16 Time History Response of Multi-Degree-of-Freedom Systems

The nonlinear analysis of a single-degree-of-freedom system using the step-by-step linear acceleration method was presented in Chap. 6. The extension of this method with a modification known as the Wilson-θ method, for the solution of structures modeled as multidegree-of-freedom systems is developed in this chapter. The modification introduced in the method by Wilson et al. 1973 serves to assure the numerical stability of the solution process regardless of the magnitude selected for the time step; for this reason, such a method is said to be *unconditionally stable*. On the other hand, without Wilson's modification, the step-by-step linear acceleration method is only conditionally stable and for numerical stability of the solution it may require such an extremely small time step as to make the method impractical if not impossible. The development of the necessary algorithm for the linear and nonlinear multidegree-of-freedom systems by the step-by-step linear acceleration method parallels the presentation for the single-degree-of-freedom system in Chap. 6.

Another well-known method for step-by-step numerical integration of the equations of motion of a discrete system is the Newmark beta method. This method which also may be considered a generalization of the linear acceleration method is presented later in this chapter after discussing in detail the Wilson-θ method.

16.1 Incremental Equations of Motion

The basic assumption of the Wilson-θ method is that the acceleration varies linearly over the time interval from t to $t + \theta\Delta\, t$. Where $\theta \geq 1.0$. The value of the factor θ is determined to obtain optimum stability of the numerical process and accuracy of the solution. It has been shown by Wilson that, for $\theta \geq 1.38$, the method becomes unconditionally stable.

The equation of motion evaluated at time t_i for a mutidegree-of-freedom system in matrix notation is given by

$$\mathbf{M}\ddot{\mathbf{u}}_\mathbf{i}+\mathbf{C}(\dot{\mathbf{u}})\dot{\mathbf{u}}_\mathbf{i}+\mathbf{K}(\mathbf{u})\mathbf{u}_\mathbf{i}=\mathbf{F}_\mathbf{i}(t) \tag{16.1}$$

where **M, C** and **K** are, respectivly the mass, damping and stiffness matrices of the system; $\mathbf{u}_\mathbf{i}, \dot{\mathbf{u}}_\mathbf{i}, \ddot{\mathbf{u}}_\mathbf{i}$ the displacement, velocity and acceleration vectors; and $\mathbf{F}_\mathbf{i}$(t) the external force vector.

M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_16

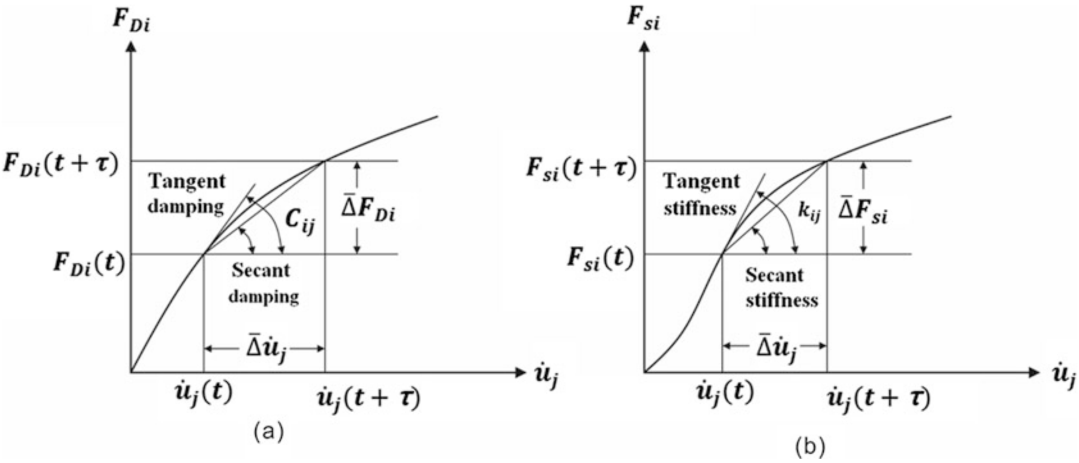

Fig. 16.1 Definition of influence coefficients, (**a**) Nonlinear viscous damping, c_{ij}. (**b**) Nonlinear stiffness, k_{ij}

Then the equations expressing the incremental equilibrium conditions for a multidegree-of-freedom system can be derived as the matrix[1] equivalent of the incremental equation of motion for the single degree-of-freedom system, Eq. (6.6). Thus taking the difference between dynamic equilibrium conditions defined at time t_i and at $t_i + \tau$, where $\tau = \theta\Delta \ t$, we obtain the incremental equations of motion:

$$\mathbf{M}\widehat{\boldsymbol{\Delta}}\ddot{\mathbf{u}}_\mathbf{i}+\mathbf{C}(\dot{\mathbf{u}})\widehat{\boldsymbol{\Delta}}\dot{\mathbf{u}}_\mathbf{i}+\mathbf{K}(\mathbf{u})\widehat{\boldsymbol{\Delta}}\mathbf{u}_\mathbf{i}=\widehat{\boldsymbol{\Delta}}\mathbf{F}_\mathbf{i} \tag{16.2}$$

in which the circumflex over Δ indicates that the increments are associated with the extended time step $\tau = \theta\Delta \ t$. Thus

$$\widehat{\Delta}u_i = u(t_i + \tau) - u(t_i) \tag{16.3}$$

$$\widehat{\Delta}\dot{u}_i = \dot{u}(t_i + \tau) - \dot{u}(t_i) \quad \ldots \tag{16.4}$$

$$\widehat{\Delta}\ddot{u}_i = \ddot{u}(t_i + \tau) - \ddot{u}(t_i) \tag{16.5}$$

and

$$\widehat{\Delta}F_i = F(t_i + \tau) - F(t_i) \tag{16.6}$$

In writing Eq. (16.2), we assumed, as explained in Chap. 6 for single-degree-of-freedom systems, that the stiffness and damping are obtained for each time step as the initial values of the tangent to the corresponding curves as shown in Fig. 16.1 rather than the slope of the secant line which requires iteration. Hence the stiffness coefficient is defined as

$$k_{ij} = \frac{dF_{si}}{du_j} \tag{16.7}$$

and the damping coefficient as

[1] Matrices and vectors are denoted with boldface lettering throughout this chapter.

$$c_{ij} = \frac{dF_{Di}}{d\dot{u}_j} \tag{16.8}$$

in which F_{si}, and F_{Di} are, respectively, the elastic and damping forces at nodal coordinate i and u_j and u_j are, respectively, the displacement and velocity at nodal coordinate j,.

16.2 The Wilson-θ Method

The integration of the nonlinear equations of motion by the step-by-step linear acceleration method with the extended time step introduced by Wilson is based, as has already been mentioned, on the assumption that the acceleration may be represented by a linear function during the time step $\tau = \theta\Delta\, t$ as shown in Fig. 16.2. From this figure we can write the linear expression for the acceleration during the extended time step as

$$\ddot{u}(t) = \ddot{u}_i + \frac{\widehat{\Delta}\ddot{u}_i}{\tau}(t - t_i) \tag{16.9}$$

in which $\widehat{\Delta}\ddot{u}_i$ is given by Eq. (16.5). Integrating Eq. (16.9) twice between limits t_i and t yields

$$\dot{u}\,(t) = \dot{u}_i + \ddot{u}_i(t - t_i) + \frac{1}{2}\frac{\widehat{\Delta}\ddot{u}_i}{\tau}(t - t_i)^2 \tag{16.10}$$

and

$$u(t) = u_i + \dot{u}_i(t - t_i) + \frac{1}{2}\ddot{u}_i(t - t_i)^2 + \frac{1}{6}\frac{\widehat{\Delta}\ddot{u}_i}{\tau}(t - t_i)^3 \tag{16.11}$$

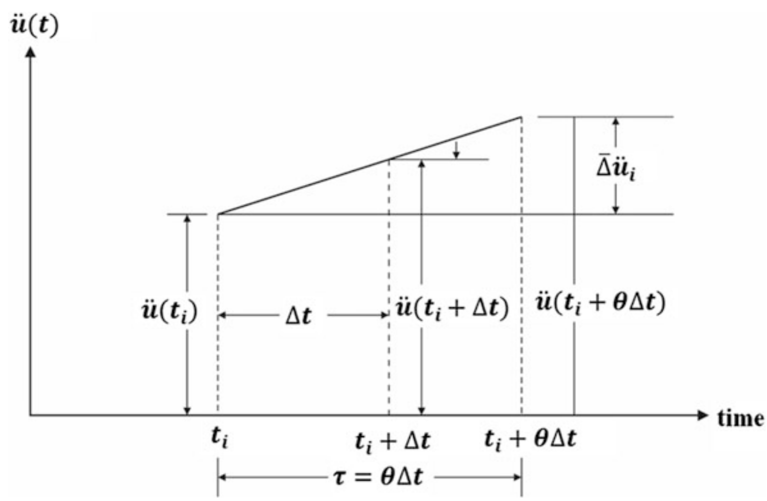

Fig. 16.2 Linear acceleration assumption in the extended time interval

Evaluation of Eqs. (16.10) and (16.11) at the end of the extended interval $t = t_i + \tau$ gives

$$\widehat{\Delta}\dot{u}_i = \ddot{u}_i\tau + \frac{1}{2}\widehat{\Delta}\ddot{u}_i\tau \tag{16.12}$$

and

$$\widehat{\Delta} u_i = \dot{u}_i \tau + \frac{1}{2} \ddot{u}_i \tau^2 + \frac{1}{6} \widehat{\Delta} \ddot{u}_i \tau^2 \quad (16.13)$$

in which $\widehat{\Delta} u_i$ and $\widehat{\Delta} \dot{u}_i$ are defined by Eqs. (16.3) and (16.4), respectively.

Now Eq. (16.13) is solved for the incremental acceleration $\widehat{\Delta} \ddot{u}_i$ and substituted in Eq. (11.12) to obtain:

$$\widehat{\Delta} \ddot{u}_i = \frac{6}{\tau^2} \widehat{\Delta} u_i - \frac{6}{\tau} \dot{u}_i - 3\ddot{u}_i \quad (16.14)$$

and

$$\widehat{\Delta} \dot{u}_i = \frac{3}{\tau} \widehat{\Delta} u_i - 3\dot{u}_i - \frac{\tau}{2} \ddot{u}_l \quad (16.15)$$

Finally, substituting Eqs. (16.14) and (16.15) into the incremental equation of motion, Eq. (16.2), results in an equation for the incremental displacement $\widehat{\Delta} u_i$, which may be conveniently written as

$$\bar{K}_i \widehat{\Delta} u_i = \overline{\widehat{\Delta} F_i} \quad (16.16)$$

where

$$\bar{K}_i = K_i + \frac{6}{\tau^2} M + \frac{3}{\tau} C_i \quad (16.17)$$

and

$$\overline{\widehat{\Delta} F_i} = \widehat{\Delta} F_i + M\left(\frac{6}{\tau} \dot{u}_i + 3\ddot{u}_i\right) + C_i\left(3\dot{u}_i + \frac{\tau}{2} \ddot{u}_i\right) \quad (16.18)$$

the matrix equation (16.16) has the same form as the static incremental equilibrium equation and may be solved for the incremental displacements $\widehat{\Delta} u_i$ by simply solving a system of linear equations.

To obtain the incremental accelerations $\widehat{\Delta} \ddot{u}_i$ for the extended time interval, the value of $\widehat{\Delta} u_i$ obtained from the solution of Eq. (16.16) is substituted into Eq. (16.14). The incremental acceleration $\widehat{\Delta} \ddot{u}_i$ for the normal time interval Δt is then obtained by a simple linear interpolation. Hence

$$\Delta \ddot{u} = \frac{\widehat{\Delta} \ddot{u}}{\theta} \quad (16.19)$$

To calculate the incremental velocity $\Delta \dot{u}_i$ and incremental displacement Δu_i, corresponding to the normal interval Δt, use is made of Eqs. (16.12) and (16.13) with the extended time interval parameter τ substituted for Δt, that is,

$$\Delta \dot{u}_i = \ddot{u}_i \Delta t + \frac{1}{2} \Delta \ddot{u}_i \Delta t \quad (16.20)$$

and

$$\Delta u_i = \dot{u}_i \Delta t + \frac{1}{2} \ddot{u}_i \Delta t^2 + \frac{1}{6} \Delta \ddot{u}_i \Delta t^2 \quad (16.21)$$

Finally, the displacement u_{i+1} and velocity $\dot{u}_{i+1}$ at the end of the normal time interval are calculated by

$$u_{i+1} = u_i + \Delta u_i \quad (16.22)$$

and

$$\dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i \tag{16.23}$$

As mentioned in Chap. 6 for the single degree-of-freedom system, the initial acceleration for the next step is calculated from the condition of dynamic equilibrium at the time $t + \Delta t$; thus

$$\ddot{u}_{i+1} = M^{-1}\left[F_{i+1} - F_D\left(\dot{u}_{i+1}\right) - F_S(u_{i+1})\right] \tag{16.24}$$

in which $F_D\left(\dot{u}_{i+1}\right)$ and $F_S(u_{i+1})$ represent, respectively, the damping force and stiffness force vectors evaluated at the end of the time step $t_{i+1} = t_i + \Delta t$. Once the displacement, velocity, and acceleration vectors have been determined at time $t_{i+1} = t_i + \Delta t$, the outlined procedure is repeated to calculate these quantities at the next time step $t_{i+2} = t_{i+1} + \Delta t$ and the process is continued to any desired final time.

The step-by-step linear acceleration, as indicated in the discussion for the single degree-of-freedom system, involves two basic approximations: (1) the acceleration is assumed to vary linearly during the time step, and (2) the damping and stiffness characteristics of the structure are evaluated at the initiation of the time step and are assumed to remain constant during this time interval. The algorithm for the integration process of a linear system by the Wilson-8 method is outlined in the next section. The application of this method to linear structures is then developed in the following section.

16.3 Algorithm for Step-by-Step Solution of a Linear System Using the Wilson-θ Method

16.3.1 Initialization

1. Assemble the system stiffness matrix $\boldsymbol{K}$**,** mass matrix $\boldsymbol{M}$**,** and damping matrix **C**.
2. Set initial values for displacement u_0, velocity $\dot{u}_0$, and forces F_0.
3. Calculate initial acceleration $\ddot{u}_0$ from Eq. (16.1) as

$$M\ddot{u}_0 = F_0 - C\dot{u}_0 - Ku_0$$

4. Select a time step Δt, the factor θ (usually Taken as 1.4), and calculate the constants τ, a1, a2, a3 and a4 from the relationships:

$$\tau = \theta\Delta t, \quad a_1 = \frac{3}{\tau}, \quad a_2 = \frac{6}{\tau}, \quad a_3 = \frac{\tau}{2}, \quad a_4 = \frac{6}{\tau^2},$$

5. Form the effective stiffness matrix $\boldsymbol{K}$ [Eq. (16.17)], namely,

$$\bar{K} = K + a_4 M + a_1 C$$

16.3.2 For Each Time Step

1. Calculate by linear interpolation the incremental load $\widehat{\Delta F}_i$ for the time interval t_i to $t_i + \tau$, from the relationship

$$\widehat{\Delta} F_i = F_{i+1} + (F_{i+2} - F_{i+1})(\theta - 1) - F_i$$

2. Calculate the effective incremental load $\overline{\widehat{\Delta} F_i}$; for the time interval ti to ti,+ τ, from Eq. (16.18) as

$$\overline{\widehat{\Delta} F_i} = \widehat{\Delta} F_i + (a_2 M + 3C)\dot{u}_i + (3M + a_3 C)\ddot{u}_i$$

3. Solve for the incremental displacement $\widehat{\Delta} u_i$, from Eq. (16.16) as

$$\bar{K}\widehat{\Delta} u_i = \overline{\widehat{\Delta} F_i}$$

4. Calculate the incremental acceleration for the extended time interval τ, from the relation Eq. (16.14) as

$$\widehat{\Delta}\ddot{u}_i = a_4 \widehat{\Delta} u_i - a_2 \dot{u}_i - 3\ddot{u}_i$$

5. Calculate the incremental acceleration for the normal interval from Eq. (16.19) as

$$\Delta\ddot{u} = \frac{\widehat{\Delta}\ddot{u}}{\theta}$$

6. Calculate the incremental velocity $\Delta\dot{u}_i$, and the incremental displacement Δu_i, from time ti to ti + Δt from Eqs. (16.20) and (16.21) as

$$\Delta\dot{u}_i = \ddot{u}_i \Delta t + \frac{1}{2}\Delta\ddot{u}_i \Delta t$$

$$\Delta u_i = \dot{u}_i \Delta t + \frac{1}{2}\ddot{u}_i \Delta t^2 + \frac{1}{6}\Delta\ddot{u}_i \Delta t^2$$

7. Calculate the displacement and velocity at time $ti + 1 = ti + \Delta t$ using

$$u_{i+1} = u_i + \Delta u_i$$

$$\dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i$$

8. Calculate the acceleration $\ddot{u}_{i+1}$ at time $t_{i+1} = t_i + \Delta t$ directly from the equilibrium equation of motion, namely,

$$M\ddot{u}_{i+1} = F_{i+1} - C\dot{u}_{i+1} - Ku_{i+1}$$

Illustrative Example 16.1

Calculate the displacement response for a two-story shear building of Fig. 16.3 subjected to a suddenly applied force of 10 Kip at the level of the second floor. Neglect damping and assume elastic behavior.

Solution

The equations of motion, in matrix notation, for this structure are:

$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

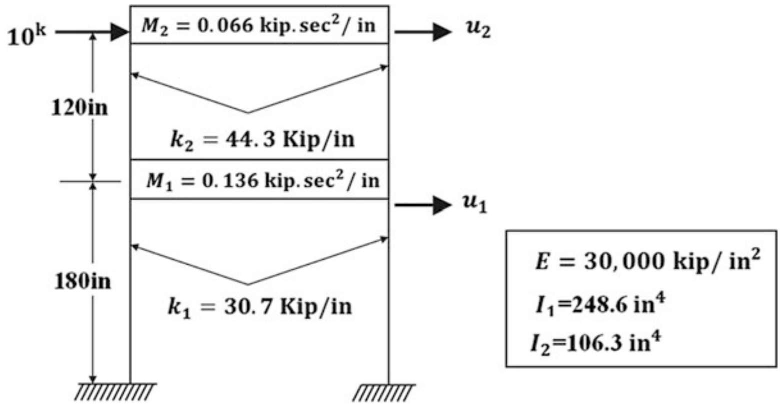

Fig. 16.3 Two-story shear building for Illustrative Examples 16.1 and 16.2

which, for free vibration, become

$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

Substitution of $y_i = a_i \, sin \, \omega t$ results in the eigenproblem:

$$\begin{bmatrix} 75.0 - 0.136\omega^2 & -44.3 \\ -44.3 & 44.3 - 0.066\omega^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

which requires for a nontrivial solution that

$$\begin{vmatrix} 75.0 - 0.136\omega^2 & -44.3 \\ -44.3 & 44.3 - 0.0666\omega^2 \end{vmatrix} = 0$$

Expansion of this determinant yields

$$\omega^4 - 1222.68\omega^2 + 151516 = 0$$

which has the roots

$$\omega_1^2 = 139.94 \qquad \text{and} \qquad \omega_2^2 = 1082.0$$

Hence, the natural frequencies are

$$\omega_1 = 11.83 \text{ rad/ sec}, \qquad \text{and} \qquad \omega_2 = 32.90 \text{ rad/ sec},$$

or

$$f_1 = 1.883 \text{ cps}, \qquad \text{and} \qquad f_2 = 5.237 \text{ cps}$$

and the natural periods

$$T_1 = 0.531 \text{ sec}, \qquad \text{and} \qquad T_2 = 0.191 \text{ sec}$$

The initial acceleration at the nodal coordinates is calculated from Eq. (16.1) after setting the initial displacement and velocity equal to zero. Thus we obtain:

$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{10} \\ \ddot{u}_{20} \end{Bmatrix} \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

giving

$$\ddot{u}_{10} = 0$$

$$\ddot{u}_{20} = 151.51 \text{ in}/\sec^2$$

Conveniently, we select $\Delta t = 0.02$ sec and $\theta = 1.4$, so $\tau = \theta\Delta t = 0.028$, and calculate the constants to obtain:

$$a_1 = \frac{3}{\tau} = 107.14, \qquad a_2 = \frac{6}{\tau} = 0.014$$

$$a_3 = \frac{\tau}{2} = 214.28, \qquad a_4 = \frac{6}{\tau^2} = 7653$$

The effective stiffness is then

$$\bar{K} = K + a_4 M + a_1 C \qquad (C = 0 \quad \text{foranundampedsysteem})$$

$$\bar{K} = \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} + 7653 \begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} 1115.8 & -44.3 \\ -44.3 & 549.4 \end{bmatrix}$$

and the effective force

$$\overline{\widehat{\Delta} F_i} = \widehat{\Delta} F_i + (a_2 M + 3C)\dot{u}_i + (3M + a_3 C)\ddot{u}_i$$

$$\overline{\widehat{\Delta} F_i} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + 21428 \begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + 3 \begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} 0 \\ 151.51 \end{Bmatrix}$$

$$\overline{\widehat{\Delta} F_i} = \begin{Bmatrix} 0 \\ 30 \end{Bmatrix}$$

Solving for $\widehat{\Delta} u$ from $\bar{K}\widehat{\Delta} u = \overline{\widehat{\Delta} F}$ yields

$$\begin{bmatrix} 1115.8 & -44.3 \\ -44.3 & 549.4 \end{bmatrix} \begin{Bmatrix} \widehat{\Delta} u_1 \\ \widehat{\Delta} u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30 \end{Bmatrix}, \qquad \widehat{\Delta} u = \begin{Bmatrix} 0.002175 \\ 0.054780 \end{Bmatrix}$$

Solving for $\widehat{\Delta} \ddot{u}$ from Eq. (16.14) we obtain

$$\widehat{\Delta} \ddot{u} = 7656 \begin{Bmatrix} 0.002175 \\ 0.054780 \end{Bmatrix} - 214.28 \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - 3 \begin{Bmatrix} 0 \\ 151.51 \end{Bmatrix}, \quad \widehat{\Delta} \ddot{u} = \begin{Bmatrix} 16.645 \\ -35.299 \end{Bmatrix}$$

Then

$$\Delta \ddot{u} = \frac{\widehat{\Delta} \ddot{u}}{\theta} = \frac{1}{1.4} \begin{Bmatrix} 16.647 \\ -35.299 \end{Bmatrix} = \begin{Bmatrix} 18.891 \\ -25.21 \end{Bmatrix}$$

From Eq. (16.20), it follows that

$$\Delta \dot{u} = \begin{Bmatrix} 0 \\ 151.51 \end{Bmatrix} (0.02) + \frac{0.02}{2} \begin{Bmatrix} 11.891 \\ -25.21 \end{Bmatrix} = \begin{Bmatrix} 0.1189 \\ 2.7781 \end{Bmatrix}$$

From Eq. (16.21),

$$\Delta u = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}(0.02) + \frac{(0.02)^2}{2}\begin{Bmatrix} 0 \\ 151.61 \end{Bmatrix} + \frac{(0.02)^2}{6}\begin{Bmatrix} 11.891 \\ -25.21 \end{Bmatrix} = \begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix}$$

From Eqs. (16.21) and (16.23)

$$\{u\}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix} = \begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix} \tag{a}$$

and

$$\{\dot{u}\}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.1189 \\ 2.7781 \end{Bmatrix} = \begin{Bmatrix} 0.1189 \\ 2.7781 \end{Bmatrix} \tag{b}$$

From Eq. (16.23),

$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix}\{\ddot{u}\}_1 = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} - \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix}\begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix}$$

Which gives

$$\{\ddot{u}\}_1 = \begin{Bmatrix} 8.875 \\ 132.85 \end{Bmatrix} \tag{c}$$

The results given in Eqs. (a), (b), and (c) for the displacement, velocity, and acceleration, respectively, at time $t_1 = t_0 + \Delta t$ complete a first cycle of the integration process. The continuation in determining the response for this structure is given in Illustrative Example 16.2 with the use of the computer program described in the next section.

16.4 Response by Step Integration Using MATLAB

MATLAB performs the step-by-step integration of the equations of motion for a linear system using the linear acceleration method with the Wilson-θ modification. The program requires previous modeling of the structure to determine the stiffness matrix and the mass matrix of the system.

The program performs a linear interpolation between the load data points, which result in the magnitude of the applied forces at each nodal coordinate calculated at increments of time equal to the time step Δt. The output consists of the response for each nodal coordinate in terms of displacement, velocity, and acceleration at increments of time Δt up to the maximum time specified by the duration of the forces including, if desired, an extension with forces set to zero.

Illustrative Example 16.2

Use MATLAB to determine the response of the two-story shear building shown in Fig. 16.3. The first cycle of the integration process for this structure has been hand calculated in Illustrative Example 16.1. This is confirmed by the MATLAB program as follows:

Solution:

```
clc
clear all
close all

%___________________________________________________________________________
% Inputs:
%   M, K
%   F = forcing function
%   t = Time period
%   u0 = initial displacement
%   v0 = initial velocity
%   w0 = initial acceleration
%___________________________________________________________________________

%%%%-GIVEN VALUES-%%%%

deltat = 0.02;  % ?t=0.02

%%%Define Mass Matrix
M = [136 0;
      0 66];

%%%Define Stiffness Matrix
 k1=30700;
 k2=44300;

 K = [k1+k2 -k2;
     -k2 k2];

%%%Define Damping Matrix
C =0;

[n,n]= size(M);

%%%Define the load vector
F = zeros(n,1); F(2) =10000;
%__________________________________________________________________________
% Initial conditions
%__________________________________________________________________________

u0 = zeros(n,1); u0(1) =0;
v0 = zeros(n,1); v0(1) =0;
w0 = inv(M)*(F-K*u0-C*v0);             % acceleration at t = 0

  t = 0;

u_t1(:,1)=u0(1,:)';
u_t2(:,1)=u0(1,:)';

v_t1(:,1)=v0(1,:)';
v_t2(:,1)=v0(2,:)';

w_t1(:,1)=w0(1,:)';
w_t2(:,1)=w0(2,:)';

%__________________________________________________________________________
% For each time step apply the Wilson-? Method to calculate u, v, and w
%__________________________________________________________________________
```

```
for i=2:100

[u,v,w,t] = STI(t, deltat, M, K, C, F, u0, v0, w0);

ti(:,i)=t(:,1)';

u_t1(:,i) = u(1,:)';
u_t2(:,i) = u(2,:)';

v_t1(:,i) = v(1,:)';
v_t2(:,i) = v(2,:)';

w_t1(:,i) = w(1,:)';
w_t2(:,i) = w(2,:)';

u0=u;                       %u_(i) for calculating u_(i+1)
v0=v;                       %v_(i) for calculating v_(i+1)
w0=w;                       %w_(i) for calculating w_(i+1)
F=F;                        %F(i) for calculating F(i+1)
t=t;                        %t(i) for calculating t(i+1)
end

figure(1)

subplot(3,2,1)
plot (ti, u_t1);
title ('1DOF'); xlabel ('Time (sec)'); ylabel ('u_1(in.)'); grid on

subplot(3,2,2)
plot (ti, u_t2);
title ('2DOF'); xlabel ('Time (sec)'); ylabel ('u_2(in.)'); grid on

subplot(3,2,3)
plot (ti, v_t1);
title ('1DOF'); xlabel ('Time (sec)'); ylabel ('v_1(in./sec)'); grid on

subplot(3,2,4)
plot (ti, v_t2);
title ('2DOF'); xlabel ('Time (sec)'); ylabel ('v_2(in./sec)'); grid on

subplot(3,2,5)
plot (ti, w_t1);
title ('1DOF'); xlabel ('Time (sec)'); ylabel ('w_1(in./sec^2)'); grid on

subplot(3,2,6)
plot (ti, w_t2);
title ('2DOF'); xlabel ('Time (sec)'); ylabel ('w_2(in./sec^2)'); grid on

u_1max= max(abs(u_t1))          %Max. displacement @ 1DOF (or 1st story)
u_2max= max(abs(u_t2))          %Max. displacement @ 2DOF (or 2nd story)

v_1max= max(abs(v_t1))          %Max. velocity @ 1DOF (or 1st story)
v_2max= max(abs(v_t2))          %Max. velocity @ 2DOF (or 2nd story)

w_1max= max(abs(w_t1))          %Max. acceleration @ 1DOF (or 1st story)
w_2max= max(abs(w_t2))          %Max. acceleration @ 2DOF (or 2nd story)
```

For each time step, the function file of STI.m is used to calculate the displacement, velocity, and acceleration using the Wilson-θ Method (Sect. 16.3).

```
function [u,v,w,t]=STI(t, deltat, M, K, C, F,u0,v0,w0)

%______________________________________________________________________________
%  The Wilson-? Method
%   t = Time period
%   u = displacement
%   v = velocity
%   w = acceleration
%______________________________________________________________________________

%%%Calculate the constants ?, a1, a2, a3 and a4 from the relationships
theta = 1.4;
tau = theta*deltat;

a1 = 3/tau; a2 = 6/tau; a3 = tau/2; a4 = 6/tau^2;          %16.3.1 (4)

K_bar = K+a1.*C+a4.*M;                                      %16.3.1.(5)

%%%For each time step (Ch 16.3.2)
F_eff0 = F+(F-F)*(theta-1)-F;                               %16.3.2.(1)
F_eff = F_eff0+(a2.*M+3*C)*v0+(3.*M+a3.*C)*w0;              %16.3.2.(2)

delta_u= inv(K_bar)*F_eff;                                  %16.3.2.(3)

delta_w =1./theta*(a4.*delta_u-a2.*v0-3*w0);                %16.3.2.(4)&(5)

delta_v = deltat.*w0+deltat./2.*delta_w;                    %16.3.2.(6)
delta_u = deltat.*v0+deltat^2./2.*w0+deltat^2./6.*delta_w;  %16.3.2.(6)

u = u0+delta_u;                                             %16.3.2.(7)
v = v0+delta_v;
w = inv(M)*(F-K*u-C*v);
t = t+deltat;
```

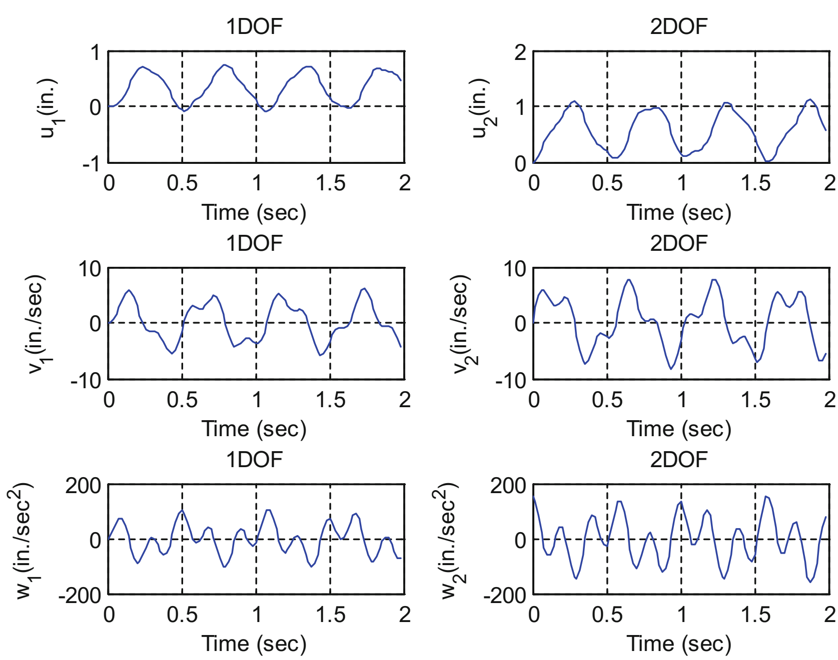

Fig. 16.4 Responses of displacement, velocity, and acceleration

16.5 The Newmark Beta Method

The Newmark beta method may be considered a generalization of the linear acceleration method. It uses a numerical parameter designated as β. The method, as originally proposed by Newmark (1959), contained in addition to β, a second parameter γ. These parameters replace the numerical coefficients $\frac{1}{2}$ and $\frac{1}{6}$ of the terms containing the incremental acceleration $\Delta\ddot{u}_i$, in Eqs. (16.20) and (16.21), respectively. Thus, replacing by γ the coefficient $\frac{1}{2}$ of $\Delta\ddot{u}_i$, in Eq. (16.20) and by β the coefficient $\frac{1}{6}$ also of $\Delta\ddot{u}_i$, in Eq. (16.20), we have

$$\Delta\dot{u}_i = \ddot{u}_i\Delta t + \gamma\Delta\ddot{u}_i\Delta_t \tag{16.25}$$

and

$$\Delta u_i = \dot{u}_i\Delta t + \frac{1}{2}\ddot{u}_i\Delta t^2 + \beta\Delta\ddot{u}_i\Delta t^2 \tag{16.26}$$

It has been found that for values of γ different than $\frac{1}{2}$, the method introduces a superfluous damping in the system. For this reason this parameter is generally set as $\gamma = \frac{1}{2}$. The solution of Eq. (16.26) for $\Delta\ddot{u}_i$, and its substitution into Eq. (16.25) after setting $\gamma = \frac{1}{2}$ yield

$$\Delta\dot{u}_i = \frac{1}{\beta\Delta t^2}\Delta u_i - \frac{1}{\beta\Delta t}\dot{u}_i - \frac{1}{2\beta}\ddot{u}_i \tag{16.27}$$

$$\Delta\dot{u}_i = \frac{1}{2\beta\Delta t}\Delta u_i - \frac{1}{2\beta}\dot{u}_i + \left(1 - \frac{1}{4\beta}\right)\Delta t\ddot{u}_i \tag{16.28}$$

Then the substitution of Eqs. (16.27) and (16.28) into the incremental equation of motion

$$M\Delta\ddot{u}_i + C_i\Delta\dot{u}_i + K_i\Delta u_i = \Delta F_i \tag{16.29}$$

results in an equation to calculate the incremental displacement Δu_i, namely

$$\bar{K}_i\Delta u_i = \Delta\bar{F}_i \tag{16.30}$$

where the effective stiffness matrix $\bar{K}_i$, and the effective incremental force vector $\Delta\bar{F}_i$ are given respectively by

$$\bar{K}_i = K_i + \frac{M}{\beta\Delta t^2} + \frac{C_i}{2\beta\Delta t} \tag{16.31}$$

and

$$\Delta\bar{F}_i = \Delta F_i + \frac{M}{\beta\Delta t}\dot{u}_i + \frac{C_i}{2\beta}\dot{u}_i + \frac{M}{2\beta}u_i - C_i\Delta t\left(1 - \frac{1}{4\beta}\right)\ddot{u}_i \tag{16.32}$$

In these equations C_i and K_i are respectively the damping and stiffness matrices with coefficients evaluated at the initial time t_i of the time step $\Delta t = t_{i+1} - t_i$.

In the implementation of the Newmark beta method, the process begins by selecting a numerical value for the parameter β. Newmark suggested a value in the range $\frac{1}{6} \leq \beta \leq \frac{1}{2}$. For $\beta = \frac{1}{6}$ the method is exactly equal to the linear acceleration method and is only conditionally stable.

For $\beta = \frac{1}{4}$ the method is equivalent to assuming that the velocity varied linearly during the time step, which would require that the mean acceleration is maintained for the interval. In this last case, that is, $\beta = \frac{1}{4}$, the Newmark beta method is unconditionally stable and it provides the satisfactory accuracy.

16.6 Elastoplastic Behavior of Framed-Structures

The dynamic analysis of beams and frames having linear elastic behavior was presented in the preceding chapters. To extend this analysis to structures whose members may be strained beyond the yield point of the material, it is necessary to develop the member stiffness matrix for the assumed elastoplastic behavior. The analysis is then carried out by a step-by-step numerical integration of the differential equations of motion. Within each short time interval Δt, the structure is assumed to behave in a linear elastic manner, but the elastic properties of the structure are changed from one interval to another as dictated by the response. Consequently, the nonlinear response is obtained as a sequence of linear responses of different elastic systems. For each successive interval, the stiffness of the structure is evaluated based on the moments in the members at the beginning of the time increment.

The changes in displacements of the linear system are computed by integration of the differential equations of motion over the finite interval and the total displacements by addition of the incremental displacement to the displacements calculated in the previous time step. The incremental displacements are also used to calculate the increment in member end forces and moments from the member stiffness equation. The magnitude of these end moments relative to the yield conditions (plastic moments) determines the characteristics of the stiffness and mass matrices to be used in the next time step.

16.7 Member Stiffness Matrix

If only bending deformations are considered, the force-displacement relationship for a uniform beam segment (Fig. 16.4) with elastic behavior (no hinges) is given by Eq. (10.20). This equation may be written in incremental quantities as follows:

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \Delta\delta_1 \\ \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \end{Bmatrix} \tag{16.33}$$

in which ΔP_i and $\Delta\delta_i$, are, respectively, the incremental forces and the incremental displacements at the nodal coordinates of the beam segment. When the moment at one end of the beam reaches the value of the plastic moment M_p, a hinge is formed at that end. Under the assumption of an elastoplastic relation-between the bending moment and the angular displacement as depicted in Fig. 16.5, the section that has been transformed into a hinge cannot support a moment higher than the plastic moment M_p but it may continue to deform plastically at a constant moment M_p. The relationship reverses to an elastic behavior when the angular displacement begins to decrease as shown in Fig. 16.5. We note the complete similarity for the behavior between an elastoplastic spring (Fig. 6.5) in a single degree-of-freedom system and an elastoplastic section of a beam (Fig. 16.5).

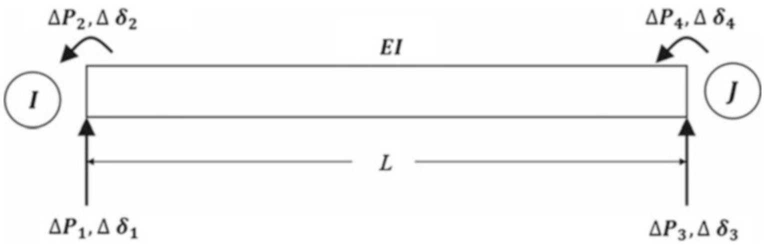

Fig. 16.5 Beam segment indicating incremental end forces and corresponding incremental displacements

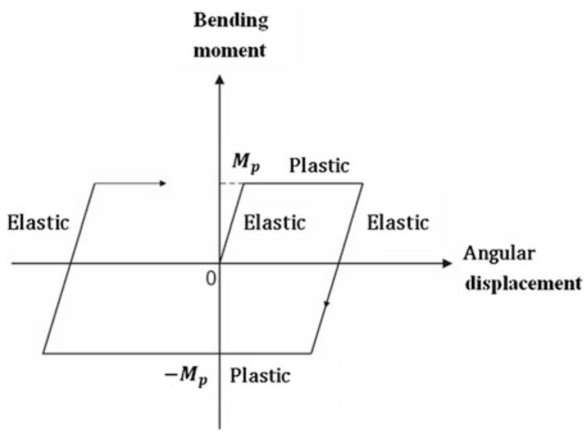

Fig. 16.6 Elastoplastic relationship between bending moment and angular displacement at a section of a beam

The stiffness matrix for a beam segment with a hinge at one end (Fig. 16.6) may be obtained by application of Eq. (10.16) which is repeated here for convenience, namely

$$k_{ij} = \int_0^L EI\,\mathrm{N}_i''(x)\mathrm{N}_j''(x)dx \tag{16.34}$$

Where $\psi_i(x)$ and $\psi_j(x)$ are displacement functions. For a uniform beam in which the formation of the plastic hinge takes place at end Θ as shown in Fig. 16.6, the deflection functions corresponding to unit displacement at one of the nodal coordinates δ_1, δ_2, δ_3, or δ_4 are given respectively by

$$N_1(x) = 1 - \frac{3x}{2L} + \frac{x^3}{2L^3} \tag{16.35a}$$

$$N_2(x) = 0 \tag{16.35b}$$

$$N_3(x) = \frac{3x}{2L} + \frac{x^3}{2L^3} \tag{16.35c}$$

$$N_4(x) = -\frac{x}{2} + \frac{x^3}{2L^2} \tag{16.35d}$$

For example, to calculate k_{11}, we substitute the second derivative $N_1''(x)$ from Eq. (16.35a) into Eq. (16.34) and obtain

$$k_{11} = EI\int_0^L \left(\frac{3x}{L^3}\right)^2 dx = \frac{3EI}{L^3} \tag{16.36}$$

Similarly, all the other stiffness coefficients for the case in which the formation of the plastic hinge takes place at end Θ of a beam segment are determined using Eq. (16.34) and the deflection functions given by Eqs. (16.35). The resulting stiffness equation in incremental form is

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 3 & 0 & -3 & 3L \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 3 & -3L \\ 3L & 0 & -3L & 3L^2 \end{bmatrix}\begin{Bmatrix} \Delta\delta_1 \\ \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \end{Bmatrix} \tag{16.37}$$

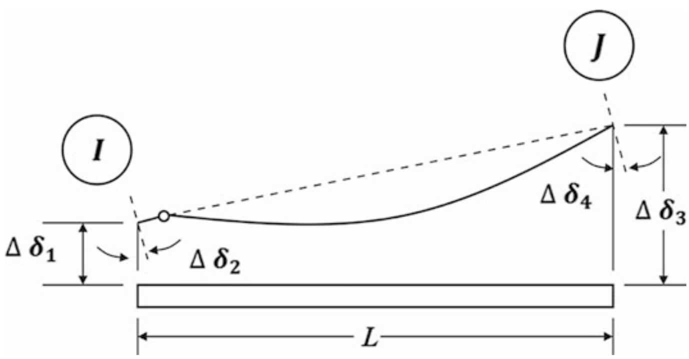

Fig. 16.7 Beam geometry with a plastic hinge at joint i

It should be pointed out that $\Delta\delta_2$ is the incremental rotation of joint Θ at the frame and not the increase in rotation at end Θ of the beam under consideration. The incremental rotation of the plastic hinge is given by the difference between $\Delta\delta_2$ and the increase in rotation of the end Θ of the member. Hinge rotation may be calculated for the various cases with formulas developed in the next section. Analogous to Eq. (16.37), the following equation gives the relationship between incremental forces and incremental displacements for a uniform beam with a hinge at end Θ (Fig. 16.7), namely,

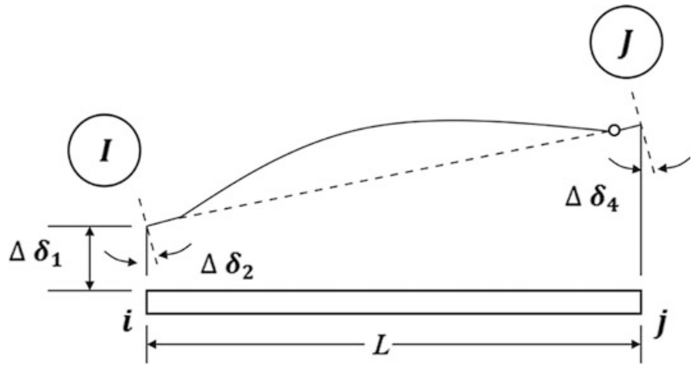

Fig. 16.8 Beam geometry with a plastic hinge at joint Θ

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 3 & 3L & -3 & 0 \\ 3L & 3L^2 & -3L & 0 \\ -3 & -3L & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \Delta\delta_1 \\ \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \end{Bmatrix} \tag{16.38}$$

Finally, if hinges are formed at both ends of the beam, the stiffness matrix becomes null. Hence in this case the stiffness equation is

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \Delta\delta_1 \\ \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \end{Bmatrix} \tag{16.39}$$

16.8 Member Mass Matrix

The relationship between forces and accelerations at the nodal coordinates of an elastic uniform member considering flexural deformation is given by Eq. (10.34). This equation written in incremental quantities is

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{\bar{m}L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}\begin{Bmatrix} \Delta\ddot{\delta}_1 \\ \Delta\ddot{\delta}_2 \\ \Delta\ddot{\delta}_3 \\ \Delta\ddot{\delta}_4 \end{Bmatrix} \tag{16.40}$$

where ΔP_i, and $\Delta \ddot{\delta}_i$ are, respectively, the incremental forces and the incremental accelerations at the nodal coordinates, L is the length of the member, and $\bar{m}$ is its mass per unit length. Assuming elastoplastic behavior, when the moment at an end of the beam segment reaches the magnitude of the plastic moment M_p and a hinge is formed, the consistent mass coefficients are determined from Eq. (10.33) using the appropriate deflection curves. For a uniform beam in which the formation of the plastic hinge develops at end I as shown in Fig. 16.6, the deflection functions corresponding to a unit displacement of the nodal coordinates are given by Eqs. (16.35). Analogously, the deflection functions of a beam segment with a plastic hinge at end J as shown in Fig. 16.8 for unit displacement at nodal coordinates δ_1, δ_2, δ_3, or δ_4, are respectively given by

$$N_1(x) = 1 + \frac{x^3}{2L^3} - \frac{3x^2}{2L^2}$$

$$N_2(x) = \frac{x^3}{2L^2} - \frac{3x^2}{2L} + x$$

$$N_3(x) = -\frac{x^3}{2L^3} - \frac{3x^2}{2L^2}$$

$$N_4(x) = 0 \tag{16.41}$$

The mass coefficients for a beam segment with a hinge at one end are then obtained by application of Eq. (10.33) which is repeated here, namely,

$$m_{ij} = \int_0^L \bar{\mathrm{m}}\mathrm{N}_i(x)N_j(x)dx \tag{16.42}$$

where $N_i\,(x)$ and $N_j\,(x)$ are the corresponding displacement functions from Eqs. (16.35) or (16.41).

Application of Eq. (16.42) and the use of displacement functions (16.34) results in the mass matrix for a beam segment with a hinge at the "j" end. The resulting mass matrix relates incremental forces and accelerations at the nodal coordinates, namely,

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{\bar{m}L}{420}\begin{bmatrix} 204 & 0 & 58.5 & -16.5L \\ 0 & 0 & 0 & 0 \\ 58.5 & 0 & 99 & -36L \\ -16.5L & 0 & -36L & 8L^2 \end{bmatrix}\begin{Bmatrix} \Delta\ddot{\delta}_1 \\ \Delta\ddot{\delta}_2 \\ \Delta\ddot{\delta}_3 \\ \Delta\ddot{\delta}_4 \end{Bmatrix} \tag{16.43}$$

Analogously to Eq. (16.43), the following equation gives the relationship between incremental forces and incremental accelerations for a uniform beam segment with a hinge at the "j" end:

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{\bar{m}L}{420}\begin{bmatrix} 99 & 36L & 58.5 & 0 \\ 36L & 8L^2 & 16.5L & 0 \\ 58.5 & 16.5L & 204 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \Delta\ddot{\delta}_1 \\ \Delta\ddot{\delta}_2 \\ \Delta\ddot{\delta}_3 \\ \Delta\ddot{\delta}_4 \end{Bmatrix} \tag{16.44}$$

Finally, if hinges are formed at both ends of the beam segment, the deflection curves are given by

$$\begin{aligned} N_1(x) &= -\frac{x}{L} + 1 \\ N_2(x) &= 0 \\ N_3(x) &= \frac{x}{L} \\ N_4(x) &= 0 \end{aligned} \tag{16.45}$$

and the corresponding relationship in incremental form by

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{\bar{m}L}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\ddot{\delta}_1 \\ \Delta\ddot{\delta}_2 \\ \Delta\ddot{\delta}_3 \\ \Delta\ddot{\delta}_4 \end{Bmatrix} \tag{16.46}$$

16.9 Rotation of Plastic Hinges

In the solution process, at the end of each step interval it is necessary to calculate the end moments of every beam segment to check whether or not a plastic hinge has been formed. The calculation is done using the element incremental moment-displacement relationship. It is also necessary to check if the plastic deformation associated with a hinge is compatible with the sign of the moment. The plastic hinge is free to rotate in one direction only, and in the other direction the section returns to an elastic behavior. The assumed moment rotation characteristics of the member are of the type illustrated in Fig. 16.5. The conditions implied by this model are: (1) the moment cannot exceed the plastic moment; (2) if the moment is less than the plastic moment, the hinge cannot rotate; (3) if the moment is equal to the plastic moment, then the hinge may rotate in the direction consistent with the sign of the moment; and (4) if the hinge starts to rotate in a direction inconsistent with the sign of the moment, the hinge is removed.

The incremental rotation of a plastic hinge is given by the difference between the incremental joint rotation of the frame and the increase in rotation of the end of the member at that joint. For example, with a hinge at end "i" only (Fig. 20.6), the incremental joint rotation is $\Delta\delta_2$ and the increase in rotation of this end due to rotation $\Delta\delta_4$ is $-\Delta\delta_4/2$ and that due to the displacements $\Delta\delta_1$ and $\Delta\delta_3$ is 1.5 $\left(\Delta\delta\underline{3} - \Delta\delta\right)/L$ Hence the increment in rotation $\Delta\rho_i$ of a hinge formed at end i is given by

$$\Delta\rho_i = \Delta\delta_2 + \frac{1}{2}\Delta\delta_4 - 1.5\frac{\Delta\delta_3 - \Delta\delta_1}{L} \tag{16.47}$$

Similarly, with a hinge formed at end "j" only (Fig. 16.7), the increment in rotation of this hinge is given by

$$\Delta\rho_j = \Delta\delta_4 + \frac{1}{2}\Delta\delta_2 - 1.5\frac{\Delta\delta_3 - \Delta\delta_1}{L} \tag{16.48}$$

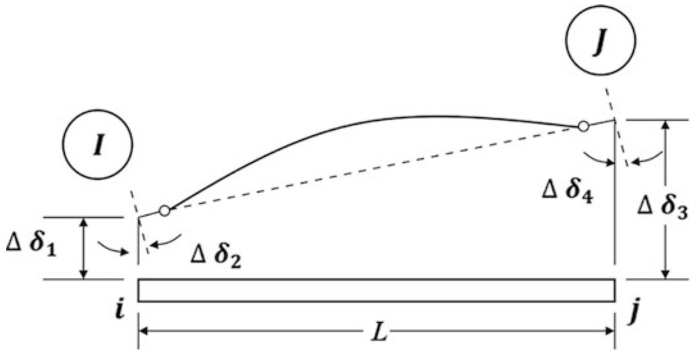

Fig. 16.9 Beam geometry with plastic hinges at both ends

Finally, with hinges formed at both ends of a beam segment (Fig. 16.8), the rotations of the hinges are given by

$$\Delta\rho_i = \Delta\delta_1 - \frac{\Delta\delta_3 - \Delta\delta_1}{L} \tag{16.49}$$

$$\Delta\rho_j = \Delta\delta_2 - \frac{\Delta\delta_3 - \Delta\delta_1}{L} \tag{16.50}$$

16.10 Calculation of Member Ductility Ratio

Nonlinear beam deformation are expressed in terms of the member ductility ratio, which is defined as the maximum total end rotation of the member to the end rotation at the elastic limit. The elastic limit rotation is the angle developed when the member is subjected to antisymmetric yield moments M_x as shown in Fig. 16.9. In this case the relationship between the end rotation and end moment is given by

$$\phi_y = \frac{M_y L}{6EI} \tag{16.51}$$

The member ductility ratio μ is then defined as

$$\mu = \frac{\phi_y + \rho_{\max}}{\phi_y} \tag{16.52}$$

which from Eq. (16.50) becomes

$$\mu = 1 + \frac{6EI}{M_y L}\rho_{\max} \tag{16.53}$$

where $\rho_{\max}$ is the maximum rotation of the plastic hinge (Fig. 16.10).

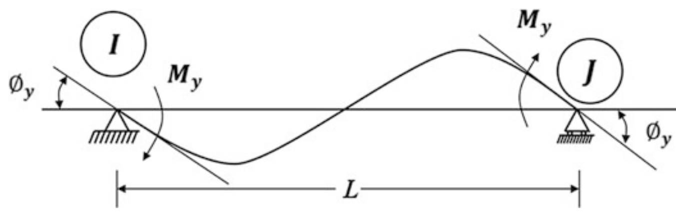

Fig. 16.10 Definition of yield rotation for beam segment

16.11 Summary

The determination of the nonlinear response of multidegree-of-freedom structures requires the numerical integration of the governing equations of motion. There are many methods available for the solution of these equations. The step-by-step linear acceleration method with a modification known as the Wilson-θ method was presented in this chapter. This method is unconditionally stable, that is, numerical errors do not tend to accumulate during the integration process regardless of the magnitude selected for the time step. The basic assumption of the Wilson-θ method is that the acceleration varies linearly over the extended interval $\tau = \theta\Delta t$ in which $\theta \geq 1.38$ for unconditional stability.

In the final sections of this chapter, stiffness and mass matrices for elastoplastic behavior of framed structures are presented. Formulas to determine the plastic rotation of hinges and to calculate the corresponding ductility ratios are also presented in this chapter.

16.12 Problems

Problem 16.1

The stiffness and the mass matrices for a certain structure modeled as a two-degree-of-freedom system are

$$[K] = \begin{bmatrix} 100 & -50 \\ -50 & 50 \end{bmatrix} (\text{Kip/in}), \qquad [M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \left(\text{Kip} \cdot \text{sec}^2/\text{in}\right)$$

Use Program 19 to determine the response when the structure is acted upon by the forces.

$$\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \begin{Bmatrix} 772 \\ 386 \end{Bmatrix} f(t)(\text{Kip})$$

Where *f*(*t*) is given graphically in Fig. P16.1. Neglect damping in the system.

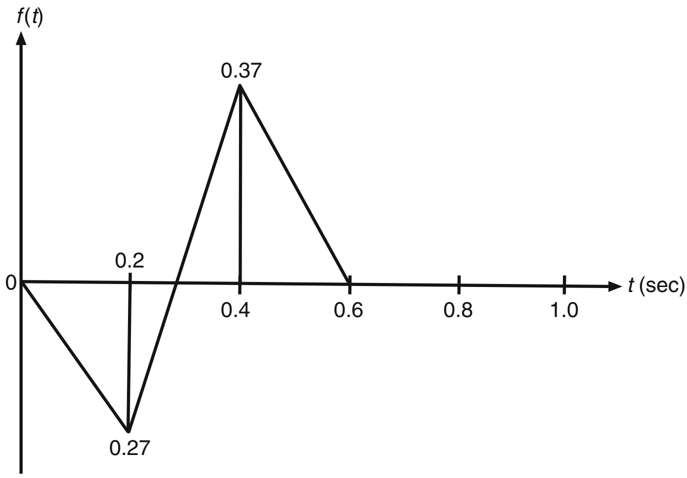

Fig. P16.1

Problem 16.2
Solve Problem 16.1 considering that the damping present in the system results in the following damping matrix:

$$[C] = \begin{bmatrix} 10 & -5 \\ -5 & 5 \end{bmatrix} (\text{Kip} \cdot \text{ sec} / \text{in})$$

Problem 16.3
Use Program 19 to determine the response of the three-story shear building subjected to the force $F_3(t)$ as depicted in Fig. P16.3 applied at the third level of the building. Neglect damping in the system.

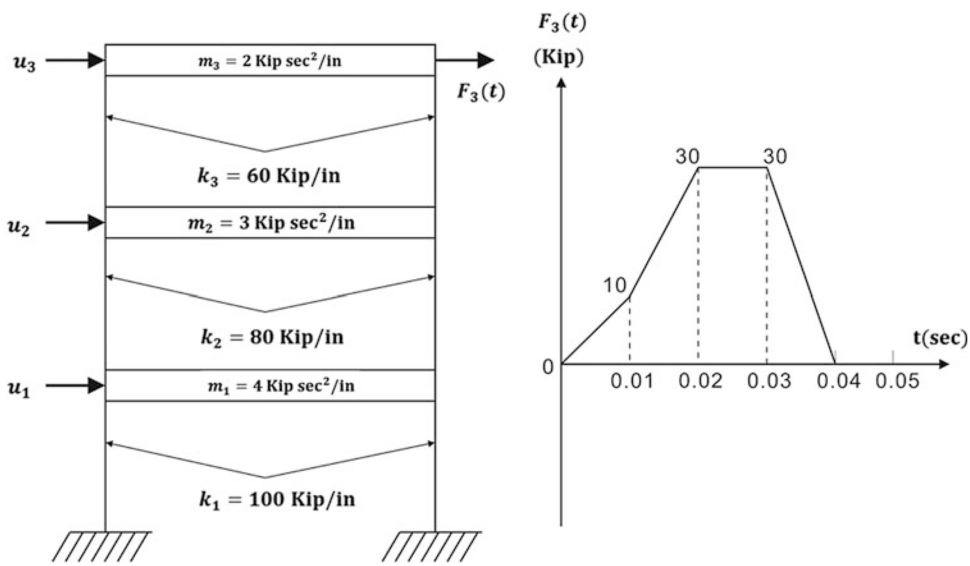

Fig. P16.3

Problem 16.4
Solve Problem 16.3 considering damping in the system of 10% in all the modes.

Problem 16.5
Use Program 19 to obtain the response in the elastic range for the structure of Problem 16.1 subjected to an acceleration at its foundation given by the function $f(t)$ shown in Fig. P16.1.

Problem 16.6
Solve Problem 16.5 considering damping in the system as indicated in Problem 16.2.

Problem 16.7
Use Program 19 to obtain the response in the elastic range of the shear building shown in Fig. P16.3 when subjected to an acceleration of its foundation given by the function $f(t)$ depicted in Fig. P16.1. Neglect damping in the system.