Random Vibration

22

The previous chapters of this book have dealt with the dynamic analysis of structures subjected to excitations which were known as a function of time. Such an analysis is said to be deterministic. When an excitation function applied to a structure has an irregular shape that is described indirectly by statistical means, we speak of a random vibration. Such a function is usually described as a continuous or discrete function of the exciting frequencies, in a manner similar to the description of a function by Fourier series. In structural dynamics, the random excitations most often encountered are either motion transmitted through the foundation or acoustic pressure. Both of these types of loading are usually generated by explosions occurring in the vicinity of the structure. Common sources of these explosions are construction work and mining. Other types of loading, such as earthquake excitation, may also be considered a random function of time. In these cases the structural response is obtained in probabilistic terms using random vibration theory.

A record of random vibration is a time function such as shown in Fig. 22.1. The main characteristic of such a random function is that its instantaneous value cannot be predicted in a deterministic sense. The description and analysis or random processes are established in a probabilistic sense for which it is necessary to use tools provided by the theory of statistics.

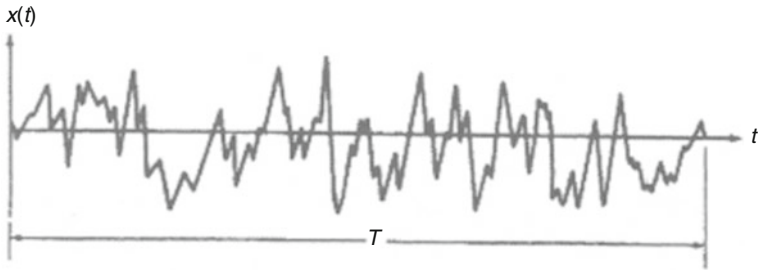

Fig. 22.1 Record of a random function of time

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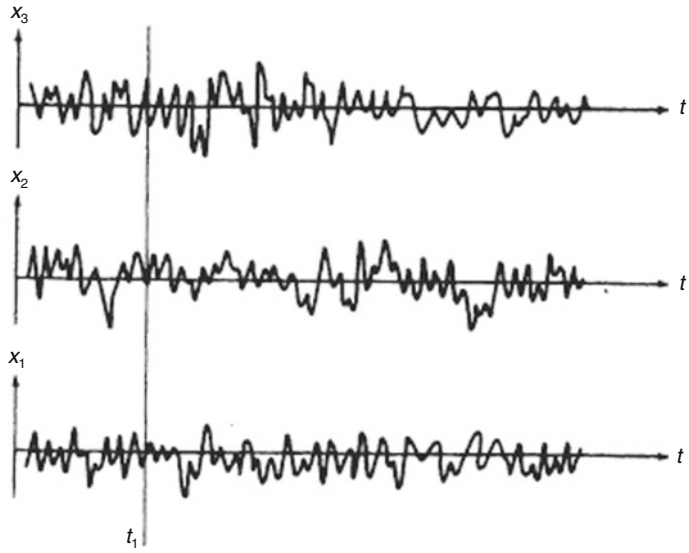

Fig. 22.2 An ensemble of random functions of time

22.1 Statistical Description of Random Functions

In any statistical method a large number of responses is needed to describe a random function. For example, to establish the statistics of the foundation excitation due to explosions in the vicinity of a structure, many records of the type shown in Fig. 22.2 may be needed. Each record is called a sample, and the total collection of samples an ensemble. To describe an ensemble statistically, we can compute at any time t_i the average value of the instantaneous displacements x_i. If such averages do not differ as we select different values of t_i, then the random process is said to be stationary. In addition, if the average obtained with respect to time for any member of the ensemble is equal to the average across the ensemble at an arbitrary time t_i, the random process is called ergodic. Thus in a stationary, ergodic process, a single record may be used to obtain the statistical description of a random function. We shall assume that all random process considered are stationary and ergodic. The random function of time shown in Fig.22.1 has been recorded during an interval of time T. Several averages are useful in describing such a random function. The most common are the mean value $\bar{x}$ which is defined as

$$\bar{x} = \frac{1}{T}\int_0^T x(t)dt \tag{22.1}$$

and the mean-square value $\overline{x^2}$ defined as

$$\overline{x^2} = \frac{1}{T}\int_0^T x^2(t)dt \tag{22.2}$$

Both the mean and the mean-square values provide measurements for the average value of the random function $x(t)$. The measure of how widely the function $x(t)$ differs from the average is given by its variance, σ_x^2, defined as

$$\sigma_x^2 = \frac{1}{T}\int_0^T [x(t) - \bar{x}]^2 dt \tag{22.3}$$

When the expression under the integral is expanded and then integrated, we find that

$$\sigma_x^2 = \overline{x^2} - \left(\bar{x}\right)^2 \tag{22.4}$$

which means that the variance can be calculated as the mean-square minus the square of the mean. Quite often the mean value is zero, in which case variance is equal to the mean square value. The root mean-square RMS_x of the random function $x(t)$ is defined as

$$RMS_s = \sqrt{\overline{x^2}} \tag{22.5}$$

The standard deviation σ_x of the random function $x(t)$ is the square root of the variance; hence from Eq. (22.4)

$$\sigma_x = \sqrt{\overline{x^2} - \left(\bar{x}\right)^2} \tag{22.6}$$

Illustrative Example 22.1

Determine the mean value $\bar{F}$, the mean-square value $\overline{F^2}$, the variance $\sigma^2{}_F$ and the root mean square values RMS_F of the forcing function $F(t)$ shown in Fig. 22.3.

Solution; Since the force $F(t)$ is periodic with period T, we can take the duration of the force equal to T; hence by Eq. (22.1) we have

$$\bar{F} = \frac{1}{T}\int_0^t F(t)dt = \frac{F_{\max}}{2}$$

and noting that

$$F(t) = \frac{2F_{\max}}{T}t \quad \text{for} \quad 0 < t < \frac{T}{2}$$

we obtain by Eq. (22.2)

$$\overline{F^2} = \frac{2}{T}\int_0^T \left(\frac{2F_{\max}}{T}\right)^2 t^2 dt = \frac{F_{\max}^2}{3}$$

The variance may now be calculated from Eq. (22.4) as

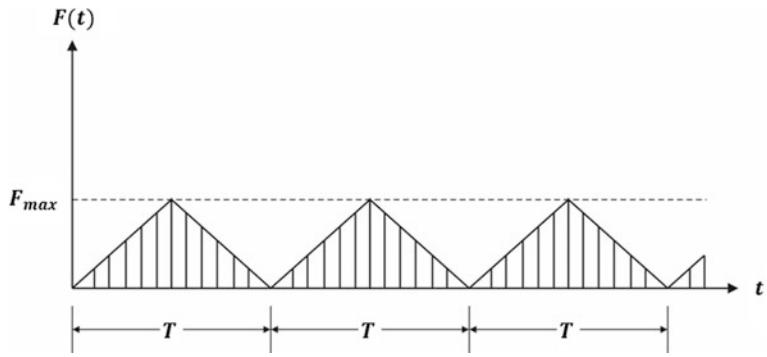

Fig. 22.3 Forcing function for Illustrative Example 22.1

$$\sigma_F^2 = \overline{F^2} - \left(\bar{F}\right)^2$$
$$\sigma_F^2 = \frac{F_{\max}^2}{3} - \frac{F_{\max}^2}{4} = \frac{F_{\max}^2}{12}$$

Finally the root mean square value of $F(t)$ is

$$RMS_F = \sqrt{\overline{F^2}} = \frac{F_{\max}}{\sqrt{3}}$$

22.2 Probability Density Function

Figure 22.4 shows a portion of a record of a random function $x(t)$. If we wish to determine the probability of x having a value in the range (x_1, x_2), we may draw horizontal lines through the values x_1 and x_2, and then measure the corresponding time intervals Δt_1. The ratio given by

$$P(x_1 \leq x \leq x_2) = \frac{\Delta t_1 + \Delta t_2 + \ldots + \Delta t_n}{T} \tag{22.7}$$

and calculated for the entire record length T, is the probability of x having a value between x_1 and x_2 at any selected time t_i during the random process.

Similarly, the probability of $x(t)$ being smaller than a value x can be expressed as

$$P(x) = P[x(t) < x] = \lim_{T-x} \frac{1}{T} \sum_i \Delta t_i \tag{22.8}$$

where the time intervals Δt_i are now those for which the function $x(t)$ has a value smaller than the specified x.

The function $P(x)$ in Eq. (22.8) is known as the cumulative distribution function of the random function $x(t)$. This function is plotted in Fig. 22.5a as a function of x. The cumulative distribution function is a monotonically increasing function for which

$$P(-\infty) = 0, \quad 0 \leq P(x) \leq 1, \quad P(\infty) = 1$$

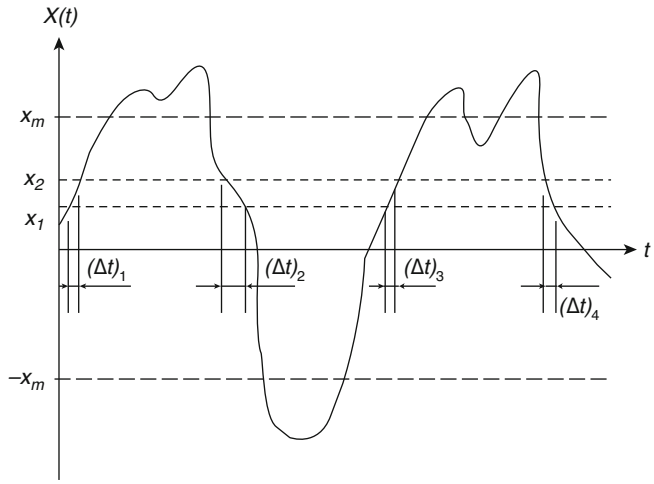

Fig. 22.4 Portion of random record showing determination of probabilities

Now, the probability that the value of the random variable is smaller than the value $x + \Delta x$ is denoted by $P(x + \Delta x)$ and that $x(t)$ takes a value between x and $x + \Delta x$ is $P(x + \Delta x) - P(x)$. This allows us to define the probability density function as

$$p(x) = \lim_{\Delta x \to 0} \frac{P(x - \Delta x) - P(x)}{\Delta x} = \frac{dP(x)}{dx} \tag{22.9}$$

Thus, the probability density function $p(x)$ is represented geometrically by the slope of the cumulative probability function $P(x)$. The functions $P(x)$ and $p(x)$ are shown in Figs. 22.5a, b, respectively. From Eq. (22.9) we conclude that the probability that a random variable $x(t)$ has a value between x and $x + dx$ is given by $p(x)\, dx$, where $p(x)$ is the probability density junction. Having prescribed $p(x)$, for example, as the function plotted in Fig. 22.5b, the probability of x being in the range (x_1, x_2) at any selected time is given by

$$P(x_1 \leq x \leq x_2) = \int_{x1}^{x_2} p(x)dx \tag{22.10}$$

and is equal to the shaded area shown between x_1 and x_2 in Fig. 22.5b. Similarly, the probability of x being greater than x_m, that is, $P\ (|x| > x_m)$ can be represented as the two shaded "tail" areas in Fig. 22.5b. Since every real x lies in the interval $(-\infty,\infty)$, the area under the entire probability density function is equal to 1, that is,

$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{22.11}$$

Thus as x tends to infinity in either direction, $p(x)$ must asymptotically diminish to zero.

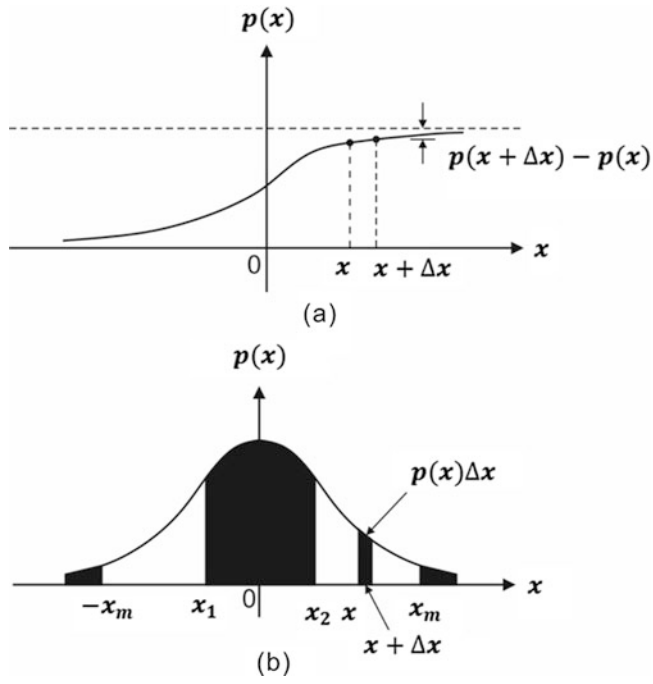

Fig. 22.5 (**a**) Cumulative probability function $P(x)$ and (**b**) Probability density function $p(x)$ of the random variable $x = x(t)$

22.3 The Normal Distribution

The most commonly used probability density function is the normal distribution, also referred to as the Gaussian distribution, expressed by

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(x-\bar{x}\right)^2/\sigma^2} \tag{22.12}$$

Figure 22.6 shows the shape of this function. It may be observed that the normal distribution function is symmetric about the mean value $\bar{x}$. In Fig. 22.7 the standard normal distribution is plotted non-dimensionally in terms of $(x-\bar{x})/\sigma$. Values of $P(-\infty, x_2)$ [$x_1 = -\infty$ in Eq. (22.10)] are tabulated in many sources including mathematical handbook.[1] The probability of x being between $x - \lambda\sigma$ and $x + \lambda\sigma$, where λ is any positive number, is given by the equation.

$$P\left[\bar{x} - \lambda\sigma < x < \bar{x} + \lambda\sigma\right] = \frac{1}{\sqrt{2\pi}\sigma} \int_{\bar{x}-\lambda\sigma}^{\bar{x}+\lambda\sigma} e^{-\frac{1}{2}\left(x-\bar{x}\right)^{2/\sigma^2}} dx \tag{22.13}$$

Equation (22.13) represents the probability that x lies within λ standard deviations from $\bar{x}$. The probability of x lying more than λ standard deviations from $\bar{x}$ is the probability of $|x-\bar{x}|$ exceeding $\lambda\sigma$, which is 1.0 minus the value given by Eq. (22.13). The following table presents numerical values for the normal distribution associated with $\lambda = 1$, 2, and 3:

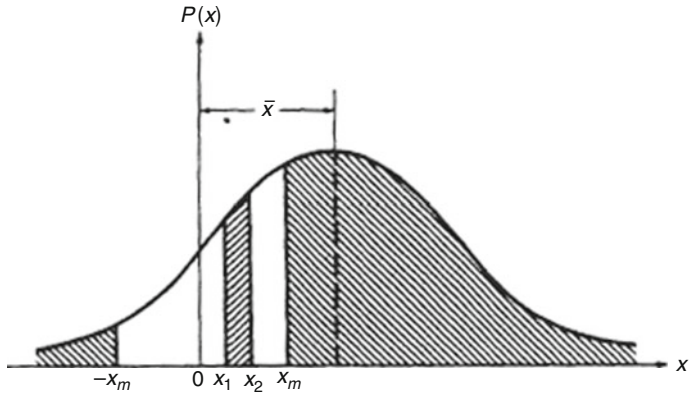

Fig. 22.6 Normal probability density function

λ	$P[\bar{x} - \lambda\,\sigma < x < \bar{x} + \lambda\,\sigma]$	$P[x - \bar{x}] > \lambda\,\sigma]$
1	68.3%	31.7%
2	95.4%	4.6%
3	99.7%	0.3%

[1] *Standard mathematical Tables,* The Chemical Rubber Co. (CRC) 20th Ed. 1972, pp. 566–575.

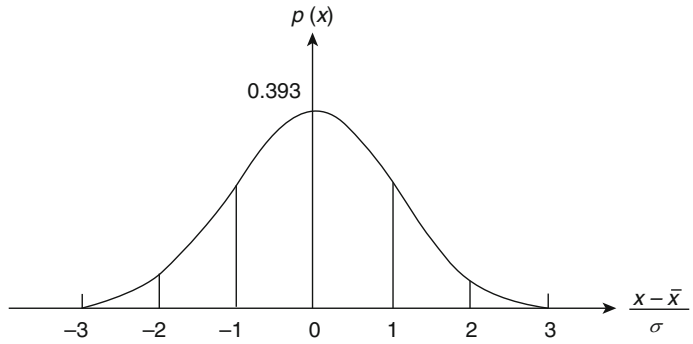

Fig. 22.7 Standard normal probability density function

22.4 The Rayleigh Distribution

Variables that are positive, such as the absolute value A of the peaks of vibration of a random function $x(t)$, often tend to follow the Rayleigh distribution, which is defined by the equation

$$p(A) = \frac{A}{\sigma^2} e^{-A^2/2\sigma^2}, \quad A > 0 \tag{22.14}$$

where σ is a parameter that may be interpreted as the standard deviation of a function $x(t)$.

The probability density $p(A)$ is zero for $A < 0$ and has the shape shown in Fig. 22.8 for positive values A.

The mean and mean square values for the Rayleigh distribution function are given by Eqs. (22.1) and (22.2)as

$$\bar{A} = \int_0^\chi A p(A) dA = \int_0^\chi \frac{A^2}{\sigma^2} e^{-A^2/2\sigma^2} dA = \sqrt{\frac{\pi}{2}}\sigma$$

$$\overline{A^2} = \int_0^\chi A^2 p(A) dA = \int_0^\chi \frac{A^3}{\sigma^2} e^{-A^2/2\sigma^2} dA = 2\sigma^2$$

Therefore, the Root Mean Square value of the random variable A (RMS_A) is

$$RMS_A = \sigma\sqrt{2} \tag{22.15}$$

The variance associated with the Rayleigh distribution function is, by Eq. (22.4),

$$\sigma_A^2 = \overline{A^2} - \left(\bar{A}\right)^2 = \frac{4-\pi}{2}\sigma^2 = 0.429\sigma^2 \tag{22.16}$$

The probability of A exceeding a specified value $\lambda\sigma$, $P\ (A > \lambda\sigma)$ is defined as

$$P(A > \lambda\sigma) = \int_{\lambda\sigma}^{\infty} p(A) dA$$

Which after substituting $p(A)$ from Eq. (22.14) result in

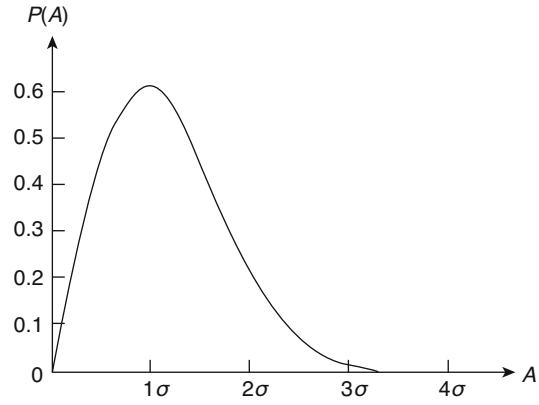

Fig. 22.8 Rayleigh probability density function

$$P(A > \lambda\sigma) = \int_{\lambda\sigma}^{\infty} \frac{A}{\sigma^2} e^{-A^2/2\sigma^2} dA$$

which may be written as

$$P(A > \lambda\sigma) = -\int_{\lambda\sigma}^{\infty} e^{\left(-A^2/2\sigma^2\right)} d\left(-A^2/2\sigma^2\right)$$

Thus,

$$P(A > \lambda\sigma) = e^{-A^2/2} \tag{22.17}$$

Also, the probability of A exceeding a specified value $\lambda\sigma$, $P\ (A > \lambda\sigma)$ is given by

$$P(A > \lambda\sigma_A) = \int_{\lambda\sigma_A}^{\infty} \frac{A}{\sigma^2} e^{-A^2/2\sigma^2} dA$$

Substituting from Eq. (22.16) $\sigma^2 = \sigma^2{}_A/0.429$ and integrating results in

$$P(A > \lambda\sigma_A) = e^{-0.429A^2/2} \tag{22.18}$$

The following table presents values for the probability of A, calculated from Eqs. (22.17) and (22.18), of exceeding $\lambda\sigma$, or $\lambda\sigma_A$ for $\lambda = 1,2,3$, and 4.

λ	$P[A > \lambda\sigma]\%$	$P[A > \lambda\sigma_A]\%$
1	60.65	80.69
2	13.53	42.40
3	1.11	14.51
4	0.03	3.23

22.5 Correlation

Correlation is a measure of the dependence between two random processes. Consider the two records shown in Fig. 22.9. The *correlation* between them is calculated by multiplying the coordinates of these two records at each time t_i and computing the average over all values of t. It is evident that the

correlation so found will be larger when the two records are similar. For dissimilar records with mean zero, some products will be positive and others negative. Hence their average product will approach zero.

We consider now the two records shown in Fig. 22.10 in which $x_1(t)$ is identical to $x(t)$ but shifted to the left in the amount τ, that is, $x_1(t) = x(t + \tau)$. The correlation between $x(t)$ and $x_1(t) = x(t + T)$ is known as the autocorrelation $R(\tau)$ and is given by

$$R(\tau) \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)x(t+\tau)dt \tag{22.19}$$

When $\tau = 0$, the autocorrelation reduces to the mean square value, that is,

$$R(0) = \lim_{T\to\infty} \frac{1}{T} \int_0^T [x(t)]^2 dt = \overline{x^2} \tag{22.20}$$

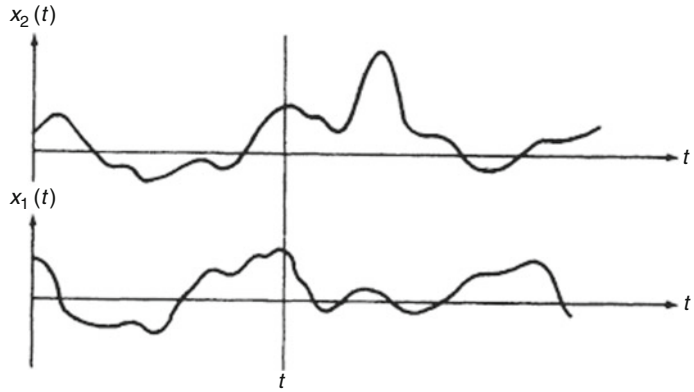

Fig. 22.9 Correlation between x_1 (t) and x_2 (t)

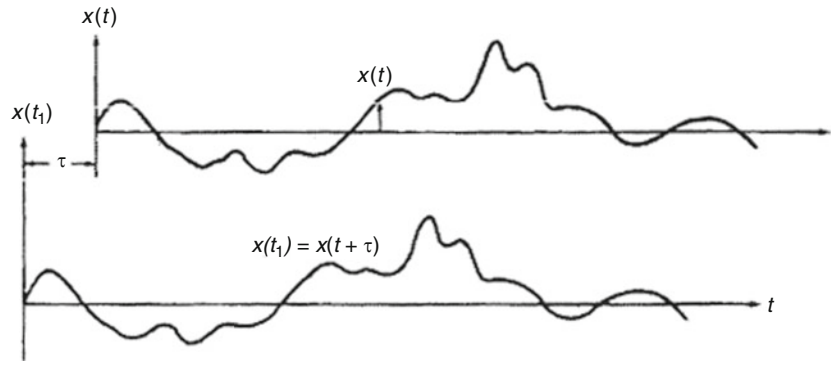

Fig. 22.10 Autocorrelation between x (t) and x $(t + \tau)$

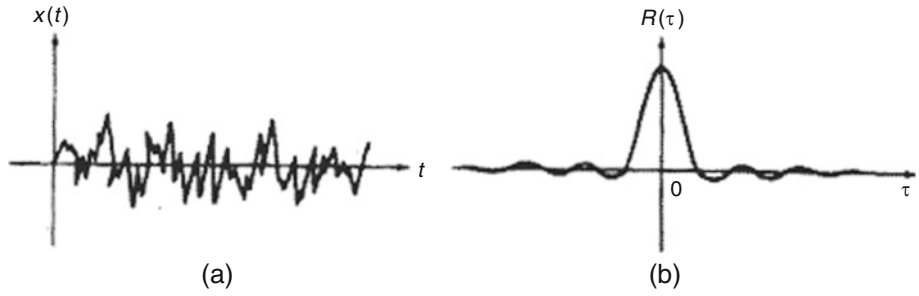

Fig. 22.11 Wide-band random process $x\ (t)$ and its autocorrelation function $R(\tau)$

Since the second record of Fig. 22.10 can be considered to be delayed with respect to the first record, or the first record advanced with respect to the second record, it is evident that $R(\tau) = \mathrm{R}(-\tau)$ is symmetric about the R axis and that $\mathrm{R}(\tau)$ is always less than R(0).

Highly random or wide-band functions such as the one shown in Fig. 22.11a lose their similarity within a short time shift. The autocorrelation of such a function, therefore, is a sharp spike at $\tau = 0$ that drops off rapidly as τ moves away from zero, as shown in Fig. 22.11b. For the narrow-band record containing a dominant frequency as shown in Fig. 22.12a, the autocorrelation has the characteristics indicated in Fig. 22.12b in that it is a symmetric function with a maximum at $\tau = 0$ and frequency ω_0 corresponding to the dominant frequency of $x(t)$.

Illustrative Example 22.2

For the function depicted in Fig. 22.13, determine: (a) the mean, (b) the mean square value, and (c) the autocorrelation function. Also, plot the autocorrelation function.

Solution: because the function $x(t)$ is periodic, averages calculated over a long time approach those calculated over a single period. Considering the period between 0 and T, the function is described analytically by

$$\begin{aligned} x(t) &= x_0 \qquad 0 < x < T/2 \\ &= 0 \qquad T/2 < x < T \end{aligned} \tag{a}$$

(a) Mean value: by Eq. (22.1)

$$\bar{x} = \frac{1}{T}\int_0^{\frac{T}{2}} x_0 dt = \frac{x_0}{2} \tag{b}$$

(b) Mean square value: by Eq. (22.2)

$$\overline{x^2} = \frac{1}{T}\int_0^{\frac{T}{2}} x_0^2 dt = \frac{x_0^2}{2} \tag{c}$$

(c) Autocorrelation function:

To calculate the autocorrelation function, it is necessary to distinguish between the time shift

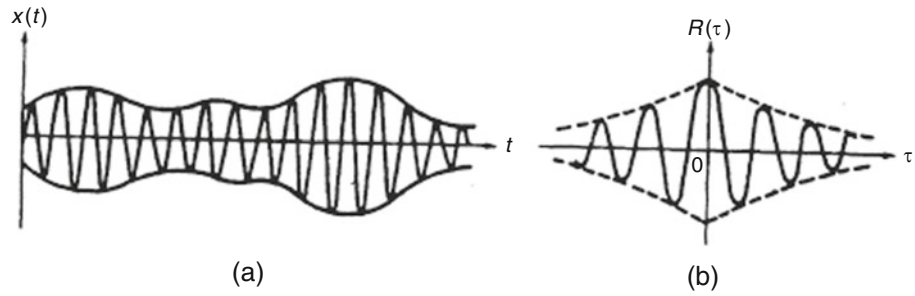

Fig. 22.12 Narrow-band random process $x(t)$ and its autocorrelation function $R(T)$

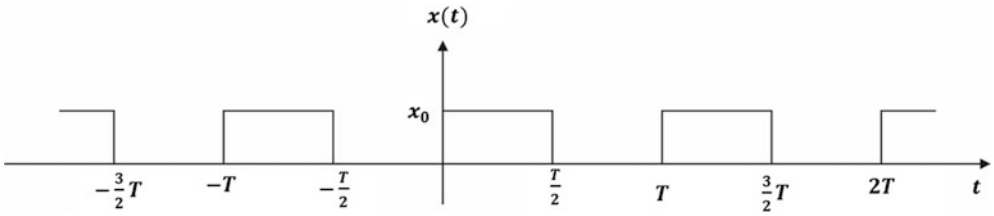

Fig. 22.13 Periodic function $x(t)$ of Illustrative Example 22.2.

$0 < \tau < T/2$ and $T/2 < \tau < T$ as shown, respectively, in Fig. 22.14a, b.

(d) Using Eq. (22.19) and considering Fig. 22.14a, we obtain for $0 < \tau < T/2$

$$R(\tau) = \frac{1}{T}\int_0^T x(t)\cdot x(t+\tau)dt = \int_0^{\frac{T}{2}-\tau} x_0 \cdot x_0 dt$$
$$= \frac{x_0^2}{T}\left(\frac{T}{2}-\tau\right) = x_0^2\left(\frac{1}{2}-\frac{\tau}{T}\right) \quad \text{for} \quad 0 > \tau > \frac{T}{2} \tag{d}$$

in which the limits of integration are given by the overlapping portions of $x(t)$ and $x(t+T)$ as indicated by the shaded area in Fig.22.14a.

When $T/2 < \tau < T$, we obtain from Fig. 22.14b.

$$R(\tau) = \frac{1}{T}\int_{T-\tau}^{\frac{T}{2}} x_0 . x_0 dt = \frac{x_0^2}{T}\left[\frac{T}{2} - T + \tau\right]$$
$$= x_0^2\left(-\frac{1}{2}+\frac{\tau}{T}\right) \quad \text{for} \quad \frac{T}{2} < \tau < T \tag{e}$$

From examination of Fig. 22.14, it can be concluded that the autocorrelation function $R(\tau)$ for the function $x(t)$ must be periodic in t with period T. Therefore, from Eq. (d), (e), and the fact that $R(\tau)$ is periodic, we can plot the autocorrelation function as shown in Fig. 22.15.

22.6 The Fourier Transform

In Chap. 19, we used Fourier series to obtain the frequency components of periodic functions of time.

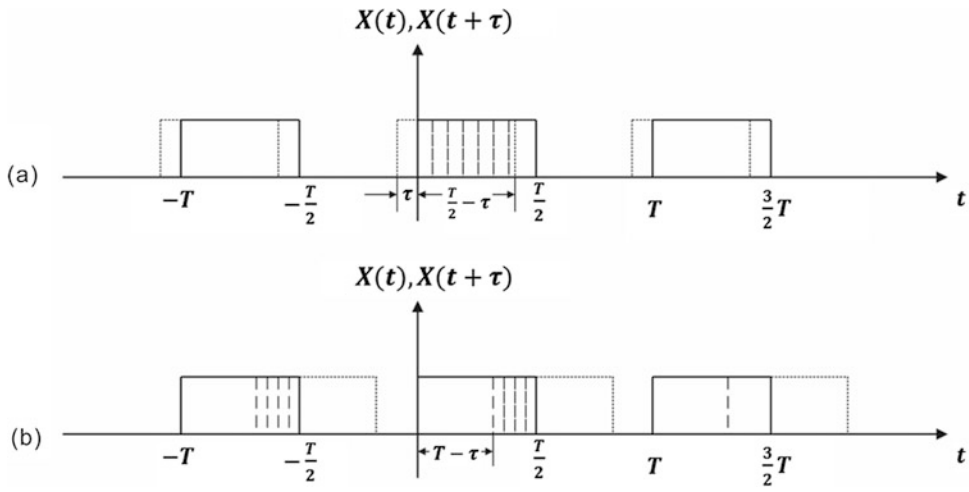

Fig. 22.14 Plot of $x(t)$ and $x(t+\tau)$; (**a**) for $0 < \tau < T/2$. (**b**) for $T/2 < \tau < T$

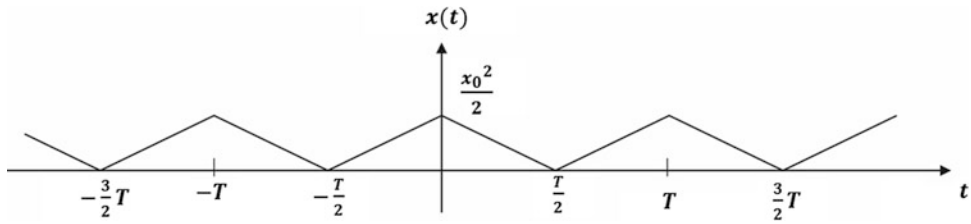

Fig. 22.15 Autocorrelation function $R(\tau)$ for the function of Illustrative Example 22.2

In general, random vibrations are not periodic, and the frequency analysis requires the extension of Fourier series to the Fourier integral for non-periodic functions. Fourier transforms, which result from Fourier integrals, enable a more extensive treatment of the random vibration problem.

We begin by showing that the Fourier integral can be viewed as a limiting case of the Fourier series as the period goes to infinity. Toward this objective we consider the Fourier series in exponential form and substitute the coefficient C_n given by Eq. (19.20) into Eq. (19.19) to obtain:

$$F(t) = \sum_{n=-\chi}^{\chi} \frac{1}{T} \int_{-T/2}^{T/2} F(\tau) \cdot e^{-in\varpi\tau} \cdot e^{in\varpi t} d\tau \tag{22.21}$$

In Eq. (22.21) we have selected the integration period from $-T/2$ to $T/2$ and substituted the symbol τ for t as the dummy variable of integration. The frequency $\omega = n\varpi$ is specified here at discrete, equally spaced values separated by the increment

$$\Delta\omega = (n+1)\varpi - n\varpi = \varpi = \frac{2\pi}{T}$$

We substitute in Eq. (22.21) ω for $n\varpi$ and $\Delta\omega/2\pi$ for $1/T$) and notice that as $T \to \infty$, $\Delta\omega \to d\omega$. Thus, in the limit, Eq. (22.21) becomes

$$\mathrm{F(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} F(\tau)e^{-i\omega\tau}d\tau\right\}e^{i\omega t}d\omega \tag{22.22}$$

which is the Fourier integral of $F(t)$.

Since the function within the inner braces is a function of only ω, we can write this equation in two parts as

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt \tag{22.23}$$

and

$$F(t) = \int_{-\infty}^{\infty} C(\omega)e^{i\omega t}d\omega \tag{22.24}$$

The validity of these relations, according to classical Fourier transform theory, is subject to the condition that

$$\int_{-\infty}^{\infty} |F(t)|dt < \infty \tag{22.25}$$

The function $C(\omega)$ in Eq. (22.23) is the Fourier transform of $F(t)$ and the function $F(t)$ in Eq. (22.24) is the inverse Fourier transform of $C(\omega)$. The pair of functions $F(t)$ and $C(\omega)$ is referred to as a Fourier transform pair. Equation (22.23) resolves the function $F(t)$ into harmonic components $C(\omega)$, whereas Eq. (22.24) synthesizes $F(t)$ from these harmonic components. In practice it is more convenient to use frequency f in cps rather than the angular frequency ω in rad/sec. Mathematically, since $\omega = 2\pi f$ and $d\omega = 2\pi df$, this also has the advantage of reducing the Fourier transform pair into a more symmetric form, namely,

$$F(t) = \int_{-\infty}^{\infty} C(f)e^{i2\pi ft}df \tag{22.26}$$

and

$$C(f) = \int_{-\infty}^{\infty} F(t)e^{i2\pi ft}dt \tag{22.27}$$

22.7 Spectral Analysis

We have seen in Chap. 19 that the application of Fourier analysis to a periodic function yields the frequency components of the function given by either trigonometric terms [Eq. (19.2)] or exponential terms [Eq. (19.19)]. When the periodic function is known at N discrete, equally spaced times, the frequency components are then given by the terms in Eq. (19.28). Our purpose in this section is to relate the Fourier analysis for a given function $x(t)$ to its mean square value $\overline{x^2}$.

The contributions of the frequency components of $x(t)$ to the value $\overline{x^2}$ are referred to as the spectral function of $x(t)$. Hence spectral analysis consists in expressing $\overline{x^2}$ in terms of either the coefficients of the Fourier series (i.e., a_n and b_n. or equivalently C_n) when $x(t)$ is periodic, or in terms of the Fourier transform [i.e., $C(\omega)$] when $x\ (t)$ is not a periodic function.

We begin by performing the spectral analysis of a periodic function $x(t)$ expressed in Fourier series, Eq. (19.2), which is

$$x(t) = a_0 + \sum_{n-1}^{\infty} \{a_n \cos n\,\varpi t + b_n \sin n\,\varpi t\} \tag{22.28}$$

where the coefficients are given by Eq. (19.3) as

$$\begin{aligned} a_0 &= \frac{1}{T}\int_0^T x(t)dt \\ a_n &= \frac{2}{T}\int_0^T x(t)\cos n\,\varpi dt \\ b_n &= \frac{2}{T}\int_0^T x(t)\sin n\,\varpi dt \end{aligned} \tag{22.29}$$

In Eq. (22.29), T is the period of the function and $\varpi = 2\pi/T$ its frequency. The substitution of $x(t)$ from Eq. (22.28) for one of the factors in the definition of mean square value gives

$$\begin{aligned} \overline{x^2} &= \frac{1}{T}\int_0^T x^2(t)dt \\ &= \frac{1}{T}\int_0^T x(t)\left\{a_0 + \sum_{n=1}^{\infty}(a_n \cos n\varpi t + b_n \sin n\varpi t)\right\}dt \\ &= \frac{a_0}{T}\int_0^T x(t)dt + \sum_{n=1}^{\infty}\left[\frac{a_n}{T}\int_0^T x(t)\cos n\varpi t dt + \frac{b_n}{T}\int_0^T x(t)\sin n\varpi t dt\right] \end{aligned}$$

Finally substituting the integral expressions from Eqs. (22.29) we get the desired result

$$\overline{x^2} = a_0^2 + \sum_{n=1}^{\infty}\left[\frac{a_n^2}{2} + \frac{b_n^2}{2}\right] \tag{22.30}$$

The spectrum of the function $x(t)$ is then given by the terms of the series in Eq. (22.30). Each term of this series is the contribution of the corresponding frequency to the mean square value of $x(t)$.

We now consider a discrete time function $F(t_j)$ expressed as a discrete Fourier transform [Eq. (19.28)], that is, as

$$F(t_j) = \sum_{n=0}^{N-1} C_n e^{2\pi i(nj/N)} \tag{22.31}$$

where C_n is given by Eq. (19.27) as

$$C_n = \frac{1}{N}\sum_{j=0}^{N-1} F(t_j)e^{-2\pi i(nj/N)} \tag{22.32}$$

As indicated in Chap. 19 [Eq. (19.29) or (19.30)], harmonic components of the function $F(t_j)$ higher than the Nyquist frequency, $\omega_{N\,2} = \pi/\Delta t$ rad/sec or $f_{N\,2} = 1/\Delta t$ cps, are not included in the discrete Fourier transform [Eq. (22.31)]. Also as noted in Chap. 19, if there are harmonic components in $F(t_j)$ higher than this limiting value, these higher frequencies introduce distorting contributions to the lower harmonic frequencies. Hence it is imperative that the value of N be selected sufficiently large to include the frequencies that contribute significantly to the original function. To be certain that this condition is satisfied, one may filter the signal of the function electronically to remove all frequencies higher than the Nyquist frequency.

The mean square value of a discrete function $F(t_j)$ ($j = 0, 1, 2, \ldots, N–1$) is obtained from Eq. (22.2) as

$$\overline{F^2} = \frac{1}{T}\sum_{j=0}^{N-1} F^2(t_j)\Delta t \tag{22.33}$$

Substituting $\Delta t/T$ for $1/N$ and using Eqs. (22.31) and (22.32) for one factor $F(t_j)$ in Eq. (22.33), we obtain

$$\begin{aligned}\overline{F^2} &= \frac{1}{N}\sum_{j=0}^{N-1} F(t_j)\sum_{n=0}^{N-1} C_n e^{2\pi i(nj/N)} \\ &= \sum_{n=0}^{N-1} C_n\left[\frac{1}{N}\sum_{j=0}^{N-1} F(t_j)e^{2\pi i(nj/N)}\right] \\ &= \sum_{j=0}^{N-1} C_n C_n^*\end{aligned} \tag{22.34}$$

where C_n^* is the complex conjugate[2] of C_n. Hence

$$\overline{F^2} = \sum_{n=0}^{N-1} |C_n|^2 = |C_0|^2 + |C_1|^2 + |C_2|^2 + \ldots + |C_{N-1}|^2 \tag{22.35}$$

The terms of the summation in Eq. (22.35) are the required spectrum of the discrete function $F(t_j)$; that is, these terms are the frequency contributions to the mean square value $\overline{F^2}$. As we can see in Eq. (22.35), the contribution of each frequency is equal to the square of the modulus of the corresponding complex coefficient C_n which is given by Eq. (22.32).

Illustrative Example 22.3

[2] Since $F(t_j)$ is a real function, its conjugate $F(t_j)^* = F(t_j)$ and also $[e^{-2\pi(nj/N)}]^* = e^{2\pi(nj/N)}$, it follows that $1/N\sum_{j=0}^{N-1} F(t_j)e^{2\pi(nj/N)} = C_n^*$

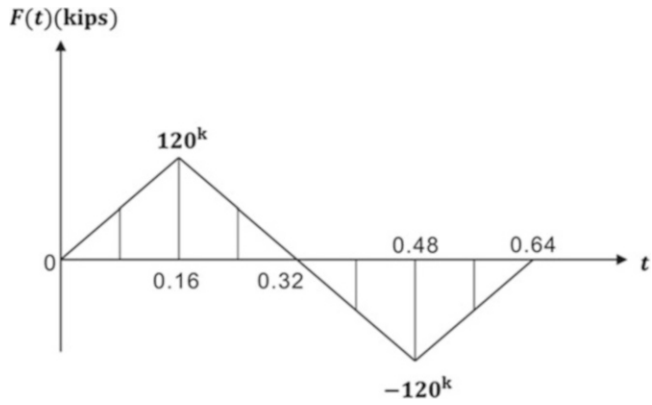

Fig. 22.16 Forcing function for Illustrative Example 22.3

Determine the spectrum of the forcing function $F(t)$ shown in Fig. 22.16. Assume that the function is defined at eight equally spaced time intervals. Use the spectrum to estimate the mean square value $\overline{F^2}$ and compare this result with the mean square calculated directly from the definition of $\overline{F^2}$.

Solution: We use computer Program 4 with $N = 8$ to determine the discrete Fourier coefficients C_n. The values thus obtained are shown in Table 22.1 together with the calculation needed to obtain the spectrum using Eq. (22.35). The summation of the spectral values for $F(t)$ shown in the last column of Table 22.1 is $\overline{F^2} = 0.5400$ E10. We check this value by calculating the mean square value of $F(t)$ directly from the definition Eq. (22.2), namely,

$$\overline{F^2} = \frac{1}{T}\int_0^T F^2(t)dt$$

in which we substitute

$$F(t) = \frac{120,000}{0.16}t$$

$$\overline{F^2} = \frac{4}{0.64}\int_{0.}^{0.16}\left[\frac{120,000}{0.16}\cdot t\right]^2 dt = 0.4800E10$$

Table 22.1 Spectral analysis for the function F(t) in Fig. 22.16

N	Fourier Coefficients C_n Re(Cn)	Im(Cn)	$\lvert C_n\rvert$	Spectrum of $F(t)$ $\lvert C_n\rvert^2$
0	0	0		0
1	0	−51,210	51,210	0.2623 $E10$
2	0	0		0
3	0	8787	8787	0.7721$E8$
4	0	0		0
5	0	8787	8787	0.7721$E8$
6	0	0		0
7	0	51,210	51,210	0.2623$E10$
				$\overline{F^2} = 0.5400\ E10$

Table 22.2 Mean square value for function $F(t)$ in Fig. 22.16

Exponent M	Time intervals $N = 2^M$	Mean square value $\overline{F^2}$
3	8	0.5400 *E* 10
4	16	0.4900 *E* 10
5	32	0.4838 *E* 10
6	64	0.4809 *E* 10
7	128	0.4802 *E* 10
		Exact $\overline{F^2} = 0.4800\ E10$

Considering that we have used in this example a relatively small number of intervals or sampled points ($N = 8$), the value of $\overline{F^2} = 0.5400$ E10 obtained from the spectrum of $F(t)$ in Table 22.1 compares fairly well with the exact mean square value $\overline{F^2} = 0.4800$ E10. As discussed before, errors are introduced in the calculations when the number of time intervals N is not large enough to include the higher frequency components of $F(t)$. In order to improve the calculation of the spectrum, it is necessary to use more time intervals in the discrete Fourier Series. Table 22.2 shows results obtained using Program 4 with $M = 3, 4, 5, 6, 7$, corresponding to $N = 2^M = 8, 16, 32, 64, 128$ time intervals for the function $F(t)$ shown in Fig. 22.16. It may be observed that values displayed in the last column of the table are converging to the exact value $\overline{F^2} = 0.4800$ E10 as the number of time intervals, N, is increased.

22.8 Spectral Density Function

If a random process $x(t)$ is normalized (or adjusted) so that the mean value of the process is zero, then, provided that $x(t)$ has no periodic components, the autocorrelation function $R_x(\tau)$ approaches zero as τ increases, that is.

$$\lim_{\tau \to \infty} R_x(\tau) = 0$$

We therefore expect that $R_x(\tau)$ should satisfy the condition in Eq. (22.25). We can then use Eqs. (22.23) and (22.24) to obtain the Fourier transform $S_x(\omega)$ of the autocorrelation function $R_x(\tau)$ and its inverse as

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega\tau} d\tau \tag{22.36a}$$

and

$$R_x(\tau) = \int_{-\infty}^{\infty} s_x(\omega) e^{i\omega\tau} d\omega \tag{22.36b}$$

In Eq. (22.36a), $S_x(\omega)$ is called the spectral density function of $x(t)$. The most important property of $S_x(\omega)$ becomes apparent by letting $\tau = 0$ in Eq. (22.36b). In this case

$$R_x(0) = \int_{-\infty}^{\infty} S_x(\omega) d\omega) \tag{22.37}$$

which by Eq. (22.20) is equal to mean square value of the function $x(t)$, that is,

$$\overline{x^2} = \int_{-\infty}^{\infty} S_x(\omega) d\omega \tag{22.38}$$

The mean square value of a random process is therefore given by the area under the graph of the spectral density function as shown in Fig. 22.17. Consequently, the contribution of an incremental frequency $\Delta\omega$ to the mean square value is

$$\Delta\overline{x^2} = S_x(\omega)\Delta\omega \tag{22.39}$$

The Fourier transform pair for the discrete autocorrelation function $R_j = R(\tau)$ and for the discrete spectral density function $S_n = S(\omega_n)$ of a time function $x = x(t)$, is defined from Eqs. (19.27) and (19.28) as

$$R_j = \sum_{n=0}^{N-1} S_n e^{\pi i(nj/N)} \tag{22.40a}$$

and

$$S_n = \frac{1}{N} \sum_{j=0}^{N-1} R_j e^{-2\pi j(nj/N)} \tag{22.40b}$$

It may be seen then, that the terms in Eq. (22.40b) provide the contributions, $\Delta\overline{x^2}$, to mean square value $\overline{x^2}$, at the frequency as

$$\Delta\overline{x^2} = S_n \tag{22.41}$$

The contribution to the mean square value may also be expressed, in terms of the Fourier transform coefficient C_n and its conjugate C_n^*, using Eq. (22.34) as

$$\Delta\overline{x^2} = C_n C_n^* \tag{22.42}$$

Therefore, it follows from Eqs. (22.41) and (22.42) that.

$$S_n = C_n C_n^* \tag{22.43}$$

The spectral density of a given record can be obtained electronically by an instrument called a frequency analyzer or spectral density analyzer. The output of an accelerometer or other vibration transducer is fed into the instrument, which is essentially a variable frequency narrow-band filter with a spectral meter to display the filtered output. With this instrument the experimenter searches for the predominant frequencies present in a vibration signal. The output of the spectral density analyzer is the contribution to the mean square value $\overline{x^2}$ of the input signal $x(t)$ for a small range $\Delta\omega$ around the set frequency.

When dealing with theory, the natural unit for the frequency is rad /sec. However, in most practical problems the frequency is expressed in cycles per second or Hertz (abbreviated Hz). In the latter case, we rewrite Eq. (22.39) as

$$\Delta\overline{x^2} = S_x(f)\Delta f \tag{22.44}$$

where f is the frequency in Hertz. Since $\Delta\omega = 2\pi\Delta f$, it follows from Eqs. (22.39) and (22.44) that

$$S_x(f) = 2\pi S_x(\omega) \tag{22.45}$$

When the spectral density function for the excitation is known, its mean-square value may be determined from Eq. (22.44) as

$$\overline{x^2} = \int_{-\infty}^{\infty} S_x(f)df \tag{22.46}$$

The spectral density function $S_x(f)$ is expressed in square units of x per Hertz. Since the autocorrelation function $R_x(\tau)$ is real and even, the use of Euler's relationship

$$e^{i\pi\tau} = \cos\,\omega\tau + i\,\sin\,\omega\tau$$

in Eq. (22.36a) yields the cosine transform:

$$S_x(\omega) = \frac{1}{2\pi}\int_{-x}^{x} R_x(\tau)\cos\,\omega\tau d\tau \tag{22.47}$$

It is clear from Eq. (22.47) that $S_x(\omega)$ is also an even function of ω; hence Eq. (22.36b) may be written as

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\cos\,\omega(\tau)d\omega \tag{22.48}$$

Alternatively, Eqs. (22.47) and (22.48) may be written as

$$S_x(\omega) = \frac{1}{\pi}\int_{0}^{\infty} R_x(\tau)\cos\,\omega\tau\,d\tau \tag{22.49}$$

and

$$R_x(\tau) = 2\int_{0}^{\infty} S_x(\omega)\cos\,\omega\tau\,d\omega \tag{22.50}$$

These are the celebrated Wiener-Kinchin equations, which describe how the spectral density function can be determined from the autocorrelation function and vice versa.

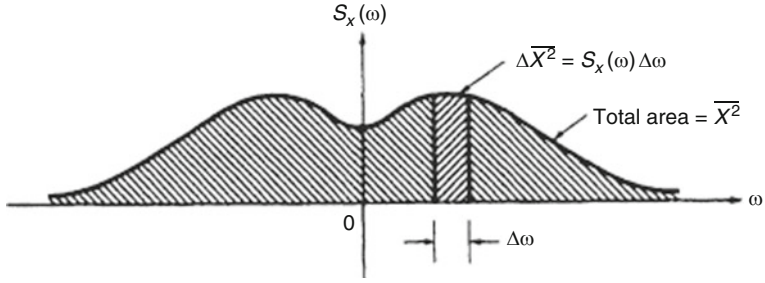

Fig. 22.17 Spectral density function showing area equal to mean-square value

22.9 Narrow-Band and Wide-Band Random Processes

A process whose spectral density function has nonzero values only in a narrow frequency range as shown in Fig. 22.18 is called a narrow band process. In contrast, a wide-band process is one whose spectral density function is nonzero over a broad range of frequencies. The time history of such a process is then made up of the superposition of the whole band of frequencies as shown in Fig. 22.19a. In the limit, when the frequency band extends from $\omega_1 = 0$ to $\omega_2 = \infty$, this spectrum is called white noise. From Eq. (22.38) the mean square value of a white noise process must be infinite; therefore, the white noise process is only a theoretical concept. In practice a process is called white noise when the bandwidth of its frequencies extends well beyond all the frequencies of interest.

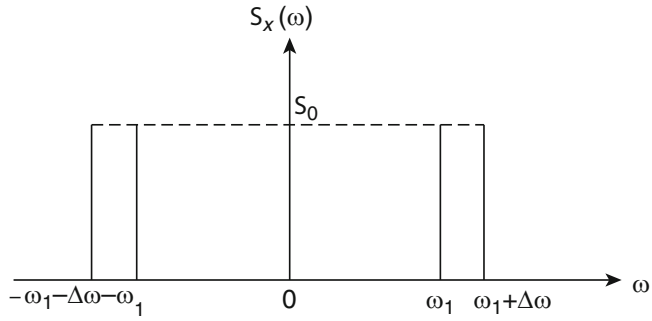

Fig. 22.18 Spectral density function for a narrow-band random process

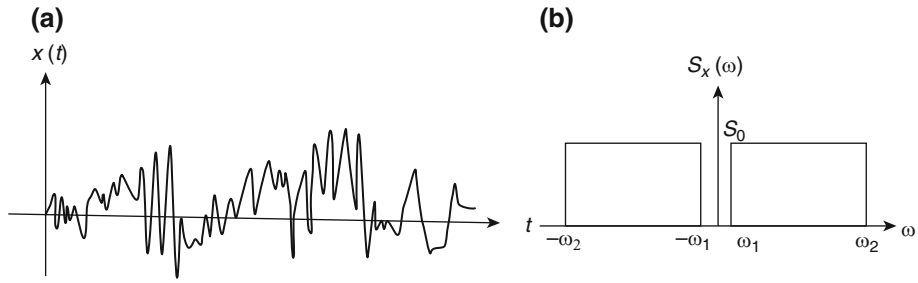

Fig. 22.19 Wide-band process. (**a**) Time history. (**b**) Spectral density function

Illustrative Example 22.4

Determine the mean square value and the autocorrelation function for the narrow-band random process $x(t)$ whose spectral density function is shown in Fig. 22.18.

Solution: From Eq. (22.38)

$$\overline{x^2} = \int_{-\infty}^{\infty} S_x(\omega)d\omega = 2S_0\Delta\omega$$

and from Eq. (22.50)

$$
\begin{aligned}
R_x(\tau) &= 2\int_0^{\infty} S_x(\omega)\cos\omega\tau d\omega \\
&= 2\int_{\omega_1}^{\omega_1+\Delta\omega} S_0\cos\omega\tau d\omega \\
&= \frac{2S_0}{\tau}\,|\sin\omega\tau|_{\omega_1}^{\omega_1+\Delta\omega}
\end{aligned}
\tag{a}
$$

$$
\begin{aligned}
&= \frac{2S_0}{\tau}\,[\sin(\omega_1+\Delta\omega)\tau - \sin\omega_1\tau] \\
R_x(\tau) &= \frac{4S_0}{\tau}\cos\left(\omega_1+\frac{\Delta\omega}{2}\right)\tau\cdot\sin\frac{\Delta\omega}{2}\tau
\end{aligned}
\tag{b}
$$

The autocorrelation function for a narrow-band random process given by Eq. (b) has the form shown in Fig. 22.20, where the predominant frequency of $R_x(\tau)$ is the average value ($\omega_1 + \Delta\omega/2$). The autocorrelation for such a process has a maximum of $2S_0\,\Delta\omega$ when $\tau = 0$ and decreases like a cosine graph as τ moves away from $\tau = 0$.

The autocorrelation function $R_x(\tau)$ for a wide-band random process whose spectral density function extends in the range ω_1 to ω_2 as shown in Fig. 22.19b can be obtained from the result of Illustrative Example 22.4. In this case, letting the lower frequency $\omega_1 = 0$ and the upper frequency $\Delta\omega = \omega_2$ we obtain from Eq. (b) of Illustrative Example 22.4

$$
R_x(\tau) = \frac{4S_0}{\tau}\cos\left(\frac{\omega_2\tau}{2}\right)\sin\left(\frac{\omega_2\tau}{2}\right) = \frac{2S_0}{\tau}\sin\omega_2\tau \tag{22.51}
$$

which has the form shown in Fig. 22.22a.

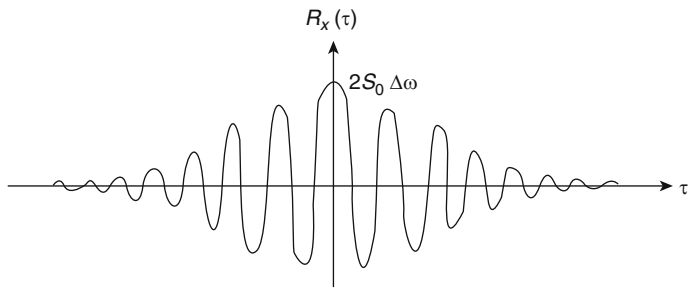

Fig. 22.20 Autocorrelation for a narrow-band random process

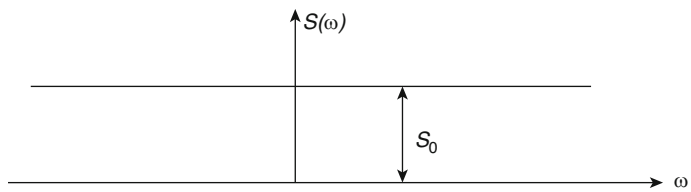

Fig. 22.21 Spectral density function for Illustrative Example 22.5

Illustrative Example 22.5

Determine the autocorrelation function for constant spectral density function $S(\omega) = S_0$ (white noise) for all value of the frequency ω as shown graphically in Fig. 22.21.

The autocorrelation function for white noise may be obtained from Eq. (22.51) by letting $\omega_2 \to \infty$. In this case adjacent cycles come closer together, resulting in a high peak at $\tau = 0$ and zero value elsewhere as shown in Fig. 22.22b.

This high peak will be of infinite height, zero width, but of a finite area. Such behavior may be described mathematically using Dirac's delta function $\delta(\tau)$. The delta function $\delta(\tau)$ is defined as having zero value everywhere except at $\tau = 0$ in such a way that

$$\int_{-\infty}^{\infty} \delta(\tau) f(\tau)\, d\tau = f(0) \tag{22.52}$$

for any function of time $f(\tau)$ defined at $\tau = 0$.

Using the delta function $\delta(\tau)$ we can express the autocorrelation function for white noise as

$$R_x(\tau) = C\delta(\tau) \tag{22.53}$$

where C must be determined from Eq. (22.36a) using the fact that $S_x(\omega)$ must be the constant S_0. Substituting Eq. (22.53) into the Eq. (22.36a) gives

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C\delta(\tau) e^{-j\omega\tau} d\tau$$

and using Eq. (22.52) yields

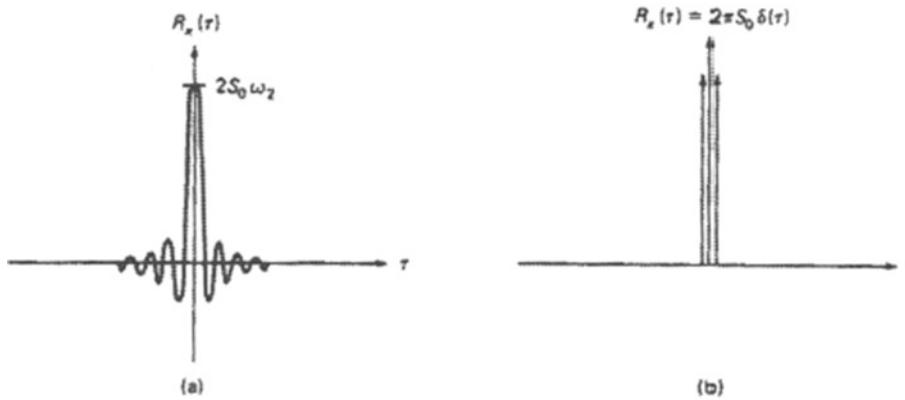

Fig. 22.22 Autocorrelation for a wide-band random process becomes a delta function for white noise

$$S_x(\omega) = \frac{1}{2\pi} C e^{-i\omega 0} = \frac{C}{2\pi} \tag{22.54}$$

Hence

$$S_0 = \frac{C}{2\pi} \tag{22.55}$$

Finally, solving for C and substituting into Eq. (22.53), we obtain the autocorrelation function for white noise shown in Fig. 22.22b, as

$$R(\tau) = 2\pi S_0 \delta(\tau) \tag{22.56}$$

22.10 Response to Random Excitation: Single-Degree-of-Freedom System

To determine the response of a structural system subjected to a random excitation, we need to examine the frequency content of the excitation function. We are mostly interested in estimating the spectral function or the spectral density function of the excitation.

Until recently, the procedure for estimating the spectrum of a discrete time series has been to first determine the autocorrelation function [Eq. (22.19)] and then apply the Fourier transform to this function to obtain the required spectrum [Eq. (22.36a) or Eq. (22.49)]. However, the method of calculation has changed since the development of Fast Fourier Transform (usually abbreviated FFT). As has been indicated in Chap. 19, the FFT is a remarkably efficient method for calculating the Fourier transform of a time series. Rather than estimate the spectrum by first determining the autocorrelation function and then calculating the Fourier transform, it is now more efficient and more accurate to calculate spectra directly from the original time series.

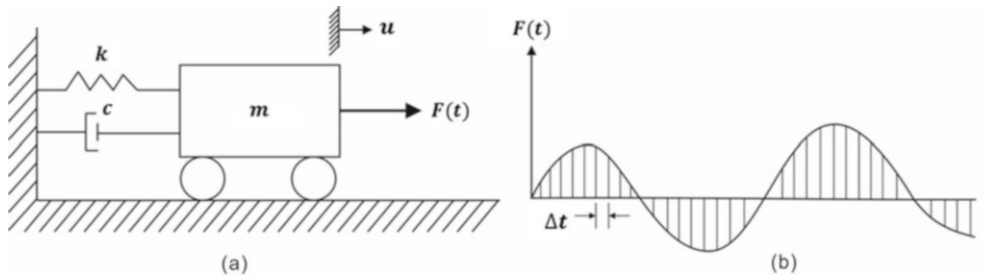

Fig. 22.23 Single-degree-of-freedom system subject to a random force sampled at regular time intervals

Consider the damped, single-degree-of-freedom system shown in Fig.22.23a subjected to a random force $F(t)$, a sample of which is shown in Fig. 22.23b. We will assume that this force is known at N discrete equally spaced time $t_j = j\Delta t$ ($j = 0, 1, 2, \ldots, N-1$). Fourier analysis of $F(t)$ results in the frequency component as given by Eqs. (22.31) and (22.32). By superposition the response of the single degree-of-freedom system to the harmonic components of $F(t)$ is given by Eq. (19.35) as

$$u(t_j) = \sum_{n=0}^{N-1} \frac{C_n e^{2\pi(nj/N)}}{k(1 - r_n^2 + 2ir_n\xi_n)} \tag{22.57}$$

in which, as discussed in Sect. 19.6,

$$\begin{aligned} \omega_n &= n\varpi \qquad \text{for } n \le N/2 \\ \omega_n &= -(N-n)\varpi \quad \text{for } n > N/2 \end{aligned} \tag{22.58}$$

$$r_n = \frac{\omega_n}{\omega}, \quad \varpi = \frac{2\pi}{T}, \quad \omega = \sqrt{\frac{k}{m}} \tag{22.59}$$

In these formulas, T is the time duration of the excitation, N the number of equal intervals of the excitation, and ξ_n the damping ratio corresponding to the frequency ω_n.

Equation (22.57) may conveniently be written as

$$u(t_j) = \sum_{n=0}^{N-1} H_{ij} C_n e^{2\pi(nj/N)} \tag{22.60}$$

where the frequency response fonction, H_n is given by

$$H_n = \frac{1}{k} \cdot \frac{1}{1 - r_n^2 + 2ir_n\xi_n} \tag{22.61}$$

The mean square value $\overline{u^2}$ of the response can then be obtained from Eq. (22.35) as

$$\overline{u^2} = \sum_{n=0}^{N-1} |H_n|^2 |C_n|^2 \tag{22.62}$$

Alternatively, Eq. (22.62) may be expressed in terms of the spectral density function. The frequency contributions $\Delta\overline{F^2}$ to the mean square value $\overline{F^2}$ are given by Eq. (22.35) as

$$\Delta\overline{F^2} = |C_n|^2 \tag{22.63}$$

which by Eq. (22.39), may be expressed as

$$\Delta\overline{F^2} = S_F(\omega_n)\Delta\omega \tag{22.64}$$

Now, using Eqs. (22.63) and (22.64), we may write Eq. (22.62) as

$$\overline{u^2} = \sum_{n=0}^{N-1} |H_n|^2 S_F(\omega_n)\Delta\omega \tag{22.65}$$

or as

$$\overline{u^2} = \sum_{n=0}^{N-1} S_u(\omega_n)\Delta\omega \tag{22.66}$$

where

$$S_u(\omega_n) = |H_n|^2 S_F(\omega_n) \tag{22.67}$$

When the frequency is expressed in cps, we may write Eq. (22.66) as

$$\overline{u^2} = \sum_{n=0}^{N-1} S_u(f_n)\Delta f \tag{22.68}$$

where

$$S_u(f_n) = |H_n|^2 S_F(f_n) \tag{22.69}$$

Equation (22.69) states the important result that, when the frequency response function, H_n is known, the spectral density $S_u(f_n)$ for the response can be calculated from the spectral density, $S_F(f_n)$, of the excitation.

Illustrative Example 22.6

Determine the mean square value of the response for the single-degree-of-freedom system in which $k = 100{,}000$ lb/in, $m = 100$ lb · sec^2/in, $c = 632$ lb · sec/in subjected to the $F(t)$ shown in Fig. 22.16. Choose $N = 8$ for the number of intervals.

Solution: The mean square value $\overline{u^2}$ of the response is given by Eq. (22.62) as

$$\overline{u^2} = \sum_{n=0}^{N-1} |H_n|^2 |C_n|^2$$

From Eq. (22.61)

$$|H_n|^2 = \frac{1}{k^2} \cdot \frac{1}{\left(1 - r_n^2\right)^2 + \left(2r_n\xi_n\right)^2}$$

where ω_n/ω is is the frequency ratio and $\xi_n = c_n/c_{cr}$ is the damping ratio.

The natural frequency is calculated as $\omega = \sqrt{k/m} = 31.62$ rad/ sec and the frequency components are given by Eq. (22.58) as

$$\omega_n = n\omega_1, \qquad n \le N/2$$
$$\omega_n = -(N-n)\omega_n, \quad n > N/2$$

in which $\omega_1 = 2\pi/T = 2\pi/0.64 = 9.8175$ rad/sec, since the duration of the applied force is $T = 0.64$ sec.

Values of $|C_n|$, for the function $F(t)$ shown in Fig. 22.16, have been determined in Illustrative Example 22.3 and are shown in Table 22.1. The necessary computations to determine $\overline{u^2}$ are conveniently shown in Table 22.3. From this table the mean square value of the response is $\overline{u^2} = 0.9352$.

The MATLAB program can be used to obtain the response of 0.9352.

```
clear all
close all
clc

%%%%-GIVEN VALUES-
m=100;                  %Mass (lb.sec^2/in.)
k =100000;              %Stiffness (lb/in.)

omega = sqrt(k/m);      %Natural frequency
c=632;                  %Damping coefficient. (lb.sec/in.)
xi =c/(2*m*omega);      %Damping ratio

T = 0.64;               %Time period, T(sec)
omega_bar = 2*pi/T;     %Excitation frequency (rad/sec)
M= 3;                   %Select M, M=3
N = 2^M;                %The number of time increments N

t=0:0.08:0.64;          %Time ranging from 0 to 0.64 sec with deltat = 0.08 sec
Dt = t(2)-t(1);         %Deltat = 0.08
tt= length(t);          %Total number of calculation

for i= 1:tt-1
%%%Define the function of N harmonic force
if t(i)<=0.16
       F(i) = 120000*t(i)/0.16;
elseif t(i) <=0.48
       F(i) =-750000*(t(i)-0.16)+120000;
else
       F(i)=min(0, 750000*(t(i)-0.64)) ;
end

%%%Define the discrete Fourier transform of the series
Cn=fft(F/N);                                          %Eq.19.27

%%%Calculate frequency ratio, r_n
ifi<=N/2
omega_n(i) = (i-1)*omega_bar;                         %Eq.22.58
else
omega_n(i) = -(N-(i-1))*omega_bar;                    %Eq.22.58
end

rn(i)=omega_n(i)/omega;                               %Eq.22.59

    H(i)=sqrt(1/k^2/((1-rn(i)^2)^2+(2*rn(i)*xi)^2));  %Eq.22.61

    Cn2=abs(Cn);

uu = H.^2.*Cn2.^2;                                    %Eq.22.62 (ith)

end
uu_sum = sum(uu)                                      %Eq.18.62
```

Analogous to Eq. (22.65), when the excitation is a random acceleration applied to the support of the structure, the mean-square acceleration response $\overline{a_p^2}$ at a point P of the structure is given by

$$\overline{a_p^2} = \sum S(\omega_n)|H_n|^2 \Delta\omega \tag{22.70}$$

where H_n is now the frequency response in terms of the acceleration at point P resulting from a unit harmonic acceleration at the support of the structure.

The response of a single-degree-of-freedom structure subjected to a single point random excitation can be determined by a simple numerical calculation provided that the spectral function or the spectral density function of the excitation and the frequency response of the structure are known.

The frequency response H_n may be obtained experimentally by applying a sinusoidal excitation of varying frequency at the foundation and measuring the response at the desired point in the structure. The necessary calculations are explained in the following numerical example.

Illustrative Example 22.7

Determine the response at a point P of the structure modeled as a shear building shown schematically in Fig. 22.24a when subjected to a random acceleration at its foundation. The spectral density function of the excitation is known and shown in Fig. 22.24b. The frequency response a_p/a_0 of the structure at point P, obtained experimentally, when the foundation is excited by a sinusoidal acceleration of amplitude a_0 and varying frequency f_n, is shown in Fig. 22.24c.

Table 22.3 Calculation of $\overline{u^2}$ for Illustrative Example 22.6

n	ω_n (rad/sec)	$r = \omega_n / \omega$	$\lvert H_n \rvert$ (in/lb)	$/C_n/$ (lb)	$\Delta\overline{u^2} = \lvert H_n \rvert^2 \lvert C_n \rvert^2$ (in^2)
0	0	0	1.000 E-5	0	0
1	9.8175	0.3105	1.104 E-5	0.5121 E 5	0.3197
2	19.6350	0.6209	1.595 E-5	0	0
3	29.4524	0.9314	4.376 E-5	0.8787 E 4	0.1479
4	39.2699	1.2418	1.677 E-5	0	0
5	−29.4524	−0.9314	4.376 E-5	0.8787 E4	0.1479
6	−19.6350	−0.6209	1.595 E-5	0	0
7	−9.8175	−0.3105	1.104 E-5	0.5121 E 5	0.3197
					$\overline{u^2} = 0.9352$

Solution: The mean square value $\overline{a_p^2}$ of the response at the point P is calculated from Eq. (22.70) as

$$\overline{a_p^2} = \sum S(f_n)|H_n|^2 \Delta f \tag{a}$$

where

$$| H_n | = a_p/a_0$$

Table 22.4 summarizes the computational procedure. By Eq. (a) we obtain from the sum in the last column of this table the mean square value of the response $\bar{a}^2 = 1.8100\text{g}^2$ and by Eq. (22.3) (assuming mean value $\bar{a} = 0$)

$$\sigma = \sqrt{1.8100}\, g = 1.345\, g$$

The probability of exceeding specified accelerations (see Table in Sect. 22.3) can now be found assuming the normal distribution for $/a_p/ > \sigma = 1.345\ g$ and for $/a_p/ > 3\sigma = 4.041\ g$, respectively, as

$$P\left[|a_p| > 1.345g\right] = 31.7\%$$

and

$$P[|a_P| > 4.041g] = 0.3\%$$

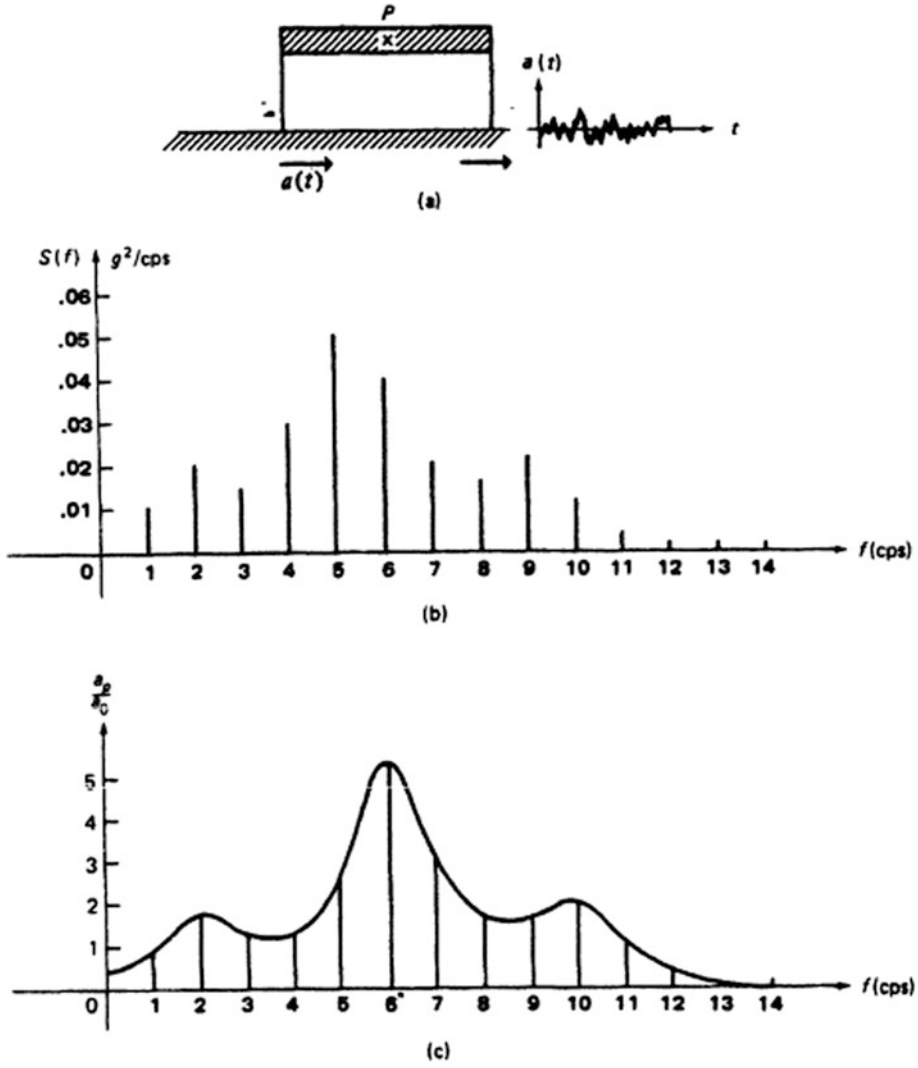

Fig. 22.24 (**a**) Structure subjected to random acceleration at the base. (**b**) Spectral density function of the excitation. (**c**) Relative frequency response at point *P*

Similarly, the probability that the peak acceleration A_p at point P will exceed a specified value (1 σ or 3 σ) is found using the Rayleigh distribution (see Table in Sect. 22.4) as

$$P\left[A_p > 1.345g\right] = 60.65\%$$
$$P\left[A_p > 4.041g\right] = 1.11\%$$

Table 22.4 Calculations of the response for Illustrative Example 22.7

f_n (cps)	Δf (cps)	$S(f_n)$(g^2/cps)	$\lvert H_n\rvert$ (in/lb)	$\lvert H_n\rvert^2\, S(f_n)\Delta f$ (g^2 units)
0	1.0	0	0.5	0
1.0	1.0	0.010.0	1.0	0.100
2.0	1.0	0.020	1.8	0.0648
3.0	1.0	0.015	1.3	0.0253
4.0	1.0	0.030	1.4	0.0588
5.0	1.0	0.050	2.2	0.2420
6.0	1.0	0.040	5.2	1.0816
7.0	1.0	0.020	3.0	0.1800
8.0	1.0	0.015	1.8	0.0486
9.0	1.0	0.020	1.7	0.0578
10.0	1.0	0.010	1.9	0.0361
11.0	1.0	0.005	1.0	0.0050
12.0	1.0	0	0.4	0
13.0	1.0	0	0	0
14.0	1.0	0	0	0
				Sum = 1.8100

22.11 Response to Random Excitation: Multiple-Degree-of-Freedom System

The extension for determining the response to random vibration from a single-degree-of-freedom system to a multiple-degree-of-freedom system can readily be accomplished using the modal superposition method. This method, as we have seen in previous chapters, transforms a system of coupled differential equations into a set of independent or uncoupled differential equations of only one dependent variable in each equation. Thus, each equation of this set is equivalent to the differential equation for a single-degree-of-freedom system and consequently can be solved by the method presented in the preceding section for a single-degree-of-freedom system.

We present first, the relationship between the complex frequency response and the unit impulse response followed by the development of random vibration for a system of two degrees of freedom and the generalization to multiple-degree-of-freedom systems.

22.11.1 Relationship Between Complex Frequency Response and Unit Impulse Response

Consider a linear dynamic system of a single-degree-of-freedom represented by the damped simple oscillator shown in Fig. 22.25.

The random force $F(t)$ acting on this system may be expressed in terms of its frequency content by means of the inverse Fourier transform given by Eq. (22.24) as

$$F(t) = \int_{-\infty}^{\infty} C(\omega) e^{i\omega t}\, d\omega \quad \text{(repeated)} \tag{22.24}$$

where the coefficient $C(\omega)$ is given by Fourier transform, Eq. (22.23) as

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-i\omega t}\,d\omega \quad \text{(repeated)} \tag{22.23}$$

The differential equation of motion for the system shown in Fig. 22.25 is

$$m\ddot{u} + c\dot{u} + ky = F(t) \tag{22.71}$$

which for an excitation of a unit exponential function, $F(t) = e^{i\ \omega t}$, as presented in Chap. 3, has a steady-state solution, $u_1(t)$ of the form

$$u_1(t) = H(\omega)e^{i\omega t} \tag{22.72}$$

in which

$$H(\omega) = \frac{1}{k - m\omega^2 + ic\omega}$$

Consequently, the response $u(t)$ due to the force expressed by the inverse of Fourier transform in Eq. (22.24) is given by

$$u(t) = \int_{-\infty}^{\infty} C(\omega)H(\omega)e^{i\omega t}\,dt \tag{22.73}$$

Consider now the special case where the total excitation consists of a single impulse of magnitude 1 applied at time $t = 0$.

In this case, $C(\omega)$ in Eq. (22.23) becomes

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-i\omega t}\,dt = \frac{1}{2\pi} \tag{22.74}$$

because $e^{-i\omega t} = 1$ at $t = 0$ and

$$\int_{-\infty}^{\infty} F(t)dt = 1 \tag{22.75}$$

for a unit impulse.

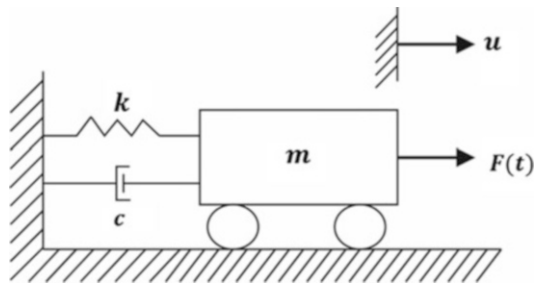

Fig. 22.25 Model of a damped single-degree-of-freedom system

The expression given by Eq. (22.75) is known as the Dirac's δ function. Thus, from Eqs. (22.73) and (22.74), the response, $h(t)$, to unit impulse is given by

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t}\, dt \tag{22.76}$$

which is recognized as the inverse Fourier transform of

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, d\omega \tag{22.77}$$

We observe that the unit impulse response $h(t)$ of Eq. (22.76) and the complex frequency response $H(\omega)$ of Eq. (22.77) form a Fourier transform pair.

22.11.2 Response to Random Excitation: Two-Degree-of-Freedom System

The concepts and methodology for determining the response of multiple-degree-of-freedom systems can be adequately presented through the treatment of a two-degree-of-freedom system. We begin by presenting a two-degree-of-freedom system subjected to a single random excitation. This case is then extended by subjecting this two-degree-of-freedom system to a second excitation, in addition to the first random excitation. Finally, we present the general case of a multiple-degree-of-freedom system subjected to multiple random excitations.

We consider in Fig. 22.26a the structural model of a two-degree-of-freedom system subjected to a random stationary force $F_2(t)$ applied at coordinate u_2. The equations obtained by establishing the dynamic equilibrium in the free-body-diagram in Fig. 22.26b are:

$$\begin{aligned} m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_1 &= 0 \\ m_2\ddot{u}_2 - k_1u_1 + k_2u_2 &= F_2(t) \end{aligned} \tag{22.78}$$

The natural frequencies and modal shapes are then found, as presented in Chap. 11, by setting $F_2(t) = 0$ in Eq. (22.78) and substituting a trial solution $u_1 = a_1 \sin \omega t$ and $u_2 = a_2 \sin \omega t$. Hence, we obtain

$$\begin{aligned} (k_1 + k_2 - m_1\omega^2)a_1 - k_2a_2 &= 0 \\ -k_2a_1 + (k_2m_2\omega_2)a_2 &= 0 \end{aligned} \tag{22.79}$$

For a nontrivial solution of Eq. (22.79), we set equal to zero the determinant of the coefficients:

$$\begin{vmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_1\omega^2 \end{vmatrix} \tag{22.80}$$

The expansion of the determinant in Eq. (22.80) results in the following quadratic equation in ω^2:

$$m_1m_2\omega^4 - [(k_1 + k_2)m_2 + k_2m_1]\omega^2 + k_1k_2 = 0 \tag{22.81}$$

Equation (22.81) provides the roots ω_1^2 and ω_2^2 which are the square values of the natural frequencies for this two-degree-of-freedom system. The substitution of ω_1^2 into one of the equations in Eq. (22.79) results in the first mode a_{11}, a_{21}, and the subsequent substitution of ω_2^2, in the second mode a_{12}, a_{22}. These modal shapes are conveniently normalized by dividing the components a_{11} and a_{21} of the first mode by $\sqrt{m_1a_{11}^2 + m_2a_{21}^2}$ and the components, a_{12} and a_{22}, of the second mode by j

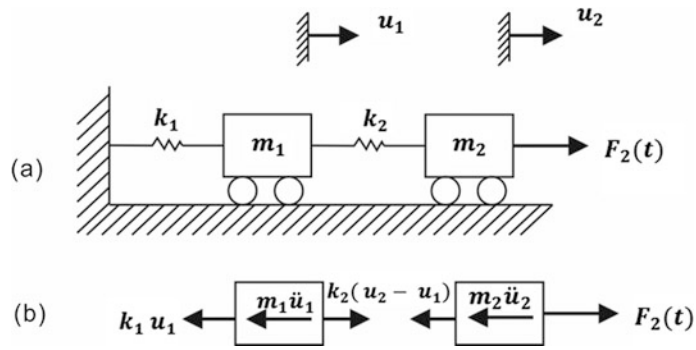

Fig. 22.26 Two-degree-of-freedom system. (**a**) Mathematical model. (**b**) Free-body-diagrams

$\sqrt{m_1 a_{12}^2 + m_2 a_{22}^2}$ to obtain the modal matrix given by Eq. (22.82) in which the normalized modes are arranged in the columns of this matrix.

$$[\phi] = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \tag{22.82}$$

The uncoupled modal equation, as presented in Chap. 6, is obtained by introducing in Eq. (22.78) the linear transformation

$$\begin{aligned} u_1(t) &= \phi_{11} z_1(t) + \phi_{12} z_2(t) \\ u_2(t) &= \phi_{21} z_1(t) + \phi_{22} z_2(t) \end{aligned} \tag{22.83}$$

resulting in

$$\begin{aligned} \ddot{z}_1 + \omega_1^2 z_1 &= \phi_{21} F_2(t) \\ \ddot{z}_2 + \omega_2^2 z_2 &= \phi_{22} F_2(t) \end{aligned} \tag{22.84}$$

in which z_1 and z_2 are the modal displacement functions.

Damping in the system may readily be included by adding in Eqs. (22.84) damping terms expressed in function of the modal damping ratios ξ_1 and ξ_2. Hence,

$$\begin{aligned} \ddot{z}_1 + 2\xi_2 \omega_1 \dot{z}_1 + \omega_1^2 z_1 &= \phi_{21} F_2(t) \\ \ddot{z}_2 + 2\xi_2 \omega_2 \dot{z}_2 + \omega_2^2 z_2 &= \phi_{22} F_2(t) \end{aligned} \tag{22.85}$$

The solution of Eqs. (22.85) may be expressed by Duhamel's integral which for a damped system is given from Eq. (4.24) as

$$z_1(t) = \int_{-\infty}^{\infty} \phi_{21} F_2(t - \lambda_1) h_1(\lambda_1) d\lambda_1 \tag{22.86}$$

and

$$z_2(t) = \int_{-\infty}^{\infty} \phi_{22} F_2(t - \lambda_2) h_2(\lambda_2) d\lambda_2$$

in which the unit impulse functions $h_1(t)$ and $h_2(t)$ are given for $t \geq 0$ by

$$h_1(t) = \frac{e^{-\xi_1 \omega_1 t}}{\omega_1 \sqrt{1 - \xi_1^2}} \sin\left[\omega_1 \sqrt{1 - \xi_1^2}\, t\right] \tag{22.87}$$

and

$$h_2(t) = \frac{e^{-\xi_2 \omega_2 t}}{\omega_2 \sqrt{1 - \xi_2^2}} \sin\left[\omega_2 \sqrt{1 - \xi_2^2}\, t\right] \tag{22.88}$$

and for $t < 0$ by

$$h_1(t) = h_2(t) = 0$$

The response is then obtained in terms of the displacements $u_1(t)$ and $u_2(t)$ by substituting the solution of the modal equations, $z_1(t)$ and $z_2(t)$, into Eq. (22.83).

We proceed now to the determination of the mean square value of the response, $u_1(t)$ in order to estimate its standard deviation and its confidence interval, due to random excitation. The mean square value of the response may be calculated from the autocorrelation function R_{u1} using Eqs. (22.19) and (22.20) as

$$R_{u_1}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u_1(t) u_1(t+\tau)\, dt$$

or using a more convenient notation as

$$R_{u_1}(\tau) = E[u_1(t) u_1(t + \tau)] \tag{22.89}$$

in which the letter E denotes expected or mean value of the function in the bracket. The substitution into (22.89) of $u_1(t)$ and $u_1(t + \tau)$ from Eq. (22.83) results in

$$R_{u_1}(\tau) = E[\phi_{11} z_1(t) + \phi_{12} z_2(t)][\phi_{11} z_1(t + \tau) + \phi_{12} z_2(t + \tau)]$$

or

$$R_{u_1}(\tau) = E\left[\phi_{11}^2 z_1(t) z_1(t + \tau) + \phi_{11}\phi_{12} z_1(t) z_2(t + \tau) + \phi_{12}\phi_{11} z_2(t) z_1(t + \tau) + \phi_{12}^2 z_2(t) z_2(t + \tau)\right] \tag{22.90}$$

The substitution of Eq. (22.86) into the expected value for the product of $z_1(t)\, z_1(t + \tau)$ in Eq. (22.90) results in

$$E[z_1(t) z_1(t + \tau)] = E\left[\int_{-\infty}^{\infty} \phi_{21} F_2(t - \lambda_1) h_1(\lambda_1) d\lambda_1 \int_{-\infty}^{\infty} \phi_{21} F_2(t + \tau - \lambda_2) h_1(\lambda_2) d\lambda_2\right]$$

Interchanging the symbols for expected value and integration, we obtain

$$E[z_1(t)z_1(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[F_2(t-\lambda_1)F_2(t+\tau-\lambda_2)]h_1(\lambda_1)\,h_1(\lambda_2)\phi_{21}^2 d\lambda_1 d\lambda_2$$

or

$$E[z_1(t)z_1(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_2(\tau-\lambda_2+\lambda_1)h_1(\lambda_1)\,h_1\left(\lambda_2\phi_{21}^2 d\lambda_1\lambda_2\right.$$

since the expected value, $E\,[F_2(t-\lambda_1)F_2(t+\tau-\lambda_2)]$ is equal to the autocorrelation of the function $F_2\,(t-\lambda_2+\lambda_1)$.

Analogously, for the other terms in Eq. (22.88), we obtain

$$E[z_1(t)z_2(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_2(\tau-\lambda_2+\lambda_1)h_1(\lambda_1)h_2(\lambda_2)\phi_{21}\phi_{22}d\lambda_1 d\lambda_2$$

$$E[z_2(t)z_1(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_2(\tau-\lambda_2+\lambda_1)h_1(\lambda_2)h_2(\lambda_1)\phi_{22}\phi_{21}d\lambda_1 d\lambda_2$$

$$E[z_2(t)z_2(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_2(\tau-\lambda_2+\lambda_1)h_2(\lambda_1)h_2(\lambda_2)\phi_{22}^2 d\lambda_1 d\lambda_2$$

which substituted into Eq. (22.90) yields

$$\begin{aligned} R_{u_1}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_2(\tau-\lambda_2+\lambda_1)\big[&\phi_{11}^2\phi_{21}^2 h_1(\lambda_1)h_2(\lambda_2) + \phi_{11}\phi_{12}\phi_{21}\phi_{22}h_1(\lambda_1)h_2(\lambda_2) \\ &+ \phi_{12}\phi_{11}\phi_{22}\phi_{21}h_1(\lambda_2)h_2(\lambda_1) + \phi_{12}^2\phi_{22}^2 h_2(\lambda_1)h_2(\lambda_2)\big]d\kappa_1\, d\lambda_2 \end{aligned} \tag{22.91}$$

The spectral density function $S_{u1}(\omega)$ for the response is then given by Eq. (22.36a) as

$$S_{u_1}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{u_1}(\omega)e^{-i\omega t}\,d\tau \tag{22.92}$$

The multiplication of Eq. (22.91) by

$$\frac{1}{2\pi}e^{-i\omega t}d\tau = \frac{1}{2\pi}e^{i\omega(\tau-\lambda_2+\lambda_1)}e^{i\omega(\lambda_2-\lambda_1)}\,d\tau$$

followed by integration yields

$$\begin{aligned} S_{u_1}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}R_2(\tau-\lambda_2+\lambda_1)e^{i\omega(\tau-\lambda_2+\lambda_1)}\big[&\phi_{11}^2\phi_{21}^2 h_1(\lambda_1)h_2(\lambda_1) \\ &+ \phi_{11}\phi_{12}\phi_{21}\phi_{22}h_1(\lambda_1)h_2(\lambda_2)\big] + \phi_{12}\phi_{11}\phi_{22}\phi_{21}h_1(\lambda_2)h_2(\lambda_1) \\ &+ \phi_{12}^2\phi_{22}^2 h_2(\lambda_1)h_2(\lambda_2)\big]e^{i\omega\left((\lambda_2-\lambda_1)\right.}\,d\tau d\lambda_1 d\lambda_2 \end{aligned}$$

After using Eq. (22.77), $S_{u1}(\omega)$ may be expressed as

$$S_{u_1}(\omega) = \sum_{j=1}^{2}\sum_{k=1}^{2}\phi_{1j}\phi_{1k}\phi_{2j}\phi_{2k}S_2(\omega)H_j(\omega)H_k^*(\omega) \tag{22.93}$$

(22.93)

in which $H_k^*(\omega)$ is the complex conjugate of $H_k(\omega)$ and

$$S_2(\omega) = \frac{1}{2\pi}\int R_2(\tau - \lambda_2 + \lambda_1)e^{i\omega(\tau-\lambda_2+\lambda_1)}\,d\tau$$

Equation (22.93) provides the spectral density function, $S_{u1}(\omega)$, for the response at coordinate 1 of a two-degree-of-freedom system excited by a random force at coordinate 2 expressed by the spectral density function $S_{u2}(\omega)$. Now we use Eq. (22.43) to obtain the discrete spectral function, $S_{u1}(\omega_n)$, evaluated at frequency ω_n as

$$S_{u_1}(\omega) = \sum_{j=1}^{2}\sum_{k=1}^{2}\phi_{1j}\phi_{1k}\phi_{2j}\phi_{2k}C_2(\omega_n)C_2^*(\omega_n)H_j(\omega_n)H_k^*(\omega_n) \tag{22.94}$$

where $C_2(\omega_n)$ is the discrete Fourier transform coefficient of the random force at coordinate 2 and $C_2{}^*(\omega_n)$ its complex conjugate.

If the single excitation is the random force $F_1(t)$ at coordinate 1 instead of $F_2(t)$ at coordinate 2, the expression for the spectral density function for the response at coordinate 1 will be

$$S_{u_1}(\omega_n) = \sum_{j=1}^{2}\sum_{k=1}^{2}\phi_{1j}^2\phi_{1k}^2C_1(\omega_n)C_1^*(\omega_n)H_j(\omega_n)H_k^*(\omega_n) \tag{22.95}$$

Then, the generalization to a system of N degrees of freedom subjected to multiple random excitations leads to the following equation:

$$S_{u_1}(\omega_n) = \sum_{j=1}^{2}\sum_{k=1}^{2}\phi_{ij}\phi_{ik}\sum_{L=1}^{2}\sum_{m=1}^{2}\phi_{Lj}\phi_{mk}C_L(\omega_n)C_m^*(\omega_m)H_j(\omega_n)H_k^*(\omega_n) \tag{22.96}$$

where

$S_{ui}\,(\omega_n)$ is the discrete spectral density function of the response at coordinate i
$C_L(\omega_n)$ is the Discrete Fourier Transform coefficient for the force at coordinate L
$C_m^*(\omega_n)$ is the complex conjugate of the coefficient $C_m\,(\omega_n)$

22.11.3 Response to Random Excitation: *N* Degree of Freedom System

A program written in MATLAB has been developed to evaluate using Eq. (22.96) the spectral density function of the response $S_{ui}\,(\omega)$ at coordinate i of a multidegree-of-freedom system subjected to multiple random stationary excitations.

Illustrative Example 22.8

Determine the response for a two-degree-of-freedom structure modeled by system shown in Fig. 22.26 for which the natural frequencies have bee determined as $\omega_1 = 10.10$ rad/sec and $\omega_2 = 12.32$ rad/sec with corresponding modal shapes $\{\Phi\} = \{0.707, 0.707\}_1$ and $\{\Phi\}_2 = \{0.707, -0.707\}_2$, respectively. Samples of the applied random forces, $F_1(t)$ and $F_2(t)$ are depicted in Fig. 22.27.

Use MATLAB to determine the spectral functions of the response at the two coordinates of this system.

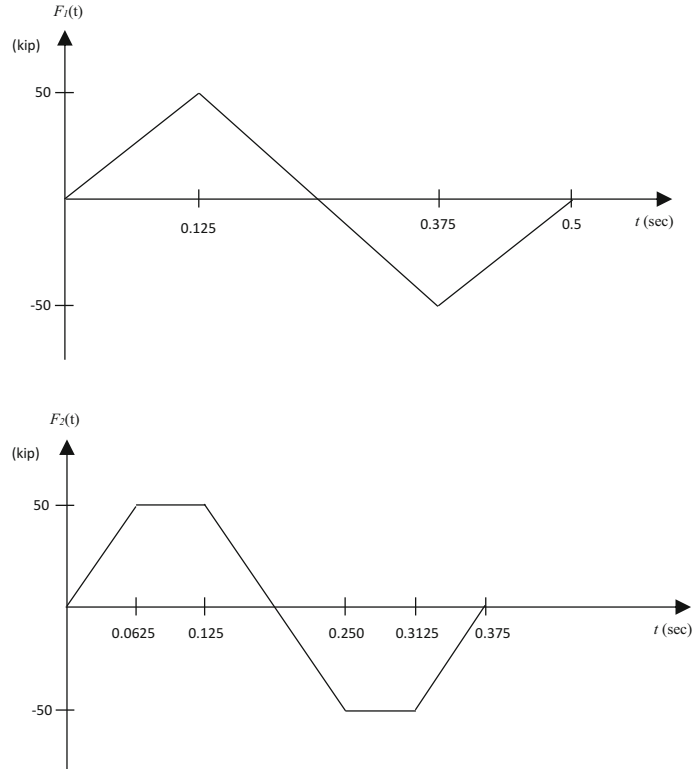

Fig. 22.27

Solution:

```
clear all
close all
clc

w1 = 10;                        %Natural frequency @ 1 mode
w2 = 12.32;                     %Natural frequency @ 2 mode
xi = 0;

omega = [w1 w2];
phi1 = [0.707; 0.707];          %Mode shape @ mode 1
phi2 = [0.707; -0.707];         %Mode shape @ mode 2

T = 1.0;                        %Time period, T(sec)
omega_bar = 2*pi/T;             %Excitation frequency (rad/sec)
M= 3;                           %Select M, M=3
N = 2^M;                        %The number of time increments N

Dt = T/N;                       %Deltat
t=0:Dt:T;                       %Time ranging from 0 to 0.64 sec with deltat

tt= length(t);                  %Total number of calculation


for i= 1:tt-1
%%%Define the function of force F1
if t(i)<=0.125
       F1(i) = 100000*t(i)/0.125;
elseif t(i) <=0.3125
       F1(i) =-800000*(t(i)-0.125)+100000;
else
       F1(i)=min(0, 800000*(t(i)-0.50)) ;
end
```

```
%%%Define the function of force F2
if t(i)<0.0625
        F2(i) = 100000*t(i)/0.125;
elseif t(i)<=0.125 && t(i)>= 0.0625
        F2(i) = 50000;
elseif t(i)<=0.250 && t(i) > 0.125
        F2(i) = -800000*(t(i)-0.250)-50000;
elseif t(i)>0.250 && t(i) <0.3125
        F2(i) = -50000;
elseif t(i)>=0.3125 && t(i) <0.375
        F2(i) = 800000*(t(i)-0.3125)-50000;
else
        F2(i) = 0;
end

%%%Define the discrete Fourier transform of the series
   Cn1 = fft(F1'/N)/1000;
   Cn2 = fft(F2'/N)/1000;

%%%Calculate frequency ratio, r_n
ifi<=N/2
omega_n(i) = (i-1)*omega_bar;       %Eq.22.58
else
omega_n(i) = -(N-(i-1))*omega_bar;  %Eq.22.58
end
for j=1:2
rn(i,j)=omega_n(i)/omega(j);        %Eq.22.59
      H(i,j)=1/(1-rn(i,j)^2);
end

     H1 = 1/omega(:,1)^2*H(:,1);            %Eqs.22.93 and 22.94 for H1
     H2 = 1/omega(:,2)^2*H(:,2);            %Eqs.22.93 and 22.94 for H2

    Su1 =
(Cn1.*conj(Cn1)).*(phi1(1,1)^2*phi1(1,1)^2*H1.*conj(H1)+phi2(1,1)^2*phi1(1,1)^2*H1.*conj(
H2)+...
         phi1(1,1)^2*phi2(1,1)^2*
H2.*conj(H1)+phi2(1,1)^2*phi2(1,1)^2*H2.*conj(H2))            %Eq.22.93

    Su2 =
(Cn2.*conj(Cn2)).*(phi1(2,1)^2*phi1(2,1)^2*H1.*conj(H1)+phi2(2,1)^2*phi1(2,1)^2*H1.*conj(
H2)+...
         phi1(2,1)^2*phi2(2,1)^2*
H2.*conj(H1)+phi2(2,1)^2*phi2(2,1)^2*H2.*conj(H2))            %Eq.22.94

end

%%%Discrete spectral density function of the response at coordinate i

Su1f = [omega_n',Su1];                      %Eq.22.96 (i=1)
Su2f = [omega_n',Su2];                      %Eq.22.96 (i=2)
```

22.12 Summary

The objective of this chapter was to introduce the fundamentals of the theory of random vibrations for application in structural dynamics. In structural dynamics the most common source of random vibration is due to explosions occurring in the vicinity of the structure. The response of a structure to earthquakes may also be predicted using random vibration theory.

A random process is described by a function of time whose value at any time is known only as a set of sample records known as an ensemble. Such a function can only be described in probabilistic terms

using the tools of statistics. The most important statistics of a random process $x(t)$ are its mean value $\bar{x}$, its mean square value $\overline{x^2}$, and its variance σ_x^2 given, respectively, by Eqs. (22.1), (22.2), and (22.3).

The most commonly used probability distribution for a random process is the normal distribution. However, when the random variable can only assume positive values (e.g., the absolute values of the peaks of vibration), the process tends to follow the Rayleigh distribution.

The autocorrelation $R_x(\tau)$ of a random variable $x(t)$ is defined by Eq. (22.19). The spectral density function $S_x(\omega)$ is defined as the Fourier transform of the autocorrelation function $R(\tau)$ [Eq. (22.36a)]. Although the spectrum of $x(t)$ can be obtained from $R_x(\tau)$, it is now more efficient to determine the spectrum of a random function from its discrete Fourier series [Eq. (22.35)] using the fast Fourier transform (FFT).

If the spectrum of the excitation function and the frequency response of a dynamic system are known, it is a simple matter to calculate the mean-square value of the response using Eq. (22.62). Knowing the mean-square value of the response and using standard probability functions (such as the Normal or the Rayleigh distributions), we can predict the response in probabilistic terms.

In this introductory chapter on random vibration, the presentation has been given in detail for single-degree-of-freedom systems and extended to multidegree-of-freedom systems using the modal superposition method. This method, as we have seen in previous chapters, transforms a system of differential equations into a set of independent or uncoupled differential equations. Each equation of this set is equivalent to the differential equation for a single degree-of-freedom system and consequently can be solved for random vibration excitation by the methods presented in this chapter.

22.13 Problems

Problems 22.1–22.5

Determine the mean and mean square values for the functions shown in Figs. P22.1, P22.2, P22.3, P22.4 and P22.5.

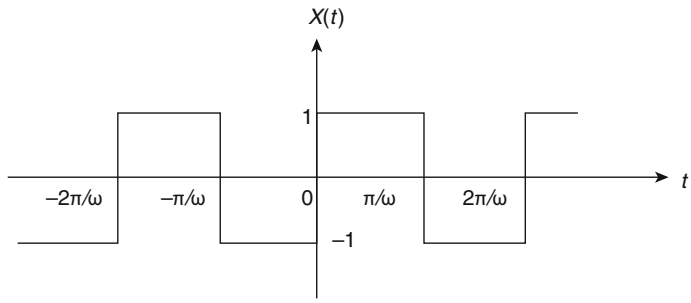

Fig. P22.1

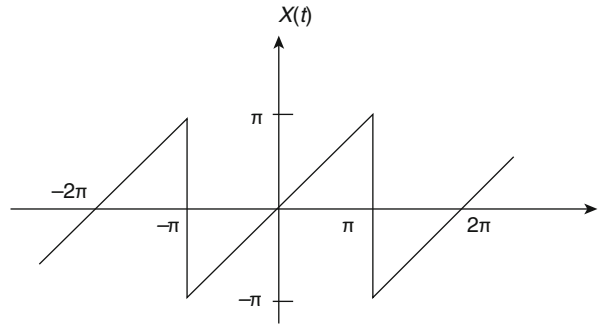

Fig. P22.2

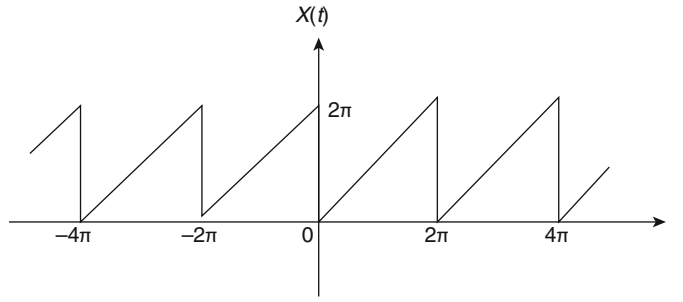

Fig. P22.3

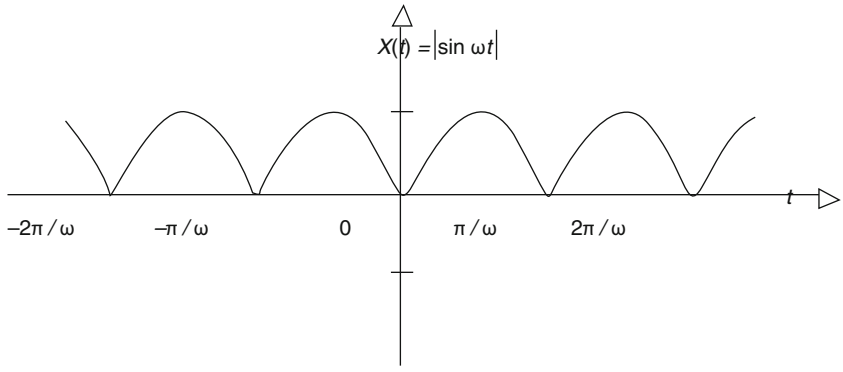

Fig. P22.4

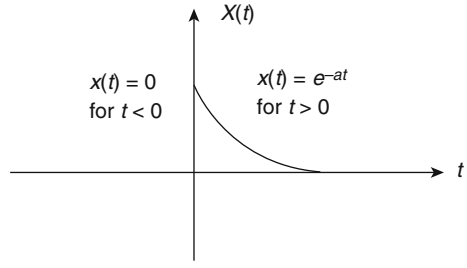

Fig. P22.5

Problems 22.6–22.10
Determine the Fourier series expansions for the periodic functions shown in Figs. P22.1, P22.2, P22.3, P22.4 and P22.5.

Problems 22.11–22.15
Determine and plot the spectral functions for the functions shown in Figs. P22.1, P22.2, P22.3, P22.4 and P22.5.

Problem 22.16
A sine wave with a steady-state component is given by

$$x(t) = A_0 + A_1 \sin \varpi t$$

Determine the mean value j and the mean-square value $\overline{x^2}$.

Problem 22.17
Determine the Fourier coefficients C_n and the spectral function for the periodic function shown in Fig. P22.17.

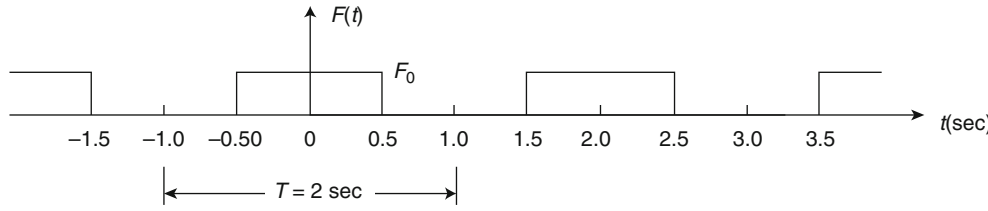

Fig. P22.17

Problem 22.18
A random force has a mean value $F = 2$ Kips and spectral density function shown in Fig. P22.18. Determine its standard deviation σ_F and its root mean square RMS_F.

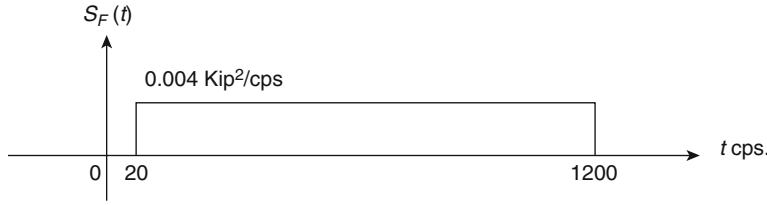

Fig. P22.18

Problem 22.19
Calculate the autocorrelation function for a stationary ergodic random process x (t), for which each sample function is a square wave of.

Each sample function is a square wave of amplitude a and period T.

Problem 22.20

Determine Fourier transform and Fourier integral representation for the function shown in Fig. P22.20.

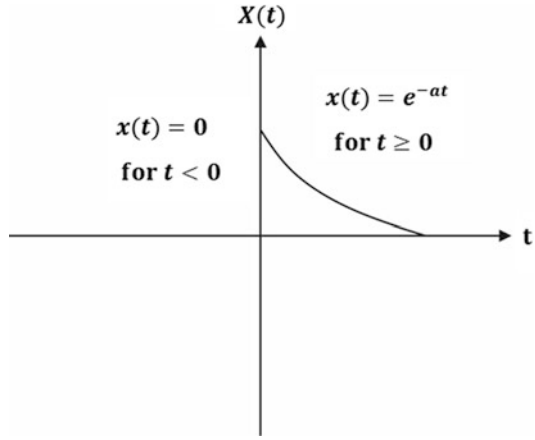

Fig. P22.20

Problem 22.21

A single degree-of freedom system with mass 1.0 lb sec^2/in, stiffness 100 lb/in, and damping $\xi = 0.20$ is excited by the force

$$F(t) = 1000\cos 5t + 1000\cos 10t + 1000\cos 15t\,(\text{lb})$$

1. Determine the spectral function and the mean square value of the response.
2. Also plot the spectra for the input force and the output response displacement.