9 Reduction of Dynamic Matrices

In the discretization process it is sometimes necessary to divide a structure into a large number of elements because of changes in geometry, loading, or material properties. When the elements are assembled for the entire structure, the number of unknown displacements, that is, the number of degrees of freedom, may be quite large. As a consequence, the stiffness, mass, and damping matrices will be of large dimensions. The solution of the corresponding eigenproblem to determine natural frequencies and modal shapes will be difficult and, in addition, expensive. In such cases it is desirable to reduce the size of these matrices in order to make the solution of the eigenproblem more manageable and economical. Such reduction is referred to as condensation.

A popular method of reduction is the Static Condensation Method This method, though simple to apply, is only approximate and may produce relatively large errors in the results when applied to dynamic problems. An improved method for condensation of dynamic problems, which gives virtually exact results, has recently been proposed. This method, called the Dynamic Condensation Method, will be presented in this chapter after the introduction of the Static Condensation Method.

9.1 Static Condensation

A practical method of accomplishing the reduction of the stiffness matrix is to identify those degrees of freedom to be condensed as dependent or secondary degrees of freedom, and express them in the term of the remaining independent or primary degrees of freedom. The relationship between the secondary or primary degrees of freedom is found by establishing the static relation between them, hence the name Static Condensation Method (Guyan 1965). This relationship provides the means to reduce the stiffness matrix. This method is also used in the static problems to eliminate unwanted degrees of freedom such as the internal degrees of freedom of an element used with the Finite Element Method. In order to describe the Static Condensation Method, let us assume that those (secondary) degrees of freedom to be reduced or condensed are arranged as the first s coordinates, and the remaining (primary) degrees of freedom are the last ρ coordinates. With this arrangement, the stiffness equation for the structure may be written using partition matrices as

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$$\begin{bmatrix} [K_{ss}] & \vdots & [K_{sp}] \\ \cdots & \vdots & \cdots \\ [K_{ps}] & \vdots & [K_{pp}] \end{bmatrix} \begin{Bmatrix} \{u_s\} \\ \cdots \\ u_p \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \cdots \\ \{F_p\} \end{Bmatrix} \tag{9.1}$$

where $\{u_s\}$ is the displacement vector corresponding to the s degrees of freedom to be reduced and $\{u_p\}$ is the vector corresponding to the reining p independent degrees of freedom. In Eq. (8.1), it is assumed that the external forces were zero at the dependent (i.e., secondary) degrees of freedom; this assumption is not mandatory (Gallagher 1975), but serves to simplify explanations without affecting the final results. A simple multiplication of the matrices on the left side of Eq. (9.1) expands this equation into two matrix equations, namely,

$$[K_{ss}]\{u_s\} + K_{sp}\{u_p\} = \{0\} \tag{9.2}$$

$$K_{ps}\{u_s\} + K_{pp}\{u_p\} = \{F_p\} \tag{9.3}$$

Equation (9.2) is equivalent to

$$\{u_s\} = [\bar{T}]\{u_p\} \tag{9.4}$$

where $[\bar{T}]$ is the transformation matrix given by

$$[\bar{T}] = -[K_{ss}]^{-1}[K_{sp}] \tag{9.5}$$

Substituting Eq. (9.4) and using Eq. (9.5) in Eq. (9.3) results in the reduced stiffness equation relating forces and displacements at the primary coordinates, that is,

$$[\bar{K}]\{u_p\} = \{F_p\} \tag{9.6}$$

where $[\bar{K}]$ is the reduced stiffness matrix given by

$$[\bar{K}] = [K_{pp}] - [K_{ps}][K_{ss}]^{-1}[K_{sp}] \tag{9.7}$$

Equation (9.4), which expresses that static relationship between the secondary coordinates $\{u_s\}$ and primary coordinates $\{u_p\}$, may also be written using the identity $\{u_p\} = [I]\{u_p\}$ as

$$\begin{Bmatrix} \{u_s\} \\ \cdots \\ \{u_p\} \end{Bmatrix} = \begin{bmatrix} [\bar{T}] \\ \cdots \\ [I] \end{bmatrix} \{u_p\}$$

or

$$\{u\} = [T]\{u_p\} \tag{9.8}$$

where

$$\{u\} = \begin{Bmatrix} \{u_s\} \\ \{u_p\} \end{Bmatrix} \quad \text{and} \quad [T] = \begin{bmatrix} [T] \\ [I] \end{bmatrix} \tag{9.9}$$

Substituting Eqs. (9.8) and (9.9) into Eq. (9.1) and pre-multiplying by the transpose of $[T]$ results in

$$[T]^T[K][T]\{u_p\} = \left[[T]^T[I]\right]\left\{\begin{array}{c}\{0\}\\ \{F_p\}\end{array}\right\}$$

or

$$[T]^T[K][T]\{u_p\} = \{F_p\}$$

and using Eq. (9.6)

$$\left[\bar{K}\right] = [T]^T[K][T] \tag{9.10}$$

thus showing that the reduced stiffness matrix $\left[\bar{K}\right]$ can be expressed as a transformation of the system stiffness matrix $[K]$.

It may appear that the calculation of the reduced stiffness matrix $\left[\bar{K}\right]$ given by Eq. (9.7) requires the inconvenient calculation of the inverse matrix $[K_{ss}]^{-1}$. However, the practical application of the Static Condensation Method does not require a matrix inversion. Instead, the standard Gauss-Jordan elimination process, as it used in the solution of a system of linear equations, is applied systematically on the system's stiffness matrix $[K]$ up to the elimination of the secondary coordinates $\{u_s\}$. At this stage of the elimination process the stillness Eq. (9.1) has been reduced to

$$\begin{bmatrix} [I] & -\left[\bar{T}\right] \\ [0] & \left[\bar{K}\right] \end{bmatrix}\left\{\begin{array}{c}\{u_s\}\\ \{u_p\}\end{array}\right\} = \left\{\begin{array}{c}\{0\}\\ \{F_p\}\end{array}\right\} \tag{9.11}$$

It may be seen by expanding Eq. (9.11) that the partition matrices $\left[\bar{T}\right]$ and $\left[\bar{K}\right]$ are precisely the transformation matrix and the reduced stiffness matrix defined by Eqs. (9.4) and (9.6), respectively. In this way, the Gauss-Jordan elimination process yields both the transformation matrix $\left[\bar{T}\right]$ and the reduced stiffness matrix $\left[\bar{K}\right]$. There is thus no need to calculate $[K_{ss}]^{-1}$ in order to reduce the secondary coordinates of the system.

Illustrative Example 9.1

Consider the two-degree-of-freedom system represented by the model shown in Fig. 9.1 and use static condensation to reduce the first coordinate.

Solution:

For this system the equations of equilibrium are readily obtained as

$$\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\left\{\begin{array}{c}u_1\\ u_2\end{array}\right\} = \left\{\begin{array}{c}0\\ F_2\end{array}\right\} \tag{9.12}$$

The reduction of u_1 using Gauss elimination leads to

$$\begin{bmatrix} 1 & -1/2 \\ 0 & k/2 \end{bmatrix}\left\{\begin{array}{c}u_1\\ u_2\end{array}\right\} = \left\{\begin{array}{c}0\\ F_2\end{array}\right\} \tag{9.13}$$

Comparing Eq. (9.13) with Eq. (9.11), we identify in this example

$$\begin{aligned}\left[\bar{T}\right] &= \frac{1}{2}\\ \left[\bar{K}\right] &= k/2\end{aligned} \tag{9.14}$$

Consequently, from Eq. (9.9) the transformation matrix is

$$[T] = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \tag{9.15}$$

We can now check Eq. (9.10) by simple performing the indicated multiplications, namely

$$[\bar{K}] = [1/2 \;\; 1]\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \frac{k}{2} \tag{9.16}$$

which agrees with the result given in Eq. (9.14).

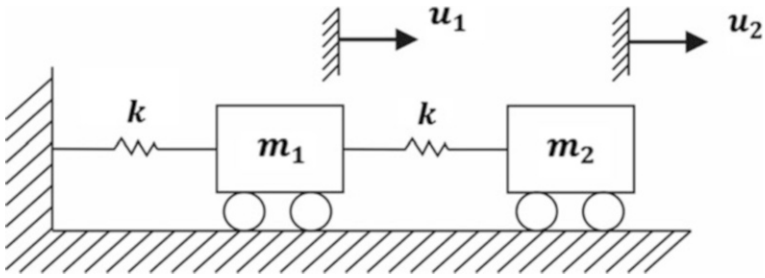

Fig. 9.1 Mathematical model for a two-degree-of-freedom system

9.2 Static Condensation Applied to Dynamic Problems

In order to reduce the mass and the damping matrices, it is assumed that the same static relationship between the secondary and primary degrees of freedom remains valid in the dynamic problem. Hence the same transformation based on static condensation for the reduction of the stiffness matrix is also used in reducing the mass and damping matrices. In general this method of reducing the dynamic problem is not exact and introduces errors in the results. The magnitude of these errors depends on the relative number of degrees of freedom reduced as well as on the specific selection of these degrees of freedom for a given structure.

We consider first the case in which the discretization of the mass results in a number of massless degrees of freedom selected to be condensed. In this case it is only necessary to carry out the static condensation of the stiffness matrix and to delete from the mass matrix the rows and columns corresponding to the massless degrees of freedom. The Static Condensation Method in this case does not alter the original problem, and thus results in an equivalent eigenproblem without introducing any error.

In the general case, that is, the case involving the condensation of degrees of freedom to which the discretization process has allocated mass, the reduced mass and damping matrices are obtained using transformations analogous to Eq. (9.10). Specifically, if $[M]$ is the mass matrix of the system, then the reduced mass matrix is given by

$$[\bar{M}] = [T]^T[M][T] \tag{9.17}$$

where $[T]$ is the transformation matrix defined by Eq. (9.9).

Analogously, for a damped system, the reduced damping matrix is given by

$$[\bar{C}] = [T]^T[C][T] \tag{9.18}$$

where $[C]$ is the damping matrix of the system.

This method of reducing the mass and damping matrices may be justified as follows: The potential elastic energy V and the kinetic energy KE of the structure may be written, respectively, as

$$V = \frac{1}{2}\{u\}^T[K]\{u\} \tag{9.19}$$

$$KE = \frac{1}{2}\{\dot{u}\}^T[M]\{\dot{u}\} \tag{9.20}$$

Analogously, the virtual work of δW_d done by the damping forces $F_d = [C]\{\dot{u}\}$ corresponding to the virtual displacement $\{\delta u\}$ may be expressed as

$$\delta W_d = \{du\}^T[C]\{\dot{u}\} \tag{9.21}$$

Introduction of the transformation Eq. (9.8) in the above equation results in

$$V = \frac{1}{2}\{u_p\}^T[T]^T[K][T]\{u_p\} \tag{9.22}$$

$$KE = \frac{1}{2}\{\dot{u}_p\}^T[T]^T[M][T]\{\dot{u}_p\} \tag{9.23}$$

$$\delta W_d = \{\delta u_p\}^T[T]^T[C][T]\{\dot{u}_p\} \tag{9.24}$$

The respective substitution of $[\bar{K}]$ $[\bar{M}]$ and $[\bar{C}]$ from Eqs. (9.10), (9.17), and (9.18) for the product of the three central matrices in the Eqs. (9.22), (9.23), and (9.24) yields

$$V = \frac{1}{2}\{u_p\}^T[\bar{K}][T]\{u_p\} \tag{9.25}$$

$$KE = \frac{1}{2}\{\dot{u}_p\}^T[\bar{M}]\{\dot{u}_p\} \tag{9.26}$$

$$\delta W_d = \{\delta u_p\}^T[\bar{C}]\{\dot{u}_p\} \tag{9.27}$$

These last three equations express the potential energy, the kinetic energy, and the virtual work of the damping forces in terms of the primary coordinates $\{u_p\}$. Hence the matrices $[\bar{K}]$, $[\bar{M}]$ and $[\bar{C}]$ may be interpreted, respectively, as the stiffness, mass, and damping matrices of the structure corresponding to the primary degrees of freedom $\{u_p\}$ resulting in the same potential energy, kinetic energy and virtual work calculated with all the original nodal coordinate.

Illustrative Example 9.2

Determine the natural frequencies and modal shapes for the three-degree-of-freedom shear building shown in Fig. 9.2; then condense the first degree of freedom and compare the resulting values obtained for natural frequencies and modal shapes. The stiffness of each story and the mass at each floor level are indicated in the figure. MATALB program is used to demonstrate the natural frequencies using static condensation method.

Solution:

1. Calculation of Natural Frequencies and Modal Shapes:

The equation of motion in free vibration for this structure is given by Eq. (8.3) with the force vector $\{F\} = \{0\}$, namely,

$$[M]\{\ddot{u}\} + [K]\{u\} = 0 \tag{a}$$

where the matrices $[M]$ and $[K]$ are given, respectively, by Eqs. (7.4) and (7.5). Substituting the corresponding numerical value in Eq. (a) yields

$$\begin{bmatrix} 25 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} 40,000 & -10,000 & 0 \\ -10,000 & 20,000 & -10,000 \\ 0 & -10,000 & 10,000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Upon substitution $u_i = U_i \sin \omega t$ and cancellation of the factor $\sin \omega t$, we obtain the following system of equations:

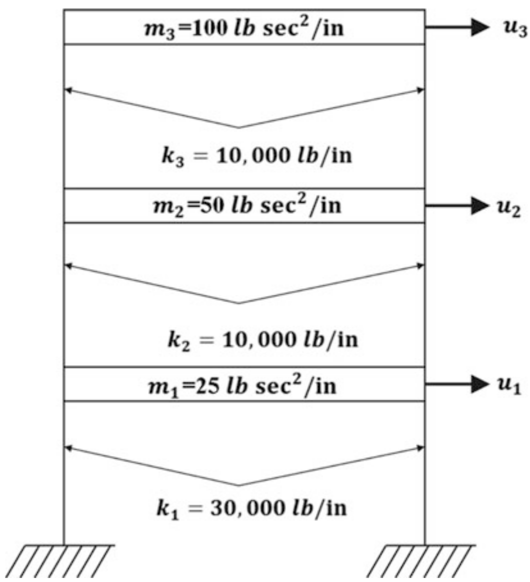

Fig. 9.2 Shear building for Illustrative Example 9.2

$$\begin{bmatrix} 40,000 - 25\omega^2 & -10,000 & 0 \\ -10,000 & 20,000 - 50\omega^2 & -10,000 \\ 0 & -10,000 & 10,000 - 100\omega^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{b}$$

which for a nontrivial solution requires that the determinant of the coefficients of the unknowns be equal to zero, that is,

$$\begin{vmatrix} 40,000 - 25\omega^2 & -10,000 & 0 \\ -10,000 & 20,000 - 50\omega^2 & -10,000 \\ 0 & -10,000 & 10,000 - 100\omega^2 \end{vmatrix} = 0$$

The expansion of this determinant results in a third degree equation in terms of ω^2 having the following roots:

$$\omega_1^2 = 36.1 \qquad \omega_2^2 = 400.0 \qquad \omega_3^2 = 1664.0 \tag{c}$$

The natural frequencies are calculated as $f = \omega/2\pi$, so that

$$f_1 = 0.96\,\text{cps} \quad f_2 = 3.18\,\text{cps} \quad f_3 = 264.8\,\text{cps}$$

The modal shapes are then determined by substituting in turn each of the values for the natural frequencies into Eq. (b), deleting a redundant equation, and solving the remaining two equations for two of the unknowns in terms of the third. As we mentioned previously, in solving for these unknowns it is expedient to set the first nonzero unknown equal to one. Performing these operations, we obtain from Eqs. (b) and (c) the following values for the modal shapes:

$$\begin{array}{lll} U_{11} = 1.00, & U_{12} = 1.00, & U_{13} = 1.00 \\ U_{21} = 3.91, & U_{22} = 3.00, & U_{23} = 3.338 \\ U_{31} = 6.11, & U_{32} = -1.00, & U_{33} = -2.025 \end{array}$$

in which the second sub-index in U refers to the modal order.

2. Condensation of Coordinate u_1:

The stiffness matrix for this structure is

$$\begin{bmatrix} 40,000 & -10,000 & 0 \\ -10,000 & 20,000 & -10,000 \\ 0 & -10,000 & 10,000 \end{bmatrix}$$

Gauss elimination applied to the first row gives

$$\left[\begin{array}{c:cc} 1 & -0.25 & 0 \\ \hdashline 0 & 17,500 & -10,000 \\ 0 & -10,000 & 10,000 \end{array}\right] \tag{d}$$

A comparison of Eq. (d) with Eq. (9.11) indicates that

$$\begin{aligned} [\bar{T}] &= [0.25 \quad 0] \\ [\bar{K}] &= \begin{bmatrix} 17,500 & -10,000 \\ -10,000 & 10,000 \end{bmatrix} \end{aligned} \tag{e}$$

Then from Eq. (9.9)

$$[T] = \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{f}$$

To check results, we use Eq. (9.10) to compute $[\bar{K}]$. Hence

$$[\bar{K}] = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 40,000 & -10,000 & 0 \\ -10,000 & 20,000 & -10,000 \\ 0 & -10,000 & 10,000 \end{bmatrix} \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\bar{K}] = \begin{bmatrix} 17,500 & -10,000 \\ -10,000 & 10,000 \end{bmatrix}$$

which checks with Eqs. (e).

The reduced mass matrix is calculated by substituting matrix $[T]$ and its transpose from Eq. (f) into Eq. (9.17), so that

$$[\bar{M}] = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 25 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which results in

$$[\bar{M}] = \begin{bmatrix} 51.6 & 0 \\ 0 & 100 \end{bmatrix}$$

The condensed dynamic problem is then

$$\begin{bmatrix} 51.6 & 0 \\ 0 & 100 \end{bmatrix} \begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} 17,500 & -10,000 \\ -10,000 & 10,000 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The natural frequencies and modal shapes are determined from the solution of the following reduced eigenproblem:

$$\begin{bmatrix} 17,500 - 51.6\omega^2 & -10,000 \\ -10,000 & 10,000 - 100^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{g}$$

Equating to zero the determinant of the coefficient matrix in Eq. (g) and solving the resulting quadratic equation in ω^2 gives

$$\omega_1^2 = 36.1 \qquad \omega_2^2 = 403.3 \tag{h}$$

from which

$$f_1 = \sqrt{36.1/2\pi} = 0.95\,\text{cps}$$
$$f_2 = \sqrt{403.3/2\pi} = 3.20\,\text{cps}$$

Corresponding modal shapes are obtained from Eq. (g) after substituting in turn the numerical values for ω_1^2 or ω_2^2 and solving the first equation for U_3 with $U_2 = 1$,. We then obtain

$$U_{21} = 1.00, \quad U_{22} = 1.00$$
$$U_{31} = 1.56, \quad U_{32} = -0.33$$

in which the second subindexes in U serve to indicate the mode 1 or 2.

Application of Eq. (9.8) for the first mode gives

$$\begin{Bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{Bmatrix} = \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1.00 \\ 1.56 \end{Bmatrix} = \begin{Bmatrix} 0.25 \\ 1.00 \\ 1.56 \end{Bmatrix}$$

or, after normalizing so that the first component is 1,

$$U_{11} = 1.00, \quad U_{21} = 4.00, \quad U_{31} = 6.24$$

and analogously for the second mode

$$U_{12} = 1.00, \quad U_{22} = 4.00, \quad U_{32} = -1.32$$

For this system of only three degrees of freedom, the reduction of one coordinate gives natural frequencies that compare rather well for the first two modes [Eqs. (h) and (c)]. However, experience shows that static condensation may produce large errors in the calculation of eigenvalues and eigenvectors obtained from the reduced system. A general recommendation given by users of this method is to assume that the static condensation process results in an eigenproblem that provides acceptable approximations of only about a third of the calculated eigenvalues (natural frequencies) and eigenvectors (modal shapes).

The MATLAB program is used to obtain the natural frequencies in Eq. (h).

```
clc
clear all
close all

%___________________________________________________________________________
% Inputs:
%   M, K
%   m =  Number of row to apply the elimination process
%____________________________________________________________________________


%%%%-GIVEN VALUES-%%%%

%%%Mass Matrix
M = [25 0 0; 0 50 0; 0 0 100]

%%%Stiffness Matrix
K = [40000, -10000,0; -10000, 20000, -10000; 0, -10000, 10000]

%%%Elimination of the first row: m=1
m = 1;

N= length(K);    %Total number of row

K_full = K;      %[K] before the elimination

%%%Partition Matrix for Static Condensation (Eq.9.1)
K_pp_full = K_full(m+1:N,m+1:N);
K_ps_full = K_full(m+1:N,1:m);
K_ss_full = K_full(1:m,1:m);
K_sp_full = K_full(1:m,m+1:N);

K_bar_full = K_pp_full-K_ps_full*inv(K_ss_full)*K_sp_full;  %Eq.9.7

T_bar_full = -inv(K_ss_full)*K_sp_full;                     %Eq.9.5
T = [T_bar_full; eye(N-m,N-m)];

K_bar = T'*K_full*T                                         %Eq.9.10
M_bar_full = T'*M*T                                         %Eq.9.17

%______________________________________________________________________
% Solve the eignevalue problem and normalized eigenvectors
%______________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(K_bar, M_bar_full)
```

Illustrative Example 9.3

Figure 9.3 shows a uniform four-story shear building. For this structure, determine the following: (a) the natural frequencies and corresponding modal shapes as a four-degree-of-freedom system, (b) the natural frequencies and modal shapes after static condensation of coordinates u_1 and u_3.

Solution:

(a) Natural Frequencies and Modal Shapes as a Four-Degree-of Freedom System:

The stiffness and the mass matrices for this structure are respectively

$$[K] = 327.35 \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{(a)}$$

and

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(b)}$$

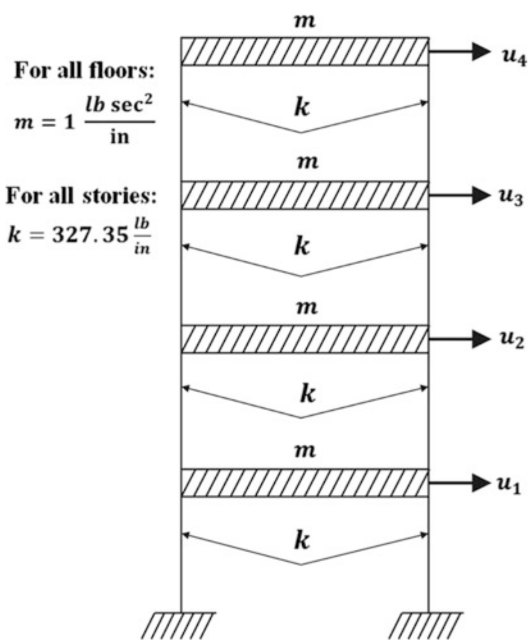

Fig. 9.3 Uniform four-story shear building for Illustrative Example 9.3

Substituting Eqs. (a) and (b) into Eq. (9.3) and solving the corresponding eigenvalue problem (using Program 8) yields

$$\omega_1^2 = 39.48, \quad \omega_2^2 = 327.35, \quad \omega_3^2 = 768.3, \quad \text{and} \quad \omega_4^2 = 1156.00$$

corresponding to the natural frequencies

$$\begin{aligned} f_1 &= \frac{\omega_1}{2\pi} = 1.00\,\text{cps} \quad f_2 = \frac{\omega_2}{2\pi} = 2.88\,\text{cps} \\ f_3 &= \frac{\omega_3}{2\pi} = 4.41\,\text{cps} \quad f_4 = \frac{\omega_4}{2\pi} = 5.41\,\text{cps} \end{aligned} \tag{c}$$

and the normalized modal matrix (see Sect. 7.2)

$$[\Phi] = \begin{bmatrix} 0.2280 & 0.5774 & -0.6565 & 0.4285 \\ 0.4285 & 0.5774 & 0.2280 & -0.6565 \\ 0.5774 & 0 & 0.5774 & 0.5774 \\ 0.6565 & -0.5774 & 0.4285 & -0.2280 \end{bmatrix} \tag{d}$$

(b) Natural Frequencies and Modal Shapes after Reduction to Two-Degree-of Freedom System:

To reduce coordinates u_1 and u_3, for convenience, we first rearrange the stiffness matrix in Eq. (a) to have the coordinates in order u_1, u_3, u_2, u_4

$$[K] = 327.35 \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{e}$$

Applying Gauss-Jordan elimination to the first two rows of the matrix in Eq. (e) result in

$$[D] = \left[\begin{array}{cc:cc} 1 & 0 & -0.5 & 0 \\ 0 & 1 & -0.5 & -0.5 \\ \hdashline 0 & 0 & 327.35 & -163.70 \\ 0 & 0 & -163.70 & 163.70 \end{array}\right] \tag{f}$$

A comparison of Eq. (f) with Eq. (9.11) reveals that

$$[\bar{T}] = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} \tag{g}$$

and

$$[\bar{K}] = \begin{bmatrix} 327.35 & -163.70 \\ -163.70 & 163.70 \end{bmatrix} \tag{h}$$

Use of Eq. (9.9) gives

$$[T] = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The reduced mass matrix can now be calculated by Eq. (9.17) as

$$[\bar{M}] = [T]^T [M][T] = \begin{bmatrix} 1.5 & 0.25 \\ 0.25 & 1.25 \end{bmatrix} \tag{i}$$

The condensed eigenproblem is then

$$\begin{bmatrix} 327.35 - 1.5\omega^2 & -163.70 - 0.25\omega^2 \\ -163.70 - 0.25\omega^2 & 163.70 - 1.25\omega^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{j}$$

and its solution is

$$\omega_1^2 = 40.39, \quad \omega_2^2 = 365.98 \tag{k}$$

$$[U]_p = \begin{bmatrix} 0.4380 & 0.7056 \\ 0.6723 & -0.6128 \end{bmatrix} \tag{l}$$

where $[U]_p$ is the modal matrix corresponding to the primary degrees of freedom.

The eigenvectors for the four-degree-of-freedom system are calculated for the first mode using Eq. (9.8) as

$$\begin{Bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{Bmatrix}_1 = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.4380 \\ 0.6723 \end{Bmatrix} = \begin{Bmatrix} 0.2190 \\ 0.5552 \\ 0.4380 \\ 0.6723 \end{Bmatrix}_1$$

or rearranging the component of the first modal shape:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.2190 \\ 0.4380 \\ 0.5552 \\ 0.6723 \end{Bmatrix}_1 \tag{m}$$

and for the second mode

$$\begin{Bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{Bmatrix}_2 = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.7056 \\ -0.6128 \end{Bmatrix} = \begin{Bmatrix} 0.3528 \\ 0.0464 \\ 0.7056 \\ -0.6128 \end{Bmatrix}_2$$

or

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.3528 \\ 0.7056 \\ 0.0464 \\ -0.6128 \end{Bmatrix}_2$$

The MATLAB program is used to obtain the natural frequencies in Eq. (k).

```
clc
clear all
close all

%___________________________________________________________________________
% Inputs:
%   M, K
%   m =  Number of row to apply the static elimination process
%____________________________________________________________________________


%%%%-GIVEN VALUES-%%%%

%%%Mass Matrix
M = [1, 0 0 0; 0, 1, 0, 0; 0, 0, 1,0; 0,0,0, 1];

%%%Stiffness Matrix
K = 327.35*[2,0,-1,0; 0,2,-1,-1; -1,-1,2,0; 0,-1,0,1];

%%%Elimination of the first and second rows: m=2
m = 2;


N= length(K);       %Total number of row


K_full = K;         %[K] before the elimination

%%%Partition Matrix for Static Condensation (Eq.9.1)
K_pp_full = K_full(m+1:N,m+1:N);
K_ps_full = K_full(m+1:N,1:m);
K_ss_full = K_full(1:m,1:m);
K_sp_full = K_full(1:m,m+1:N);

K_bar_full = K_pp_full-K_ps_full*inv(K_ss_full)*K_sp_full;      %Eq.9.7

T_bar_full = -inv(K_ss_full)*K_sp_full;                         %Eq.9.5
T = [T_bar_full; eye(N-m,N-m)];

K_bar = T'*K_full*T;                                            %Eq.9.10
M_bar_full = T'*M*T;             %Eq.9.17

%_____________________________________________________________________
% Solve the eignevalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(K_bar, M_bar_full)
```

Illustrative Example 9.4

The shear building of Illustrative Example 9.3 is subjected to an earthquake motion at its foundation. For design purposes, use the response spectrum of Fig. 5.10 Sect. (5.4) and determine the maximum horizontal displacements of the structure at the level of the floors.

Solution:
The participation factor for the ith mode of a shear building with N stories is given by Eq. (8.40) as

$$\Gamma_i = \text{-}\sum_{j=1}^{N}\left(m_j\phi_{ji}\right) \tag{a}$$

where m_j is the mass at the jth floor and the ϕ_{ji} the jth element of the normalized ith eigenvector of the mode.

(a) Response Considering Four Degrees of Freedom:

The substitution into Eq. (a) of the corresponding numerical results from Illustrative Example 9.3 gives

$$\Gamma_1 = -1.890, \quad \Gamma_2 = -0.5775, \quad \Gamma_3 = -0.2797, \quad \text{and} \quad \Gamma_4 = -0.1213 \tag{b}$$

The spectral displacements corresponding to the values of the natural frequencies of this building [Eq. (c) of Illustrative Example 9.3] are obtained from the response spectrum, Fig. 5.10, as

$$S_{D1} = 14.32, \quad S_{D2} = 3.240, \quad S_{D3} = 1.433, \quad \text{and} \quad S_{D4} = 0.969 \tag{c}$$

The maximum displacements at the floor levels relative to the displacement at the base of the building are calculated using Eq. (8.41), namely,

$$u_{i\max} = \sqrt{\sum_{j=1}^{n}\left(T_j S_{Dj}\phi_{ij}\right)^2} \tag{d}$$

to obtain

$$u_{1\max} = 6.274\,\text{in}, \quad u_{2\max} = 11.65\,\text{in}, \quad u_{3\max} = 15.64\,\text{in}, \quad \text{and} \quad u_{4\max} = 17.81\,\text{in}$$

(b) Response Considering the System Reduce to Two Degrees of Freedom:

The natural frequencies, calculated from Eq. (k) in Illustrative Example 9.3, are

$$\begin{aligned} f_1 &= \sqrt{40.39}/2\pi = 1.011\,\text{cps} \\ f_2 &= \sqrt{365.98}/2\pi = 3.044\,\text{cps} \end{aligned} \tag{e}$$

Upon introducing, into Eq. (a), the corresponding eigenvectors give in Eqs. (m) and (n) of Illustrative Example 9.3, we obtain the participation factors

$$\Gamma_1 = -1.884 \qquad \Gamma_2 = -0.492$$

The values of spectral displacements corresponding to the frequencies in Eq. (e) can be read from Fig. 5.10:

$$S_{D1} = 14.16 \qquad S_{D4} = 2.913$$

Use of Eq. (d) gives the relative maximum displacements at the level of the floors as

$$u_{1\max} = \sqrt{(1.884 \times 14.16 \times 0.2190)^2 + (0.4920 \times 2.913 \times 0.3528)^2} = 5.864\,\text{in}$$

$$u_{2\max} = \sqrt{(1.884 \times 14.16 \times 0.4380)^2 + (0.4920 \times 2.913 \times 0.7056)^2} = 11.73\,\text{in}$$

$$u_{3\max} = \sqrt{(1.884 \times 14.16 \times 0.5552)^2 + (0.4920 \times 2.913 \times 0.0464)^2} = 14.81\,\text{in}$$

$$u_{4\max} = \sqrt{(1.884 \times 14.16 \times 0.6723)^2 + (0.4920 \times 2.913 \times 0.6128)^2} = 17.97\,\text{in}$$

9.3 Dynamic Condensation

A method of reduction that may be considered an extension of the Static Condensation Method has been proposed (Paz 1984). The algorithm for this method starts by assigning an approximate value (e.g., zero) to the first eigenvalue ω_1^2, applying dynamic condensation to the dynamic matrix of the system $[D_1] = [K] - \omega_1^2[M]$ and then solving the reduced eigenproblem to determine the first and second eigenvalues, ω_1^2 and ω_2^2. The process continues in this manner, with one virtually exact eigenvalue and an approximate value for the next order eigenvalue calculated at each step.

The Dynamic Condensation Method requires neither matrix inversion nor series expansion. To demonstrate this fact, consider the eigenvalues problem of a discrete structural system for which it is desired to reduce the secondary degrees of freedom $\{u_o\}$ and retain the primary degrees of freedom $\{u_p\}$. In this case, the equations of free motion may be written in partitioned matrix form as

$$\begin{bmatrix} [M_{ss}] & [M_{sp}] \\ [M_{ps}] & [M_{pp}] \end{bmatrix} \begin{Bmatrix} \{\ddot{u}_s\} \\ \{\ddot{u}_p\} \end{Bmatrix} + \begin{bmatrix} [K_{ss}] & [K_{sp}] \\ [K_{ps}] & [K_{pp}] \end{bmatrix} \begin{Bmatrix} \{u_s\} \\ u_p \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{9.28}$$

The substitution of $\{u\} = \{U\}\sin\omega_i t$ in Eq. (9.28) results in the generalized eigenproblem

$$\begin{bmatrix} [K_{ss}] - \omega_i^2[M_{ss}] & [K_{sp}] - \omega_i^2[M_{sp}] \\ [K_{ps}] - \omega_i^2[M_{ps}] & [K_{pp}] - \omega_i^2[M_{pp}] \end{bmatrix} \begin{Bmatrix} \{u_s\} \\ \{u_p\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{9.29}$$

where ω_i^2 is the approximation of the ith eigenvalues which was calculated in the preceding step of the process. To start the process one takes an approximate or zero value for the first eigenvalue ω_1^2.

The following three steps are executed to calculate the ith eigenvalue ω_i^2 and the corresponding eigenvector $\{U\}_i$ as well as an approximation of the eigenvalue of the next order $\omega_{i=1}^2$:Step 1: The approximation of ω_i^2 is introduced in Eq. (9.29). Gauss-Jordan elimination of the secondary coordinates $\{U_s\}$ is then used to reduce Eq. (9.29) to

$$\begin{bmatrix} [I] & -[\bar{T}_i] \\ [0] & [D_i] \end{bmatrix} \begin{Bmatrix} \{U_s\} \\ \{U_p\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \end{Bmatrix} \tag{9.30}$$

The first equation in Eq. (9.30) can be written as

$$\{U_s\} = [\bar{T}_i]\{U_p\} \tag{9.31}$$

Consequently, the ith modal shape $\{U\}_i$ can be expressed as

$$\{U\}_\mathrm{i} = [T]\{U_p\} \tag{9.32}$$

where

$$[T_i] = \begin{bmatrix} [\bar{T}] \\ [\bar{I}] \end{bmatrix} \quad \text{and} \quad \{U_i\} = \begin{Bmatrix} \{U_s\} \\ \{U_p\} \end{Bmatrix}_i \tag{9.33}$$

Step 2: The reduces mass matrix $[\bar{M}_i]$ and the reduced stiffness matrix $[\bar{K}_i]$ are calculated as

$$[\bar{M}_i] = [T_i]^T[M][T_i] \tag{9.34}$$

and

$$[\bar{K}_i] = [\bar{D}_i] + \omega_i^2[\bar{M}_i] \tag{9.35}$$

where the transformation matrix $[T_i]$ is given by Eq. (9.33) and the reduced dynamic matrix $[D_i]$ is defined Eq. (9.30).Step 3: The reduced eigenproblem

$$[[\bar{K}_i] - \omega^2[\bar{M}_i]]\{U_p\} = \{0\} \tag{9.36}$$

is then solved to obtain improved eigenvalues ω_i^2, its corresponding eigenvector $\{Up\}_i$, and also an approximation for the next order eigenvalue ω_{i+1}^2.

These three-step process may be applied iteratively; that is, the value of ω_i^2 obtained in step 3 may be used as an improved approximate value in step 1 to obtain a further improved value of ω_i^2, in step 3. Experience has shown that one or two such iterations will produce virtually exact eigensolutions.

Illustrative Example 9.5

Repeat Example 9.3 of Sect. 9.2 using the Dynamic Condensation Method.

Solution:

The stiffness matrix and the mass matrix with the coordinates in the order u_1 u_3, u_2, u_4 are given, respectively, by Eqs. (e) and (b) of Illustrative Example 9.3. Substitution of these matrices into Eq. (9.29) results in the dynamic matrix for the system:

$$[D]=\left[\begin{array}{cc:cc} 654.70-\omega_i^2 & 0 & -327.35 & 0 \\ 0 & 654.70-\omega_i^2 & -327.35 & -327.35 \\ \hdashline -327.35 & -327.35 & 654.70-\omega_i^2 & 0 \\ 0 & -327.35 & 0 & 327.35-\omega_i^2 \end{array}\right] \tag{a}$$

Step 1: Assuming we have no initial approximation of ω_1^2, we start step 1 by setting $\omega_1^2 = 0$ and substituting this value into Eq. (a):

$$[D]=\left[\begin{array}{cc:cc} 654.70 & 0 & -327.35 & 0 \\ 0 & 654.70 & -327.35 & -327.35 \\ \hdashline -327.35 & -327.35 & 654.70 & 0 \\ 0 & -327.35 & 0 & 327.35 \end{array}\right] \tag{b}$$

Application of the Gauss-Jordan elimination process to the first two rows gives

$$\left[\begin{array}{cc:cc} 1 & 0 & -0.5 & 0.0 \\ 0 & 1 & -0.5 & -0.5 \\ \hdashline 0 & 0 & 327.35 & -163.67 \\ 0 & 0 & -163.67 & 163.67 \end{array}\right]$$

from which, by Eqs. (9.30) and (9.33)

$$[T_1] = \begin{bmatrix} 0.5 & 0.0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[\bar{D}_1] = \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix}$$

Step 2: The reduced mass and stiffness matrices by Eqs. (9.34) and (9.35), are

$$[\bar{M}_1] = [T_1]^T[M][T_1] = \begin{bmatrix} 1.5 & 0.25 \\ 0.25 & 1.25 \end{bmatrix}$$

and

$$[\bar{K}_1] = [\bar{D}_1] + \omega_1^2[\bar{M}_1] = \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix}$$

Step 3: The solution of the reduced eigenproblem $\left[[\bar{K}_1] - \omega^2[\bar{M}_1]\right]\{U_p\}_1 = \{0\}$ yields

$$\omega_1^2 = 40.39 \quad \text{and} \quad \omega_2^2 = 365.98$$

These values for ω_1^2 and ω_2^2 may be improved by iterating the calculations, that is, by introducing $\omega_1^2 = 40.39$ into Eq. (9.29). This substitution results in

$$[D_1] = \left[\begin{array}{cc:cc} 614.31 & 0 & -327.35 & 0 \\ 0 & 614.31 & -327.35 & -327.35 \\ \hdashline -327.35 & -327.35 & 614.31 & 0 \\ 0 & -327.35 & 0 & 286.96 \end{array}\right]$$

Applications of Gauss-Jordan elimination to the first two rows gives

$$\left[\begin{array}{cc:cc} 1 & 0 & -0.533 & 0.0 \\ 0 & 1 & -0.533 & -0.533 \\ \hdashline 0 & 0 & 265.44 & -174.44 \\ 0 & 0 & -174.44 & 112.53 \end{array}\right]$$

from which

$$[T_1] = \begin{bmatrix} 0.533 & 0.0 \\ 0.533 & 0.533 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[\bar{D}_1] = \begin{bmatrix} 265.44 & -174.44 \\ -174.44 & 112.53 \end{bmatrix}$$

The reduced mass and stiffness matrices are then

$$[\bar{M}_1] = [T_1]^T[M][T_1] = \begin{bmatrix} 1.568 & 0.284 \\ 0.284 & 1.284 \end{bmatrix}$$

and

$$[\bar{K}_1] = \bar{D}_1 + \omega_1^2[\bar{M}_1] = \begin{bmatrix} 328.76 & -162.97 \\ -162.67 & 164.39 \end{bmatrix}$$

The solution of the reduced eigenproblem,

$$[[\bar{K}_1] - \omega^2[\bar{M}_1]]\{U_p\} = \{0\}$$

yields the eigenvalues

$$\omega_1^2 = 39.48, \quad \omega_2^2 = 360.21 \tag{c}$$

and corresponding eigenvectors

$$\{U_p\}_1 = \begin{bmatrix} 0.4283 \\ 0.6562 \end{bmatrix} \qquad \{U_p\}_2 = \begin{bmatrix} 0.6935 \\ -0.6171 \end{bmatrix} \tag{d}$$

The same process is now applied to the second mode, starting by substituting into Eq. (9.29) the approximate eigenvalues $\omega_2^2 = 360.21$ calculated for the second mode in Eq. (c). In this case we obtain

$$[D_2] = \left[\begin{array}{cc:cc} 294.49 & 0 & -327.35 & 0 \\ 0 & 294.49 & -327.35 & -327.35 \\ \hdashline -327.35 & -327.35 & 294.49 & 0 \\ 0 & -327.35 & 0 & -32.86 \end{array}\right]$$

Gauss-Jordan elimination of the first two rows yields

$$\left[\begin{array}{cc:cc} 1 & 0 & -1.112 & 0.0 \\ 0 & 1 & -1.112 & -1.112 \\ \hdashline 0 & 0 & -433.27 & -363.88 \\ 0 & 0 & -363.88 & -396.74 \end{array}\right]$$

from which

$$[T_2] = \begin{bmatrix} 1.112 & 0.0 \\ 1.112 & 1.112 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [\bar{D}_2] = \begin{bmatrix} -433.27 & -363.88 \\ -363.88 & -396.74 \end{bmatrix}$$

The reduced mass and stiffness matrices are

$$[\bar{M}_2] = [T_2]^T[M][T_2] = \begin{bmatrix} 3.471 & 1.236 \\ 1.236 & 2.236 \end{bmatrix}$$

$$[\bar{K}_2] = [\bar{D}_2] + \omega_2^2[\bar{M}_2] = \begin{bmatrix} 817.12 & 81.21 \\ 81.21 & 408.56 \end{bmatrix}$$

The solution of the reduced eigenproblem $\left[[\bar{K}_2] - \omega^2[\bar{M}_2^2]\right]\{U_p\}_2 = \{0\}$ yields for the second mode

$$\omega_2^2 = 328.61$$

An iteration is performed by introducing $\omega_2^2 = 328.61$ into Eq. (9.29) to obtain the following:

$$[D_2]=\left[\begin{array}{cc:cc} 326.09 & 0 & -327.35 & 0 \\ 0 & 326.09 & -327.35 & -327.35 \\ \hdashline -327.35 & -327.35 & 326.09 & 0 \\ 0 & -327.35 & 0 & -1.26 \end{array}\right]$$

Applying Gauss-Jordan elimination to the first two rows yields

$$\left[\begin{array}{cc:cc} 1 & 0 & -1.004 & 0.0 \\ 0 & 1 & -1.004 & -1.004 \\ \hdashline 0 & 0 & -331.14 & -328.62 \\ 0 & 0 & -328.62 & -329.88 \end{array}\right]$$

from which

$$[T_2]=\begin{bmatrix} 1.004 & 0.0 \\ 1.004 & 1.004 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[\bar{D}_2]=\begin{bmatrix} -331.14 & -328.62 \\ -328.62 & -329.88 \end{bmatrix}$$

The reduced mass and stiffness matrices are:

$$[\bar{M}_2]=[T_2]^T[M][T_2]=\begin{bmatrix} 3.015 & 1.008 \\ 1.008 & 2.008 \end{bmatrix}$$

and

$$[\bar{K}_2]=[\bar{D}_2]+\omega_2^2[\bar{M}_2]=\begin{bmatrix} 659.78 & 2.54 \\ 2.54 & 329.89 \end{bmatrix}$$

The solution of the reduced eigenproblem $\left[[\bar{K}_2]-\omega^2[\bar{M}_2]\right]\{U_p\}_2=\{0\}$ now gives for the second mode

$$\{U_p\}_2=\begin{Bmatrix} 0.5766 \\ -0.5766 \end{Bmatrix} \quad \omega_2^2=327.35 \tag{e}$$

Therefore, from Eqs. (c), (d), and (e) we have obtained for the first two eigenvalues

$$\omega_1^2 = 39.48 \quad \text{and} \quad \omega_2^2 = 327.35 \tag{f}$$

and corresponding eigenvectors

$$\{U_p\}_1 = \begin{Bmatrix} 0.4283 \\ 0.6562 \end{Bmatrix}, \quad \{U_p\}_2 = \begin{Bmatrix} 0.5766 \\ -0.5766 \end{Bmatrix}$$

The eigenvectors of the system are then computed using Eq. (9.32) as follows:

$$\begin{bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0.533 & 0.0 \\ 0.533 & 0.533 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.4283 \\ 0.6562 \end{Bmatrix} = \begin{Bmatrix} 0.2283 \\ 0.5780 \\ 0.4283 \\ 0.6562 \end{Bmatrix}_1 \tag{g}$$

Hence ordering the elements of the modal vector:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.2283 \\ 0.4283 \\ 0.5780 \\ 0.6562 \end{Bmatrix}_1 \tag{h}$$

and

$$\begin{Bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{Bmatrix}_2 = \begin{bmatrix} 1.004 & 0.0 \\ 1.004 & 1.004 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.5766 \\ -0.5766 \end{Bmatrix} = \begin{Bmatrix} 0.5789 \\ 0.0 \\ 0.5766 \\ -0.5766 \end{Bmatrix}_2$$

Hence ordering the modal vector:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.5789 \\ 0.5766 \\ 0.0 \\ -0.5766 \end{Bmatrix}_2 \tag{i}$$

The eigenvalues and eigenvectors [Eqs. (f), (h), and (i)] calculated for the first two modes in this example using dynamic condensation are virtually identical to the exact solution determined in Eqs. (c) and (d) of Illustrative Example 9.3.

It should be noted that normalization of the eigenvectors is not needed in Eqs. (h) and (i) if the reduced vectors are normalized with respect to the reduced mass of the system, that is, if a reduced eigenvector $\{U_p\}$ satisfies the normalizing equation

$$\{U_p\}^T [\bar{M}] \{U_p\} = 1$$

then by Eq. (9.34)

$$\{U_p\}^T [T]^T [M] [T] \{U_p\} = 1 \tag{9.37}$$

and since by Eq. (9.32)

$$\{U\}_i = [T]\{U_p\}$$

and

$$\{U_i^T = \{U\}_p^T [T]^T$$

Upon substitution of $\{U\}_\mathrm{I}$ and $\{U\}_i^T$ into Eq. (9.37) results in

$$\{U\}_i^T [M] \{U\}_i = 1$$

thus demonstrating that the eigenvector $\{U\}_i$ is normalized with respect to the mass matrix [M] of the system, if $\{U_p\}$ has been normalized with respect to the reduced mass matrix $[\bar{M}]$.

The MATLAB program is used to obtain the natural frequencies in Eq. (e).

```
clear all
close all

%___________________________________________________________________________
% Inputs:
%   M, K
%   m =  Number of row to apply the dynamic elimination process
%____________________________________________________________________________

%%%%-GIVEN VALUES-%%%%

%%%Mass Matrix
M = [1, 0 0 0; 0, 1, 0, 0; 0, 0, 1,0; 0,0,0, 1];

%%%Stiffness Matrix
K = 327.35*[2,0,-1,0; 0,2,-1,-1; -1,-1,2,0; 0,-1,0,1];
% m = 0;          %Exact Solution;

%%%Elimination of the first and second rows: m=2
m = 2;            %Dynamic Condensation

N= length(K);    %Total number of row

s=0;
for i=1:2
W =[];
w(i)= sqrt(s(1,1));
W = diag(w(i)^2);

    D = K - W.*M;

    %%%Partition Matrix for Dynamic Condensation (Eq.9.28)
    D_pp = D(m+1:N,m+1:N);
    D_ps = D(m+1:N,1:m);
    D_ss = D(1:m,1:m);
    D_sp = D(1:m,m+1:N);


    D_bar = D_pp-D_ps*inv(D_ss)*D_sp;
    T_bar = -inv(D_ss)*D_sp;
    T = [T_bar; eye(N-m,N-m)];
    M_bar = T'*M*T;
    K_bar = D_bar+W.*M_bar;

    %%%% Solve the eigneproblem (Eq.9.36)
    [a, s] = eig(K_bar, M_bar);

    s;

    a;
end
```

```
s1 = s(1,1)           % Eigenvalue, w1 in Eq.(c)
s2 = s(2,2)           % Eigenvalue, w2 in Eq.(c)
a1 = a(:,1)           % Eigenvector, {U_p}_1 in Eq.(d)

% Use s(2,2) in Eq.(c) for the following iterations

for i=1:2
w(i)= sqrt(s(2,2));
W = diag(w(i)^2);

    D = K - W.*M;

    %%%Partition Matrix for Dynamic Condensation (Eq.9.28)
    D_pp = D(m+1:N,m+1:N);
    D_ps = D(m+1:N,1:m);
    D_ss = D(1:m,1:m);
    D_sp = D(1:m,m+1:N);


    D_bar = D_pp-D_ps*inv(D_ss)*D_sp;
    T_bar = -inv(D_ss)*D_sp;
    T = [T_bar; eye(N-m,N-m)];
    M_bar = T'*M*T;
    K_bar = D_bar+W.*M_bar;

    %%%% Solve the eigenproblem (Eq.9.36)
    [a, s] = eig(K_bar, M_bar);

    s;
    a;
end

s2 = s(2,2);            % Eigenvalue, w2 in Eq.(f)
a2 = a(:,2);            % Eigenvalue, w2 in Eq.(c)

aa = [a1,a2]            % Eigenvalues in Eq.(f)
ss = [s1,s2]            % Eigenvectors in Eq.(f)
```

9.4 Modified Dynamic Condensation

The dynamic condensation method requires the application of elementary operations, as it is routinely done to solve a linear system of algebraic equations, using the Gauss-Jordan elimination process. The elementary operations are required to transform Eq. (9.29) to the form given by Eq. (9.30). However, the method also requires the calculation of the reduced mass matrix by Eq. (9.34). This last calculation involves the multiplication of three matrices of dimensions equal to the total number of coordinates in the system. Thus, for a system defined with many degrees of freedom, the calculation of the reduced mass matrix $[M]$ requires a large number of numerical operations. A modification (Paz 1989), recently proposed, obviates such large number of numerical operations. This modification consists of calculating the reduced stiffness matrix $[\bar{K}]$ only once by simple elimination of s displacements in Eq. (9.29) after setting $\omega^2 = 0$, thus making unnecessary the repeated calculation of $[\bar{K}]$ for each mode using Eq. (9.35). Furthermore, it also eliminates the time consumed in calculating the reduced mass matrix $[\bar{M}]$ using Eq. (9.34). In the modified method, the reduced mass matrix for any mode i is calculated from Eq. (9.35) as

$$[\bar{M}_i] = \frac{1}{\omega_i^2}\left[[\bar{K}] - [\bar{D}_i]\right] \tag{9.38}$$

Where $[\bar{K}]$ is the reduced stiffness matrix, already calculated, and $[\bar{D}_i]$ is the dynamic matrix given in the partitioned matrix of Eq. (9.30).

As can be seen, the modified algorithm essentially requires, for each eigenvalues calculated, only the application of the Gauss-Jordan process to eliminate s unknowns in a linear system of equations such as the system in Eq. (9.29).

Illustrative Example 9.6

Repeat Illustrative Example 9.5 using the modified dynamic condensation method.

Solution:

The initial calculations of the modified method are the same as those in Illustrative Example 9.5. Thus, from Illustrative Example 9.5 we have

$$[\bar{K}_1] = \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix} \tag{a}$$

$$[\bar{M}_1] = \begin{bmatrix} 1.5 & 0.25 \\ 0.25 & 1.25 \end{bmatrix} \tag{b}$$

$$\omega_1^2 = 40.39 \qquad \omega_2^2 = 365.98, \tag{c}$$

$$[\bar{D}_i] = \begin{bmatrix} 265.44 & -174.44 \\ -174.44 & 112.53 \end{bmatrix} \tag{d}$$

and

$$[\bar{T}_1] = \begin{bmatrix} 0.533 & 0 \\ 0.533 & 0.533 \end{bmatrix} \tag{e}$$

Now, the reduced mass matrix $[\bar{M}_1]$ is calculated from Eq. (9.38), after substitution in this equation of $[\bar{K}_1]$ from Eq. (a) and $[\bar{D}_1]$ from Eq. (d), as

$$[\bar{M}_1] = \frac{1}{\omega_1^2}\left[[\bar{K}_1] - [\bar{D}_1]\right] = \begin{bmatrix} 1.530 & 0.267 \\ 0.267 & 1.266 \end{bmatrix} \tag{f}$$

Then the reduced stiffness and mass matrix given by Eqs. (a) and (f) are used to solve the reduced eigenproblem:

$$\left[[\bar{K}_1] - \omega^2[\bar{M}_1]\right]\{U_p\}_1 = \{0\}$$

to obtain the eigenvalues

$$\omega_1^2 = 39.46 \qquad \omega_2^2 = 363.67 \tag{g}$$

and the eigenvector for the first mode

$$\begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.43359 \\ 0.66424 \end{Bmatrix}_1 \tag{h}$$

The eigenvector for the first mode, in terms of the original four coordinate, is then obtained from Eq. (9.31) as

$$\begin{Bmatrix} U_1 \\ U_3 \end{Bmatrix} = \begin{bmatrix} 0.533 & 0 \\ 0.533 & 0.533 \end{bmatrix} \begin{Bmatrix} 0.43359 \\ 0.66424 \end{Bmatrix} = \begin{Bmatrix} 0.23110 \\ 0.58514 \end{Bmatrix} \tag{i}$$

The combination of Eqs. (h) and (i) give the eigenvector for the first mode as

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.23110 \\ 0.43359 \\ 0.58514 \\ 0.66424 \end{Bmatrix}_1 \tag{j}$$

Analogously, for the second mode, we substitute $\omega_2^2 = 363.67$ from Eqs. (g) into (a) of Illustrative Example 9.5, to obtain the following matrices after reducing the first two coordinates:

$$[\bar{T}_2] = \begin{bmatrix} 1.1248 & 0 \\ 1.1248 & 1.1248 \end{bmatrix} \tag{k}$$

$$[\bar{D}_2] = \begin{bmatrix} -445.43 & -368.20 \\ -368.20 & -404.54 \end{bmatrix} \tag{l}$$

The reduced mass matrix $[\bar{M}_2]$ is the calculated from Eq. (9.37) as

$$\begin{gathered} [\bar{M}_2] = \frac{1}{363.67}\left\{ \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix} - \begin{bmatrix} -445.43 & -368.20 \\ -368.20 & -404.54 \end{bmatrix} \right\} \\ [\bar{M}_2] = \begin{bmatrix} 2.1250 & 0.5624 \\ 0.5624 & 1.5624 \end{bmatrix} \end{gathered} \tag{m}$$

Then, for the second mode, the solution of the corresponding reduced eigenproblem gives

$$\omega_2^2 = 319.41, \qquad \begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.61894 \\ -0.63352 \end{Bmatrix} \tag{n}$$

$$\begin{Bmatrix} U_1 \\ U_3 \end{Bmatrix}_2 = \begin{bmatrix} 1.1248 & 0 \\ 1.1248 & 1.1248 \end{bmatrix} \begin{Bmatrix} 0.61894 \\ -0.63352 \end{Bmatrix} = \begin{Bmatrix} 0.69618 \\ -0.01640 \end{Bmatrix} \tag{o}$$

and

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.69618 \\ 0.61894 \\ -0.01640 \\ -0.63352 \end{Bmatrix}_2 \tag{p}$$

The results obtained for this example using the modified method, although sufficiently approximate, are not as close to the exact solution as those obtained in Example 9.5 by the direct application of the dynamic condensation method. Table 9.1 shows and compares eigenvalues calculated in Illustrative Examples 9.5 and 9.6 with the exact solution obtained previously in Illustrative Example 9.3.

Table 9.1 Comparison of results in Illustrative Examples 9.5 and 9.6 using dynamic condensation and modified dynamic condensation

Eigenvalue					
Mode	Exact solution	Dynamic condensation	Error %	Modified method	Error %
1	39.48	39.48	0.00	39.46	0.05
2	327.35	327.35	0.00	319.41	2.42

The following MATLAB program is to reproduce the solution of Example 6.5 using the modified dynamic condensation method.

```
clc
clear all
close all

%_____________________________________________________________________________
% Inputs:
%   M, K
%   m =  Number of row to apply the dynamic elimination process
%_____________________________________________________________________________

%%%%-GIVEN VALUES-%%%%

%%%Mass Matrix
M = [1, 0 0 0; 0, 1, 0, 0; 0, 0, 1,0; 0,0,0, 1];

%%%Stiffness Matrix
K = 327.35*[2,0,-1,0; 0,2,-1,-1; -1,-1,2,0; 0,-1,0,1]; %u1,u3,u2,u4
% m = 0;        % Exact Solution;

%%%Elimination of the first and second rows: m=2
m = 2;          % Dynamic Condensation

N= length(K);   %Total number of row

s=0;
for i=1:1
W =[];
w(i)= sqrt(s(1,1));
W = diag(w(i)^2);

    D = K - W.*M;

    %%%Partition Matrix for Dynamic Condensation (Eq.9.28)
    D_pp = D(m+1:N,m+1:N);
    D_ps = D(m+1:N,1:m);
    D_ss = D(1:m,1:m);
    D_sp = D(1:m,m+1:N);


    D_bar = D_pp-D_ps*inv(D_ss)*D_sp;
    T_bar = -inv(D_ss)*D_sp;
    T = [T_bar; eye(N-m,N-m)];
    M_bar = T'*M*T;
    K_bar = D_bar+W.*M_bar;

    %%%% Solve the eigneproblem (Eq.9.36)
    [a, s] = eig(K_bar, M_bar);

    a;

    s;

end


w = sqrt(s(1,1));
M_bar = M_bar;
K_bar = K_bar;


for i=1:1
W = diag(w^2);

    D = K - W.*M;

    %%%Partition Matrix for Dynamic Condensation (Eq.9.28)
    D_pp = D(m+1:N,m+1:N);
    D_ps = D(m+1:N,1:m);
    D_ss = D(1:m,1:m);
    D_sp = D(1:m,m+1:N);

    D_bar = D_pp-D_ps*inv(D_ss)*D_sp;
    T_bar = -inv(D_ss)*D_sp;


end
```

```
D_bar = D_bar;                     % Eq.(d)
T_bar = T_bar;                     % Eq.(e)

%%%Modified Dynamic Condensation (Eq.9.38)
M_bar = 1/W.*(K_bar-D_bar);        % Eq.(f)

[a_1, s_1] = eig(K_bar, M_bar);

s_11 = s(1,1)                      % Eq.(g)
a_11 = (a_1(:,1));                 % Eq.(h) for u2, u4
a_12 = T_bar*(a_11);               % Eq.(i) for u1, u3

a_1 = [a_11; a_12]                 % Eq.(j) u2,u4,u1,u3


%%%Analogously, for the second mode

w = sqrt(s_1(2,2));
M_bar = M_bar;
K_bar = K_bar;

for i=1:1
W = diag(w^2);

    D = K - W.*M;

    D_pp = D(m+1:N,m+1:N);
    D_ps = D(m+1:N,1:m);
    D_ss = D(1:m,1:m);
    D_sp = D(1:m,m+1:N);


    D_bar = D_pp-D_ps*inv(D_ss)*D_sp;
    T_bar = -inv(D_ss)*D_sp;


end

D_bar = D_bar;                     % Eq.(l)             for mode 2
T_bar = T_bar;                     % Eq.(k)             for mode 2


M_bar = 1/W.*(K_bar-D_bar);        % Eq.(m)             for mode 2

[a_2, s_2] = eig(K_bar, M_bar);

s_22 = s_2(2,2)                    % Eq.(n)             for mode 2

a_21 = (a_2(:,2));                 % Eq.(n) for u2, u4  for mode 2
a_22 = T_bar*(a_21);               % Eq.(o) for u1, u3  for mode 2

a_2 = [a_21; a_22]                 % Eq.(p) u2,u4,u1,u3 for mode 2
```

9.5 Summary

The reduction of secondary or dependent degrees of freedom is usually accomplished in practice by the Static Condensation Method. This method consists of determining, by a partial Gauss-Jordan elimination, the reduced stiffness matrix corresponding to the primary degrees of freedom and the transformation matrix relating the secondary and primary degrees of freedom. The same transformation matrix is used in an orthogonal transformation to reduce the mass and damping matrices of the system. Static condensation introduces errors when applied to the solution structural dynamics problems. However, as is shown in this chapter, the application of the Dynamic Condensation Method substantially reduces or eliminates these errors. Furthermore, the Dynamic Condensation Method converges rapidly to the exact solution when iteration is applied.

9.6 Problems

Problem 9.1

The stiffness and mass matrices of a certain structure are given by

$$[K] = \begin{bmatrix} 10 & -2 & -1 & 0 \\ -2 & 6 & -3 & -2 \\ -1 & -3 & 12 & -1 \\ 0 & -2 & -1 & 8 \end{bmatrix}, \quad [M] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(a) Use the Static Condensation Method to determine the transformation matrix and the reduced stiffness and mass matrices corresponding to the elimination of the first two degrees of freedom (the massless degrees of freedom).

(b) Determine the natural frequencies and corresponding normal modes for reduced system.

Problem 9.2

Repeat (a) and (b) of Problem 9.1 for a structure having stiffness matrix as indicated in that problem, but mass matrix given by

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Problem 9.3

Determine the natural frequencies and modal shape of the system in Problem 9.2 in terms of its four original coordinates; find the errors in the two modes obtained in Part (b) of Problem 9.2.

Problem 9.4

Consider the shear building shown in Fig. P9.4.

(a) Determine the transformation matrix and the reduced stiffness and mass matrices corresponding to the static condensation of the coordinates u_1 u_3, and u_4 as indicated in the figure.

(b) Determine the natural frequencies and normal modes for the reduced system obtained in (a).

(c) Use the results of (b) to determine the modal shapes, described by the five original coordinates, corresponding to the two lowest frequencies.

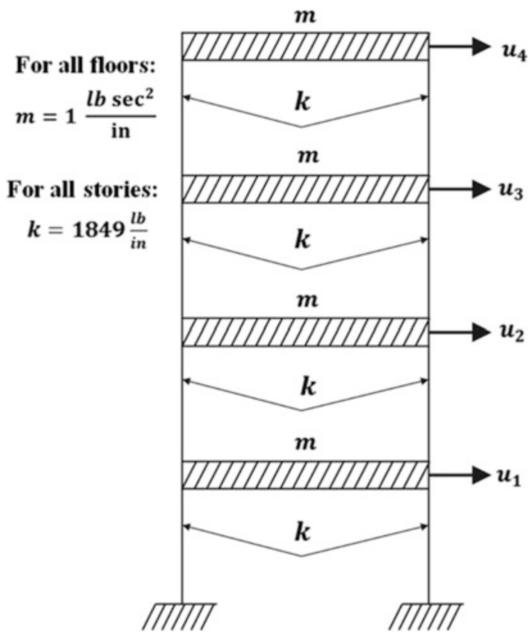

Fig. P9.4

Problem 9.5

Use the results obtained in Problem 9.4 to determine the maximum shear forces in the stories of the building in Fig. P9.4 when subjected to an earthquake for which the response spectrum is given in Fig. 5.10 of Sect. 5.4.

Problem 9.6

Use the results in Problem 9.4 to determine the maximum shear forces in the stories of the building in Fig. P9.4 when subjected to an earthquake the response spectrum is given in Fig. 5.10 of Sect. 5.4.

Problem 9.7

Repeat Problem 9.2 using the Dynamic Condensation Method.

Problem 9.8

Repeat Problem 9.4 using the Dynamic Condensation Method and compare results with the exact solution.

Problem 9.9

Consider the five-story shear building of Fig. P9.4 subjected at its foundation to the time-acceleration excitation depicted in Fig. P9.9. Use static condensation of the coordinates u_1, u_3, and u_4 and determine:

(a) The two natural frequencies and corresponding modal shapes of the reduced system.
(b) The displacements at the floor levels considering two modes.
(c) The shear forces in the columns of the structure also considering two modes.

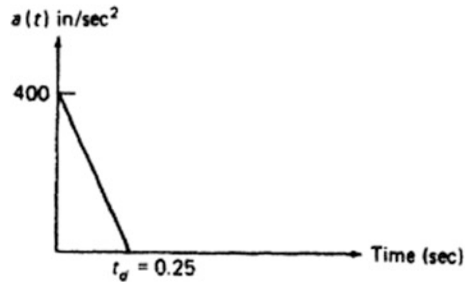

Fig. P9.9

Problem 9.10

Solve Problem 9.9 using the Dynamic Condensation Method. Problem 9.11.

Problem 9.11

The stiffness and mass matrices for a certain structure are

$$[K] = 10^6 \begin{bmatrix} 0.906 & 0.294 & 0.424 \\ 0.294 & 0.318 & 0.176 \\ 0.424 & 0.176 & 80.000 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 288 & -8 & 1566 \\ -8 & 304 & 644 \\ 1566 & 644 & 80,000 \end{bmatrix}$$

Calculate the fundamental natural frequency of the system after reduction of the first coordinate by the following methods: (a) Static condensation; and (b) Dynamic condensation. Also obtain the natural frequencies as a three-degrees-of-freedom system and compare results for the fundamental frequency.

Problem 9.12

Repeat Problem 9.11 using the Modified Dynamic Condensation Method and compare results with the solution obtained with no condensation.

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