3 Response of One-Degree-of-Freedom System to Harmonic Loading

In this chapter, we will study the motion of structures idealized as single-degree-of-freedom systems excited harmonically, that is, structures subjected to forces or displacements whose magnitudes may be represented by a sine or cosine function of time. This type of excitation results in one of the most important motions in the study of mechanical vibrations as well as in applications to structural dynamics. Structures are very often subjected to the dynamic action of rotating machinery which produces harmonic excitations due to the unavoidable presence of mass eccentricities in the rotating parts of such machinery. Furthermore, even in those cases when the excitation is not a harmonic function, the response of the structure may be obtained using the Fourier Method, as the superposition of individual responses to the harmonic components of external excitation. This approach will be dealt with in Chap. 20 as a special topic.

3.1 Harmonic Excitation: Undamped System

The impressed force $F(t)$ acting on the simple oscillator in Fig. 3.1 is assumed to be harmonic and equal to $F_0 \sin \overline{\omega} t$ where F_0 is the peak amplitude and $\overline{\omega}$ is the frequency of the force in radians per second.

The differential equation obtained by summing all the forces in the free body diagram of Fig. 3.1b is

$$m\ddot{u} + ku = F_0 \sin \overline{\omega} t \tag{3.1}$$

The solution of Eq. (3.1) can be expressed as

$$u(t) = u_c(t) + u_P(t) \tag{3.2}$$

where $u_c(t)$ is the complementary solution satisfying the homogeneous equation, that is, Eq. (3.1) with the left hand-side set equal to zero; and $u_p(t)$ is the particular solution based on the solution satisfying the nonhomogeneous differential Eq. (3.1). The complementary solution, $u_c(t)$, is given by Eq. (1.17) as

$$u_c(t) = A \cos \omega t + B \sin \omega t \tag{3.3}$$

where

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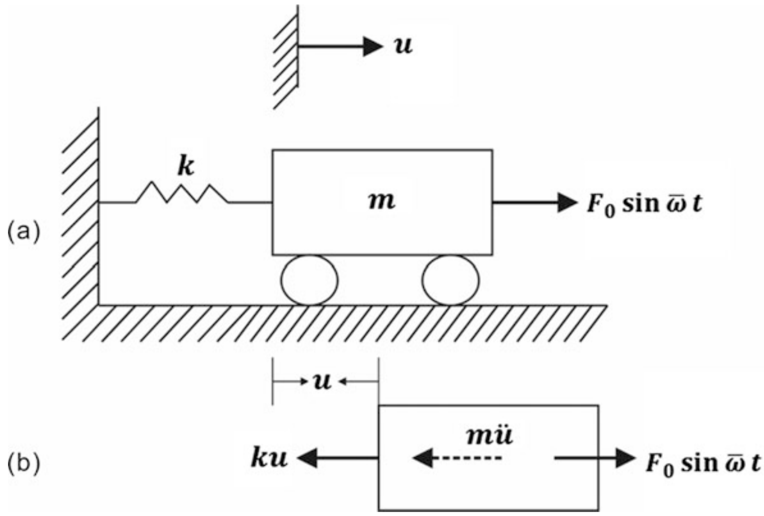

Fig. 3.1 (**a**) Undamped oscillator harmonically excited, (**b**) Free body diagram

$$\omega = \sqrt{k/m}$$

The nature of the forcing function in Eq. (3.1) suggests that the particular solution be taken as

$$u_p(t) = U \sin \overline{\omega} t \tag{3.4}$$

where U is the amplitude of the particular solution. The substitution of Eq. (3.4) into Eq. (3.1) followed by cancellation of common factors gives

$$-m\overline{\omega}^2 U + kU = F_0$$

or

$$U = \frac{F_0}{k - m\overline{\omega}^2} = \frac{F_0/k}{1 - r^2} \tag{3.5}$$

in which r represents the ratio (frequency ratio) of the applied forced frequency to the natural frequency of vibration of the system, that is,

$$r = \frac{\overline{\omega}}{\omega} \tag{3.6}$$

Combining Eqs. (3.3) through (3.5) with Eq. (3.2) yields

$$U(t) = A \cos \omega t + B \sin \omega t + \frac{F_0/k}{1 - r^2} \sin \overline{\omega} t \tag{3.7}$$

If the initial conditions for the displacement and for the velocity at time $t = 0$ are taken as zero ($u_0 = 0$, $v_0 = 0$), the constants of integration determined from Eq. (3.7) are:

$$A = 0 \qquad \text{and} \qquad B = -r\frac{F_0/k}{1 - r^2}$$

which, upon substitution in Eq. (3.7), results in

$$u(t) = \frac{F_0/k}{1 - r^2}\left(\sin\overline{\omega}t - r\sin\omega t\right) \tag{3.8}$$

As we can see from Eq. (3.8), the response is given by the superposition of two harmonic terms of different frequencies. The resulting motion is not harmonic; however, in the practical case, damping forces will always be present in the system and will cause the last term, i.e., the free frequency term in Eq. (3.8), to eventually vanish. For this reason, this term is said to represent the transient response. The forcing frequency term in Eq. (3.8), namely

$$u(t) = \frac{F_0/k}{1 - r^2}\sin\overline{\omega}t \tag{3.9}$$

is referred to as the steady-state response. It is clear from Eq. (3.8) that in the case of no damping in the system, the transient will not vanish and the response is then given by Eq. (3.8). It can also be seen from Eq. (3.8) or Eq. (3.9) that when the forcing frequency is equal to natural frequency ($r = 1.0$), the amplitude of the motion becomes infinitely large. A system acted upon by an external excitation of frequency coinciding with the natural frequency is said to be at resonance. In this circumstance, the amplitude will increase gradually to infinite. However, materials that are commonly used in practice are subjected to strength limitations and in actual structures failures occur long before extremely large amplitudes can be attained.

3.2 Harmonic Excitation: Damped System

Now consider the case of the one-degree-of-freedom system in Fig. 3.2a vibrating under the influence of viscous damping. The differential equation of motion is obtained by equating to zero the sum of the forces in the free body diagram of Fig. 3.2b. Hence

$$m\ddot{u} + c\dot{u} + ku = F_0\sin\overline{\omega}t \tag{3.10}$$

The complete solution of this equation again consists of the complementary solution $u_c(t)$ and the particular solution $u_p(t)$.The complementary solution is given for the underdamped case ($c < c_{cr}$) by Eqs. (2.15) after using (2.19) as

$$u_c(t) = e^{-\xi\omega t}(A\cos\omega_D t + B\sin\omega_D t) \tag{3.11}$$

The particular solution may be found by substituting u_p in this case assumed to be of the form

$$u_p(t) = C_1\sin\overline{\omega}t + C_2\cos\overline{\omega}t \tag{3.12}$$

into Eq. (3.10) and equating the coefficients of the sine and cosine functions. The unknowns C_1 and C_2 are found with plugging $u_p(t)$ into $m\ddot{u}_p + c\dot{u}_p + ku_p = F_0\sin\overline{\omega}t$.

$$C_1 = \frac{F_0}{k}\left(\frac{1 - r^2}{(1 - r^2)^2 + (2\xi r)^2}\right) \quad \text{and} \quad C_2 = \frac{F_0}{k}\left(\frac{-2\xi r}{(1 - r^2)^2 + (2\xi r)^2}\right) \tag{3.13}$$

In addition, here we follow a more elegant approach using Euler's relation, namely

$$e^{i\overline{\omega}t} = \cos\overline{\omega}t + i\sin\overline{\omega}t$$

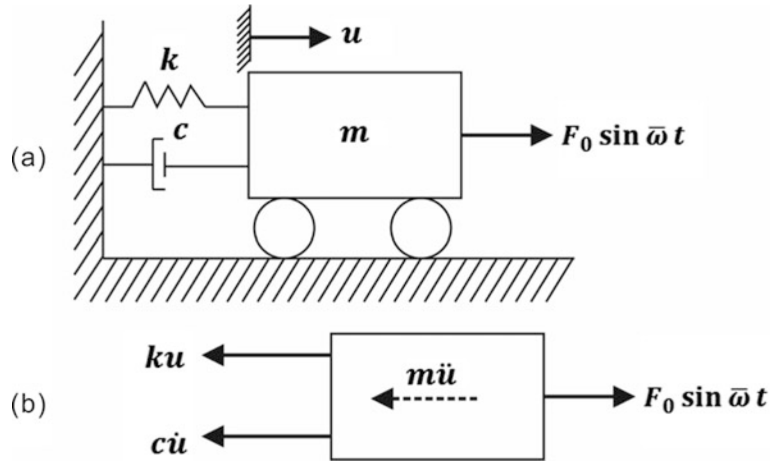

Fig. 3.2 (**a**) Damped oscillator harmonically excited, (**b**) Free body diagram

For this purpose, the reader should realize that we can write Eq. (3.10) as

$$m\ddot{u} + c\dot{u} + ku = F_0 e^{i\bar{\omega} t} \tag{3.14}$$

with the understanding that only the imaginary component of $F_0 e^{i\bar{\omega}t}$, i.e., the force component of $F_0 \sin \bar{\omega} t$, is acting and, consequently, the response will then consist only of the imaginary part of the total solution of Eq. (3.14). In other words, we obtain the solution of Eq. (3.14) which has real and imaginary components, and disregard the real component.

It is reasonable to expect that the particular solution of Eq. (3.14) will be of the form

$$u_p = Ce^{i\bar{\omega} t} \tag{3.15}$$

Substitution of Eq. (3.14) into Eq. (3.13) and cancellation of the factor $e^{i\bar{\omega} t}$ gives

$$-m\bar{\omega}^2 C + ic\bar{\omega}C + kC = F_0$$

or

$$C = \frac{F_0}{k - m\bar{\omega}^2 + ic\bar{\omega}}$$

and

$$u_p = \frac{F_0 e^{i\bar{\omega}t}}{k - m\bar{\omega}^2 + ic\bar{\omega}} \tag{3.16}$$

By using polar coordinate form, the complex denominator in Eq. (3.16) may be written as

$$u_p = \frac{F_0 e^{i\bar{\omega}t}}{\sqrt{\left(k - m\bar{\omega}^2\right)^2 + \left(c\bar{\omega}\right)^2} e^{i\theta}}$$

or

$$u_p = \frac{F_0 e^{i\left[\overline{\omega} t - \theta\right]}}{\sqrt{\left(k - m\overline{\omega}^2\right)^2 + \left(c\overline{\omega}\right)^2}} \tag{3.17}$$

where

$$\tan\theta = \frac{c\overline{\omega}}{k - m\overline{\omega}^2} \tag{3.18}$$

The response to the force in $F_0 \sin\overline{\omega} t$ (the imaginary component of $F_0 e^{i\overline{\omega} t}$) is then the imaginary component of Eq. (3.17), namely

$$u_P = \frac{F_0 \sin\left(\overline{\omega} t - \theta\right)}{\sqrt{\left(k - m\overline{\omega}\right)^2 + \left(c\overline{\omega}\right)^2}} \tag{3.19}$$

or

$$u_p = U \sin\left(\overline{\omega} t - \theta\right) \tag{3.20}$$

where

$$U = \frac{F_0}{\sqrt{\left(k - m\overline{\omega}\right)^2 + \left(c\overline{\omega}\right)^2}}$$

is the amplitude of the steady-state motion. Equations (3.19) and (3.18) may conveniently be written in terms of dimensionless ratios as

$$u(t) = \frac{u_{st} \sin\left(\overline{\omega} t - \theta\right)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}} \tag{3.21}$$

and

$$\tan\theta = \frac{2\xi r}{1 - r^2} \tag{3.22}$$

where $u_{\mathrm{st}} = F_0/k$ is seen to be the static deflection of the spring acted upon by the force F_0, $\xi = c\,/\,c_{cr}$ the damping ratio, and $r = \overline{\omega}/\omega$ the frequency ratio. The total response is then obtained by combining the complementary solution (transient response) from Eq. (3.11) and the particular solution (steady-state response) from Eq. (3.21), that is,

$$u(t) = e^{-\xi\omega t}(A\cos\omega_D t + B\sin\omega_D t) + \frac{u_{st}\sin\left(\overline{\omega} t - \theta\right)}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} \tag{3.23}$$

The reader should be warned that the constants of integration A and B must be evaluated from initial conditions using the total response given by Eq. (3.23) and not from just the transient component of the response given in Eq. (3.11). By examining the transient component of response, it may be seen that the presence of the exponential factor $e^{-\xi\omega t}$ will cause this component to vanish, leaving only the steady-state motion which is given by Eq. (3.21).

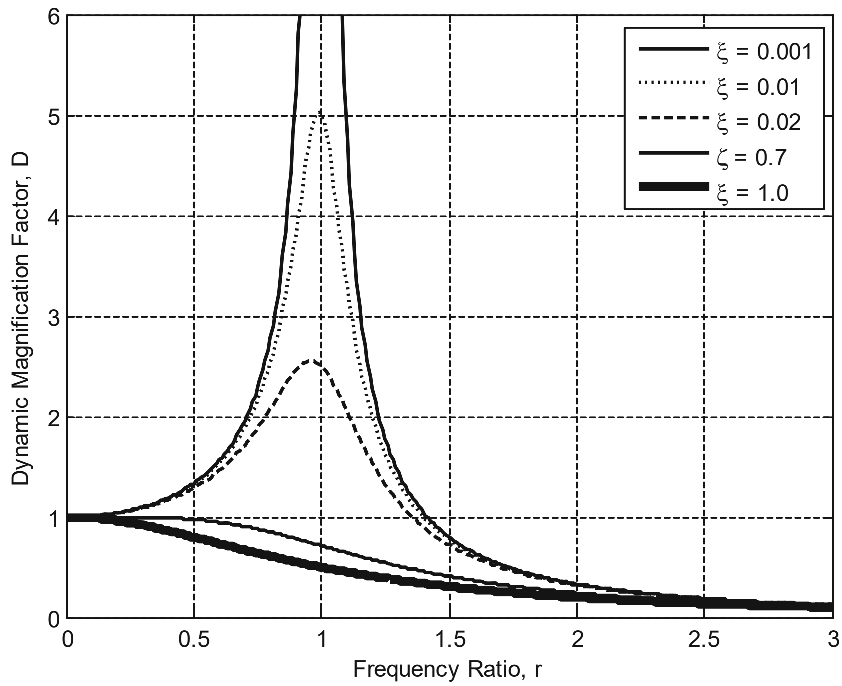

Fig. 3.3 Dynamic magnification factor as a function of the frequency ratio for various amounts of damping

The ratio of the steady-state amplitude of $u_p(t)$ to the static deflection u_{st} defined above is known as the dynamic magnification factor D, and is given from Eqs. (3.20) and (3.21) by

$$D = \frac{U}{u_{st}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} \tag{3.24}$$

It may be seen from Eq. (3.24) that the dynamic magnification factor varies with the frequency ratio r and the damping ratio ξ. Parametric plots of the dynamic magnification factor are shown in Fig. 3.3. The phase angle θ, given in Eq. (3.22), also varies with the same quantities as it is shown in the plots of Fig. 3.4. We note in Fig. 3.3 that for a lightly damped system, the peak amplitude occurs at a frequency ratio very close to r = 1; that is, the dynamic magnification factor has its maximum value virtually at resonance ($r = 1$). It can also be seen from Eq. (3.24) that at resonance the dynamic magnification factor is inversely proportional to the damping ratio, that is,

$$D(r = 1) = \frac{1}{2\xi} \tag{3.25}$$

Although the dynamic magnification factor evaluated at resonance is close to its maximum value, it is not exactly the maximum response for a damped system. However, for moderate amounts of damping, the difference between the approximate value of Eq. (3.25) and the exact maximum is negligible.

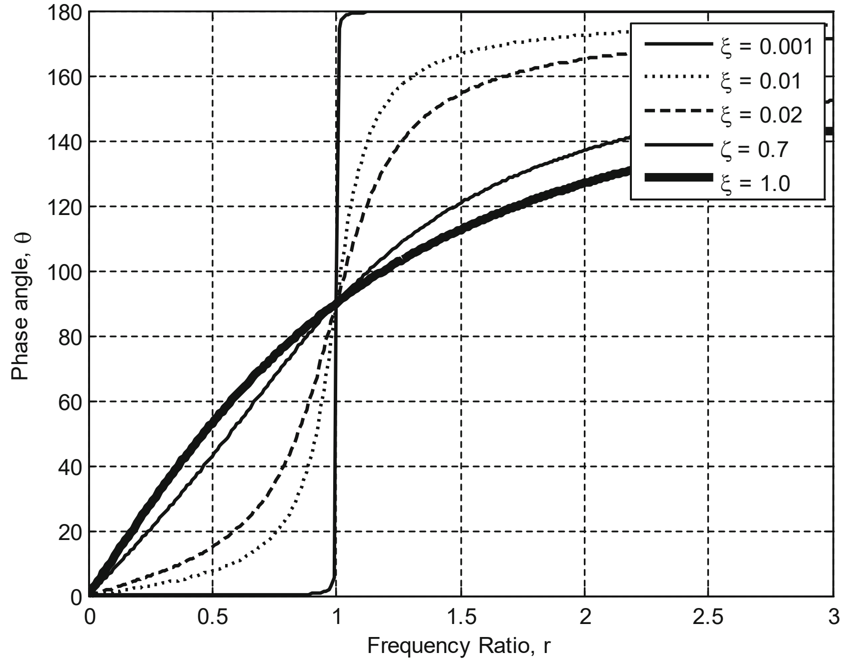

Fig. 3.4 Phase angle θ as a function of frequency ratio for various damping values

Illustrative Example 3.1

A simple beam supports at its center a machine having a weight $W = 16{,}000$ lb. The beam is made of two standard S8 × 23 sections with a clear span $L = 12$ ft and total cross-sectional moment of inertia $I = 2 \times 64.2 = 128.4$ in^4. The motor runs at 300 rpm, and its rotor is out of balance to the extent of $W' = 40$ lb at an eccentricity of $e_0 = 10$ in. What will be the amplitude of the steady-state response if the equivalent viscous damping for the system is assumed 10% of the critical?

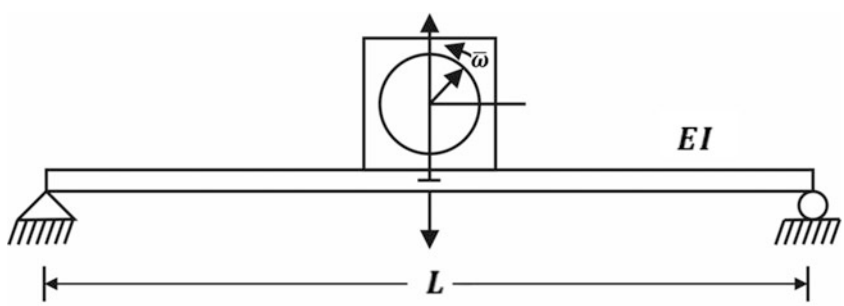

Fig. 3.5 Diagram for beam-machine system of Illustrative Example 3.1

Solution:

This dynamic system may be modeled by the damped oscillator. The distributed mass of the beam will be neglected in comparison with the large mass of the machine. Figs. 3.5 and 3.6 show, respectively, the schematic diagram of the beam-machine system and the adapted model.

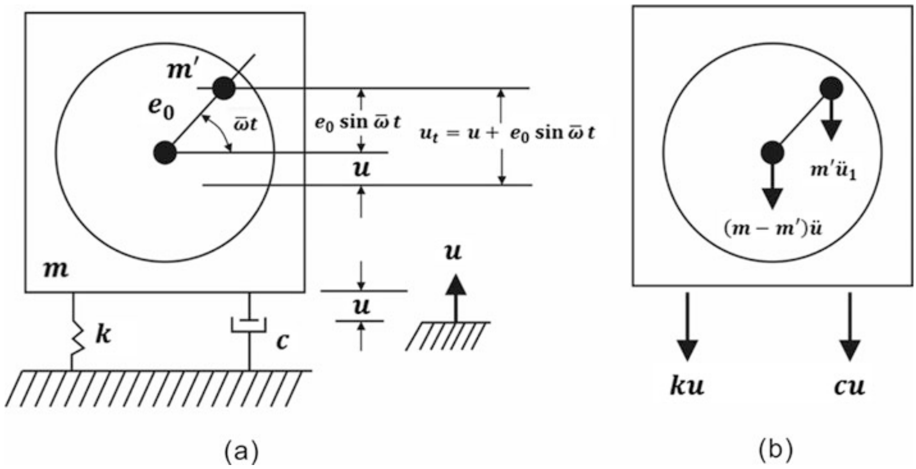

Fig. 3.6 (**a**) Analytical model for Illustrative Example 3.1; (**b**) Free body diagram

The force at the center of a simply supported beam necessary to deflect this point one unit (i.e., the stiffness coefficient) is given by the formula

$$k = \frac{48EI}{L^3} = \frac{48 \times 30 \times 10^6 \times 128.4}{(144)^3} = 61,920 \text{ lb/in}$$

The natural frequency of the system (neglecting the mass of the beam) is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{61,920}{16,000/386}} = 38.65 \text{ rad/ sec},$$

The forced frequency is

$$\varpi = \frac{300 \times 2\pi}{60} = 31.41 \text{ rad/ sec}$$

and the frequency ratio

$$r = \frac{\varpi}{\omega} = \frac{31.41}{38.65} = 0.813$$

Referring the Fig. 3.6, let *m* be the total mass of the motor and *m*' the unbalanced rotating mass. Then, if *u* is the vertical displacement from the equilibrium position of the non-rotating mass (*m* - *m*'), the displacement u_1 of the eccentric mass *m*' as shown in Fig. 3.6 is

$$u_1 = u + e_0 \sin \varpi t \tag{a}$$

The equation of motion in then obtained by summing forces along the vertical direction in the free body diagram of Fig. 3.6b, where the inertial forces of both the nonrotating mass and of the eccentric mass are also shown. This summation yields

$$(m - m')\ddot{u} + m'\ddot{u}_1 + c\dot{u} + ku = 0 \tag{b}$$

in which $m' = W'/g$ is the eccentric mass.

Substitution of $\ddot{u}_1$ obtained from Eq. (a) gives

$$(m - m')\ddot{u} + m'\left(\ddot{u} - e_0\overline{\omega}^2 \sin \overline{\omega} t\right) + c\dot{u} + ku = 0 \tag{b}$$

and with a rearrangement of terms

$$m\ddot{u} + c\dot{u} + ku = m'e_0\overline{\omega}^2 \sin \overline{\omega} t \tag{c}$$

This last equation is of the same form as the equation of motion (3.10) for the damped oscillator excited harmonically by a force of amplitude

$$F_0 = m'e_0\overline{\omega}^2 \tag{d}$$

Substituting in Eq. (d) the numerical values for this example, we obtain

$$F_0 = (40)(10)(31.41)^2/386 = 1022 \text{ lb}$$

From Eq. (3.20), the amplitude of the steady-state resulting motion is then

$$U = \frac{1022/61,920}{\sqrt{\left(1 - 0.813^2\right)^2 + (2 \times 0.813 \times 0.1)^2}}$$

$$U = 0.044 \text{ in} \tag{Ans}$$

Illustrative Example 3.2

The steel frame shown in Fig. 3.7 supports a rotating machine that exerts a horizontal force at the girder level $F(t) = 200 \sin 5.3t$ lb. Assuming 5% of critical damping, determine: (a) the steady-state amplitude of vibration and (b) the maximum dynamic stress in the columns. Assume the girder is rigid.

Solution:

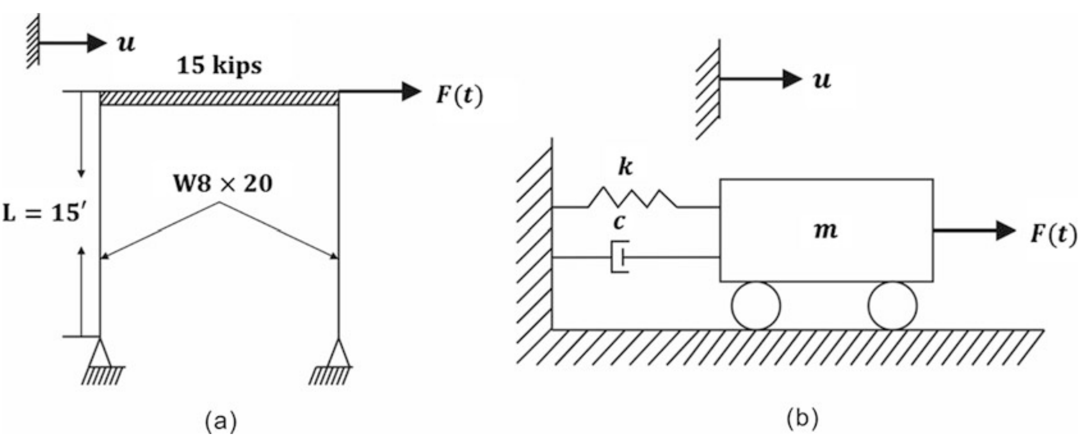

Fig. 3.7 (**a**) Diagram of the frame for Illustrative Example 3.2 (**b**) Analytical model

This structure may be modeled for dynamic analysis as the damped oscillator shown in Fig. 3.7b. The parameters in this model are computed as follows[1]:

$$k = \frac{3E(2I)}{L^3} = \frac{3 \times 30 \times 10^6 \times 2 \times 69.2}{(12 \times 15)^3} = 2136 \text{ lb/in}$$

$$\xi = 0.05$$

$$u_{st} = \frac{F_0}{k} = \frac{200}{2136} = 0.0936 \text{ in}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2136 \times 386}{15,000}} = 7.41 \text{ rad/sec}$$

$$r = \frac{\overline{\omega}}{\omega} = \frac{5.3}{7.41} = 0.715$$

The steady-state amplitude from Eqs. (3.19) and (3.20) is

$$U = \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = 0.189 \text{ in}$$

Then, the maximum shear force is the columns is

$$V_{\max} = \frac{3EIU}{L^3} = 201.8 \text{ lb}$$

The maximum bending moment

$$M_{\max} = V_{\max} L = 36,324 \text{ lb} \cdot \text{in}$$

and the maximum stress

$$\sigma_{\max} = \frac{M_{\max}}{I/c} = \frac{36,324}{17} = 2136 \text{ psi} \qquad \text{(Ans)}$$

in which I / c is the section modulus.

3.3 Evaluation of Damping at Resonance

We have seen in Chap. 2 that the free-vibration decay curve permits the evaluation of damping of a single-degree-of freedom system by simply calculating the logarithm decrement and using Eq. (2.28) or Eq. (2.29). Another technique for determining damping is based on observations of steady-state harmonic response, which requires harmonic excitations of the structure in a range of frequencies in the neighborhood of resonance. With the application of a harmonic force $F_0 \sin \overline{\omega} t$ at closely spaced values of frequencies, the response curve for the structure can be plotted, resulting in displacement amplitudes as a function of the applied frequencies. A typical response curve for such a moderately

[1] *A unit displacement at the top of pinned supported columns requires a force equal to 3 EI/L^3.

damped structure is shown in Fig. 3.8. It is seen from Eq. (3.24) that, at resonance, the damping ratio is given by

$$\xi = \frac{1}{2D(r=1)} \tag{3.25}$$

where $D(r=1)$ is the dynamic magnification factor evaluated at resonance. In practice, the damping ratio ξ is determined from the dynamic magnification factor evaluated at the maximum amplitude, namely

$$\xi = \frac{1}{2D_m} \tag{3.26}$$

where

$$D_m = \frac{U_m}{u_{st}}$$

and U_m is the maximum amplitude. The error involved in evaluating the damping ratio ξ using the approximate Eq. (3.26) is not significant in ordinary structures. This method of determining the damping ratio requires only some simple equipment to vibrate the structure in a range of frequencies that span the resonance frequency and a transducer for measuring amplitudes; nevertheless, the evaluation of the static displacement $u_{st} = F_0 \;/\; k$ may present a problem since, frequently, it is difficult to apply a static lateral load to the structure.

3.4 Bandwidth Method (Half-Power) to Evaluate Damping

An examination of the response curves in Fig. 3.3 shows that the shape of these curves is controlled by the amount of damping present in the system; in particular, the bandwidth, that is, the difference between two frequencies corresponding to the same response amplitude, is related to the damping in the system. A typical frequency amplitude curve obtained experimentally for a moderately damped structure is shown in Fig. 3.8. In the evaluation of damping it is convenient to measure the bandwidth at $1/\sqrt{2}$ of the peak amplitude as shown in this figure. The frequencies corresponding in this bandwidth f_1 and f_2 are also referred to as half-power points and are shown in Fig. 3.8. The values of the frequencies for this bandwidth can be determined by setting the response amplitude in Eq. (3.20) equal to $1/\sqrt{2}$ times the resonant amplitude given by Eq. (3.24), that is

$$\frac{u_{st}}{\sqrt{\left(1-r^2\right)^2 + (2r\,\xi)^2}} = \frac{1}{\sqrt{2}}\frac{u_{st}}{2\xi}$$

Squaring both sides and solving for the frequency ratio results in

$$r^2 = 1 - 2\xi^2 \pm 2\xi\sqrt{1+\xi^2}$$

or by neglecting ξ^2 in the square root term

$$r_1^2 \cong 1 - 2\xi^2 - 2\xi$$
$$r_2^2 \cong 1 - 2\xi^2 + 2\xi$$
$$r_1 \cong 1 - \xi - \xi^2$$
$$r_2 \cong 1 + \xi - \xi^2$$

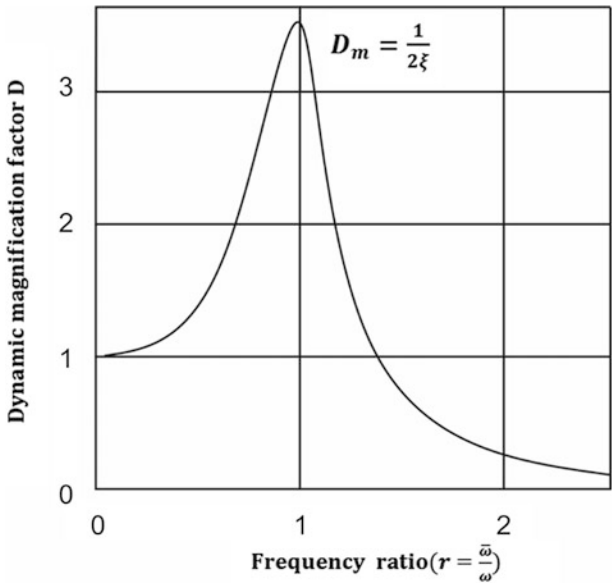

Fig. 3.8 Frequency response curve for moderately damped system

Finally, the damping ratio is given approximately by half the difference between these half-power frequency ratios, namely

$$\xi = \frac{1}{2}(r_2 - r_1)$$

or

$$\xi = \frac{1}{2}\frac{\omega_2 - \omega_1}{\omega} = \frac{f_2 - f_1}{f_2 + f_1} \tag{3.27}$$

since

$$\frac{\omega_2 - \omega_1}{2\omega} = \frac{f_2 - f_1}{2f} \qquad \text{and} \qquad f \approx \frac{f_1 + f_2}{2}$$

Illustrative Example 3.3
Experimental data for the frequency response of a single degree-of-freedom system are plotted in Fig. 3.9. Determine the damping ratio of this system.

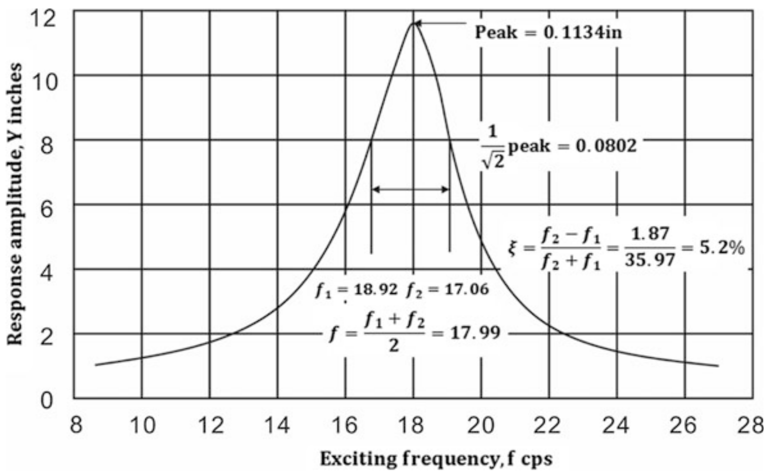

Fig. 3.9 Experimental frequency response curve of Illustrative Example 3.3

Solution:
From Fig. 3.9, the peak amplitude is 0.1134 in. hence the amplitude at half-power is equal to

$$0.1134/\sqrt{2} = 0.0802 \text{ in}$$

The frequencies at this amplitude obtained from Fig. 3.9 are

$$f_1 = 17.05 \ cps$$
$$f_2 = 18.92$$

the damping ratio is then calculated from Eq. (3.27) as

$$\xi \cong \frac{f_2 - f_1}{f_2 + f_1}$$
$$\xi \cong \frac{18.92 - 17.05}{18.92 + 17.05} = 5.2\% \qquad \text{(Ans)}$$

3.5 Energy Dissipated by Viscous Damping

The energy E_D dissipated by viscous damping during one cycle of harmonic vibration of frequency $\overline{\omega}$ is equal to the work done by the damping force $c\dot{u}$ during a differential displacement du integrated over one period of vibration $T = 2\pi/\overline{\omega}$.

Hence,

$$E_D = \int_0^{2\pi/\overline{\omega}} (c\dot{u})\, du = \int_0^{2\pi/\overline{\omega}} (c\dot{u}) \frac{du}{dt}\, dt = \int_0^{2\pi/\overline{\omega}} c\dot{u}^2\, dt \tag{3.28}$$

The velocity $\dot{u} = \dot{u}(t)$ for the damped oscillator acted upon by the harmonic force, $F = F_0 \sin \overline{\omega} t$, is given by the derivative of Eq. (3.19) as

$$\dot{u}(t) = U\overline{\omega}\cos\left(\overline{\omega}t - \theta\right) \tag{3.29}$$

which substituted in Eq. (3.28) gives

$$E_D = cU^2\overline{\omega}^2 \int_0^{2\pi/\overline{\omega}} \cos^2\left(\overline{\omega}t - \theta\right)dt = \pi c\overline{\omega}U^2$$

$$E_D = 2\pi\xi rkU^2 \tag{3.30}$$

where as previously defined

$$\xi = \frac{c}{c_{cr}}, \qquad r = \frac{\overline{\omega}}{\omega}, \qquad \omega = \sqrt{\frac{k}{m}}, \qquad \text{and} \qquad c_{cr} = 2\sqrt{km} \tag{3.31}$$

Equation (3.30) shows that the energy dissipated by viscous damping is proportional to the square of the amplitude of the motion U. It can be shown (see Problem 3.1) that during one cycle, the work W_F of the external force $F = F_0 \sin\overline{\omega}t$ is precisely the equal to the energy E_{D} dissipated by the damping force as expressed by Eq. (3.30).

3.6 Equivalent Viscous Damping

As mentioned in the introductory sections of Chap. 2, the mechanism by which structures dissipate energy during vibratory motion is usually assumed to be viscous. This assumption provides the enormous advantage that the differential equation of motion remains linear for damped dynamic systems vibrating in the elastic range. Only for some exceptionally situations such as the use of frictional devices installed in buildings to ameliorate damage resulting from strong motion earthquakes, viscous damping is usually assumed to account for frictional or damping forces in structural dynamics. The numerical value assigned to the damping coefficient is based on values obtained experimentally and the determination of an equivalent viscous damping.

The concept of equivalent viscous damping is based on test results obtained using harmonic forces. Thus, in reference to the experimental frequency response plot in Fig. 3.9, the equivalent damping ξ_{eq} may be based on the maximum relative amplitude of motion, $D_m = U_m/u_{st}$, or on the bandwidth corresponding to frequencies f_1 to f_2 at amplitude equal to $D_m/\sqrt{2}$. Thus, the equivalent viscous damping ξ_{eq} may be calculated from Eq. (3.26) as

$$\xi_{eq} = \frac{u_{st}}{2U_m} \tag{3.32}$$

or from Eq. (3.27) as

$$\xi_{eq} = \frac{f_2 - f_1}{f_2 + f_1} \tag{3.33}$$

Alternatively, the equivalent viscous damping ξ_{eq} may also be evaluated experimentally using the expression for the logarithmic decrement, δ, from Eq. (2.28) or approximately from Eq. (2.29) as

$$\xi_{eq} = \frac{\delta}{2\pi} \tag{3.34}$$

However, the most common definition of equivalent viscous damping is based on equating the energy dissipated, in a vibratory cycle of the actual structure, to the energy dissipated in an equivalent viscous system. Hence, equating the energy, E^*, dissipated in a cycle of harmonic vibration determined from experiment to the energy E_D, dissipated by an equivalent viscous system given by Eq. (3.30) we have

$$E^* = 2\pi \xi_{eq} rkU^2$$

and

$$\xi_{eq} = \frac{E^*}{2\pi rkU^2}$$

or

$$\xi_{eq} = \frac{1}{4\pi r}\frac{E^*}{E_s} \tag{3.35}$$

in which E_s, the strain energy stored at maximum displacement if the system were elastic, which is given by

$$E_s = \frac{1}{2}kU^2 \tag{3.36}$$

In the determination of the energy, E^* dissipated per cycle and the elastic energy, E_s stored at the maximum displacement, an experiment is conducted by vibrating the structure at the resonant frequency for which $r = \overline{\omega}/\omega = 1$. At this frequency, damping in the system has a maximum effect. With appropriate test equipment and measuring instrumentation, the restoring force and displacement during a cycle of vibration are measured to obtain a plot of the type shown in Fig. 3.10. The area enclosed in the loop during one cycle of vibration is equal to the energy dissipated, E^* and the triangular area corresponding to the amplitude U is equal to the strain energy, E_s. Consequently, the equivalent viscous damping is evaluated by Eq. (3.35) from the experimental results E^* and E_s, with $r = 1$, obtained from resisting force-displacement plot.

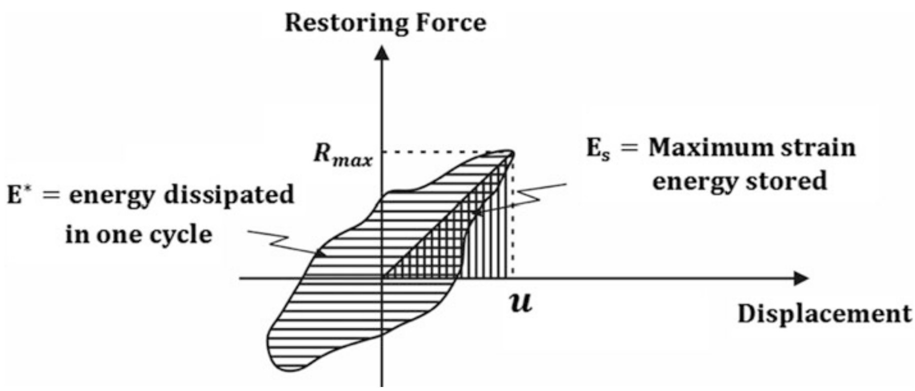

Fig. 3.10 Restoring force vs. displacement during a cycle of vibration showing the energy dissipated E^* (area within the loop) and the maximum energy stored (triangular area under maximum displacement)

Illustrative Example 3.4

Laboratory tests on a structure modeled by the damped spring-mass system [Fig. 3.11a], are conducted to evaluate equivalent viscous damping using (a) peak amplitude and (b) energy dissipated. The experimental restoring force-displacement plot at resonance is shown in Fig. 3.11b.

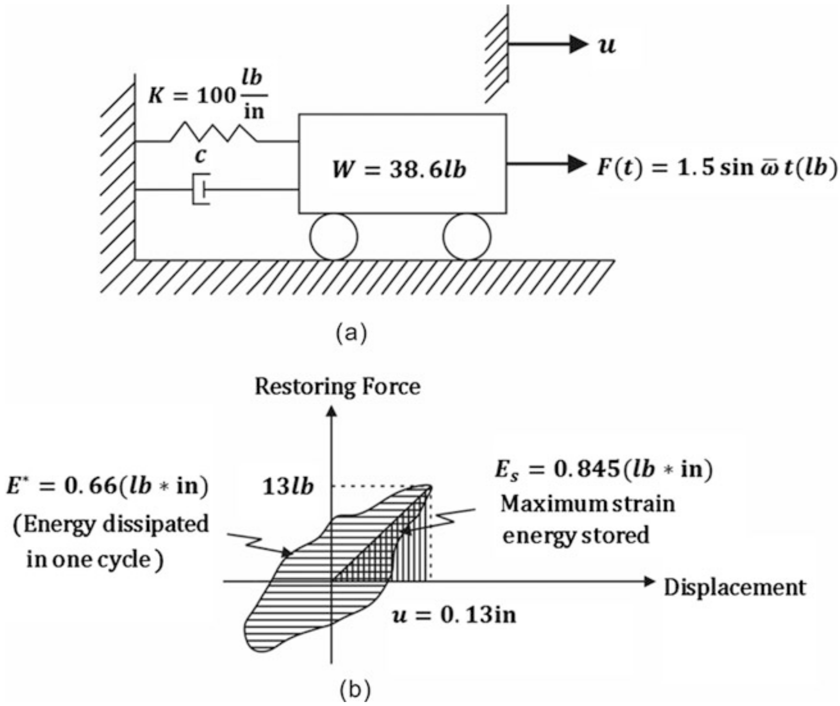

Fig. 3.11 (**a**) Analytical model for Illustrative Example 3.4; (**b**) Force-displacement plot

Solution:
The static deflection, the mass, and the natural frequency are:

$$u_{st} = \frac{F_0}{k} = \frac{1.5}{100} = 0.015 \text{ in} \qquad m = \frac{W}{g} = \frac{38.6}{386} = 0.1 \left(\text{lb} \cdot \text{sec}^2/\text{in}\right)$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{0.1}} = 31.62 \text{ rad/sec}$$

(a) Equivalent viscous damping calculated from peak amplitude:

$$\xi_{eq} = \frac{u_{st}}{2U_m} = \frac{0.015}{(2))(0.13)} = 0.0576 = 5.7\% \qquad \text{by Eq. (3.32)}$$

(b) Equivalent viscous damping calculated from energy dissipated $E^* = 0.66$ (lb.in) and elastic energy at maximum displacement, $E_s = 0.845$ (lb.in), as shown in Fig. 3.11: (at resonance, $r = 1.0$)

$$\xi_{eq} = \frac{1}{4\pi r}\frac{E^*}{E_s} = \frac{0.66}{4\pi(1.0)0.845} = 0.0621 = 6.21\% \qquad \text{by Eq. (3.35)}$$

3.7 Response to Support Motion

3.7.1 Absolute Motion

There are many actual cases where the foundation or support of a structure is subjected to time varying motion. Structures subjected to ground motion by earthquakes or other excitations such as explosions or dynamic action of machinery are examples in which support motions may have to be considered in the analysis of dynamic response. Let us consider in Fig. 3.12 the case where the support of the simple oscillator modeling the structure is subjected to a harmonic motion given by the expression

$$u_s(t) = u_0 \sin \overline{\omega} t \tag{3.37}$$

where u_0 is the maximum amplitude and $\overline{\omega}$ is the frequency of the support motion. The differential equation of motion is obtained by setting equal to zero the sum of the forces (including the inertial force) in the corresponding free body diagram shown in Fig. 3.12b. The summation of the forces in the horizontal direction gives

$$m\ddot{u} + c(\dot{u} - \dot{u}_s) + k(u - u_s) = 0 \tag{3.38}$$

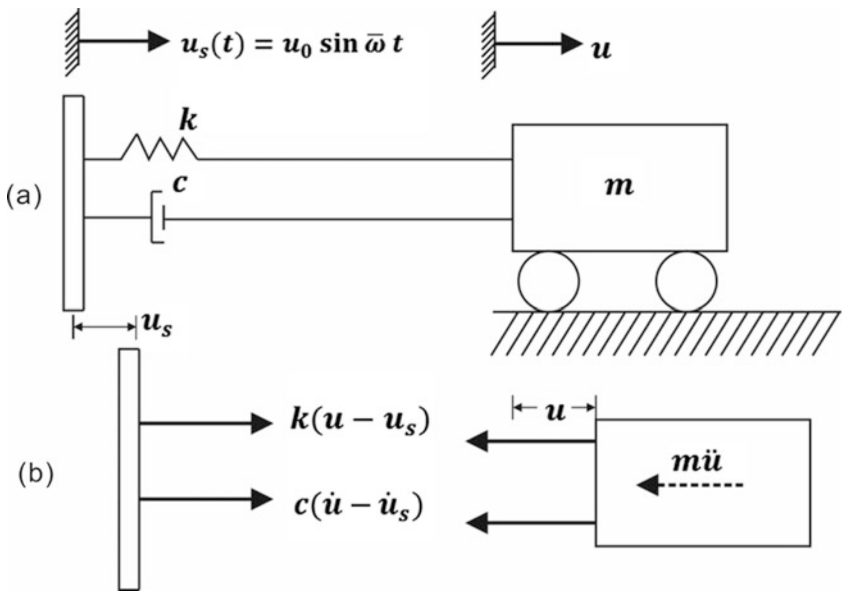

Fig. 3.12 (**a**) Damped simple oscillator harmonically excited through its support, (**b**) Free body diagram including inertial force

The substitution of Eq. (3.37) into Eq. (3.38) and the rearrangement of terms result in

$$m\ddot{u} + c\dot{u} + ku = ku_0 \sin \overline{\omega} t + c\overline{\omega} u_0 \cos \overline{\omega} t \tag{3.39}$$

The two harmonic terms of frequency $\overline{\omega}$ in the right-hand side of this equation may be combined and Eq. (3.39) rewritten [similarly to Eqs. (1.20) and (1.23)] as

$$m\ddot{u} + c\dot{u} + ku = F_0 \sin(\overline{\omega} t + \beta) \tag{3.40}$$

where

$$F_0 = u_0\sqrt{k^2 + (c\overline{\omega})^2} = u_0 k\sqrt{1 + (2r\xi)^2} \tag{3.41}$$

and

$$\tan \beta = c\varpi/k = 2r\xi \tag{3.42}$$

It is apparent that the differential Eq. (3.40) is of the same form as Eq. (3.10) for the oscillator excited by the harmonic force $F_0 \sin\left(\varpi t + \beta\right)$. Consequently, the steady-state solution of Eq. (3.40) is given as before by Eqs. (3.19) and (3.20), except for the addition of the angle β in the argument of the sine function, that is

$$u(t) = \frac{F_0/k \sin\left(\varpi t + \beta - \theta\right)}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} \tag{3.43}$$

or substituting F_0 from Eq. (3.41)

$$\frac{u(t)}{u_0} = \frac{\sqrt{1 + \left(2r\xi\right)^2}}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} \sin\left(\varpi t + \beta - \theta\right) \tag{3.44}$$

Equation (3.44) is the expression for the relative transmission of the support motion to the oscillator. This is an important problem in vibration isolation in which equipment must be protected from harmful vibrations of the supporting structure. The degree of relative isolation is known as transmissibility and is defined as the ratio of the amplitude of motion U of the oscillator to the amplitude u_0, the motion of the support. From Eq. (3.44), the transmissibility T_r is then given by

$$T_r = \frac{U}{u_0} = \frac{\sqrt{1 + \left(2r\xi\right)^2}}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} \tag{3.45}$$

Analogously to the motion transmitted, we may find the acceleration transmitted from the foundation to the mass. The acceleration transmitted to the mass is given by the second derivative of $u(t)$ in Eq. (3.44) as

$$\ddot{u}(t) = \frac{-\varpi^2 u_0 \sqrt{1 + \left(2r\xi\right)^2} \sin\left(\varpi t + \beta - \theta\right)}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} \tag{3.46}$$

while the acceleration $\ddot{u}_s(t)$ of the foundation is obtained from Eq. (3.37)

$$\ddot{u}_s(t) = -u_0 \varpi^2 \sin \varpi t \tag{3.47}$$

The acceleration transmissibility, T_r, is then given by the ratio of the amplitudes of the acceleration in Eqs. (3.46) and (3.47). Hence,

$$T_r = \frac{\sqrt{1 + \left(2r\xi\right)^2}}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)}} \tag{3.48}$$

It may be seen that the transmissibility of acceleration given by Eq. (3.48) is identical to Eq. (3.45), the transmissibility of displacements. Hence, the same expression will give either displacement or acceleration transmissibility.

A plot of transmissibility as a function of the frequency ratio and damping ratio is shown in Fig. 3.13. The curves in this figure are similar to the curves in Fig. 3.3, representing the frequency response of the damped oscillator. The major difference between these two sets of curves is that all of the curves in Fig. 3.13 pass through the same point at a frequency ratio $r = \sqrt{2}$. It can be seen in Fig. 3.13 that damping tends to reduce the effectiveness of vibration isolation for frequencies greater than this ratio, that is, for r greater than $\sqrt{2}$.

Plot Figs. 3.3, 3.4 and 3.13

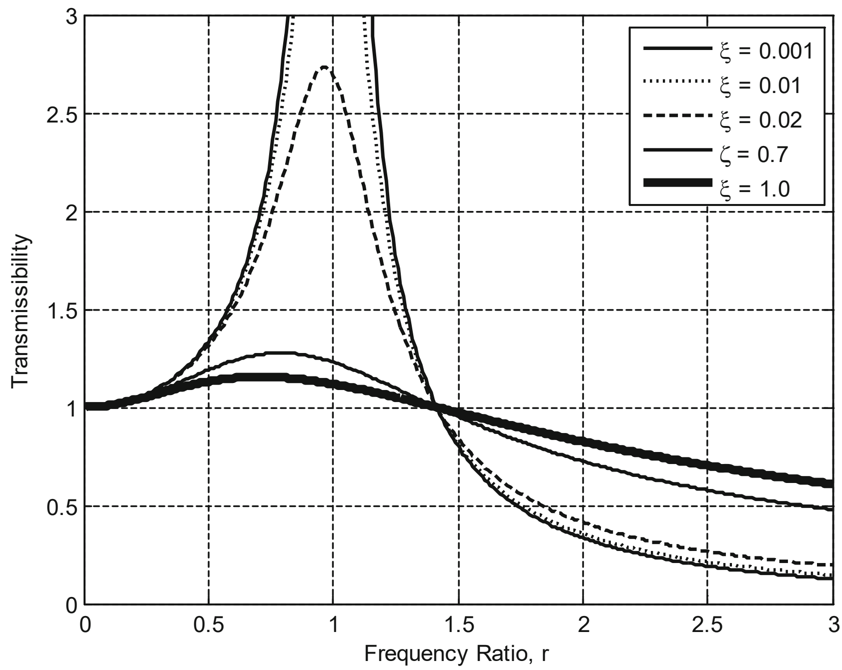

Fig. 3.13 Transmissibility versus frequency ratio for vibration isolation

```
clear all
clc
%%%%-GIVEN VALUES-%%%%
r = 0:0.01:3.0;                    %Frequency ratio ranging 0 to 3 with 0.01 intervals
xi = [0.001, 0.1, 0.2, 0.7, 1.0];   %Damping ratios of 0.001, 0.01, 0.02, 0.7, and 1.0

%%%-ESTIMATION-%%%
for i = 1:5
    z = xi(i);
    denom1 = (1-r.*r).^2;
    denom2 = (2*z*r).^2;
    denom = sqrt(denom1+denom2);
        D(i,:) = 1./denom;              %Dynamic amplification factor (Eq. 3.24)

    denom3 = 2*xi(i).*r;
    denom4 = (1-r.^2);
    theta(i,:) = atand(denom3./denom4);   %Phase angle (Eq. 3.22)

    denom5 = sqrt(1+denom3.^2);
    T(i,:) =  denom5./denom;            %Transmissibility (Eq. 3.45)
end

%%%Create figures (Figs. 3.3, 3.4, and 3.13)
figure1 = figure;
```

```
%%%Fig. 3.3
    axes1= axes('Parent', figure1);
xlim(axes1, [0 3]);                       %set x limits for the plot
ylim(axes1, [0 6]);                       %set y limits for the plot

    box (axes1,'on');
    grid (axes1,'on');
    hold (axes1, 'all');

    plot1 = plot(r, D, 'Parent', axes1, 'LineWidth', 2, 'Color', [0 0 0]);
    set(plot1(1),'LineStyle','-','DisplayName','\xi = 0.001');
    set(plot1(2),'LineStyle',':','DisplayName','\xi = 0.01');
    set(plot1(3),'LineStyle','--','DisplayName','\xi = 0.02');
    set(plot1(4),'DisplayName','\zeta = 0.7');
    set(plot1(5),'LineWidth',4,'DisplayName','\xi = 1.0');

xlabel ('Frequency Ratio, r');
ylabel ('Dynamic Magnification Factor, D');
    legend (axes1, 'show');
    grid on

figure2 = figure;

%%%Fig. 3.4
    axes1= axes('Parent', figure2);
xlim(axes1, [0 3]);                        %set x limits for the plot
ylim(axes1, [0 180]);                      %set y limits for the plot

    box (axes1,'on');
    grid (axes1,'on');
    hold (axes1, 'all');

    theta(theta<0)=theta(theta<0)+180;
    plot2 = plot(r, theta, 'Parent', axes1, 'LineWidth', 2, 'Color', [0 0 0]);
    set(plot2(1),'LineStyle','-','DisplayName','\xi = 0.001');
    set(plot2(2),'LineStyle',':','DisplayName','\xi = 0.01');
    set(plot2(3),'LineStyle','--','DisplayName','\xi = 0.02');
    set(plot2(4),'DisplayName','\zeta = 0.7');
    set(plot2(5),'LineWidth',4,'DisplayName','\xi = 1.0');

xlabel ('Frequency Ratio, r');
ylabel ('Phase angle, \theta');
    legend (axes1, 'show');
    grid on

figure3 = figure;

%%%Fig. 3.13
    axes1= axes('Parent', figure3);
xlim(axes1, [0 3]);                        %set x limits for the plot
ylim(axes1, [0 3]);                        %set y limits for the plot

    box (axes1,'on');
    grid (axes1,'on');
    hold (axes1, 'all');

    plot2 = plot(r, T, 'Parent', axes1, 'LineWidth', 2, 'Color', [0 0 0]);
    set(plot2(1),'LineStyle','-','DisplayName','\xi = 0.001');
    set(plot2(2),'LineStyle',':','DisplayName','\xi = 0.01');
    set(plot2(3),'LineStyle','--','DisplayName','\xi = 0.02');
    set(plot2(4),'DisplayName','\zeta = 0.7');
    set(plot2(5),'LineWidth',4,'DisplayName','\xi = 1.0');

xlabel ('Frequency Ratio, r');
ylabel ('Transmissibility');
legend (axes1, 'show');
grid on
```

Illustrative Example 3.5

A delicate instrument weighing 100 lb is to be mounted on a rubber pad to the floor of a test laboratory where the vertical acceleration is 0.1 g at a frequency of $f = 10$ cps. It has been determined experimentally that the ratio of the stiffness, k, to the damping coefficient, c, is equal to 100 (1/sec) for the type of rubber pad material used in the isolation. What is the stiffness of the isolation required to reduce to 0.01 g the acceleration transmitted to the instrument?

Solution:

Setting the acceleration transmissibility given by Eq. (3.48) equal to $0.01g\,/\,0.1g = 0.1$, we have

$$T_r = \frac{\sqrt{1 + (2r\xi)^2}}{\sqrt{(1-r)^2 + (2r\xi)^2}} = 0.1 \tag{a}$$

Beginning with an assumed value $\xi = 0.10$ for the damping ratio and squaring both sides of Eq. (a):

$$\frac{1 + 0.04r^2}{(1-r^2)^2 + 0.04r^2} = 0.01$$

Then solving this resulting quadratic equation yields

$$\begin{aligned}
&r^2 = 13.346\\
&r = \overline{\omega}/\omega = 3.653\\
&\overline{\omega} = 2\pi f = 2\pi\,10 = 62.83\ \text{rad/sec}\\
&\omega = \overline{\omega}/r = 17.20\ \text{rad/sec}\\
&m = 100/386 = 0.259\ \text{lb}\cdot\text{sec}^2/\text{in}\\
&k = m\omega^2 = 0.259 \times 17.20^2 = 76.64\ \text{lb/in}
\end{aligned}$$

Now, we check the value of damping contained in the rubber spring:

$$k/c = 100$$

or

$$c = k/100 = 76.22/100 = 0.766\ (\text{lb}\cdot\text{sec/in})$$

Critical damping:

$$c_{cr} = 2\sqrt{km} = 2\sqrt{76.64 \times 0.259} = 8.91\ (\text{lb}\cdot\text{sec/in})$$

Then, the calculated damping ratio is

$$\xi = c/c_{cr} = 0.766/8.91 = 0.086$$

which is somewhat less that the assumed value $\xi = 0.10$. If desired, an iterative cycle could be performed introducing $\xi = 0.086$ in Eq. (a) and repeating the calculations.

3.7.2 Relative Motion

Equation (3.43) provides the absolute response of the damped oscillator to a harmonic motion of its base. Alternatively, we can solve the differential Eq. (3.38) in terms of the relative motion between the mass m and the support given by

$$u_r = u - u_s \tag{3.49}$$

which substituted into Eq. (3.38) results in

$$m\ddot{u}_r + c\dot{u}_r + ku_r = F_{eff} \sin \overline{\omega}\, t \tag{3.50}$$

where $F_{eff} = -m\ddot{u}_s$ may be interpreted as the amplitude of the effective force acting on the mass of the oscillator with the displacement indicated by coordinate u_r. Using Eq. (3.37) to obtain $\ddot{u}_s$ and substituting in Eq. (3.50) results in

$$m\ddot{u}_r + c\dot{u}_r + ku_r = mu_0\overline{\omega}^2 \sin \overline{\omega} t \tag{3.51}$$

Again, Eq. (3.51) is of the same form as Eq. (3.10) with $F_0 = mu_0\overline{\omega}^2$. Then, from Eqs. (3.19) and (3.20), the steady-state response in terms of relative motion is given by

$$\frac{u_r(t)}{u_0} = \frac{r^2 \sin\left(\overline{\omega} t - \theta\right)}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} \tag{3.52}$$

where θ is given in Eq. (3.21),

$$r^2 = \frac{\overline{\omega}^2}{\omega^2} \quad \text{and} \quad \omega^2 = \frac{k}{m}.$$

The maximum relative amplitude of the displacement U_r,

$$U_r = \frac{u_0 r^2}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} \tag{3.53}$$

Illustrative Example 3.6

If the frame of Illustrative Example 3.2 (Fig. 3.7) is subjected to a sinusoidal ground motion $u_s(t) = 0.2 \sin 5.3t$, determine: (a) the transmissibility of motion to the girder, (b) the maximum shearing force in the supporting columns, and (c) maximum stresses in the columns.

Solution:

(a) The parameters for this system are calculated in Illustrative Example 3.2 as

$$\begin{aligned} k &= 2136 \text{ lb/in} \\ \xi &= 0.05 \\ u_0 &= 0.2 \text{ in} \\ u_{st} &= 0.0936 \text{ in} \\ \omega &= 7.41 \text{ rad/sec} \\ \overline{\omega} &= 5.3 \text{ rad/sec} \\ \mathrm{r} &= 0.715 \end{aligned}$$

The transmissibility from Eq. (3.45) is

$$T_r = \sqrt{\frac{1 + \left(2r\xi\right)^2}{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} = 2.1 \tag{Ans}$$

(b) The maximum relative amplitude of the displacement U_r is from Eq. (3.53)

$$U_r = \frac{u_0 r^2}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = 0.206 \text{ in}$$

Then the maximum shear force in each column is

$$V_{\max} = \frac{kU_r}{2} = 219.8 \text{ lb} \qquad \text{(Ans)}$$

(c) The maximum bending moment

$$M_{\max} = V_{\max} L = 39,507 \text{ lb} \cdot \text{in}$$

and the corresponding stress

$$\sigma_{\max} = \frac{M_{\max}}{I/c} = \frac{39,567}{17} = 2327 \text{ psi} \qquad \text{(Ans)}$$

in which I / c is the section modulus.

Illustrative Example 3.7

A machine having a total weight of 1800 lb, including its foundation, is to be isolated from the vibration of the ground, which is $f = 22.8$ cps, due to other machines operating nearby. Determine the stiffness of a rubber isolation spring to limit the transmitted vibration to 1/10: (a) neglect damping and (b) consider damping given by the expression $c = k/170$ obtained experimentally [units of c (lb.sec/in) and of k (lb/in)].

Solution:

(a) $\xi = 0$

By Eq. (3.45) with $\xi = 0$

$$T_r = \frac{U}{u_0} = \frac{1}{\pm(1-r^2)} = 0.1$$

$$-1 + r^2 = 10 \qquad r^2 = 11$$

$$r = 3.3166 = \frac{\overline{\omega}}{\omega}$$

$$\overline{\omega} = 2\pi f = 143.24 \text{ rad/sec}$$

$$m = \frac{1800}{386} = 4.663 \text{ lb} \cdot \text{sec}^2/\text{in}$$

$$\omega = \frac{\overline{\omega}}{r} = \frac{143.24}{3.3166} = 43.188 \text{ rad/sec}$$

Then,

$$k = \omega^2 m = (43.188)^2 (4.663) = 8698 \text{ lb/in} \qquad \text{(Ans.)}$$

(b) Assume $\xi = 0.10$
(c) Squaring Eq. (3.45) and substituting $\xi = 0.10$ gives

$$T_r^2 = \frac{1 + (0.2r)^2}{(1 - r^2)^2 + (0.2r)^2} = \left(\frac{1}{10}\right)^2$$

which results in the following quadratic equation in r^2:

$$r^4 - 5.96r^2 - 99 = 0$$

$$r^2 = 13.366 \qquad \mathrm{r} = \frac{\overline{\omega}}{\omega} = 3.65$$

and

$$\omega = \frac{\overline{\omega}}{r} = \frac{143.24}{3.65} = 39.244\ (\mathrm{rad/\,sec\,})$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{4.663}} = 39.244\ (\mathrm{rad/\,sec\,})$$

giving

$$k = 7181.3\ \mathrm{lb/in}$$

The damping is then determined from

$$c = k/170 = 7181.3/170$$

as

$$c = 42.24\ \mathrm{lb \cdot\ sec\,/in}$$

and the damping ratio as

$$c_{cr} = 2\sqrt{km} = 2\sqrt{(7181.3)(4.663)} = 348.34\ (\ \mathrm{lb \cdot\ sec\,/in})$$

$$\xi = \frac{42.24}{384.34} = 0.1212 = 12\%$$

which is slightly higher than the value $\xi = 0.10$, initially assumed. Now, repeating the calculations for $\xi = 0.11$ gives

$$r^4\ {-6.7916} - 99 = 0$$

$$r^2 = 13.909 \qquad \mathrm{r} = \frac{\overline{\omega}}{\omega} = 3.73$$

$$\omega = \frac{143.24}{3.73} = 38.40 = \sqrt{\frac{k}{m}}$$

$$k = 6876\ \mathrm{lb/in} \quad \text{and} \quad c = \frac{k}{170} = 40.45\ \mathrm{lb \cdot\ sec\,/in} \qquad \text{(Ans)}$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{6876 \times 4.663} = 358.14\ \mathrm{lb \cdot\ sec\,/in}$$

and

$$\xi = \frac{40.45}{358.14} = 0.113 = 11\%$$

This calculated value, $\xi = 11\%$ for the damping ratio, agrees with the last value tried. Therefore, the required spring constant for the damped isolation is $k = 6876$ lb/in as calculated above.

3.8 Force Transmitted to the Foundation

In the preceding section, we determined the response of the structure to a harmonic motion of its foundation. In this section we shall consider a similar problem of vibration isolation; the problem now, however, is to find the force transmitted to the foundation. Consider again the damped oscillator with a harmonic force $F(t) = F_0 \sin \overline{\omega} t$ acting on its mass as shown in Fig. 3.2. the differential equation of motion is

$$m\ddot{u} + c\dot{u} + ku = F_0 \sin \overline{\omega} t$$

with the steady-state solution, Eq. (3.19),

$$u = U \sin\left(\overline{\omega} t - \theta\right)$$

where U and θ, are given, respectively, by Eqs. (3.20) and (3.21) as

$$U = \frac{F_0/k}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} \tag{3.53}$$

and

$$\tan \theta = \frac{2\xi r}{1 - r^2}$$

The force transmitted to the support through the spring is ku and through the damping element is $c\dot{u}$. Hence the total force transmitted F_T is

$$F_T = ku + c\dot{u} \tag{3.54}$$

Differentiating Eq. (3.19) and substituting in Eq. (3.54) yields

$$F_T = U\left[k \sin\left(\overline{\omega} t - \theta\right) + c\overline{\omega} \cos\left(\overline{\omega} t - \theta\right)\right]$$

or

$$F_T = U\sqrt{k^2 + c^2\overline{\omega}^2} \sin\left(\overline{\omega} t - \theta + \beta\right) \tag{3.55}$$

$$F_T = Uk\sqrt{1 + \left(2r\xi\right)^2} \sin\left(\overline{\omega} t - \phi\right) \tag{3.56}$$

in which

$$\tan \beta = \frac{c\overline{\omega}}{k} = 2\xi r \tag{3.57}$$

and

$$\phi = \theta - \beta \tag{3.58}$$

Then, from Eqs. (3.53) and (3.56), the maximum force A_T transmitted to the foundation is

$$A_T = F_0 \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2r\xi)^2}} \tag{3.59}$$

In this case, the transmissibility T_r is defined as the ratio between the amplitude of the force transmitted to the foundation and the amplitude of the applied force. Hence from Eq. (3.59)

$$T_r = \frac{A_T}{F_0} = \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2r\xi)^2}} \tag{3.60}$$

It is interesting to note that, both, the transmissibility of motion from the foundation to the structure, Eq. (3.45), and the transmissibility of the force from the structure to the foundation, Eq. (3.60), are given by exactly the same function. Hence the curves of transmissibility in Fig. 3.13 represent either type of transmissibility. An expression for the total phase angle ϕ in Eq. (3.56) may be determined by taking the tangent function to both members of Eq. (3.58), so that

$$\tan\phi = \frac{\tan\theta - \tan\beta}{1 + \tan\theta\tan\beta}$$

Then, the substitution of $\tan\theta$ and $\tan\beta$, respectively, from Eqs. (3.21) and (3.57) results in

$$\tan\phi = \frac{2\xi r^3}{1 - r^2 + 4\xi^2 r^2} \tag{3.61}$$

Illustrative Example 3.8

A machine of weight $W = 3860$ lb is mounted on a simple supported steel beam as shown in Fig. 3.15a. A piston that moves up and down in the machine produces a harmonic force of magnitude $F_0 = 7000$ lb at a frequency of $\overline{\omega} = 60\ \text{rad/sec}$. Neglecting the weight of the beam and assuming 10% of the critical damping, determine: (a) the amplitude of the motion of the machine, (b) the force transmitted to the beam supports, and (c) the corresponding phase angle.

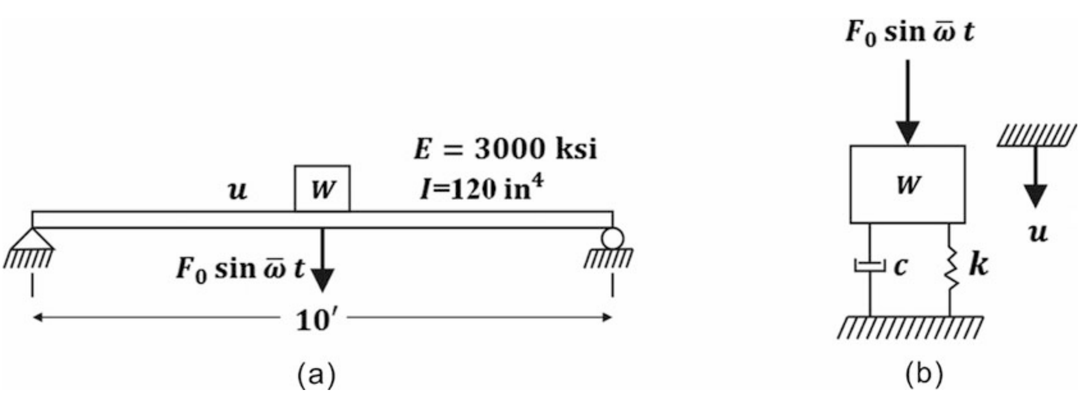

Fig. 3.14 (**a**) Beam-machine system for Illustrative Example 3.8. (**b**) Analytical Model

Solution:
The damped oscillator in Fig. 3.14b is used to model the system. The following parameters are calculated:

$$k = \frac{48EI}{L^3} = 10^5 \text{ lb/in}$$

$$\omega = \sqrt{\frac{k}{m}} = 100 \text{ rad/ sec}$$

$$\xi = 0.1$$

$$r = \frac{\overline{\omega}}{\omega} = 0.6$$

$$u_{st} = \frac{F_0}{k} = 0.07 \text{ in}$$

(a) From Eq. (3.20), the amplitude of motion is

$$U = \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = 0.1075 \text{ in} \qquad \text{(Ans)}$$

with a phase angle from Eq. (3.21)

$$\theta = \tan^{-1}\frac{2r\xi}{1-r^2} = 10.6^0$$

(b) From Eq. (3.60), the transmissibility is

$$T_r = \frac{A_T}{F_0} = \sqrt{\frac{1+(2r\xi)^2}{(1-r^2)^2 + (2r\xi)^2}} = 1.547$$

Hence the amplitude of the force transmitted to the foundation is

$$A_T = F_0 T_r = 10{,}827 \text{ lb} \qquad \text{(Ans)}$$

(c) The corresponding phase angle from Eq. (3.61) is

$$\phi = \tan^{-1}\frac{2\xi r^3}{1-r^2+(2r\xi)^2} = 3.78^\circ \qquad \text{(Ans)}$$

3.9 Seismic Instruments

When a system of the type shown in Fig. 3.15 is used for the purpose of vibration measurement, the relative displacement between the mass and the base is recorded. Such an instrument is called a seismograph and it can be designed to measure either the displacement or the acceleration of the base. The peak relative response U/u_0 of the seismograph depicted in Fig. 3.15, for harmonic motion of the base, is given from Eq. (3.52) by

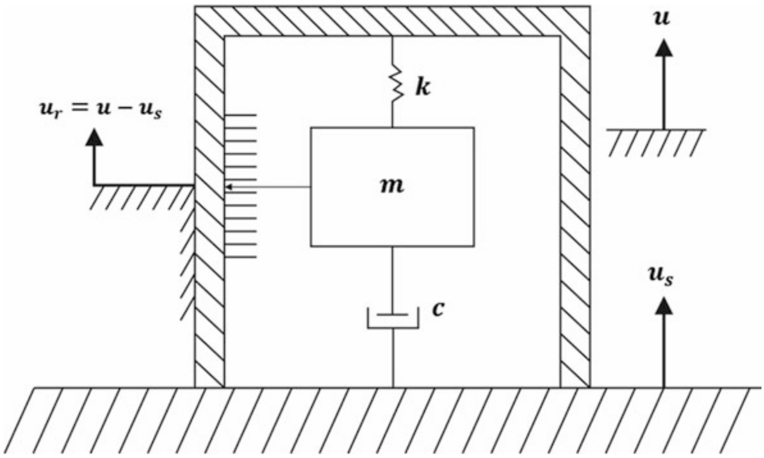

Fig. 3.15 Model of a seismograph

$$\frac{U_r}{u_0} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \tag{3.62}$$

A plot of this equation as a function of the frequency ratio and damping ratio is shown in Fig. 3.16. It may be seen from this figure that the response is essentially constant for frequency ratios $r > 1$ and damping ratio $\xi = 0.5$. Consequently, the response of a properly damped instrument of this type is essentially proportional to the base-displacement amplitude for high frequencies of motion of the base. The instrument will thus serve as a displacement meter for measuring such motions. The range of applicability of the instrument is increased by reducing the natural frequency, i.e., by reducing the spring stiffness or increasing the mass.

Now consider the response of the same instrument to a harmonic acceleration of the base $\ddot{u}_s = \ddot{u}_0 \sin \overline{\omega} t$. The equation of motion of this system is obtained from Eq. (3.50) as

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_0 \sin \overline{\omega} t \tag{3.63}$$

The steady-state response of this system expressed as the dynamic magnification factor is then given from Eq. (3.23) by

$$D = \frac{U}{m\ddot{u}_0/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \tag{3.64}$$

This equation is represented graphically in Fig. 3.3. In this case, it can be seen from this figure that for a damping ratio $\xi = 0.7$, the value of the response is nearly constant in the frequency range $0 < r < 0.6$. Thus, it is clear from Eq. (3.64) that the response indicated by this instrument will be directly proportional to the base-acceleration amplitude for frequencies up to about six-tenths of the natural frequency. Its range of applicability will be increased by increasing the natural frequency, that is, by increasing the stiffness of the spring or by decreasing the mass of the oscillator. Such an instrument is an accelerometer.

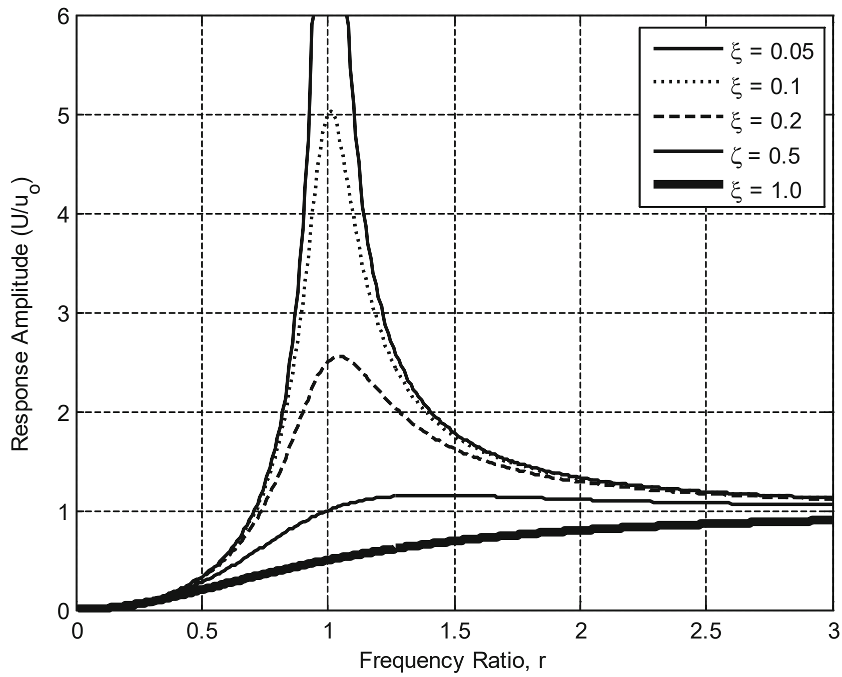

Fig. 3.16 Response of seismograph to harmonic motion of the base

3.10 Response of One-Degree-of-Freedom System to Harmonic Loading Using MATLAB

Illustrative Example 3.9

The steel frame shown in Fig. 3.17 supports a rotating machine that exerts a horizontal force at the girder level $F(t) = 200 \sin 2.3t$ (lb). The frequency of force is equal to 2.3 rad/sec. Assume 5% of critical damping and determine: (a) The maximum displacement of a total vibration between 0 to 10 sec, and (b) The maximum dynamic stress in the columns. Assume the girder is rigid. (This is the same structure in Illustrative Example 3.2) (Fig. 3.18).

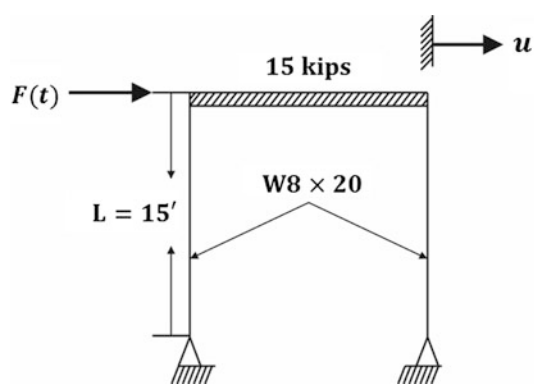

Fig. 3.17 Diagram of the frame of Illustrative Example 3.9

Solution:
The parameters for this structure were previously calculated in Illustrative Example 3.2 as:

Mass:	m = 15 × 1000/386 (lb.sec^2/in)
Stiffness:	$k = \frac{3E(2I)}{L^3} = \frac{3 \times 30 \times 10^6 \times 2 \times 69.2}{(12 \times 15)^3} = 2136\ \text{lb/in}$
Damping ratio:	$\xi = 0.05$
Amplitude Harmonic Force:	$F_0 = 200$ (lb)
Force time function:	$F(\mathrm{t}) = \sin 2.3t$
Period (of the force)	$2.3T = 2\pi$. Therefore, $T = 2.73$ sec
Time Step (Select 20 steps):	0.01 sec.

For running the files, two files are needed to save in the same folder and run main file. Ex3_9.m file will use SDOF.m file to solve partial differential equation to calculate the response. The original framework of MATLAB code is well presented in Anderson and Naeim (2012).

The approach is transform one "second order differential equation" to two "first order differential equations."

$$\frac{d\dot{u}}{dt} = -2\xi\omega\dot{u} - \omega^2 u$$

$$u = u_1; \qquad\qquad u_2 = \frac{du_1}{dt} \rightarrow u(2)\ [MATLAB]$$

$$\dot{u}_1 = u_2 = \frac{du_1}{dt} \qquad \rightarrow \qquad \frac{du_2}{dt} = -2\xi\omega u_2 - \omega^2 u_1 \rightarrow -2\xi\omega u(2) - \omega^2 u(1)\ [MATLAB]$$

$$\frac{du_2}{dt} = -2\xi\omega u_2 - \omega^2 u_1$$

Two first order differential equations are solved simultaneously in SDPF.m file.

Matlab file: Ex3_9.m

```
close all
clear
clc

%%%%-GIVEN VALUES-%%%%
%%% Initial Conditions
tspan = 0:0.01:50;              %0 to 5 secs with the interval of 0.01 sec
IC = [0 0]';                    %Initial conditions (u0=0, v0=0)

%%%Plot the displacement using ODE45
%%%Estimate response using ODE45 funtion embedded in MATALB
[t, u] = ode45(@SDOF, tspan, IC);
plot (t, u(:,1));

%%%Create xlabel
xlabel ('t(sec)');
ylabel ('u(in.)');

%%%Display maximum value of displacement response
umax=max(u(:,1))
```

MATLAB file Name: SDOF.m

```
function   u = SDOF(t, u)

%%%%-GIVEN VALUES-%%%%
m =15*1000/386;                    %Mass (lb.sec^2/in.)
k= 3*30*10^6*2*69.2/(12*15)^3;     %Stiffness (lb/in.)
xi = 0.05;                         %Damping ratio.

%%%Define the forcing function
F = 200*sin(2.3*t);                %Force as a function of time, t

omega =sqrt(k/m);                  %Natural Frequency

%%%%-ESTIMATION-%%%%
u = [u(2); -omega^2*u(1)-2*xi*omega*u(2)+F/m];
```

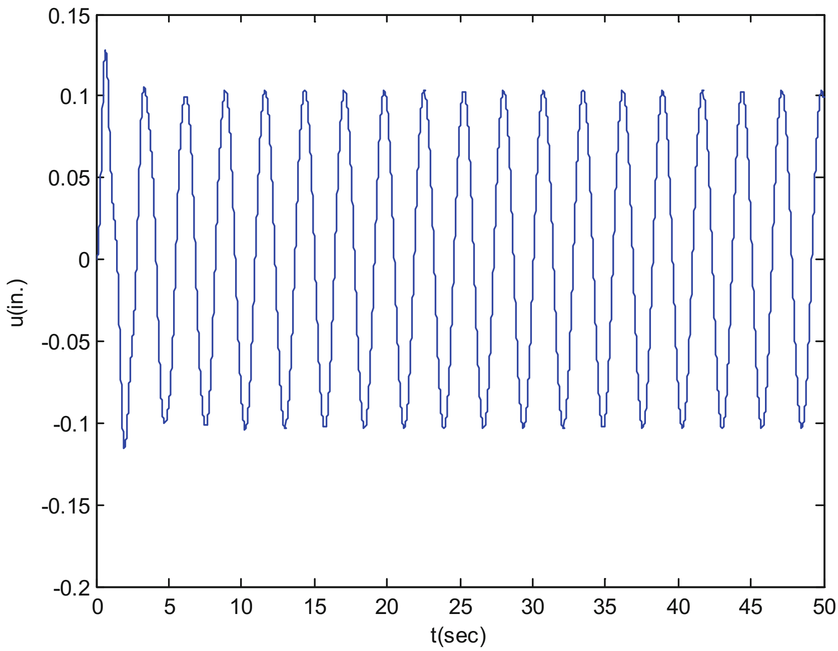

Fig. 3.18 Response of Illustrative Example 3.9

Illustrative Example 3.10

The steel frame In Illustrative Example 3.9 is supports a rotating machine that exerts a horizontal force at the girder level $F(t) = 200 \sin 7.41t$ (lb). Use MATLAB to determine the response. The solution is found using the MATLAB function based ODE 4 (A fourth-order Runge-Kutta method.). Other functions of MATLAB codes are also introduced in Anderson and Naeim (2012). More information on the numerical method can be explained in Chap. 4. In this example, the solution can be easily obtained from changes of two files (Fig. 3.19).

Note: The frequency of force is equal to the natural frequency.

Matlab file: Ex3_10.m

```
close all
clear
clc

%%%%-GIVEN VALUES-%%%%
%%% Initial Conditions
tspan = 0:0.01:50;                %0 to 5 secs with the interval of 0.01 sec
IC = [0 0]';                      %Initial conditions (u0=0, v0=0)

%%% Plot the displacement using ODE45
%%%Estimate response using ODE45 funtion embedded in MATALB
[t, u] = ode45(@SDOF1, tspan, IC);
figure (1)
plot (t, u(:,1));

% Create xlabel
xlabel ('t(sec)');
ylabel ('u(in.)');

k= 3*30*10^6*2*69.2/(12*15)^3;
F_0 = 200;

% Display maximum value of displacement response
umax=max(u(:,1))
```

MATLAB file Name: SDOF1.m

```
function  u = SDOF1(t, u)

%%%%-GIVEN VALUES-%%%%
m =15*1000/386;                 %Mass (lb.sec^2/in.)
k= 3*30*10^6*2*69.2/(12*15)^3;  %Stiffness (lb/in.)
xi = 0.05;                      %Damping coefficient. (lb.sec/in.)

%%%Define the forcing function
F_0 = 200;
F = F_0*sin(7.41*t);            %Force as a function of time, t

omega =sqrt(k/m)                %Natural frequency

u_st = F_0/k;                   %Static displacement

%%%%-ESTIMATION-%%%%
u = [u(2); -omega^2*u(1)-2*xi*omega*u(2)+F/m];
```

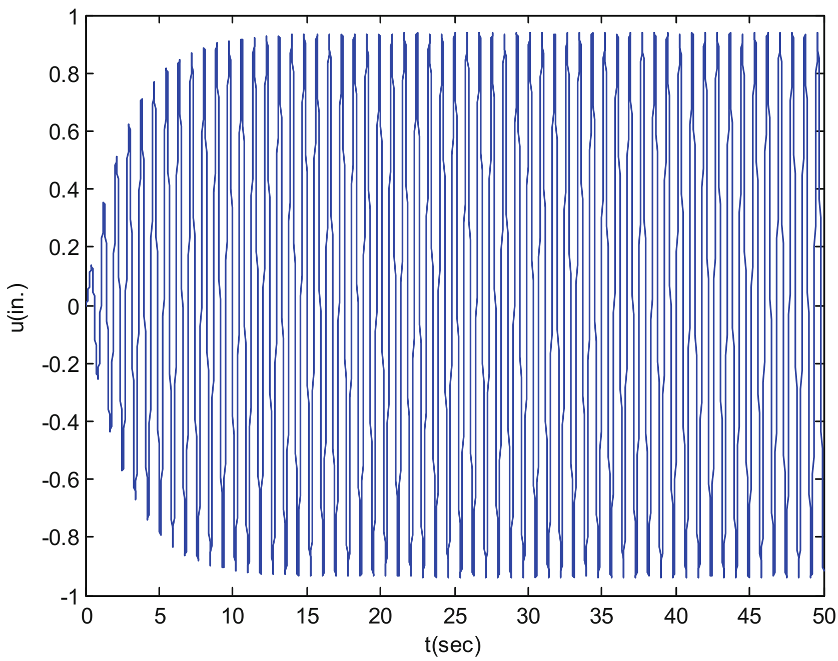

Fig. 3.19 Response of Illustrative Example 3.10

Illustrative Example 3.11

The steel frame in Illustrative Example 3.9 is now subjected to an acceleration at its base given by the function $F(t) = 0.3 \sin 5.3t$ in/sec^2. Use MATLAB to determine the response.

3.11 Summary

In this chapter, we have determined the response of a single-degree-of-freedom system subjected to harmonic loading. This type of loading is expressed as a sine, cosine, or as an exponential function and can be easily handled mathematically for the undamped or damped structure. The differential equation of motion for a linear single-degree-of-freedom system is the second-order differential equation

$$m\ddot{y} + c\dot{u} + ku = F_0 \sin \overline{\omega} t \qquad (3.10) \text{ repeated}$$

or

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = \frac{F_0}{m} \sin \overline{\omega} t$$

in which $\overline{\omega}$ is the forced frequency,

$$\xi = \frac{c}{c_{cr}} \quad \text{isthedampingratio}$$

and

$$\omega = \sqrt{\frac{k}{m}} \quad \text{is the natural frequency}$$

The general solution of Eq. (3.10) is obtained as the combination of the complementary (transient) and the particular (steady-state) solutions, namely

$$u = \underbrace{e^{-\xi\omega t}(A \cos \omega_D t + B \sin \omega_D t)}_{\text{transient solution}} + \underbrace{\frac{F_0/k \sin\left(\overline{\omega} t - \theta\right)}{\sqrt{(1-r^2)^2 + (2r\xi)^2}}}_{\text{steady state solution}}$$

in which

$$r = \frac{\overline{\omega}}{\omega} \quad \text{is the frequcency ration,}$$

$$\omega_D = \omega\sqrt{1-\xi^2} \quad \text{is the damped natural frequency,}$$

$$\theta = \tan^{-1}\left(\frac{2r\xi}{1-r^2}\right) \quad \text{is the phase angle}$$

and

A and *B* are constants of integration which can be determined from the initial conditions.

The transient part of the solution vanishes rapidly to zero because of the negative exponential factor, thus leaving only the steady-state solution. Of particular significance is the condition of resonance $\left(r = \overline{\omega}/\omega = 1\right)$ for which the amplitude of motion become very large for the damped system and tend to become infinity for the undamped system.

The response of the structure to support or foundation motion can be obtained in terms of the absolute motion of the mass or of its relative motion with respect to the support. In this latter case, the equation assumes a simpler and more convenient form, namely

$$m\ddot{u}_r + c\dot{u}_r + ku_r = F_{eff}(t) \qquad (3.50) \text{ repeated}$$

in which

$$u_r = u - u_s \quad \text{is the relative displcaement}$$

and

$$F_{eff}(t) = -m\ddot{u}_s(t) \quad \text{is the effective force}$$

For harmonic excitation of the foundation, the solution of Eq. (3.50) in terms of the relative motion is of the same form as the solution if Eq. (3.10) in which the force is acting on the mass.

In this chapter, we have also shown that the equivalent damping in the system may be evaluated experimentally either from the peak amplitude or from the bandwidth obtained from a plot of the amplitude-frequency curve when the system is forced to harmonic vibration. Most commonly, equivalent viscous damping is evaluated by equating the experimentally measured energy dissipated in the system during a vibratory cycle at the resonant frequency to the theoretically calculated energy that the system, assumed viscously damped, would dissipate in a cycle. This approach leads to the following expression for the equivalent viscous damping:

$$\xi_{eq} = \frac{1}{4\pi r}\frac{E^*}{E_s} \qquad (3.35)\ \text{repeated}$$

in which

E^* = energy dissipated in the system during a cycle of harmonic vibration at resonance
E_s = strain energy stored at maximum displacement if the system were elastic
r = ratio of forced vibration frequency to the natural frequency of the system

Two related problems of vibrating isolation were discussed in this chapter: (1) the motion transmissibility, that is, the relative motion transmitted from the foundation to the structure; and (2) the force transmissibility which is the relative magnitude of the force transmitted from the structure to the foundation. For both of these problems, the transmissibility is given by

$$T_r = \sqrt{\frac{1 + (2r\xi)^2}{(1 - r^2)^2 + (2r\xi)^2}}$$

3.12 Analytical Problem

Problem 3.1

Demonstrate that during one cycle in harmonic vibration, the work W_F of the external force $F = F_0 \sin \overline{\omega} t$ is equal to the energy E_D dissipated by the damping force as expressed by Eq. (3.30).

$$E_D = 2\pi \xi r k U^2 \qquad (3.30)\ \text{repeated}$$

Solution:

During one cycle, the work of the external force $F = F_0 \sin \overline{\omega} t$ is

$$W_F = \int_0^{2\pi/\overline{\omega}} F_0 \sin \overline{\omega} t \; dy = \int_0^{2\pi/\overline{\omega}} F_0 \sin \overline{\omega} t \frac{du}{dt} dt$$

$$= \int_0^{2\pi/\overline{\omega}} F_0 \sin \overline{\omega} t \;\; \dot{u}(t)\, dt$$

in which $\dot{u}(t)$ is given by Eq. (3.29). Hence,

$$\begin{aligned} W_F &= \int \left[F_0 \sin \overline{\omega}\ \ t\right] [\overline{\omega} U \cos \overline{\omega}(t-\theta) dt] \\ &= \pi F_0 U \sin \theta \end{aligned} \qquad \text{(a)}$$

To demonstrate that work, W_F, of the exciting force given by Eq. (a) is equal to the energy dissipated, E_D, by the viscous force in Eq. (3.30), we need to substitute the sine of the phase angle θ into Eq. (a).

Thus from Eq. (3.21), we have:

$$\begin{aligned} \tan \theta &= \frac{2\xi r}{1-r^2} \\ \frac{\sin \theta}{\cos \theta} &= \frac{2\xi r}{1-r^2} \\ \frac{\sin^2\theta}{\sin^2\theta + \cos^2\theta} &= \frac{(2\xi r)^2}{(1-r^2)^2 + (2r\xi)^2} \end{aligned} \qquad \text{(3.21) repeated}$$

Therefore

$$\sin \theta = \frac{2\xi r}{\sqrt{(1-r^2)^2 + (2r\xi)^2}}$$

Then using Eq. (3.20),

$$U = \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \qquad \text{(3.20) repeated}$$

$$\sin \theta = \frac{2\xi U}{u_{st}}$$

which substituted in Eq. (a) yields

$$W_F = \pi F_0 U^2 \frac{2\xi r}{F_0/k}$$

$$W_F = 2\pi\xi\ \ r k U^2 \qquad \text{(b)}$$

Thus, the work of external force, W_F, expressed by Eq. (b), is equal to the energy, E_D, dissipated per cycle by the viscous damping force as given by Eq. (3.30).

3.13 Problems

Problem 3.2

An electric motor of total weight $W = 1000$ lb is mounted at the center of a simply supported beam as shown in Fig. P3.2). The unbalance in the rotor is $W'e = 1$ lb.in. Determine the steady-state amplitude of vertical motion of the motor for a speed of 900 rpm. Assume that the damping in the system is 10% of the critical damping. Neglect the mass of the supporting beam.

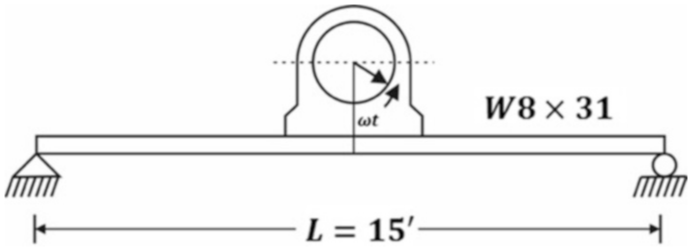

Fig. P3.2

Problem 3.3
Determine the maximum force transmitted to the supports of the beam in Problem 3.2.

Problem 3.4
Determine the steady-state amplitude for the horizontal motion of the steel frame in Fig. P3.4. Assume the horizontal girder to be infinitely rigid and neglect both the mass of the columns and damping. Using MATLAB program, determine the response of frame structure from 0 to 20 s.

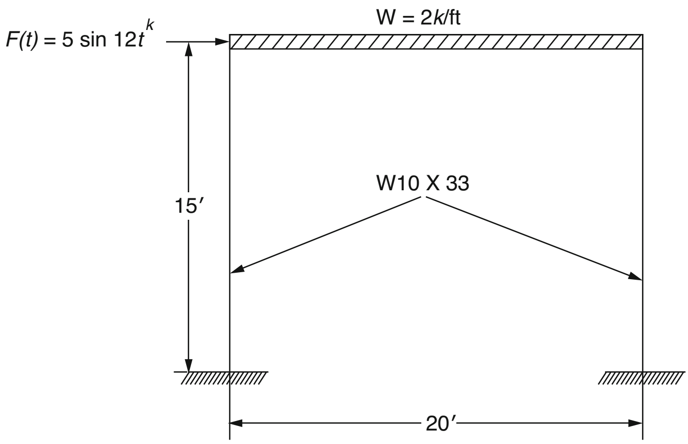

Fig. P3.4

Problem 3.5
Solve for Problem 3.4 assuming that the damping in the system is 8% of the critical damping.

Problem 3.6
For Problem 3.5 determine: (a) the maximum force transmitted to the foundation and (b) the transmissibility.

Problem 3.7
A delicate instrument is to be spring mounted to the floor of a test laboratory where it has been determined that the floor vibrates vertically with harmonic motion of amplitude 0.1 at 10 cps. If the instrument weighs 100 lb, determine the stiffness of the isolation springs required to reduce the vertical motion amplitude of the instrument to 0.01 in. Neglect damping.

Problem 3.8

Consider the water tower shown in Fig. P3.8 which is subjected to ground motion produced by a passing train in the vicinity of the tower. The ground motion is idealized as a harmonic acceleration of the foundation of the tower with an amplitude of 0.1 g at a frequency of 10cps. Determine the motion of the tower relative to the motion of its foundation. Assume an effective damping coefficient of 10% of the critical damping in the system.

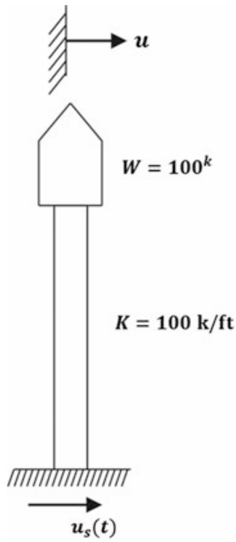

Fig. P3.8

Problem 3.9

Determine the transmissibility in Problem 3.8.

Problem 3.10

An electric motor of total weight $W = 3330$ lb is mounted on a simple supported beam with overhang as shown in Fig. P3.10. The unbalance of the rotor is $W'e = 50$ lb.·in. (a) Find the amplitudes of forced vertical vibration of the motor for speeds 800, 1000, and 1200 rpm. (b) Draw a plot of the amplitude versus rpm Assume damping equal to 10% of critical damping using MATLAB.

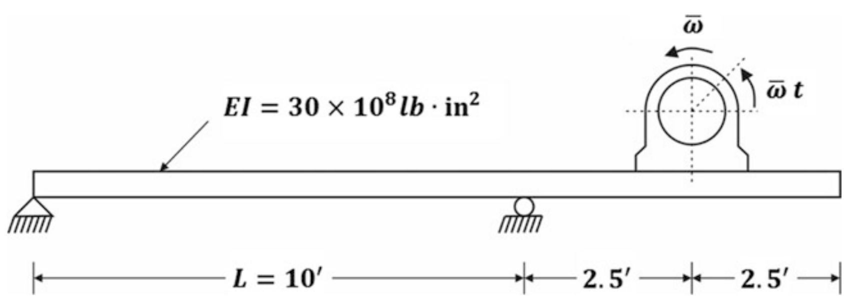

Fig. P3.10

Problem 3.11

Estimate the damping in a single-degree-of-freedom system that is excited by a harmonic force. The peak displacement amplitude at resonance was measured equal to 3 in and equal to 0.2 in at one-tenth of the natural frequency of the system.

Problem 3.12

Determine the damping in a system in which during a vibration test under a harmonic force it was observed that at a frequency 10% higher than the resonant frequency, the displacement amplitude was exactly one-half of the resonant amplitude.

Problem 3.13

Determine the natural frequency, amplitude of vibration, and maximum normal stress in the simple supported beam carrying an engine of weight $W = 30$ KN (Fig. P3.13). The engine rotates at 400 rpm and induces a vertical force $F(t) = 8\sin\overline{\omega}t$. ($E = 210 \times 10^9$ N/m^2, $I = 8950 \times 10^{-8}$ m^4, $S = 597 \times 10^6$ m^3)

(Problem contributed by Vladimir N. Alekhin and Aleksey A. Antipin of the Urals State University, Russia)

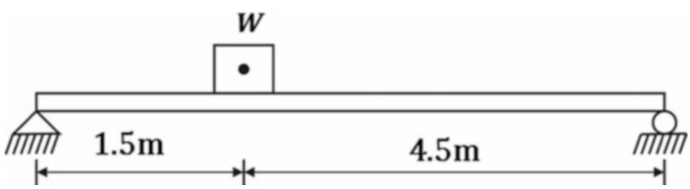

Fig. P3.13

Problem 3.14

A machine of mass m rests on an elastic floor as shown in Fig. P3.14. In order to find the natural frequency of the vertical motion, a mechanical shaker of mass m_s is bolted to the machine and run at various speeds until the resonant frequency f_r is found. Determine the natural frequency f_n of the floor-machine system in terms of f_r and the given data.

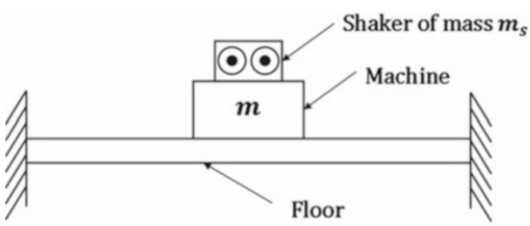

Fig. P3.14

Problem 3.15

Determine the frequency at which the peak amplitude of a damped oscillator will occur. Also, determine the peak amplitude and corresponding phase angle.

Problem 3.16

A structure modeled as a damped spring-mass system (Fig. P3.16) with $mg = 2520$ lb, $k = 89{,}000$ lb/in, and $c = 112$ lb. in/sec is subjected to a harmonic exciting force. Determine: (a) the natural frequency, (b) the damping ratio, (c) the amplitude of the exciting force when the peak amplitude of the vibrating mass is measured to be 0.37 in. and (d) the amplitude of the exciting force when the amplitude measured is at the peak frequency assumed to be the resonant frequency.

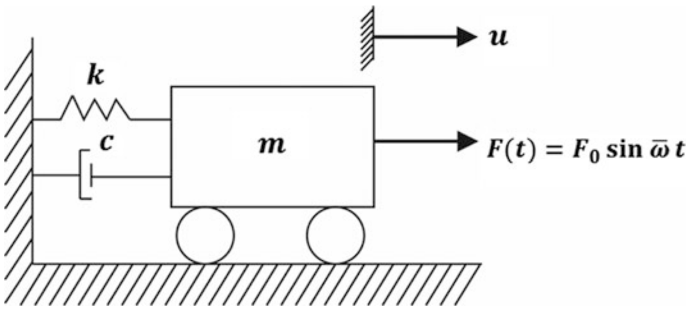

Fig. P3.16

Problem 3.17

A structural system modeled as a damped oscillator is subjected to the harmonic excitation produced by an eccentric rotor. The spring constant k and the mass m are known but not the damping and the amount of unbalance in the rotor, From measured amplitudes U_r at resonance and U_1 at a frequency ratio $r_1 \neq 1$, determine expressions to calculate the damping ratio ξ and the amplitude of the exciting force F_r at resonance.

Problem 3.18

A system is modeled by two vibrating masses m_1 and m_2 interconnected by a spring k and damper element c (Fig. P3.18). For harmonic force $F = F_0 \sin \overline{\omega} t$ acting on mass m_2 determine: (a) equation of motion in terms of the relative motion of the two masses, $u_r = u_2 - u_1$; (b) the steady-state solution of the relative motion.

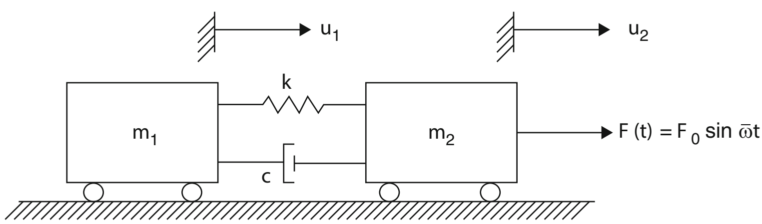

Fig. P3.18