Response to General Dynamic Loading

4

In the preceding chapter we studied the response of a single degree-of-freedom system with harmonic loading. Through this type of loading is important, real structures are often subjected to loads that are not harmonic. In this chapter we shall study the response of the single degree-of-freedom system to a general type of force. We shall see that the response can be obtained in terms of an integral that for some simple load functions can be evaluated analytically. For the general case, however, it will be necessary to resort to a numerical integration procedure.

4.1 Duhamel's Integral – Undamped System

An impulsive loading is a load which is applied during a short duration of time. The corresponding impulse of this type of load is defined as the product of the force and the time of its duration.

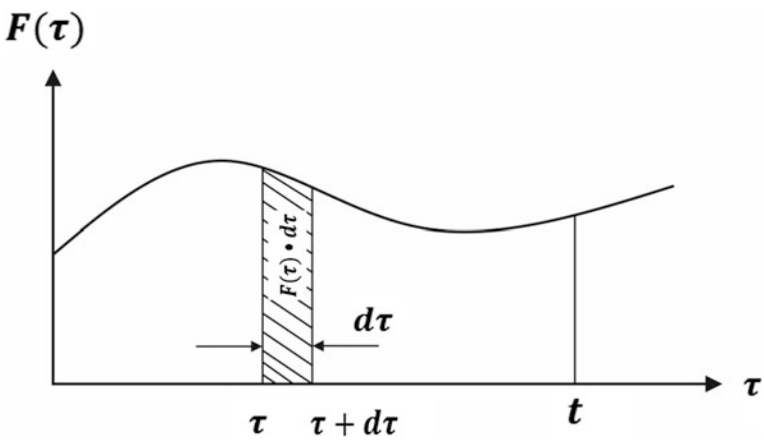

Fig. 4.1 General load function as impulsive loading

For example, the impulse of the force $F(\tau)$ depicted in Fig. 4.1 at time τ during the time interval $d\tau$ is represented by the shaded area and it is equal to $F(\tau)d\tau$. This impulse acting on a body of mass m produces a change in velocity that can be determined from Newton's Law of Motion, namely

M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_4

$$m\frac{dv}{d\tau} = F(\tau)$$

Rearrangement yields

$$dv = \frac{F(\tau)d\tau}{m} \tag{4.1}$$

where $F(\tau)d\tau$ is the impulse and dv is the incremental velocity. This incremental velocity may be considered to be an initial velocity of the mass at time τ. Let us now consider this impulse $F(\tau)d\tau$ acting on the structure represented by the undamped oscillator. At time τ the oscillator will experience a change in velocity given by Eq. (4.1). This change in velocity is then introduced in Eq. (1.20) as the initial velocity v_0 together with the initial displacement $u_0 = 0$ at time τ producing a displacement *du(t)* at a later time t given by

$$du(t) = \frac{F(\tau)d\tau}{m\omega}\sin\omega(t-\tau) \tag{4.2}$$

The loading function may then be regarded as a series of short impulses at successive incremental times $d\tau$, each producing its own differential response at time t of the form given by Eq. (4.2). Therefore, we conclude that the total displacement at time t due to the continuous action of the force $F(\tau)$ is given by the summation or integral of the differential displacements $du(t)$ from time $t = 0$ to time t, that is,

$$u(t) = \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau \tag{4.3}$$

The integral in this equation is known as Duhamel's integral. Equation (4.3) represents the total displacement produced by the exciting force $F(\tau)$ acting on the undamped oscillator; it includes both the steady-state and the transient components of the motion corresponding to zero initial conditions, $u_0 = 0$ and $v_0 = 0$. If the function $F(\tau)$ cannot be expressed analytically, the integral of Eq. (4.3) can always be evaluated approximately by suitable numerical methods. To include the effect of initial displacement u_0 and initial velocity v_0 at time $t = 0$, it is only necessary to add to Eq. (4.3) the solution given by Eq. (1.20) for the effects due to the initial conditions. Thus the total displacement of an undamped single degree-of-freedom system with an arbitrary load is given by

$$u(t) = u_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t + \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau \tag{4.4}$$

Applications of Eq. (4.4) for some simple forcing functions for which it is simple to obtain the explicit integration of Eq. (4.4) are presented below.

4.1.1 Constant Force

Consider the case of a constant force of magnitude F_0 applied suddenly to the undamped oscillator at time $t = 0$ as shown in Fig. 4.2.

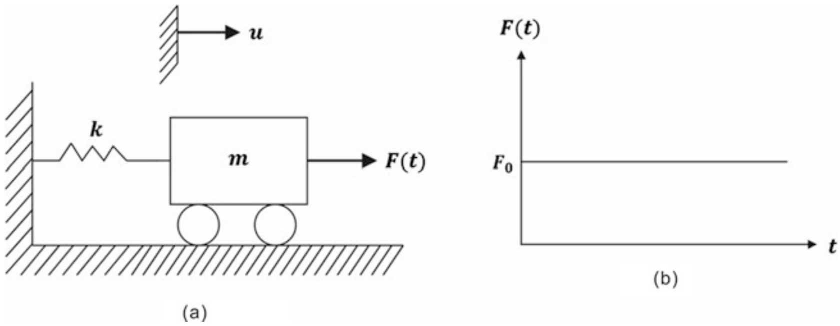

Fig. 4.2 Undamped oscillator acted upon by constant force

For both initial displacement and initial velocity equal to zero, the application of Eq. (4.4) to this case gives

$$u(t) = \frac{1}{m\omega}\int_0^t F_0 \sin\omega(t-\tau)\,d\tau$$

and the integration yields

$$\begin{aligned} u(t) &= \frac{F_0}{m\omega^2}\left|\cos\omega(t-\tau)\right|_0^t \\ u(t) &= \frac{F_0}{k}(1-\cos\omega t) = u_{st}(1-\cos\omega t) \end{aligned} \tag{4.5}$$

where $u_{st} = \frac{F_0}{k}$ is the static displacement due to a force of magnitude F_0. The response for such a suddenly applied constant load is shown in Fig. 4.3. It will be observed that this solution is very similar to the solution for the free vibration of the undamped oscillator. The major difference is that the coordinate axis t has been shifted by an amount equal to $u_{st} = \frac{F_0}{k}$. Also, it should be noted that the maximum displacement $2u_{st}$ is exactly twice the displacement that the force F_0 would produce if it were applied statically. We have found an elementary but important result; the maximum displacement of a linear elastic system for a constant force applied suddenly is twice the displacement caused by the same force applied statically (slowly). This result for displacement is also true for the internal forces and stresses in the structure.

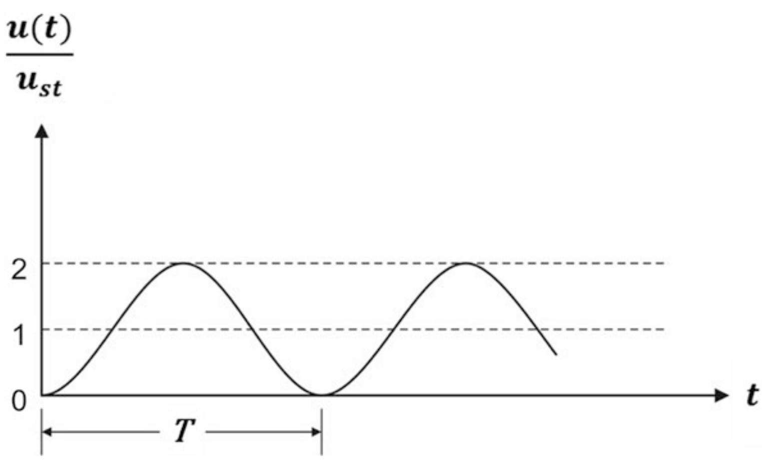

Fig. 4.3 Response of an undamped single degree-of-freedom system to a suddenly applied constant force

4.1.2 Rectangular Load

Let us consider a second case, that of a constant force F_0 suddenly applied but only during a limited time duration, t_d as shown in Fig. 4.4:

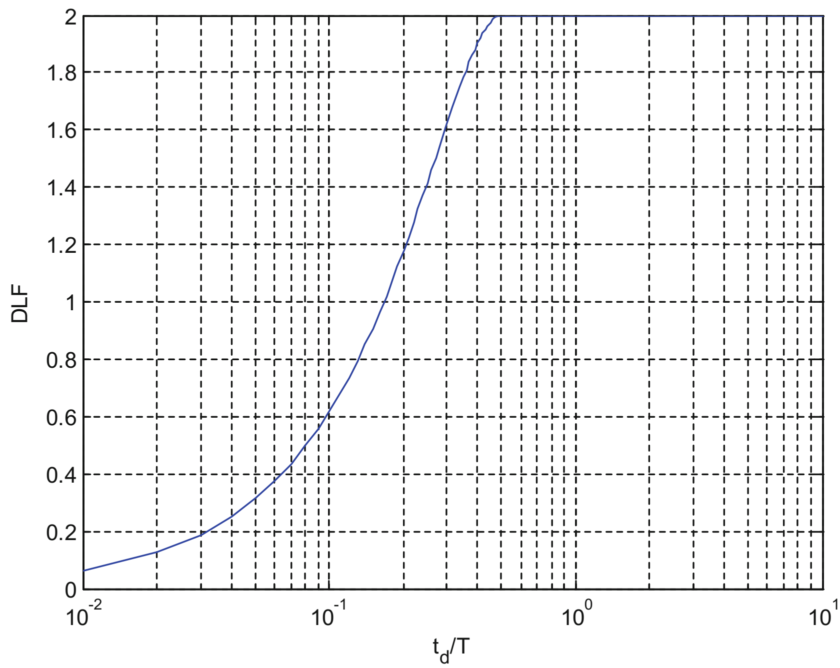

Fig. 4.4 Maximum dynamic load factor for the undamped oscillator acted on by a rectangular force

Up to the time t_d, the solution given by Eq. (4.5) applies and at that time the displacement and velocity are

$$u_d = \frac{F_0}{k}(1 - \cos\omega t_d)$$

and

$$v_d = \frac{F_0}{k}\omega \sin\omega t_d$$

For the response after time t_d we apply Eq. (1.20) for free vibration, taking as the initial conditions the displacement and velocity at t_d. After replacing t by $t - t_d$ and u_0 and v_0 for u_d and v_d, respectively, we obtain

$$u(t) = \frac{F_0}{k}(1 - \cos\omega t_d)\cos\omega(t - t_d) + \frac{F_0}{k}\sin\omega t_d \sin\omega(t - t_d)$$

which can be reduced to

$$u(t) = \frac{F_0}{k}\{\cos\omega(t - t_d) - \cos\omega t\} \tag{4.6}$$

If the dynamic load factor (DLF) is defined as the displacement at any time t divided by the static displacement $u_{st} = \frac{F_0}{k}$, we may write Eqs. (4.5 and 4.6) as

$$DLF = 1 - \cos\omega t, \qquad t \le t_d$$

and

$$DLF = \cos\omega(t - t_d) - \cos\omega t, \quad t \ge t_d \tag{4.7}$$

It is often convenient to express time as a dimensionless parameter by simply using the natural period instead of the natural frequency $\left(\omega = \frac{2\pi}{T}\right)$. Hence, Eqs. (4.7) may be written as

$$DLF = 1 - \cos 2\pi\frac{t}{T}, \qquad t \le t_d$$

and

$$DLF = \cos 2\pi\left(\frac{t}{T} - \frac{t_d}{T}\right) - \cos 2\pi\frac{t}{T}, \quad t \ge t_d \tag{4.8a}$$

or using the trigonometric identity

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

as

$$DLF = \left(2\sin\frac{\pi t_d}{T}\right)\sin\left[2\pi\left(\frac{t}{T} - \frac{t_d}{2T}\right)\right], \quad t \ge t_d \tag{4.8b}$$

The use of dimensionless parameters in Eq. (4.8a) serves to emphasize the fact that the ratio of duration of the time, that the constant force is applied, to the natural period rather than the actual value of either quantity is the important parameter. The maximum dynamic load factor $(DLF)_{max}$ obtained by maximizing Eq. (4.8b), is plotted in Fig. 4.4. It is observed from this figure that the maximum dynamic load factor for loads of duration $\left(\frac{t_d}{T} > 0.5\right)$ is the same as if the load duration had been infinite.

In general, the maximum response occurs during the application of the load, except for loadings of very short duration $\left(\frac{t_d}{T} < 0.4\right)$. In such cases, the maximum response may occur during the free vibration after the cessation of the load; it is then necessary to extend the loading time for a duration of about one period, in which the magnitude of the load is set equal to zero.

Charts, as shown in Fig. 4.4, which give the maximum response of a single degree-of-freedom system for a given loading function, are called response spectral charts. Response spectral charts for impulsive loads of short duration are often presented for the undamped system. For the short duration of the load, damping does not have a significant effect on the response of the system. The maximum dynamic load factor usually corresponds to the first peak of response and the amount of damping normally found in structures is not sufficient to appreciably decrease this value.

4.1.3 Triangular Load

We consider now a system represented by the undamped oscillator, initially at rest and subjected to a force $F(t)$ that has an initial value F_0 and that decreases linearly to zero at time t_{d} (Fig. 4.5). The response may be computed by Eq. (4.4) in two intervals. For the first interval, $\tau \leq t_d$ the force is given by

$$F(\tau) = F_0\left(1 - \frac{\tau}{t_d}\right)$$

and the initial conditions by

$$u_0 = 0, \;\; v_0 = 0$$

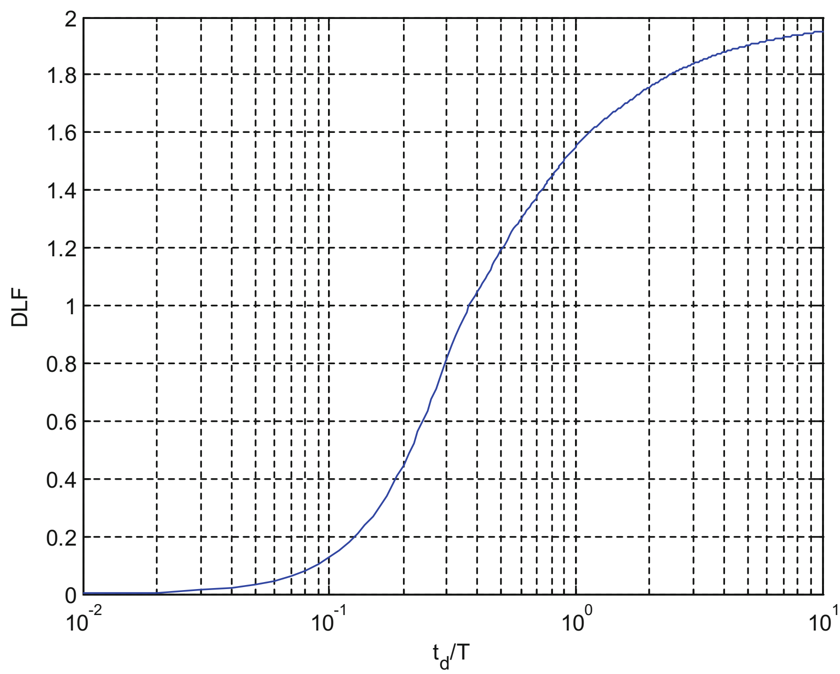

Fig. 4.5 Maximum dynamic load factor for the undamped oscillator acted upon by a triangular force

The substitution of these values in Eq. (4.4) and integration gives

$$u = \frac{F_0}{k}(1 - \cos\omega t) + \frac{F_0}{kt_d}\left(\frac{\sin\omega t}{\omega} - t\right) \tag{4.9}$$

or in terms of the dynamic load factor and dimensionless parameters

$$DLF = \frac{u}{u_{st}} = 1 - \cos\left(\frac{2\pi t}{T}\right) + \frac{\sin\left(\frac{2\pi t}{T}\right)}{\left(\frac{2\pi t_d}{T}\right)} - \frac{t}{t_d} \quad 0 \leq t \leq t_d \tag{4.10}$$

which defines the response before time t_{d}.

For the second interval ($t \geq t_d$), we obtain from Eq. (4.9) the displacement and velocity at time t_d as

$$u_d = \frac{F_0}{k}\left(\frac{\sin \omega t_d}{\omega t_d} - \cos \omega t_d\right)$$

and

$$v_d = \frac{F_0}{k}\left(\omega \sin \omega t_d + \frac{\cos \omega t_d}{t_d} - \frac{1}{t_d}\right) \tag{4.11}$$

These values may be considered as the initial conditions at time $t = t_d$ for this second interval. Replacing in Eq. (1.20) t by $t - t_d$ and u_0 and v_0 respectively, by u_d and v_d and noting that $F(\tau) = 0$ in this interval we obtain the response as

$$u = \frac{F_0}{k\omega t_d}\{\sin \omega t - \sin \omega(t - t_d)\} - \frac{F_0}{k}\cos \omega t$$

and upon dividing by $u_{st} = \frac{F_0}{k}$ gives

$$DLF = \frac{1}{\omega t_D}\{\sin \omega t - \sin \omega(t - t_d)\} - \cos \omega t \tag{4.12}$$

In terms of the dimensionless time parameter, this last equation may be written as

$$DLF = \frac{1}{\omega T_D}\left\{\sin 2\pi \frac{t}{T} - \sin 2\pi\left(\frac{t}{T} - \frac{t_d}{T}\right)\right\} - \cos 2\pi \frac{t}{T} \quad t \geq t_d \tag{4.13}$$

The plot of the maximum dynamic load factor as a function of the relative time duration t_d/T for the undamped oscillator is shown in Fig. 4.5. As would be expected, the maximum value of the dynamic load factor approaches 2 as t_d/T becomes large; that is, the effect of the decay of the force is negligible for the time required for the system to reach the maximum peak.

We have studied the response of the undamped oscillator for two simple impulse loadings; the rectangular pulse and the triangular pulse. Extensive charts have been prepared by the U. S. Army Corps of Engineers,[1] and are available for a variety of loading pulses. As already mentioned, the evaluation of Duhamel's Integral Method for a general forcing requires the use of numerical methods (See Problems 4.1).

Illustrative Example 4.1

A one-story building, shown in Fig. 4.6, is modeled as a 15-ft high frame with two steel columns fixed at the base and a rigid beam supporting a total weight of $W = 5000$ lb. Each column has a moment of inertia $I = 69.2\ \text{in}^2$ and a section modulus $S = \frac{I}{c} = 17\ \text{in}^3\left(E = 30 \times 10^6\ \text{psi}\right)$.

Determine the maximum response of the frame to a rectangular impulse of amplitude 3000 lb and duration $t_d = 0.1$ s applied horizontally to the top member of the frame. The response of interest is the horizontal displacement at the top of the frame and the bending stress in the columns.

[1] U. S. Army Corps of Engineers, *Design of Structures to Resist the Effects of Atomic Weapons*, Manuals 415, 415, 3rd 416, March 15, 1957; Manuals 417 and 419, January 15, 1958; Manuals 418, 420, 421, January 15, 1960.

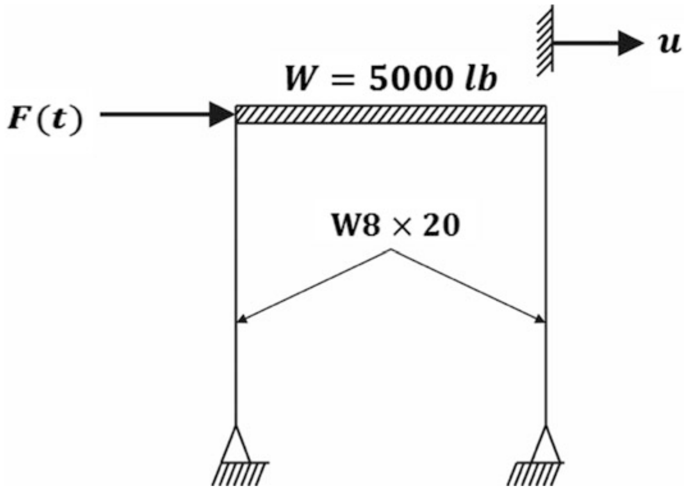

Fig. 4.6 Idealized frame for Illustrative Example 4.1

Solution:

1. Natural period.

$$k = \frac{12E(2I)}{L^3} = \frac{12 \times 30 \times 10^6 \times 2 \times 69.2}{(15 \times 12)^3} = 8544 \text{ lb/in}$$

$$m = \frac{5000}{386} = 12.9534 \text{ lb. sec}^2/\text{in}$$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{12.9534}{8544}} = 0.2446 \text{ sec}$$

2. Maximum displacement.

$$\frac{t_d}{T} = \frac{0.1}{0.2446} = 0.408$$

$$DLF_{\max} = \frac{u_{\max}}{u_{st}} = 1.9 \quad (\text{from Fig.4.4})$$

$$u_{st} = \frac{F_0}{k} = \frac{3000}{8544} = 0.3511 \text{ in}$$

$$u_{\max} = (1.9)(0.3511) = 0.667 \text{ in} \qquad \text{(Ans)}$$

3. Maximum bending stress.
   The bending moment M in the columns is given by

$$M = V_{\max}\frac{L}{2} = ku_{\max}\frac{L}{2}$$

$$= \frac{6EI}{L^2}u_{\max} = \frac{6 \times 30 \times 10^6 \times 69.2}{(15 \times 12)^2}0.667 = 256,424 \text{ (lb.in)}$$

and the maximum stress σ_{max} by

$$\sigma_{max} = \frac{M}{S} = \frac{256,424}{17} = 15,083\,\text{psi} \qquad \text{(Ans)}$$

Alternatively, the maximum response can be estimated using MATLAB file.

```
clear all
close all
clc


%%%%-GIVEN VALUES-%%%%
T = 0.2446;               %Natural period
omega = 2*pi/T;           %Natural frequency
F_0=3000;                 %Maximum force
k=8544;                   %Stiffness

t = 0:0.001:3;            %0 to 3 secs with the interval of 0.001 sec
td_T = 0.408;             %td/T ratio


%%%-ESTIMATION-%%%
td = td_T*T;              %Duration of force
u_st = F_0/k;             %Static displacement

%%%Estimate response with two functions (t<td and t>td)
for i=1:length(t)
    if t(i) <=td
        u(i) = u_st*(1-cos(omega*t(i)));
   else
    arg1=(1-cos(omega*td))*cos(omega*(t(i)-td));
    arg2=sin(omega*td)*sin(omega*(t(i)-td));
        u(i)= u_st*(arg1+arg2);

   end
end


%%%Response and DLF estimation
figure (1)
plot (t, u);
xlabel ('Time (sec)');
ylabel ('Displacement');
grid on

u_max =abs(max(u))       %u_max
D = u_max/u_st           %DLF factor
```

MATLAB yields the maximum response of 0.673 in. This value is comparable to the value using Fig. 4.4. DLF can be also calculated from MATLAB. The value using MATLAB yield the same value from the spectra chart. Using MATLAB, the response can be varied depending on t_d Fig. 4.7.

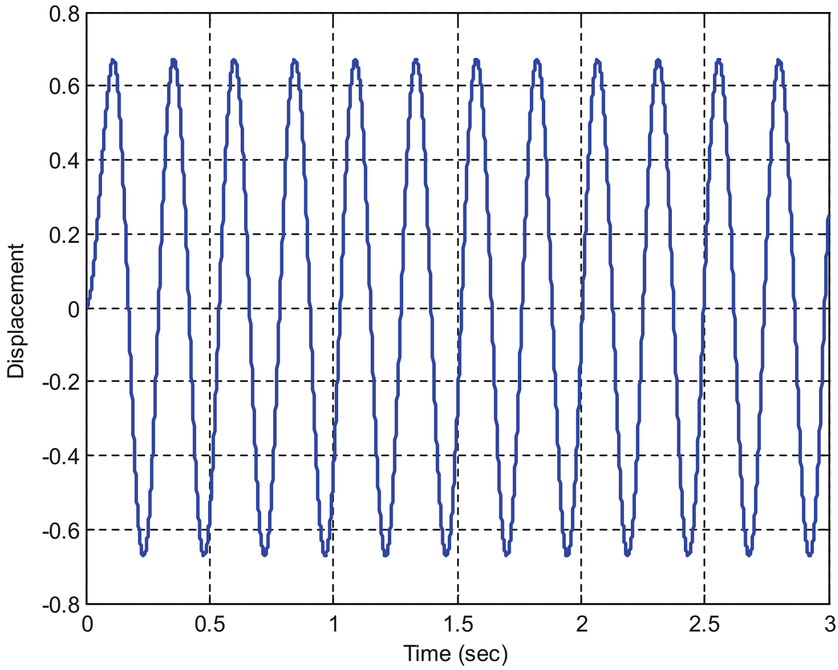

Fig. 4.7 Response of Illustrative Example 4.1

Illustrative Example 4.2

Plot Fig. 4.4 using MATLAB (Response Spectral Chart). The MATLAB code is presented below.

```
close all
clc

%%%%-GIVEN VALUES-%%%%
T = 0.2446;                 %Natural period
omega = 2*pi/T;             %Natural frequency
F_0=3000;                   %Maximum force
k=8544;                     %Stiffness

t = 0:0.01:3;               %0 to 3 secs with the interval of 0.01 sec
td_T = 0:0.01:10;           %td/T ratio


%%%-ESTIMATION-%%%
td = td_T*T;                %Duration of force
u_st = F_0/k;               %Static displacement

%%%Estimate response spectral chart
for j =1:length(td)

for i=1:length(t)
    if t(i) <=td(j)
        u(i,j) = u_st*(1-cos(omega*t(i)));
   else
    arg1=(1-cos(omega*td(j)))*cos(omega*(t(i)-td(j)));
    arg2=sin(omega*td(j))*sin(omega*(t(i)-td(j)));
        u(i,j)= u_st*(arg1+arg2);
    end

end

end
```

```
%%%DLF estimation
u_max = max(u,[],1);          %u_max
D = u_max/u_st;               %DLF factor

figure (1)
semilogx(td_T, D);
xlabel ('t_d/T');
ylabel ('DLF');
grid on
```

4.2 Duhamel's Integral-Damped System

The response of a damped system expressed by the Duhamel's integral is obtained in a manner entirely equivalent to the undamped analysis except that the impulse $F(\tau)d\tau$ producing an initial velocity $dv = F(\tau)\frac{d\tau}{m}$ is substituted into the corresponding damped free-vibration Eq. (2.20). Setting $u_0 = 0, v_0 = F(\tau)\frac{d\tau}{m}$, and substituting t for $(t-\tau)$ in Eq. (2.20), we obtain the differential displacement *du(t)* at a time t as

$$du(t) = e^{-\xi\omega(t-\tau)}\frac{F(\tau)\,d\tau}{m\,\omega_D}\sin\omega_D(t-\tau) \tag{4.14}$$

Summing or integrating these differential response terms over the entire loading interval results in

$$u(t) = \frac{1}{m\omega_D}\int_0^t F(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\,d\tau \tag{4.15}$$

which is the response for a damped system in terms of the Duhamel's integral. For numerical evaluation Eq. (4.15), we proceed as in the undamped case (See Problem 4.2).

4.3 Response by Direct Integration

The differential equation of motion for a one degree-of-freedom system represented by the damped simple oscillator, shown in Fig. 4.8a, is obtained by establishing the dynamic equilibrium of the forces in the free body diagram, Fig. 4.8b:

$$m\ddot{u} + c\dot{u} + ku = F(t) \tag{4.16}$$

in which the function F(t) represents the force applied to the mass of the oscillator.

When the structure, modeled by the simple oscillator, is excited by a motion at its support, as is shown in Fig. 4.9a, the equation of motion obtained using the free body diagram in Fig. 4.9b is

$$m\ddot{u} + c\left(\dot{u} - \dot{u}_s\right) + k(u - u_s) = 0 \tag{4.17}$$

In this case, it is convenient to express the displacement u_r of the mass relative to the displacement u_s of the support, namely

$$u_r = u - u_s \tag{4.18}$$

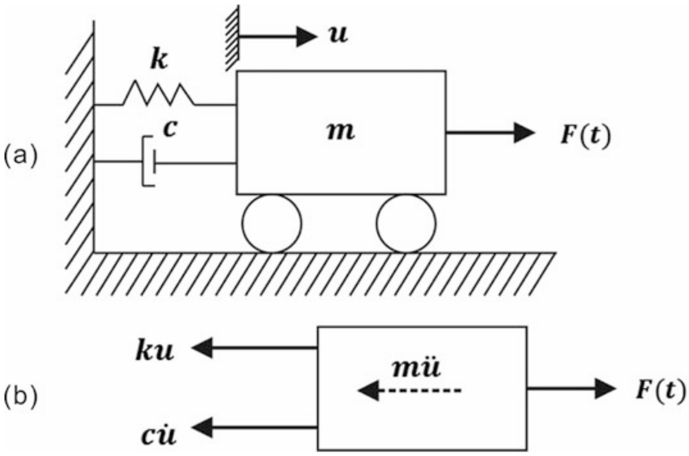

Fig. 4.8 (**a**) Damped simple oscillator excited by the force $F(t)$. (**b**) Free body diagram

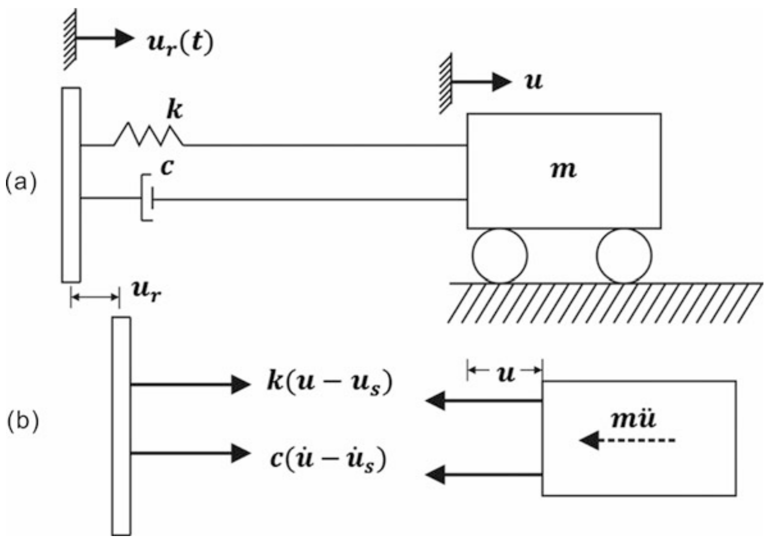

Fig. 4.9 (**a**) Damped Simple oscillator excited by the displacement $u_s(t)$ at its support. (**b**) Free body diagram

The substitution of u_r and its derivative from Eq.(4.18) into Eq. (4.17) results in

$$m\ddot{u}_r + c\dot{u}_r + ku_r = -mu_s(t) \tag{4.19}$$

Comparison of Eqs. (4.16 and 4.19) reveals that both equations are mathematically equivalent if the right-hand side of Eq. (4.19) is interpreted as the effective force

$$F_{eff}(t) = -m\ddot{u}_s(t) \tag{4.20}$$

Equation (4.19) may then be written as

$$m\ddot{u}_r + c\dot{u}_r + ku_r = F_{eff}(t) \tag{4.21}$$

Consequently, the solution of the second order differential Eq. (4.16) or Eq. (4.21) gives the response in terms of the absolute motion u for the case in which the mass is excited by a force, or in terms of the relative motion $u_r = u - u_s$, for the structure excited at the base.

4.4 Solution of the Equation of Motion

The method of solution for the differential equation of motion presented in this Section is exact for an excitation function described by linear segments. The process of solution requires for convenience that the excitation function be calculated at equal time intervals Δt. This result is accomplished by a linear interpolation between points defining the excitation. Thus, the time duration of the excitation, including a suitable extension of time after cessation of the excitation, is divided into N equal time intervals of duration Δt, For each interval Δt, the response is calculated by considering the initial conditions at the beginning of that time interval and the linear excitation during the interval. The initial conditions are, in this case, the displacement and velocity at the end of the preceding time interval. Assuming that the excitation function $F(t)$ is approximated by a piecewise linear function as shown in Fig. 4.10, we may express this function by

$$F(t) = \left(1 - \frac{t - t_i}{\Delta t}\right)F_i + \left(\frac{t - t_i}{\Delta t}\right)F_{i+1}, \qquad t_i \le t \le t_{i+1} \tag{4.22}$$

in which $t_i = i{\cdot}\Delta t$ for equal intervals of duration Δt and $i = 1,2,3,\ldots,N$. The differential equation of motion, Eq. (4.16), is then given by

$$m\ddot{u} + c\dot{u} + ku = \left(1 - \frac{t - t_i}{\Delta t}\right)F_i + \left(\frac{t - t_i}{\Delta t}\right)F_{i+1}, \qquad \mathrm{t_i} \le t \le t_{i+1} \tag{4.23}$$

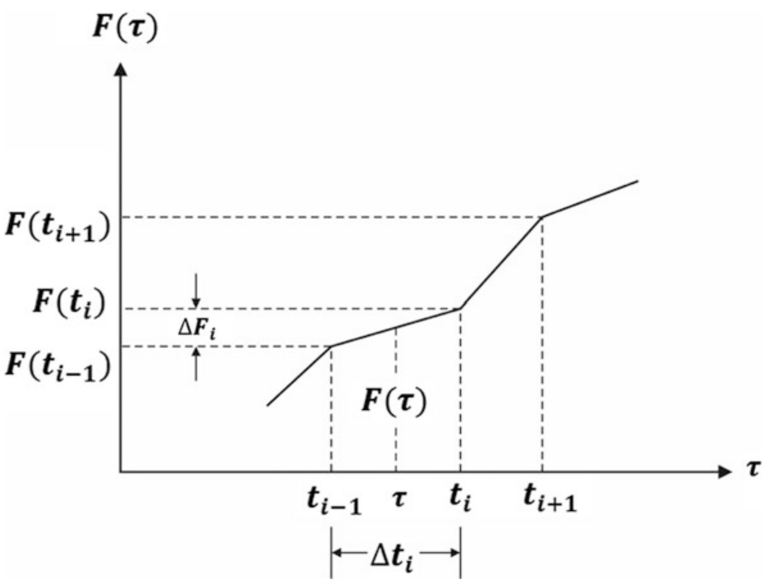

Fig. 4.10 Segmental linear loading function

The solution of Eq. (4.23) may be expressed as the sum of the complementary solution u_c for which the second member of Eq. (4.23) is set equal to zero, and the particular solution u_p, that is

$$u = u_c + u_p \tag{4.24}$$

The complementary solution is given in general by Eq. (2.15), which for the interval $t_i \leq t \leq t_i + \Delta t$ is

$$u_c = e^{\xi\omega(t-t_i)}[C_i \cos\omega_D(t-t_i) + D_i \sin\omega_D(t-t_i)] \tag{4.25}$$

On the other hand, the particular solution of Eq. (4.23) takes the form

$$u_p = B_i + A_i(t-t_i) \tag{4.26}$$

which upon its substitution into Eq. (4.23) gives

$$cA_i + k[B_i + A_i(t-t_i)] = \left(1 - \frac{t-t_i}{\Delta t}\right)F_i + \left(\frac{t-t_i}{\Delta t}\right)F_{i+1}$$

where A_{i} and B_{i} are constants for the interval $t_i \leq t \leq t_i, + \Delta t$ and where we use the notation $F_i = F(t_i)$ and $F_{i+1} = F(t + \Delta t)$. Establishing the identity of terms between the left-hand and right-hand sides, that is, between the constant terms and the terms with a factor $(t - t_i)$ and then solving the resulting equations, we obtain

$$\begin{aligned} A_i &= \frac{F_{i+1} - F_i}{k\,\Delta t} \\ B_i &= \frac{F_i - cA_i}{k} \end{aligned} \tag{4.27}$$

The substitution into Eq. (4.24) of the complementary solution u_c from Eq. (4.25) and of the particular solution u_{p} from Eq. (4.26) gives the total solution as

$$u = e^{-\xi\omega(t-t_i)}[C_i \cos\omega_D(t-t_i) + D_i \sin\omega_D(t-t_i)] + B_i + A_i(t-t_i) \tag{4.28}$$

The velocity $\dot{u}$ is then given by the derivative of Eq. (4.28) as

$$\dot{u} = e^{\xi\omega(t-t_i)}[(\omega_D D_i - \xi\omega C_i)\cos\omega_D(t-t_i) - (\omega_D C_i + \xi\omega D_i)\sin\omega_D(t-t_i)] + A_i \tag{4.29}$$

The constants of integration C_i and D_i are obtained from Eqs. (4.28 and 4.29) introducing the initial conditions for the displacement u_i and for the velocity $\dot{u}_i$, at the beginning of the interval Δt, that is, at time t_{i}. Thus, introducing into Eqs. (4.28 and 4.29) these initial conditions and solving for the constants C_i and D_i in the resulting relations yields

$$\begin{aligned} C_i &= u_i - B_i \\ D_i &= \frac{\dot{u}_i - A_i - \xi\omega C_i}{\omega_D} \end{aligned} \tag{4.30}$$

The evaluation of Eqs. (4.28 and 4.29) at time $t_i + \Delta t$ results in the displacement u_{i+1} and the velocity $\dot{u}_{i+1}$ at time t_{i+1}. Namely,

$$u_{i+1} = e^{-\xi\omega\Delta t}[C_i \cos\omega_D\Delta t + D_i \sin\omega_D\Delta t] + B_i + A_i\Delta t \tag{4.31}$$

and

$$\dot{u}_{i+1} = e^{-\xi\omega\Delta t}\left[D_i(\omega_D \cos\omega_D\Delta t - \xi\omega \sin\omega_D\Delta t) - C_i\left(\xi\omega \cos\omega_D\Delta t + \omega_D \sin\omega_D\Delta t\right] - A_i\right. \tag{4.32}$$

Finally, the acceleration at time $t_{i+1} = t_i + \Delta t$ is directly obtained after substituting u_{i+1} and $\dot{u}_{i+1}$ from Eqs. (4.31 and 4.32) into the differential Eq. (4.16) and letting $t_{i+1} = t_i + \Delta t$. Specifically,

$$\ddot{u}_{i+1} = \frac{1}{m}\left(F_{i+1} - c\dot{u}_{i+1} - ku_{i+1}\right) \tag{4.33}$$

The substitution of the coefficient A_i, B_i, C_i and D_i from Eqs. (4.27 and 4.30), together with $\varepsilon = \frac{2\xi k}{\omega}$ into Eqs. (4.31 and 4.32), results in the following formulas to calculate the displacement, velocity and acceleration at the time step $t_{i+1} = t_i + \Delta t$:

$$u_{i+1} = A' u_i + B' \dot{u}_i + C' F_i + D_i' F_{i+1} \tag{4.34}$$

$$\dot{u}_{i+1} = A'' u_i + B'' \dot{u}_i + C'' F_i + D'' F_{i+1} \quad \dot{u}_{i+1} = A'' u_i + B'' \dot{u}_i + C'' F_i + D'' F_{i+1} \tag{4.35}$$

$$\ddot{u}_{i+1} = -\omega^2 u_{i+1} - 2\xi\omega\ddot{u}_{i+1} + \frac{F_{i+1}}{m} \tag{4.36}$$

Equations (4.34, 4.35 and 4.36) are recurrence formulas to calculate, respectively, the displacement, velocity, and acceleration at the next time step $t_{i+1} = t_i + \Delta t$ from the previously calculated values for these quantities at the preceding time step t_i. Because these recurrence formulas are exact, the only restriction in selecting the length of the time step, Δt, is that it allows a close approximation to the excitation function and that equally spaced time intervals do not miss the peaks of this function. This numerical procedure is highly efficient because the coefficients in Eqs. (4.34, 4.35 and 4.36) need to be calculated only once. The final expressions to calculate the coefficients A', B', ... D" are given in Box 4.1.

Illustrative Example 4.3

Determine the dynamic response of a tower subjected to a blast loading. The idealization of the structure and the blast loading are shown, respectively, in Fig. 4.11a, b. Assume damping equal to 20% of the critical damping.

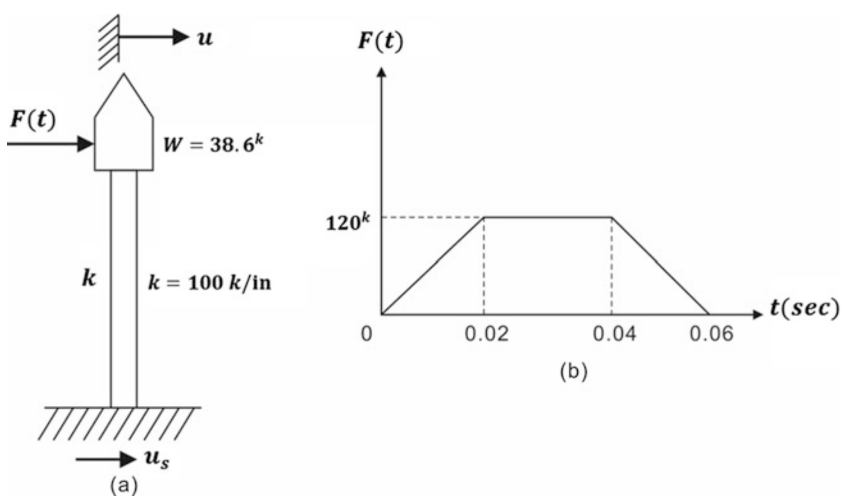

Fig. 4.11 (**a**) Idealized structure (**b**) idealized loading for Illustrative Example 4.4

Box 4.1 Coefficients in Eqs. (4.34, 4.35 and 4.36).

$$A' = e^{-\xi\omega\Delta t}\left(\frac{\xi\omega}{\omega_D}\sin\omega_D\Delta t + \cos\omega_D\Delta t\right)$$

$$B' = e^{-\xi\omega\Delta t}\left(\frac{1}{\omega_D}\sin\omega_D\Delta t\right)$$

$$C' = \frac{1}{k}\left\{e^{-\xi\omega\Delta t}\left[\left(\frac{1-2\xi^2}{\omega_D\Delta t} - \frac{\xi\omega}{\omega_D}\right)\sin\omega_D\Delta t - \left(1 + \frac{2\xi}{\omega\Delta t}\right)\cos\omega_D\Delta t\right] + \frac{2\xi}{\omega\Delta t}\right\}$$

$$D' = \frac{1}{k}\left\{e^{-\xi\omega\Delta t}\left[\left(\frac{2\xi-1}{\omega_D\Delta t}\sin\omega_D\Delta t + \frac{2\xi}{\omega\Delta t}\cos\omega_D\Delta t\right)\right] + \left(1 - \frac{2\xi}{\omega\Delta t}\right)\right\}$$

$$A'' = -e^{-\xi\omega\Delta t}\left(\frac{\omega^2}{\omega_D}\sin\omega_D\Delta t\right)$$

$$B'' = e^{-\xi\omega\Delta t}(\cos\omega_D\Delta t) - \frac{\xi\omega}{\omega_D}\sin\omega_D\Delta t$$

$$C'' = \frac{1}{k}\left\{-e^{-\xi\omega\Delta t}\left[\left(\frac{\omega^2}{\omega_D} + \frac{\omega\xi}{\Delta t\omega_D}\right)\sin\omega_D\Delta t + \frac{1}{\Delta t}\cos\omega_D\Delta t\right] - \frac{1}{\Delta t}\right\}$$

$$D'' = \frac{1}{k\Delta t}\left\{-e^{-\xi\omega\Delta t}\left(\frac{\omega\xi}{\omega_D}\sin\omega_D\Delta t + \cos\omega_D\Delta t\right) + 1\right\}$$

Solution:

Since the loading is given as a segmental linear function, the response obtained using the direct method will be exact. The necessary calculations are presented in a convenient tabular format in Table 4.1. For this system, the natural frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100,000}{100}} = 31.623 \text{ rad/ sec}$$

and damped natural frequency

$$\omega_D = \omega\sqrt{1-\xi^2} = 31.623\sqrt{1-0.2^2} = 30.984 \text{ rad/ sec}$$

Hence, the natural period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{31.623} = 0.20 \text{ sec}$$

Recommended practice is to select $\Delta t \leq \frac{T}{10}$. Specifically, we select $\Delta t = 0.02$ s. We then calculate the coefficients of Eqs. (4.34 and 4.35):

$$A' = 0.82180 \quad B' = 0.16517 \quad C' = 1.16755 \times 10^{-6} \quad D' = 6.1439 \times 10^{-7}$$

$$A'' = -16.5170 \quad B'' = 0.61286 \quad C'' = 7.60730 \times 10^{-5} \quad D'' = 8.9097 \times 10^{-5}$$

with initial conditions $u_0 = 0$ and $v_0 = 0$, we obtain from Eqs. (4.34, 4.35, and 4.36).

$$u_1 = 6.1439 \times 10^{-7} \times 1.2 \times 10^5 = 0.074 \text{ in}$$

$$\dot{u}_1 = 8.9124 \times 10^{-5} \times 1.2 \times 10^5 = 10.692 \text{ in/ sec}$$

$$\ddot{u}_1 = 31.623^2 \times 0.074 - 2 \times 0.2 \times 31.623 \times 10.692 + 1.2 \times 10^5/100 = 990.754 \text{ in/ sec}^2$$

Thus, completing the first cycle of calculations in the direct method of solution. Introducing the calculated values u_1, $\dot{u}_1$, and $\ddot{u}_1$, into the recurrence formulas Eqs. (4.34, 4.35, and 4.36), we obtain the response at time $t_2 = 0.04$ s. The continuation of this process results in the response of this system as shown in Table 4.1 up to 0.10 s.

Table 4.1 Calculation of the response for Illustrative Example 4.4

T(s)	u_i in.	$\dot{u}_i$ in./sec	$\ddot{u}_i$ in./sec^2	$F(\tau)$
0.000	0	0	0	0
0.020	0.074	10,692	990.754	120,000
0.040	0.451	25.155	430.768	120,000
0.060	0.926	17.096	−1142.511	0
0.080	1.044	−4.821	−982.581	0
0.100	0.778	−20.191	−522.555	0

Illustrative Example 4.4

Consider the tower shown in Fig. 4.11, but now subjected to a constant impulsive acceleration of magnitude $\ddot{u}_s = 0.5\ g$ during 0.5 s applied at the foundation of the tower. Determine the response of the tower in terms of the displacement and velocity of the mass relative to the motion of the foundation. Also, determine the maximum acceleration of the mass.

Solution:
The following data are obtained from Fig. 4.12:

Mass: $m = 100$ (lb · sec^2/in)

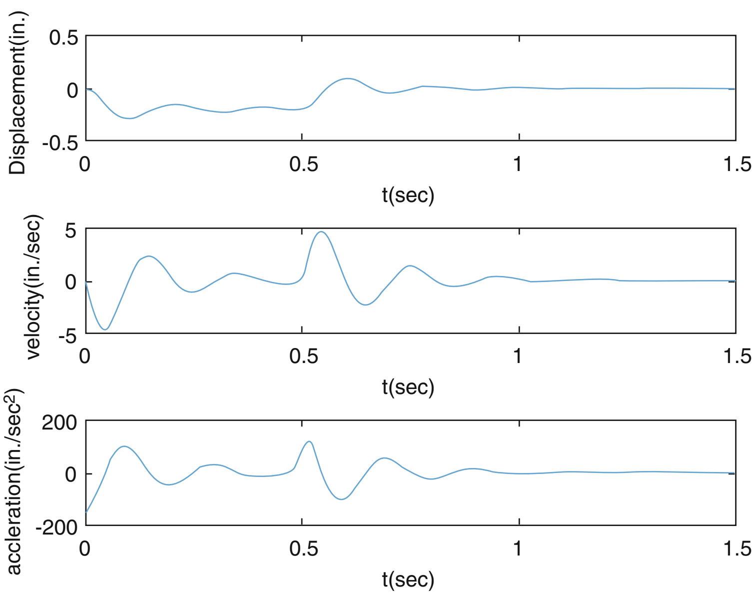

Fig. 4.12 Displacement, velocity, and acceleration (relative) for Illustrative Example 4.4

Spring constant: $k = 100{,}000$ (lb/in)
Damping coefficient: $c = 1265$ (lb · sec/in)
Acceleration of gravity: $g = 386$ (in/sec^2)
Select time step: $\Delta t = 0.002$ s
Excitation function:

Time (s)	Support acceleration (g)
0	0.5
0.5	0.5

Using ODE45 function in the MATLAB, two files can be used to create plots and determine maximum displacement, velocity, and acceleration.

Matlab file: Ex4_4.m

```
close all
clear
clc

%%%%-GIVEN VALUES-%%%%
deltat = 0.02;              %Time step
tspan=0:deltat:1.5;         %0 to 1.5 secs with the interval of 0.02 sec
IC = [0 0]';                %Initial conditions (u0=0, v0=0)

%%%Plot the displacement using ODE45
%%%-ESTIMATION-%%%
[t, u] = ode45(@SDOF2, tspan, IC);


%Display responses(displacement,velocity,acceleration)
umax= max(abs(u(:,1)))

%Velocity
udot = u(:,2);
udotmax = max(udot)

%Acceleration
udotdot = gradient(u(:,2), deltat);
udotdotmax = max(udotdot)

figure

subplot(3,1,1);
plot(t, u(:,1));
%Create x&y labels
xlabel ('t(sec)');
ylabel ('Displacement(in.)');

subplot(3,1,2);
plot(t, u(:,2));

%Create x&y labels
xlabel ('t(sec)');
ylabel ('Velocity(in./sec)');

subplot(3,1,3);
plot(t, udotdot);

%Create x&y labels
xlabel ('t(sec)');
ylabel ('Accleration(in./sec^2)');
```

Matlab file: SDOF2.m

```
function  u = SDOF2(t, u)

%%%%-GIVEN VALUES-%%%%

        m=100;                 %Mass (lb.sec^2/in.)
        k =100000;             %Stiffness (lb/in.)
        omega = sqrt(k/m);     %Natural Frequency
        c=1265;                %Damping coefficient (lb.sec/in.)
        g =386;                %Acceleration of gravity (in./sec^2)
        c_cr=2*m*omega;        %Critical damping coefficient
        xi = c/c_cr;           %Damping ratio

%%%Define the forcing function
        if t<=0.5
               F = -0.5*g*m/m;
            else
               F =0;
        end

%%%%-ESTIMATION-%%%%
u = [u(2); -omega*omega*u(1)-2*xi*omega*u(2)+F];
```

Maximum displacement and velocity are 0.2945 in. and velocity is 4.748 in./sec, respectively. The maximum acceleration is 120.07 in./sec^2. For excitation at the support: Displacement, velocity, and acceleration are relative to the support. The absolute acceleration is equal to be 120.07 + 0.5 g in./sec^2 using the relationship of $\ddot{u} = \ddot{u}_r + \ddot{u}_s$.

Illustrative Example 4.5.
A structural system modeled in Fig. 4.13a by the simple oscillator with 10% ($\xi = 0.10$) of critical damping is subjected to the impulsive load as shown in Fig. 4.13b. Polt the response (Displacement, velocity, and acceleration) using ODE45 method in MATLAB.

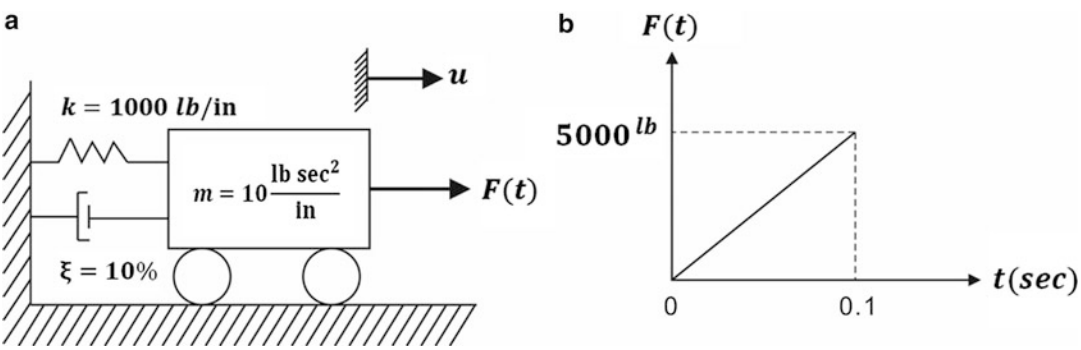

Fig. 4.13 Illustrative of Example 4.5 (a) Mathematical model, (b) Load function

Solution:
The following data are obtained from Fig. 4.14:

Mass: $m = 10\ \text{lb} \cdot \text{sec}^2/\text{in}$
Spring constant: $k = 10{,}000\ \text{lb/in}$
Damping coefficient: $c = \xi c_{cr} = 2\xi\sqrt{km} = 63.25\ \text{lb} \cdot \text{sec}/\text{in}$
Natural period: $T = 2\pi\sqrt{m/k} = 0.20\ \text{sec}$

Select time step for integration: $\Delta t = 0.02$ s
Excitation function: $F(t) = 50,000t\,\text{lb}, \quad 0 \le t \le 0.1\,\text{sec}\,.$

$$= 0 \qquad\qquad t > 0.1\,\text{sec}\,.$$

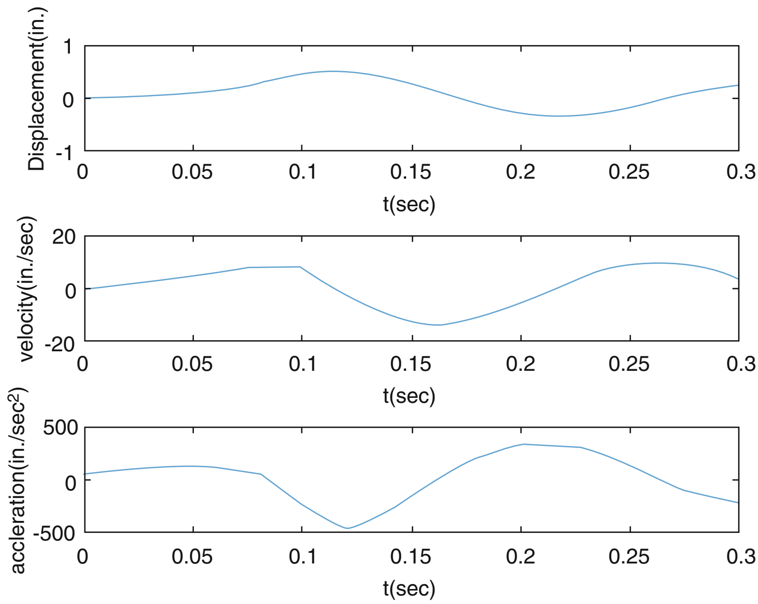

Fig. 4.14 Displacement, velocity, and acceleration for Illustrative Example 4.5 ($\Delta t = 0.02$ s)

Maximum time: $t_{\max} = 0.2$ s

4.5 Summary

In this chapter, we have shown that the differential equation of motion for a single-degree-of-freedom linear system can be solved for any forcing function $F(\tau)$ in terms of Duhamel's integral which for the undamped system is given by

$$u(t) = \frac{1}{m\omega} \int_0^t F(\tau) \sin \omega(t - \tau)\, d\tau \qquad (4.3) \text{ repeated}$$

and for a damped system by

$$u(t) = \frac{1}{m\omega_D} \int_0^t F(\tau) e^{-\xi\omega(t-\tau)} \sin \omega_D(t - \tau)\, d\tau \qquad (4.15) \text{ repeated}$$

where

$\omega = \sqrt{\frac{k}{m}}$ is the (undamped) natural frequency
$\omega_D = \omega\sqrt{1 - \xi^2}$ is the damped frequency
$\xi = \frac{c}{c_{c1}}$ is the damping ratio

The solution of the differential equation of motion may also be obtained by application of the Direct Method. In this method, it is assumed that the forcing function is given by a segmental linear function between defining points. Based on this assumption, the solution obtained is exact. The response is calculated at each time increment for the conditions existent at the end of the preceding time interval (initial conditions for the new time interval) and the action of the excitation applied during the time interval, which is assumed to be linear.

4.6 Analytical Problems

Problem 4.1

Develop a numerical method to evaluate Duhamel's Integral which gives the response of an undamped elastic one-degree-of-freedom structure modeled by the simple oscillation shown in

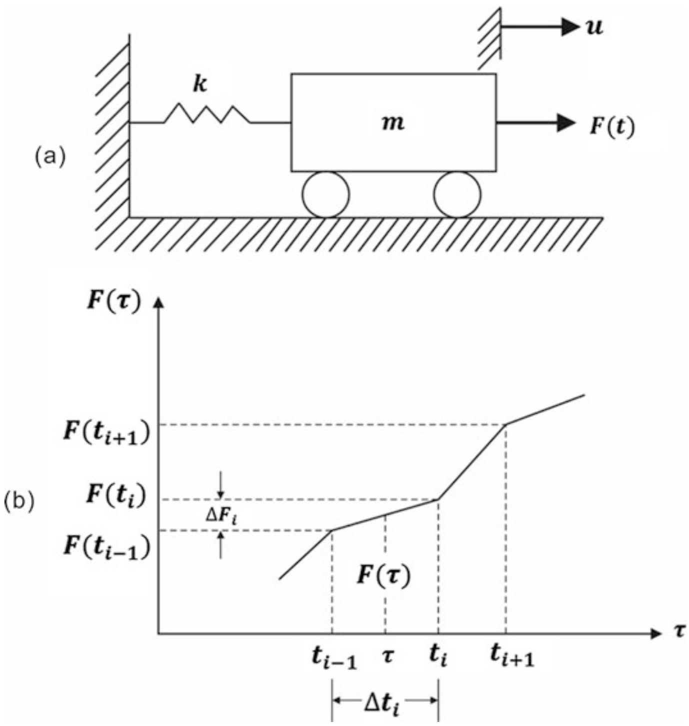

Fig. P4.1 (**a**) Structure modeled by the undamped simple oscillator. (**b**) Segmental linear forcing function

Fig. P4.1a. Assume that the forcing function $F(t)$ may be represented by a segmental linear function as shown in Fig. P4.1b.

Solution:

Assuming zero initial conditions, we obtain Duhamel's integral from Eq. (4.4) as

$$u(t) = \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau$$

Then, using the trigonometric identity

$$\sin\omega(t-\tau) = \sin\omega t\cos\omega\tau - \cos\omega t\sin\omega\tau$$

we obtain

$$u(t) = \sin\omega t\frac{1}{m\omega}\int_0^t F(\tau)\cos\omega\tau - \cos\omega t\frac{1}{m\omega}\int_0^t F(\tau)\sin\omega\tau\,d\tau$$

or

$$u(t) = \{A(t)\sin\omega t - B(t)\cos\omega t\}/m\omega \tag{a}$$

where

$$A(t) = \int_0^t F(\tau)\cos\omega\tau\,d\tau \tag{b}$$

$$B(t) = \int_0^t F(\tau)\sin\omega\tau\,d\tau \tag{c}$$

The calculation of Duhamel's integral thus requires the numerical evaluation of the integrals $A(t)$ and $B(t)$. It is more convenient to express the integrations in Eqs. (b and c) in incremental form, namely

$$A(t_i) = A(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau)\cos\omega\tau\,d\tau \tag{d}$$

$$B(t_i) = B(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau)\sin\omega\tau\,d\tau \tag{e}$$

where $A(t_i)$ and $B(t_i)$ represent the values of the integrals in Eq. (a) at time t_i. Assuming that the forcing function $F(\tau)$is approximated by a piecewise linear function as shown in Fig. P4.1b, we may write

$$F(\tau) = F(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i}(\tau - t_{i-1}), \qquad \mathrm{t_{i-1}} \le \tau \le t_i \tag{f}$$

where

$$\Delta F_i = F(t_i) - F(t_{i-1})$$

and

$$\Delta t_i = t_i - t_{i-1}$$

The substitution of Eq. (4.20) into Eq. (4.18) and integration yield

$$
\begin{aligned}
A(t_i) = & A(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right)(\sin\omega t_i - \sin\omega t_{i-1})/\omega \\
& + \frac{\Delta F_i}{\omega^2 \Delta t_i}\{\sin\omega t_i - \sin\omega t_{i-1} + \omega(t_i\cos\omega t_i - \omega t_{i-1}\cos\omega t_{i-1})\}
\end{aligned}
\tag{4.21}
$$

Analogously from Eq. (4.19),

$$
\begin{aligned}
B(t_i) = & B(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right)(\cos\omega t_{i-1} - \cos\omega t_i)/\omega \\
& + \frac{\Delta F_i}{\omega^2 \Delta t_i}\{\sin\omega t_i - \sin\omega t_{i-1} - \omega(t_i\cos\omega t_i - t_{i-1}\cos\omega t_{i-1})\}
\end{aligned}
\tag{4.22}
$$

Equations (4.21 and 4.22) are recurrent formulas for the evaluation of the integrals in Eq. (4.15) at any time $t = t_i$.

Problem 4.2
Develop a numerical method to evaluate Duhamel's integral including damping in the system.

Solution:
The response of a damped single-degree-of-freedom system in terms of Duhamel's integral is given by Eq.(4.15) as

$$
u(t) = \frac{1}{m\omega_D}\int_0^t F(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\,d\tau \quad \text{(repeated)} \tag{4.15}
$$

For numerical evaluation, we proceed as in the undamped case and obtain from Eq. (4.15)

$$
u(t) = \{A_D(t)\sin\omega_D t - B_D(t)\cos\omega_D t\}\frac{e^{-\xi\omega t}}{m\omega_D} \tag{g}
$$

where

$$
A_D(t_i) = A_D(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau)e^{\xi\omega\tau}\cos\omega_D\tau\,d\tau \tag{h}
$$

$$
B_D(t_i) = B_D(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau)e^{-\xi\omega\tau}\sin\omega_D\tau\,d\tau \tag{i}
$$

For a linear piecewise loading function, $F(\tau)$ given by Eq. (f) of Problem 4.1, is substituted into Eqs. (h and i) which require the evaluation of the following integrals:

$$
I_1 = \int_{t_{i-1}}^{t_i} e^{\xi\omega\tau}\cos\omega_D\tau\,d\tau = \frac{e^{\xi\omega\tau}}{(\xi\omega)^2 + \omega_D^2}(\xi\omega\cos\omega_D\tau + \omega_D\sin\omega_d\tau\Big|_{t_{i-1}}^{t_i} \tag{j}
$$

$$
I_2 = \int_{t_{i-1}}^{t_i} e^{\xi\omega\tau}\sin\omega_D\tau\,d\tau = \frac{e^{\xi\omega\tau}}{(\xi\omega)^2 + \omega_D^2}(\xi\omega\sin\omega_D\tau - \omega_D\cos\omega_d\tau\Big|_{t_{i-1}}^{t_i} \tag{k}
$$

$$I_3 = \int_{t_{i-1}}^{t_i} \tau e^{\xi\omega\tau} \sin \omega_D \tau \, d\tau = \left(\tau - \frac{\xi\omega}{(\xi\omega)^2 + \omega_D^2}\right) I_2' + \frac{\omega_D}{(\xi\omega)^2 + \omega_D^2} I_1' \Big|_{t_{i-1}}^{t_1} \tag{l}$$

$$I_4 = \int_{t_{i-1}}^{t_i} \tau e^{\xi\omega\tau} \cos \omega_D \tau \, d\tau = \left(\tau - \frac{\xi\omega}{(\xi\omega)^2 + \omega_D^2}\right) I_1' - \frac{\omega_D}{(\xi\omega)^2 + \omega_D^2} I_2' \Big|_{t_{i-1}}^{t_1} \tag{m}$$

where I_1' and I_2' are the integrals indicated in Eqs. (j and k) before their evaluation at the limits. In terms of these integrals, $A_D(t_i)$ and $B_D(t_i)$ may be evaluated after substituting Eq. (f) of Problem 4.1 into Eqs. (h and i) as

$$A_D(t_i) = A_D(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right) I_1 + \frac{\Delta F_i}{\Delta t_i} I_4 \tag{n}$$

$$B_D(t_i) = B_D(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right) I_2 + \frac{\Delta F_i}{\Delta t_i} I_3 \tag{o}$$

Finally, the substitution of Eqs. (n and o) into Eq. (g) gives the displacement at time t_i as

$$u_i(t_i) = \frac{e^{-\xi\omega t_i}}{m\omega_D} \{A_D(t_i) \sin \omega_D t_i - B_D(t_i) \cos \omega_D t_i\} \tag{p}$$

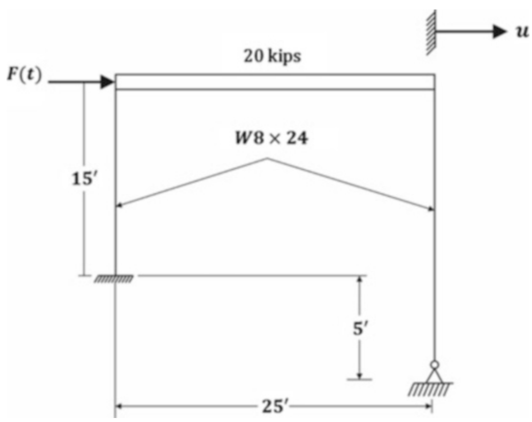

Fig. P4.3

4.7 Problems

Problem 4.3

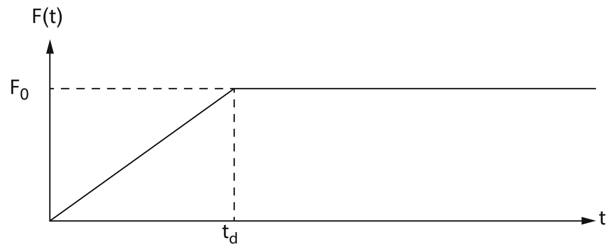

Fig. P4.5

The steel frame shown in Fig. P4.3 is subjected to a horizontal force F(t) applied at the girder level. The force decreases linearly from 5 kip at time $t = 0$ to zero at $t = 0.6$ s. Determine: (a) the horizontal deflection at $t = 0.5$ s and (b) the maximum horizontal deflection. Assume the columns massless and the girder rigid. Neglect damping.

Problem 4.4
Repeat Problem 4.3 for 10% of critical damping.

Problem 4.5
For the load-time function in Fig. P4.5, derive the expression for the dynamic load factor for the undamped simple oscillator as a function of t, ω, and t_d.

Problem 4.6
The frame shown in Fig. P4.3 is subjected to a sudden acceleration of 0.5 g applied to its foundation. Determine the maximum shear force in the columns. Neglect damping.

Problem 4.7
Repeat Problem 4.6 for 10% of critical damping.

Problem 4.8
Use Duhamel's integral to obtain the response of a damped simple oscillator of stiffness k, mass m, and damping ratio ξ, subjected to a suddenly applied force of magnitude F_0. Assume initial displacement and initial velocity equal zero.

Problem 4.9

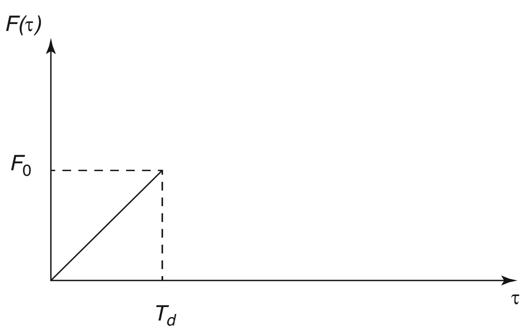

Fig. P4.11

Establish the equation of motion for the system in Problem 4.8 and solve it by superposition of complementary and [particular solutions with initial conditions for displacement and velocity equal to zero.

Problem 4.10

A trailer being pulled by a truck moving at constant speed v is idealized as a mass m connected to the truck by a spring of stiffness k. Determine the governing equation and its solution if the truck starts from rest.

Problem 4.11

Determine the response of an undamped system to a ramp force (Fig. P4.11) of maximum magnitude F_0 and duration t_d starting with zero initial conditions of displacement and velocity.

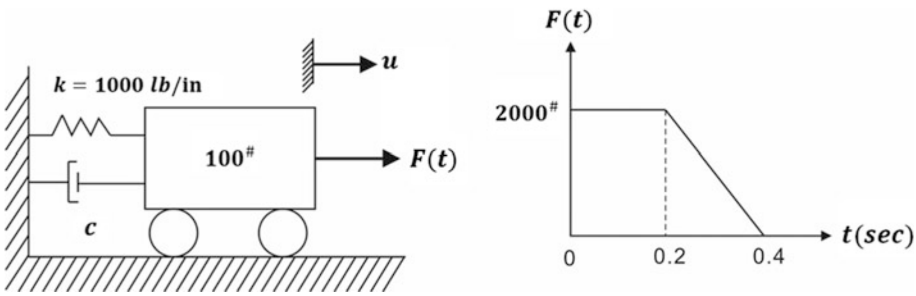

Fig. P4.14

Problem 4.12

Determine the maximum displacement at the top of the columns and maximum bending stress in the frame of Fig. P4.3 assuming that the columns are pinned at the base. Discuss the effect of base fixity.

Problem 4.13

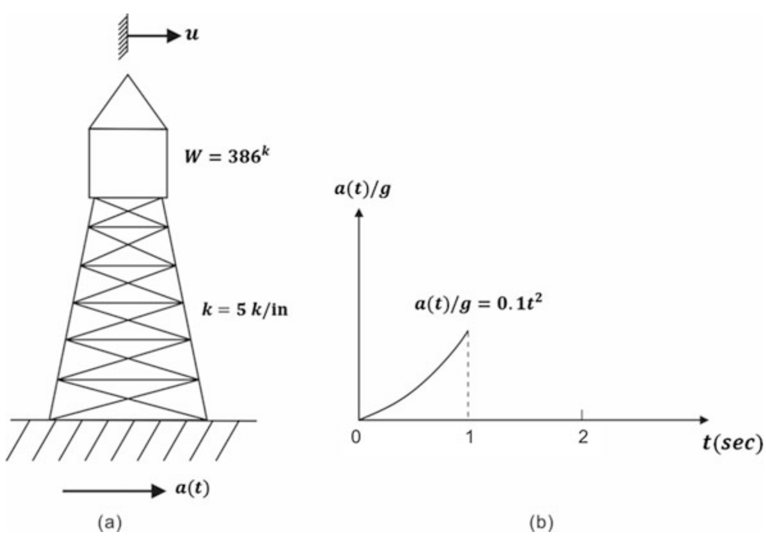

Fig. P4.16

Determine the maximum response (displacement and bending stress) for the frame of Illustrative Example 4.1 subjected to a triangular load of initial force $F_0 = 6000$ lb linearly decreasing to zero at time $t_d = 0.1$ s.

Problem 4.14.
For the dynamic system shown in Fig. P4.14, determine and plot the displacement as a function of time for the interval $0 \leq t \leq 0.5$ s. Neglect damping.

Problem 4.15
Repeat problem 4.14 for 10% critical damping.

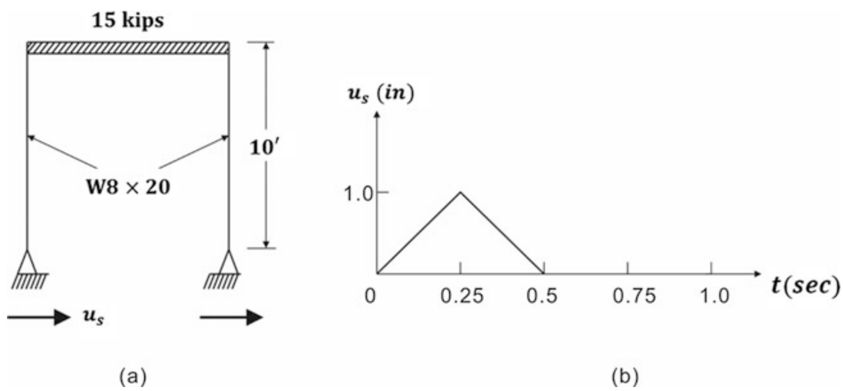

Fig. P4.19

Problem 4.16
The tower of Fig. P4.16a is subjected to horizontal ground acceleration $a(t)$ shown in Fig. P4.16b. Determine the relative displacement at the top of the tower at time $t = 1.0$ s. Neglect damping.

Problem 4.17
Repeat Problem 4.16 for 20% of critical damping.

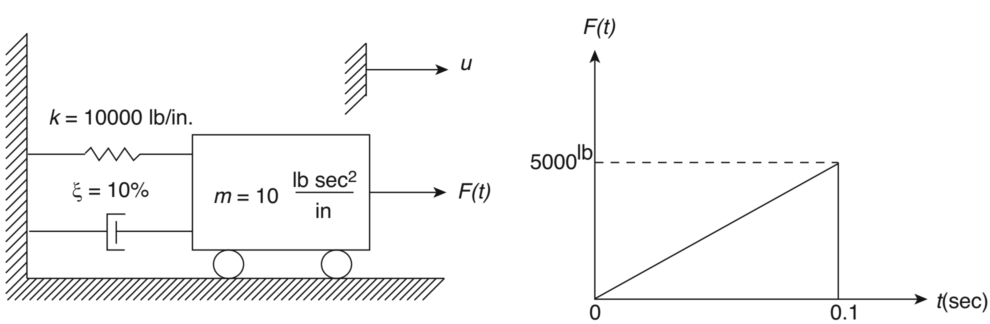

Fig. P4.21

Problem 4.18

Determine for the tower of Problem 4.17, the maximum displacement at the top of the tower relative to the ground displacement.

Problem 4.19

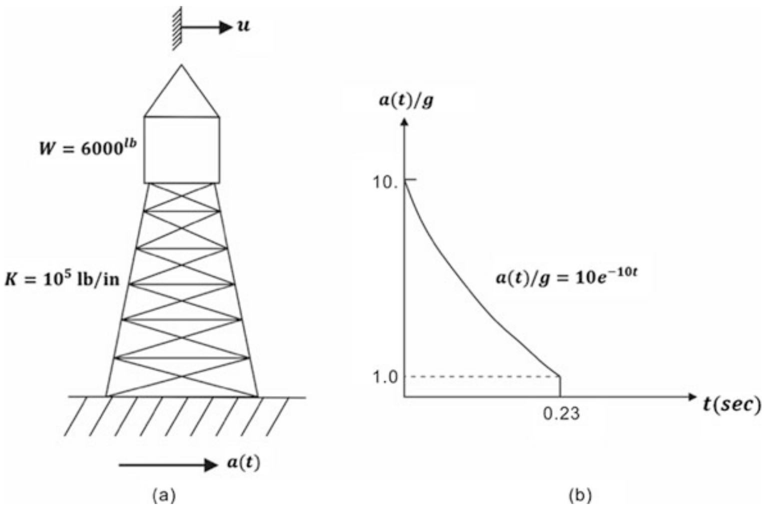

Fig. P4.22

The frame of Fig. P4.19a is subjected to horizontal support motion shown in Fig. P4.19b. Determine the maximum absolute deflection of top of the frame. Assume no damping.

Problem 4.20

Repeat Problem 4.19 for 10% of critical damping.

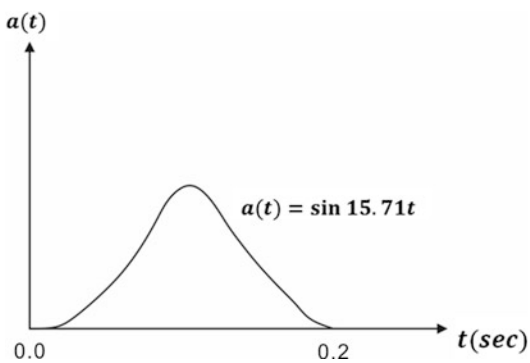

Fig. P4.24

Problem 4.21

A structural system modeled by the simple oscillator with 10% ($\xi = 0.10$) of critical damping is subjected to the impulsive load as shown in Fig. P4.21. Determine the response.

Problem 4.22

A water tower modeled as shown in Fig. P4.22a is subjected to ground shock given by the function

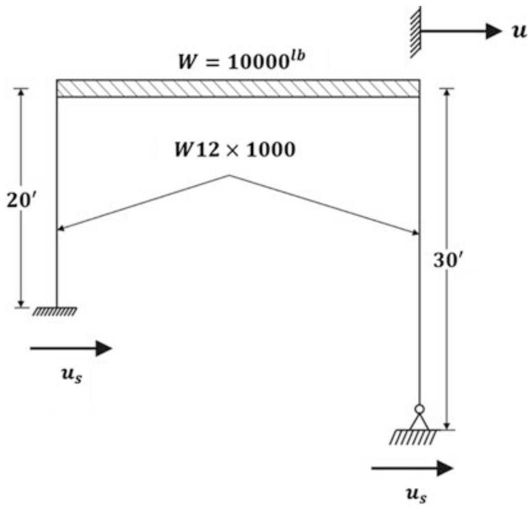

Fig. P4.25

depicted in Fig. P4.22b. Determine: (a) the maximum displacement at the top of the tower and (b) the maximum shear force at the base of the tower. Neglect damping. Use time step for integration $\Delta t = 0.005$ s.

Problem 4.23

Repeat Problem 4.22 for 20% of critical damping.

Problem 4.24

Determine the maximum response of the tower of Problem 4.22 when subjected to the impulsive ground acceleration depicted in Fig. P4.24.

Problem 4.25

The steel frame in Fig. P4.25 is subjected to the ground motion produced by a passing train in its vicinity. The ground motion is idealized as a harmonic acceleration of the foundation of the frame with amplitude 0.1 g at frequency 10 cps. Determine the maximum response in terms of the relative displacement of the girder of the frame and the motion of the foundation. Assume 10% of the critical damping.

Problem 4.26

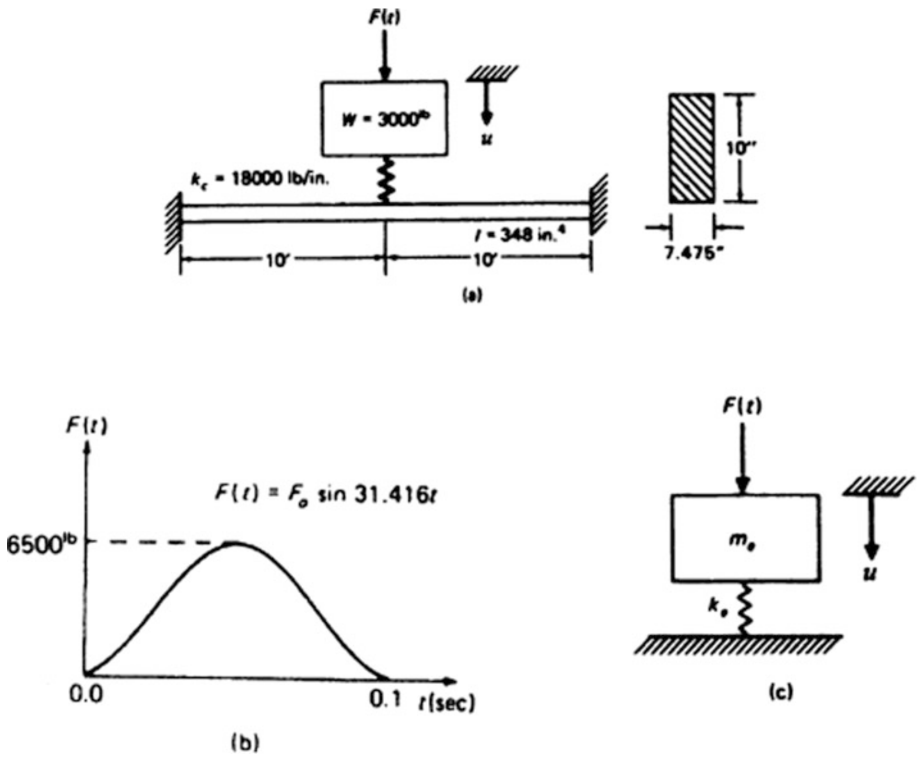

Fig. P4.27

Determine the maximum stresses in the columns of the frame in Problem 4.25 using the maximum response in terms of relative motion. Also check that the same results may be obtained using the response in terms of the maximum absolute acceleration.

Problem 4.27

A machine having a weight $W = 3000$ lb is mounted through coil springs to a steel beam of rectangular cross-section as shown in Fig. P4.27a. Due to malfunctioning, the machine produces a shock force represented in Fig. P4.27b. Neglecting the mass of the beam and damping in the system, determine the maximum displacement of the machine.

Problem 4.28

For Problem 4.27 determine: (a) the maximum tensile and compressive stresses in the beam and (b) the maximum force experienced by the coil springs during the shock.