceleration at time increments Δt. The last column of the table shows the value of the index KEY which provides information about the state of the elastoplastic system. As indicated before, KEY $= 0$ for elastic behavior and KEY $= 1$ or KEY $= -1$ for plastic behavior, respectively, in tension or in compression.

Illustrative Example 6.5

Using the MATLAB, find the response of the structure in Example 6.4. Then repeat the calculation assuming elastic behavior. Plot and compare results for the elastoplastic behavior with the elastic response.

Solution:
Problem Data (from Illustrative Example 6.4)

Spring constant	$k = 12.35$ kip/in
Damping coefficient	$c = 0.274$ kip. sec/in
Mass	$m = 0.2$ (kip. sec^2/in)
Max. restoring force (tension)	$R_t = 15$ kip
Max. restoring force (compression)	$R_c = -15$ kip
Natural period	$T = 2\pi\sqrt{m/k} = 2\pi\sqrt{0.2/12.35} = 0.8$ sec
Select time step	$\Delta t = 0.1$ sec

```
close all
clear all
clc

%%%%-GIVEN VALUES-%%%%

load ForceData.txt
t = ForceData(:,1);      %Time
F = ForceData(:,2);      %Force

Dt = t(2)-t(1);          %Time interval
m=0.2;                   %Mass
k =12.35;                %Stiffness
c =0.274;                %Damping coefficient
xi =c/(2*sqrt(m*k));     %Damping ratio
omega = sqrt(k/m);       %Natural frequency (rad/sec)
Rt = 15;                 %Forces yielding in tension
Rc = -15;                %Forces yielding in compression
```

```
%%%%Linear acceleration method (Newmark beta method[Ch.6])

gamma =1/2;
beta = 1/6;
tt= length(t);

%%%Initial calculation

u(1)=0;                                    %Initial condition; Displ.
v(1)=0;                                    %Initial condition; Velocity
a(1)=(F(1)-c*v(1)-k*u(1))/m;

A = m/(beta*Dt)+gamma*c/beta;                 %A in DFbar = DF + A*v0+B*a0 (Eq. 6.46)
B = m/(2*beta)+Dt*c*((0.5*gamma/beta)-1);    %B in DFbar = DF + A*v0+B*a0 (Eq. 6.46)
kbar = k +gamma*c/(beta*Dt)+m/(beta*Dt*Dt); %Eq.6.45
key = 0;                                      %Initial key=0 before Loop over
k_p = k;                                      %Initial stiffness before iteration

%%%Setting up for initial value of Loop over

u0=u;
v0=v;
a0=a;
t =t(1);

%%%% Calculate initial yield points %%%%
ut = Rt/k;
uc = Rc/k;
R(1)=0;

%%%%Iteration for each time step using Newmark beta method

ua =[]; va =[]; aa=[]; ta=[];
fori = 1:(tt-1)
DF=F(i+1)-F(i);
F1 = F(i+1);

[t,u,v,a, kbar, R, keyp, key, Du, k_p] =  NewmarkNon( t, DF, Dt, u0, v0, ut, uc, a0, F1, k,
c, m, Rt, Rc, R, gamma, beta, key, k_p);

keyi(:,i)=key(:,1)'; keypi(:,i)=keyp(:,1)'; F1i(:,i)=F1(:,1)'; Ri(:,i)= R(:,1)';k_pi(:,i)=
k_p(1,:)';

%%%Creating column of time,displ,velocity and acceleration
ta = [ta; t];
ua = [ua; u];
va = [va; v];
aa = [aa; a];

%%%New yielding point-Increasing displacement (Eq.6.49)
if v < 0 && v0 >0
ut = max(ua);
uc = ut-(Rt-Rc)/k;
else
ut = ut;
uc = uc;
end

%%%New yielding point-Decreasing displacement (Eq.6.50)
if v > 0 && v0 <0
uc = min(ua);
ut = uc+(Rt-Rc)/k;
else
ut = ut;
uc = uc;
end

%%%Setting up parameters for next iteration
u0 = u;
v0 = v;
a0 = a;
t = t;
R = R;

end
result = [ta,F1i',ua,keypi',keyi',va,Ri',aa,k_pi'];
```

```
%%%Plot response

figure (1)

plot (ta, ua);
grid on
xlabel ('t(sec)');
ylabel ('Displacement(in.)');
```

The function of Newmark-beta method of MATLAB is presented for considering elastoplastic behavior. The main program is the same as Newmark-beta method (Fig. 6.11).

```
function  [t,u,v,a, kbar, R, keyp, key, Du, k_p] = NewmarkNon( t, DF, Dt, u0, v0, ut, uc,

a0, F1, k, c, m, Rt, Rc, R, gamma, beta, key, k_p)

    %%%Algorithm-For each time step:(3),(4),(5),(6),(7)
    kbar = k_p +gamma*c/(beta*Dt)+m/(beta*Dt*Dt);          %Eq.6.57
    A = m/(beta*Dt)+gamma*c/beta;                          %A in DFbar (Eq. 6.46)
    B = m/(2*beta)+Dt*c*((0.5*gamma/beta)-1);              %B in DFbar (Eq. 6.46)

    DFbar = DF + A*v0+B*a0;                        %Incremental effective force (Eq.6.59)
    Du = DFbar/kbar;                               %Incremental displacement (Eq.6.60)
Dudot = gamma*Du/(beta*Dt)-gamma*v0/beta+ Dt*a0*(1-0.5*gamma/beta);  %Incremental velocity
(Eq.6.61)

    u=u0+Du ;               %Displacement at the end of time interval (Eq.6.62)
    v=v0+Dudot;             %Velocity at the end of time interval (Eq.6.63)

    %%%Algorithm-For each time step:(2)-(a)
            if u<ut && u>uc
                          keyp = 0;
                    elseif u > ut
                           keyp = 1;
                    else
                           keyp = -1;
            end
%%%Algorithm-For each time step:(2)-(b) and (2)-(c)
          if keyp == 1 && v > 0
                        key = 1;

          elseifkeyp == -1 && v < 0
                             key = -1;
                    else
                             key = 0;
            end
%%%Algorithm-For each time step:(8) (Eq. 6.65)

          if key ==0
                         if (R+(Du)*k)>=0
                             R = min(R+(Du)*k, Rt);
                           else
                             R = max(R+(Du)*k, Rc);
                           end

            elseif key ==1
                        R = Rt;
                        else
                        R = Rc;
            end
```

```
%%%Algorithm-For each time step:(3) (Eq.6.58)
                  if key == 0
                      k_p = k;
                  else
                      k_p = 0;

                  end

%%%Algorithm-For each time step:(8) (Eq.6.64)
    a=1/m*(F1-c*v-R);

    t=t+Dt;
```

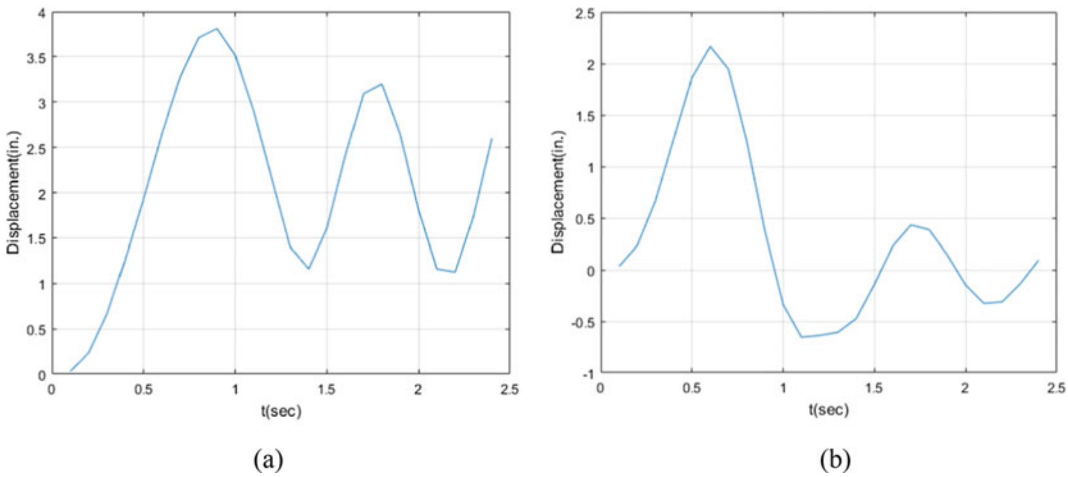

Fig. 6.11 Comparison of elastoplastic behavior with elastic response for Illustrative Example 6.2 (**a**) Nonlinear behavior; (**b**) Linear behavior

6.9 Summary

Structures are usually designed on the assumption that the structure is linearly elastic and that it remains linearly elastic when subjected to an expected dynamic excitation. However, there are situations in which the structure has to be designed for an eventual excitation of large magnitude such as the strong motion of an earthquake or the effects of nuclear explosion. In these cases, it is not realistic to assume that the structure will remain linearly elastic and it is then necessary to design the structure to withstand deformation beyond the elastic limit. The simplest and most accepted assumption for the design beyond the elastic limit is to assume an elastoplastic behavior. In this type of behavior, the structure is elastic until the restoring force reaches a maximum value (tension or compression) at which it remains constant until the motion reverses its direction and returns to an elastic behavior.

There are many methods to solve numerically the differential equation of this type of motion. The step-by-step linear acceleration presented in this chapter provides satisfactory results with relatively simple calculations. However, these calculations are tedious and time consuming when performed by hand. The use of a computer and the availability of a computer program, such as the one described in this chapter, reduce the effort to a simple routine of data preparation.

6.10 Problems

Problem 6.1

The single-degree-of-freedom of Fig. P6.1a is subjected to the foundation acceleration history in Fig. P6.1b Determine the maximum relative displacement of the columns. Assume elastoplastic behavior of Fig. P6.1c.

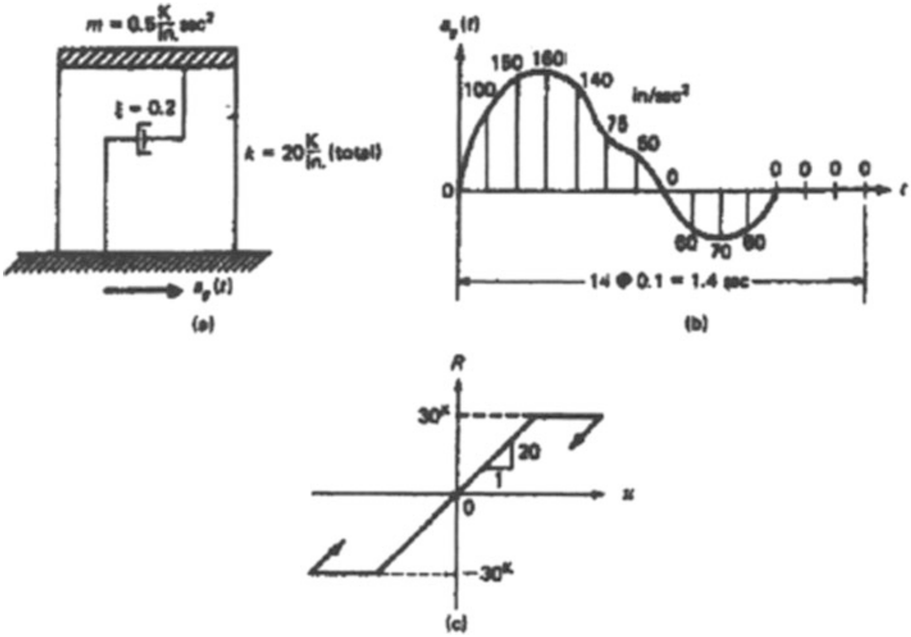

Fig. P6.1

Problem 6.2

Determine the displacement history for the structure in Fig. P6.1 when it is subjected to the impulse loading of Fig. P6.2 applied horizontally at the mass.

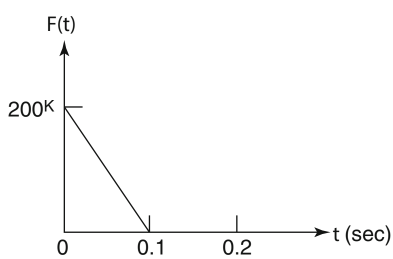

Fig. P6.2

Problem 6.3

Repeat Problem 6.2 for the impulse loading shown in Fig. P6.3 applied horizontally at the mass.

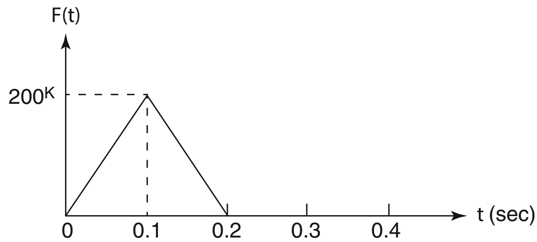

Fig. P6.3

Problem 6.4

Repeat Problem 6.2 for the acceleration history shown in Fig. P6.4 applied horizontally to the foundation.

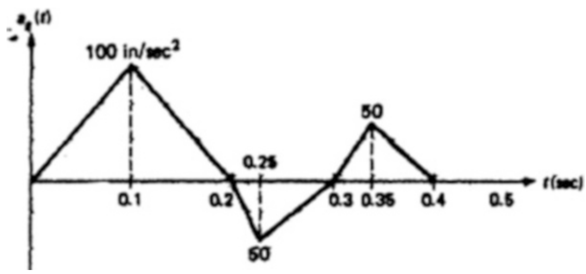

Fig. P6.4

Problem 6.5

Solve Problem 6.1 assuming elastic behavior of the structure. (Hint: Use computer Program 5 with $R_t = 200$ Kip and $R_c = -200$ Kip.)

Problem 6.6

Solve Problem 6.2 for elastic behavior of the structure. Plot the time-displacement response and compare with results from Problem 6.2.

Problem 6.7

Determine the ductility ratio from the results of Problem 6.2 (Ductility ratio is defined as the ratio of the maximum displacement to the displacement at the yield point).

Problem 6.8

A structure modeled as spring-mass shown in Fig. P6.8b is subjected to the loading force depicted in Fig. P6.8a. Assume elastoplastic behavior of Fig. P6.8c. Determine the response.

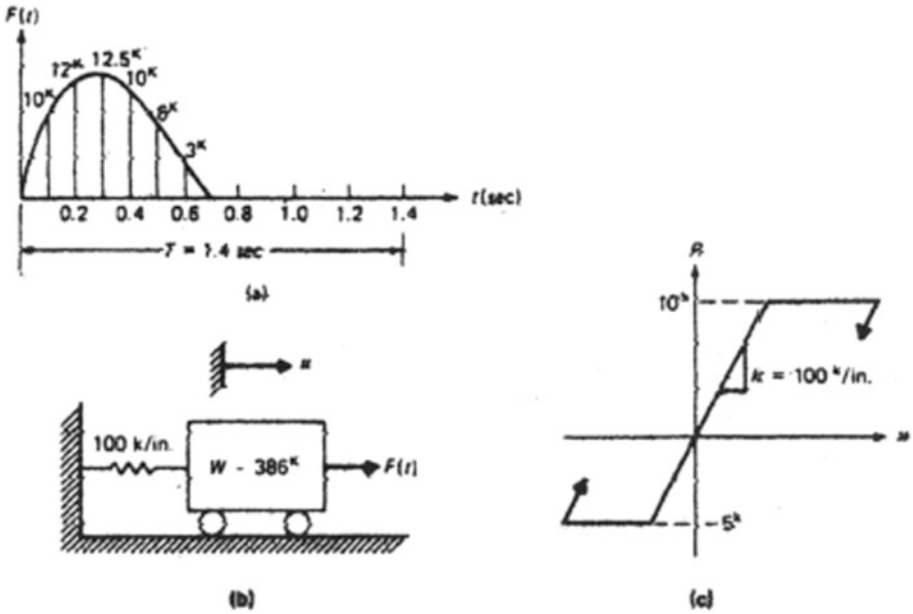

Fig. P6.8

Problem 6.9
Repeat Problem 6.8 assuming damping in the system equal to 20% of the critical damping.

Problem 6.10
Solve Problem 6.8 assuming elastic behavior of the system. (Hint: Use Program 5 with $R_T = 1000$ Kip and $R_C = -1000$ Kip.)

Problem 6.11
Solve Problem 6.9 assuming elastic behavior of the system.

Problem 6.12
A structure modeled as the damped spring-mass system shown in Fig. P6.12a is subjected to the time-acceleration excitation acting at its support. The excitation function is expressed as $a(t) = a_0 f(t)$, where $f(t)$ is depicted in Fig. P6.12b. Determine the maximum value that the factor a_0 may have for the structure to remain elastic. Assume that the structure has an elastoplastic behavior of Fig. P6.12c.

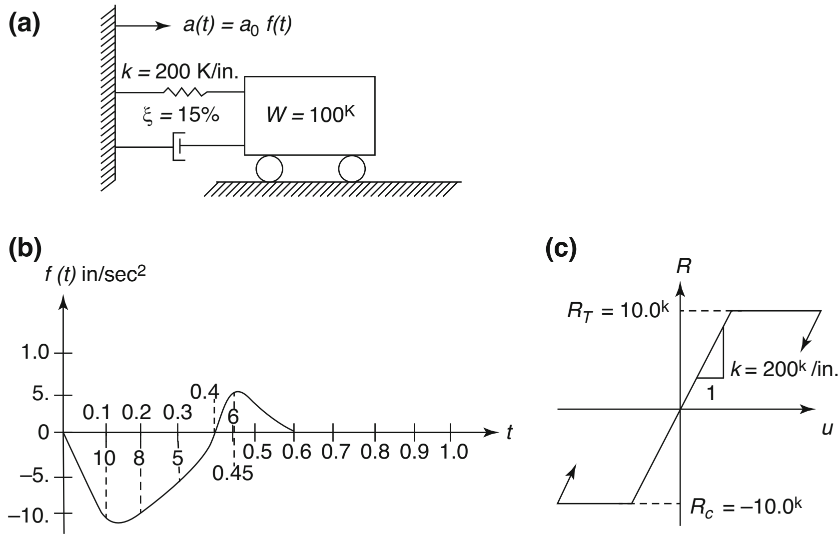
(a)
a(t) = a0 f(t)
k = 200 K/in.
ξ = 15%
W = 100K
(b)
f (t) in/sec2
1.0
5.
0
–5.
–10.
0.1 0.2 0.3
0.4
6
10 8 5
0.5 0.6 0.7 0.8 0.9 1.0
0.45
t
(c)
R
RT = 10.0k
k = 200k /in.
1
u
Rc = –10.0k

Fig. P6.12