1 Undamped Single Degree-of-Freedom System

The analysis and design of structures to resist the effect produced by time dependent forces or motions requires conceptual idealizations and simplifying assumptions through which the physical system is represented by an idealized system known as the analytical or mathematical model. These idealizations or simplifying assumptions may be classified in the following three groups:

1. Material assumptions. These assumptions or simplifications include material properties such as homogeneity or isotrophy and material behaviors such as linearity or elasticity.
2. Loading assumptions. Some common loading assumptions are to consider concentrated forces to be applied at a geometric point, to assume forces suddenly applied, or to assume external forces to be constant or periodic.
3. Geometric Assumptions. A general assumption for beams, frames and trusses is to consider these structures to be formed by unidirectional elements. Another common assumption is to assume that some structures such as plates are two-dimensional systems with relatively small thicknesses. Of greater importance is to assume that continuous structures may be analyzed as discrete systems by specifying locations (nodes) and directions for displacements (nodal coordinates) in the structures as described in the following section.

1.1 Degrees of Freedom

In structural dynamics the number of independent coordinates necessary to specify the configuration or position of a system at any time is referred to as the number of degrees of freedom. In general, a continuous structure has an infinite number of degrees of freedom. Nevertheless, the process of idealization or selection of an appropriate mathematical model permits the reduction to a discrete number of degrees of freedom. Figure 1.1 shows some examples of structures that may be represented for dynamic analysis as one-degree-of-freedom-systems, that is, structures modeled as systems with a single displacement coordinate.

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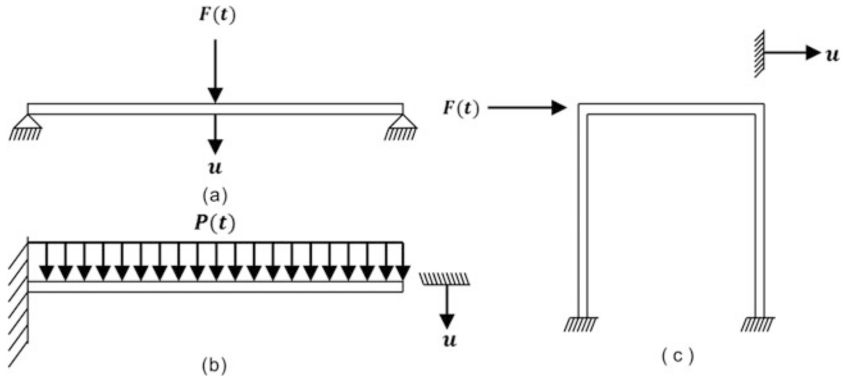

Fig. 1.1 Examples of structures modeled as one-degree-of-freedom systems

These one-degree-of-freedom systems may be conveniently described by the analytical model shown in Fig. 1.2 which has the following elements:

1. A mass element m representing the mass and inertial characteristic of the structure.
2. A spring element k representing the elastic restoring force and potential energy storage of the structure.
3. A damping element c representing the frictional characteristics and energy dissipation of the structure.
4. An excitation force $F(t)$ representing the external forces acting on the structural system.

The force $F(t)$ is written this way to indicate that it is a function of time. In adopting the analytical model shown in Fig. 1.2, it is assumed that each element in the system represents a single property; that is, the mass m represents only the property of inertia and not elasticity or energy dissipation, whereas the spring k represents exclusively elasticity and not inertia or energy dissipation. Finally, the damper c only dissipates energy. The reader certainly realizes that such "pure" elements do not exist in our physical world and that analytical models are only conceptual idealizations of real structures. As such, analytical models may provide complete and accurate knowledge of the behavior of the model itself, but only limited or approximate information on the behavior of the real physical system. Nevertheless, from a practical point of view, the information acquired from the analysis of the analytical model may very well be sufficient for an adequate understanding of the dynamic behavior of the physical system, including design and safety requirements.

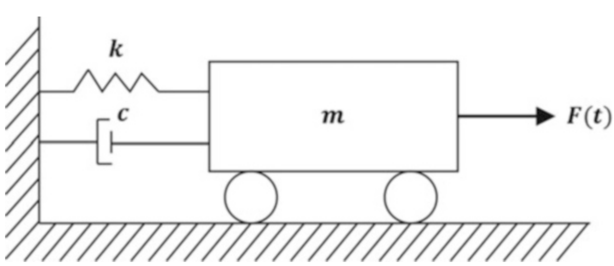

Fig. 1.2 Analytical model for one-degree-of-freedom systems

1.2 Undamped System

We start the study of structural dynamics with the analysis of a fundamental and simple system, the one-degree-of-freedom system in which we disregard or "neglect" frictional forces or damping. In addition, we consider the system, during its motion or vibration, to be free from external actions or forces. Under these conditions, the system is said to be in free vibration and it is in motion governed only by the influence of the so-called initial conditions, that is, the given displacement and velocity at time $t = 0$ when the study of the system is initiated. This undamped, one-degree-of-freedom system is often referred to as the simple undamped oscillator. It is usually represented as shown in Fig. 1.3a or Fig. 1.3b or any other similar arrangement. These two figures represent analytical models that are dynamically equivalent. It is only a matter of preference to adopt one or the other. In these models the mass m is restrained by the spring k and is limited to rectilinear motion along one coordinate axis, designated in these figures by the letter u.

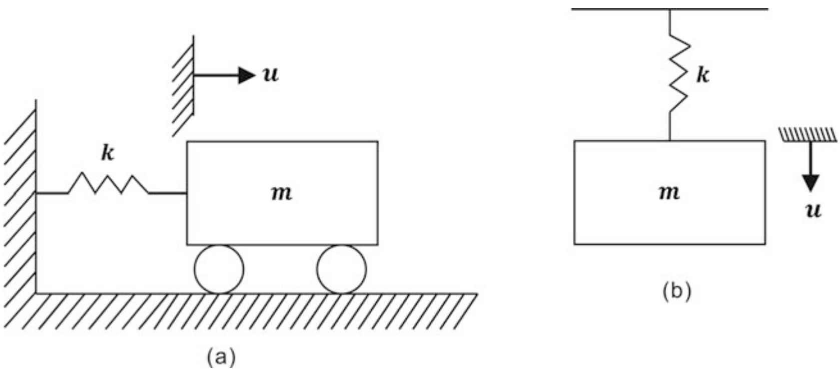

Fig. 1.3 Alternate representations of analytical models for one-degree-of-freedom systems

The mechanical characteristic of a spring is described by the relationship between the magnitude of the force F_s applied to its free end and the resulting end displacement u as shown graphically in Fig. 1.4 for three different springs.

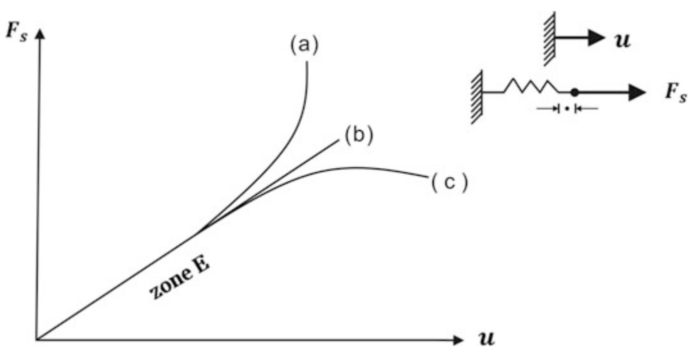

Fig. 1.4 Force-displacement relationship: (**a**) Hard spring, (**b**) Linear spring, (**c**) Soft spring

The curve labeled (a) in Fig. 1.4 represents the behavior of a hard spring in which the force required to produce a given displacement becomes increasingly greater as the spring is deformed. The second spring (b) is designated a linear spring because the deformation is directly proportional to the force and the graphical representation of its characteristic is a straight line. The constant of proportionality between the force and displacement [slope of the line (b)] of a linear spring is referred to as the stiffness or the spring constant, usually designated by the letter k. Consequently, we may write the relationship between force and displacement for a linear spring as

$$F_s = ku \tag{1.1}$$

A spring with characteristics shown by curve (c) in Fig. 1.4 is known as a soft spring. For such a spring the incremental force required to produce additional deformation decreases as the spring deformation increases. Undoubtedly, the reader is aware from his or her previous exposure to analytical modeling of physical systems that the linear spring is the simplest type to manage mathematically. It should not come as a surprise to learn that most of the technical literature on structural dynamics deals with models using linear springs. In other words, either because the elastic characteristics of the structural system are, in fact, essentially linear, or simply because of analytical expediency, it is usually assumed that the force-deformation properties of the system are linear. In support of this practice, it should be noted that in many cases the displacements produced in the structure by the action of external forces or disturbances are small in magnitude (Zone E in Fig. 1.4), thus rendering the linear approximation close to the actual structural behavior.

1.3 Springs in Parallel or in Series

Sometimes it is necessary to determine the equivalent spring constant for a system in which two or more springs are arranged in parallel as shown in Fig. 1.5a or in series as in Fig. 1.5b.

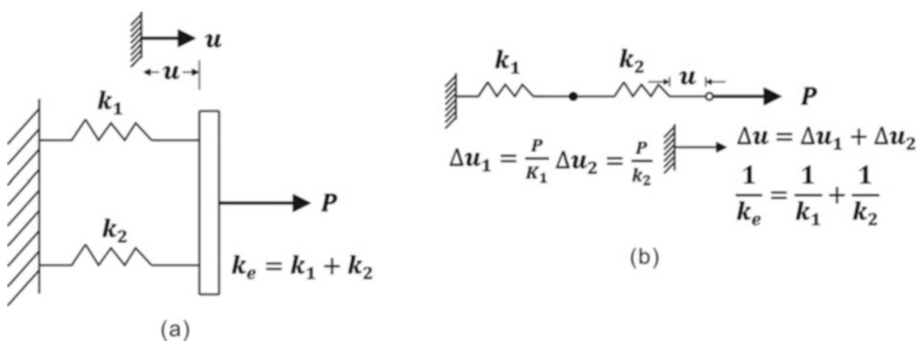

Fig. 1.5 Combination of springs: (**a**) Springs in parallel (**b**) Springs in series

For two springs in parallel the total force required to produce a relative displacement of their ends of one unit is equal to the sum of their spring constants. This total force is by definition the equivalent spring constant k_e and is given by

$$k_e = k_1 + k_2 \tag{1.2}$$

In general for n springs in parallel

$$k_e = \sum_{i=1}^{n} k_i \tag{1.3}$$

For two springs assembled in series as shown in Fig. 1.5b, the force P produces the relative displacements in the springs

$$\Delta u_1 = \frac{P}{k_1}$$

and

$$\Delta u_2 = \frac{P}{k_2}$$

Then, the total displacement u of the free end of the spring assembly is equal to $u = \Delta u_1 + \Delta u_2$, or substituting Δu_1 and Δu_2

$$u = \frac{P}{k_1} + \frac{P}{k_2} \tag{1.4}$$

Consequently, the force k_e necessary to produce one unit displacement (equivalent spring constant) is given by

$$k_e = \frac{P}{u}$$

Substituting u from this last relation into Eq. (1.4), we may conveniently express the reciprocal value of the equivalent spring constant as

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} \tag{1.5}$$

In general for n springs in series the equivalent spring constant may be obtained from

$$\frac{1}{k_e} = \sum_{i=1}^{n} \frac{1}{k_i} \tag{1.6}$$

1.4 Newton's Law of Motion

We continue with the study of the simple oscillator depicted in Fig. 1.3. The objective is to describe its motion, that is, to predict the displacement or velocity of the mass m at any tine t, for a given set of initial conditions at time $t = 0$. The analytical relation between the displacement u, and time t, is given by Newton's Second Law of Motion, which in modern notation may be expressed as

$$\boldsymbol{F} = ma \tag{1.7}$$

Where $\boldsymbol{F}$ is the resultant force acting on a particle of mass m and a its resultant acceleration. The reader should recognize that Eq. (1.7) is a vector relation and as such it can be written in equivalent form in terms of its components along the coordinate axes x, y, and z, namely,

$$\sum F_x = ma_x \tag{1.8a}$$

$$\sum F_y = ma_y \tag{1.8b}$$

$$\sum F_z = ma_z \tag{1.8c}$$

The acceleration is defined as the second derivative of the position vector with respect to time; it follows that Eqs. (1.8) are indeed differential equations. The reader should also be reminded that these equations as stated by Newton are directly applicable only to bodies idealized as particles, that is, bodies assumed to possess mass but no volume. However, as is proved in elementary mechanics, Newton's Law of Motion is also directly applicable to bodies of finite dimensions undergoing translatory motion.

For plane motion of a rigid body that is symmetric with respect to the reference plane of motion (x-y plane), Newton's Law of Motion yields the following equations:

$$\sum F_x = m(a_G)_x \tag{1.9a}$$

$$\sum F_y = m(a_G)_y \tag{1.9b}$$

$$\sum M_G = I_G \alpha \tag{1.9c}$$

In the above equations $(a_G)_x$ and $(a_G)_y$ are the acceleration components, along the x and y axes, of the center of mass G of the body; α is the angular acceleration; I_G is the mass moment of inertia of the body with respect to an axis through G, the center of mass; and $\sum M_G$ is the sum with respect to an axis through G, perpendicular to the x-y plane of the moments of all the forces acting on the body. Equations (1.9) are certainly also applicable to the motion of a rigid body in pure rotation about a fixed axis, alternatively, for this particular type of plane motion, Eq. (1.9c) may be replaced by

$$\sum M_0 = I_0 \alpha \tag{1.9d}$$

in which the mass moment of inertia I_0 and the moment of the forces M_0 are determined with respect to the fixed axis of rotation. The general motion of a rigid body is described by two vector equations, one expressing the relation between the forces and the acceleration of the mass center, and another relating the moments of the forces and the angular motion of the body. This last equation expressed in its scalar components is rather complicated, but seldom needed in structural dynamics.

1.5 Free Body Diagram

At this point, it is advisable to follow a method conducive to an organized and systematic analysis in the solution of dynamics problems. The first and probably the most important practice to follow in any dynamic analysis is to draw a free body diagram of the system, prior to writing a mathematical description of the system.

The free body diagram (FBD), as the reader may recall, is a sketch of the body isolated from all other bodies, in which all the forces external to the body are shown. For the case at hand, Fig. 1.6b depicts the FBD of the mass m of the oscillator, displaced in the positive direction with reference to coordinate u and acted upon by the spring force $F_s = ku$ (assuming a linear spring). The weight of the body mg and the normal reaction N of the supporting surface are also shown for completeness, though these forces, acting in the vertical direction, do not enter into the equation of motion written for the u direction. The application of Newton's Law of Motion gives

$$-ku = m\ddot{u} \tag{1.10}$$

where the spring force acting in the negative direction has a minus sign, and where the acceleration has been indicated by $\ddot{u}$. In this notation, double overdots denote the second derivative with respect to time and obviously a single overdot denotes the first derivative with respect to time, that is, the velocity.

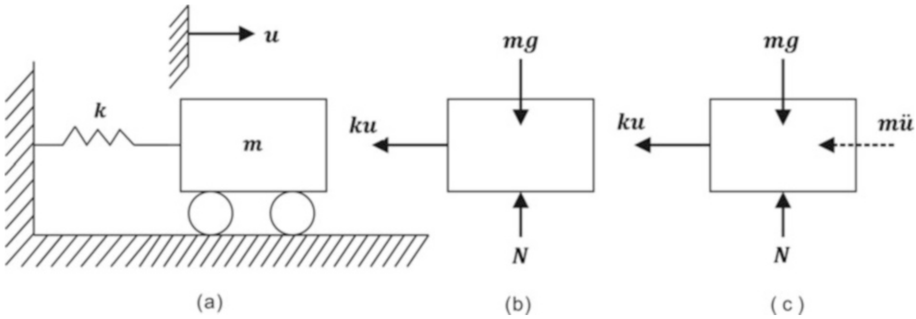

Fig. 1.6 Alternate free body diagrams: (**a**) Single degree-of-freedom system. (**b**) Showing only external forces, (**c**) Showing external and inertial forces

1.6 D'Alembert's Principle

An alternative approach to obtain Eq. (1.10) is to make use of D'Alembert's Principle which states that a system may be set in a state of dynamic equilibrium by adding to the external forces a fictitious force that is commonly known as the inertial force.

Figure 1.6c shows the FBD with inclusion of the inertial force $m\ddot{u}$. This force is equal to the mass multiplied by the acceleration, and should always be directed negatively with respect to the corresponding coordinate. The application of D'Alembert's Principle allows us to use equations of equilibrium in obtaining the equation of motion. For example, in Fig. 1.6c, the summation of forces in the u direction gives directly

$$m\ddot{u} + ku = 0 \tag{1.11}$$

which obviously is equivalent to Eq. (1.10).

The use of D'Alembert's Principle in this case appears to be trivial. This will not be the case for a more complex problem, in which the application of D'Alembert's Principle, in conjunction with the Principle of Virtual Work, constitutes a powerful tool of analysis. As will be explained later, the Principle of Virtual Work is directly applicable to any system in equilibrium. If follows then that this principle may also be applied to the solution of dynamic problems, provided that D'Alembert's Principle is used to establish the dynamic equilibrium of the system.

Illustrative Example 1.1

Show that the same differential equation is obtained for a body vibrating along a horizontal axis or for the same body moving vertically, as shown in Fig. 1.7a, b.

Solution:

The FBDs for these two representations of the simple oscillator are shown in Fig. 1.7c, e, in which the inertial forces have been included. Equating to zero in Fig. 1.7c the sum of the forces along the direction u, we obtain

$$m\ddot{u} + ku = 0 \tag{a}$$

When the body in Fig. 1.7d is in the static equilibrium position, the spring is stretched u_0 units and exerts a force $ku_0 = W$ upward on the body, where W is the weight of the body. When the body is displaced a distance u downward from this position of equilibrium the magnitude of the spring force

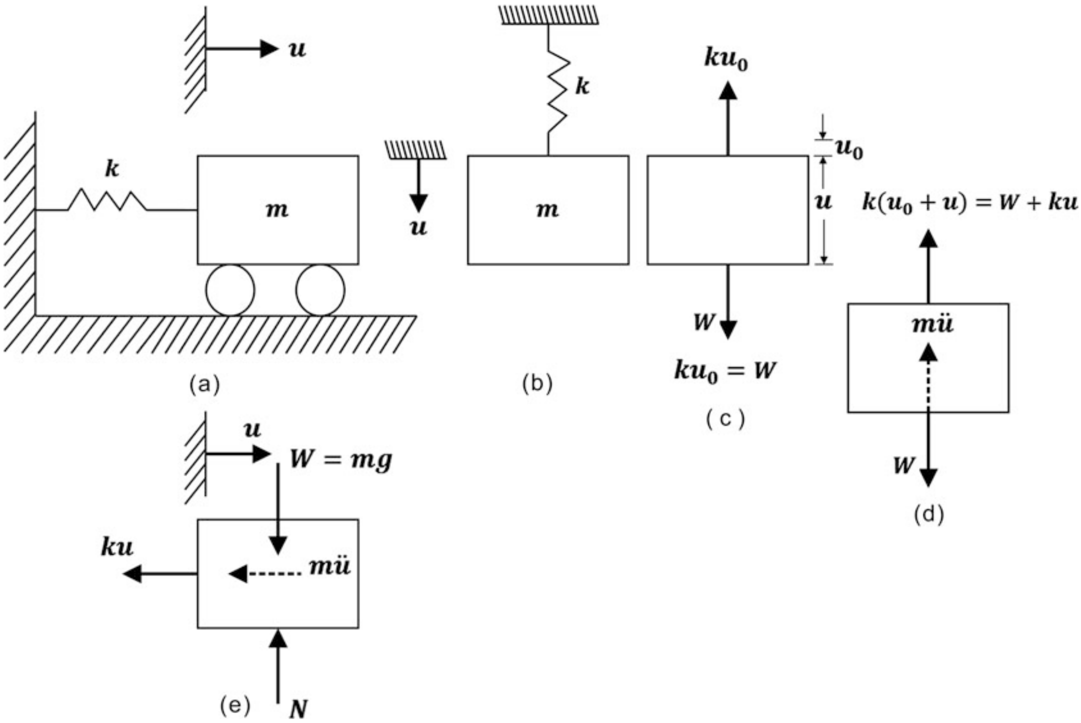

Fig. 1.7 Two representations of the simple oscillator and corresponding free body diagrams. (**a**) Idealized single degree of freedom system, (**b**) Alternative idealized single degree of freedom system, (**c**) dynamic equilibrium with inertial force, (**d**) static displacement due to gravity load, (**e**) dynamic equilibrium with inertial force of alternative model

is given by $F_s = k(u_0 + u)$ or $F_s = W + ku$ since $ku_0 = W$. Using this result and applying it to the body in Fig. 1.7e, we obtain from Newton's Second Law of Motion

$$-(W + ku) + W - m\ddot{u} = 0 \tag{b}$$

or

$$m\ddot{u} + ku = 0$$

which is identical to Eq. (a).

1.7 Solution of the Differential Equation of Motion

The next step toward our objective is to find the solution of the differential Eq. (1.11). We should again adopt a systematic approach and proceed first to classify this differential equation. Since the dependent variable u and second derivative $\ddot{u}$ appear in the first degree in Eq. (1.11), this equation is classified as linear and of second order. The fact that the coefficients of u and of $\ddot{u}$ (k and m, respectively) are constants and that the second member (right-hand side) of the equation is zero further classifies this equation as homogenous with constant coefficients. We should recall, probably with a certain degree of satisfaction, that a general procedure exists for the solution of linear differential equations (homogenous or non-homogenous) of any order. For this simple, second-order differential equation we may proceed directly by assuming a trial solution given by

$$u = A\cos\omega t \tag{1.12}$$

or by

$$u = B\sin\omega t \tag{1.13}$$

where A and B are constants depending on the initiation of the motion while ω is a quantity denoting a physical characteristic of the system as it will be shown next. The substitution of Eq. (1.12) into Eq. (1.11) gives

$$\left(-m\omega^2 + k\right)A\cos\omega t = 0 \tag{1.14}$$

If this equation is to be satisfied at any time, the factor in parentheses must be equal to zero, or

$$\omega^2 = \frac{k}{m} \tag{1.15}$$

The reader should verify that Eq. (1.13) is also a solution of the differential Eq. (1.11), with ω also satisfying Eq. (1.15).

The positive root of Eq. (1.15),

$$\omega = \sqrt{\frac{k}{m}} \tag{1.16a}$$

is known as the natural frequency of the system for reasons that will soon be apparent.

The quantity ω in Eq. (1.16a) may be expressed in terms of the static displacement resulting from the weight $W = mg$ applied to the spring. The substitution into Eq. (1.16a) of $m = W/g$ results in

$$\omega = \sqrt{\frac{kg}{W}} \tag{1.16b}$$

Hence

$$\omega = \sqrt{\frac{g}{u_{st}}} \tag{1.16c}$$

where $u_{st} = W/k$ is the static displacement of the spring due to the weight W.

Since either Eq. (1.12) or Eq. (1.13) is a solution of Eq. (1.11), and since this differential equation is linear, the superposition of these two solutions, indicated by Eq. (1.17) below, is also a solution. Furthermore, Eq. (1.17), having two constants of integration, A and B, is, in fact, the general solution for this linear second-order differential equation.

$$u = A\cos\omega t + B\sin\omega t \tag{1.17}$$

The expression for velocity, $\dot{u}$, is simply found by differentiating Eq. (1.17) with respect to time, that is,

$$\dot{u} = -A\omega\sin\omega t + B\omega\cos\omega t \tag{1.18}$$

Next, we should determine the constants of integration A and B. These constants are determined from known values for the motion of the system which almost invariably are the displacement u_0 and the velocity v_0 at the initiation of the motion, that is, at time $t = 0$. These two conditions are referred to as initial conditions, and the problem of solving the differential equation for the initial conditions is called an initial value problem.

After substituting, for $t = 0$, $u = u_0$, and $\dot{u} = v_0$ into Eqs. (1.17) and (1.18) we find that

$$u_0 = A \tag{1.19a}$$

$$v_0 = B\omega \tag{1.19b}$$

Finally, the substitution of A and B from Eq. (1.19) into Eq. (1.17) gives

$$u = u_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t \tag{1.20}$$

which is the expression of the displacement u of the simple oscillator as a function of the time variable t. Thus, we have accomplished our objective of describing the motion of the simple undamped oscillator modeling structures with a single degree of freedom.

1.8 Frequency and Period

An examination of Eq. (1.20) shows that the motion described by this equation is *harmonic* and therefore periodic, that is, it can be expressed by a sine or cosine function of the same frequency ω. The period may easily be found since the functions sine and cosine both have a period of 2π. The period T of the motion is determined from

$$\omega T = 2\pi$$

or

$$T = \frac{2\pi}{\omega} \tag{1.21}$$

The period is usually expressed in seconds per cycle or simply in seconds, with the tacit understanding that it is "per cycle". The reciprocal value of the period is the natural frequency f. From Eq. (1.21)

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{1.22}$$

The natural frequency f is usually expressed in hertz or cycles per second (cps). Because the quantity ω differs from the natural frequency f only by the constant factor 2π, ω also is sometimes referred to as the natural frequency. To distinguish between these two expressions for natural frequency, ω may be called the circular or angular natural frequency. Most often, the distinction is understood form the context or from the units. The natural frequency f is measured in cps as indicated, while the circular frequency ω should be given in radians per second (rad/sec).

Illustrative Example 1.2

Determine the natural frequency of the beam-spring system shown in Fig. 1.8 consisting of a weight of $W = 50.0$ lb attached to a horizontal cantilever beam through the coil spring k_2. The cantilever beam has a thickness $h = ¼$ in, a width $b = 1$ in. modulus of elasticity $E = 30 \times 10^6$ psi, and length $L = 12.5$ in. The coil spring has a stiffness $k_2 = 100$ (lb/in).

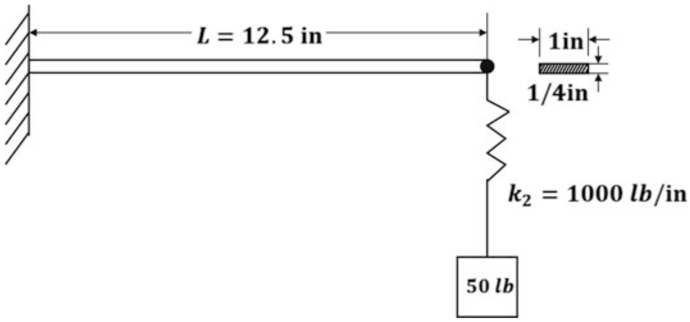

Fig. 1.8 System for Illustrative Example 1.2

Solution:
The deflection Δ at the free end of a uniform cantilever beam acted upon by a static force P at the free end is given by

$$\Delta = \frac{PL^3}{3EI}$$

The corresponding spring constant k_1 is then

$$k_1 = \frac{P}{\Delta} = \frac{3EI}{L^3}$$

where the cross-section moment of inertia $I = \frac{1}{12}bh^3$ (*for* a rectangular section). Now, the cantilever and the coil spring of this system are connected as springs in series. Consequently, the equivalent spring constant as given from Eq. (1.5) is

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{(repeated)} \tag{1.5}$$

Substituting corresponding numerical values, we obtain

$$I = \frac{1}{12} \times 1 \times \left(\frac{1}{4}\right)^3 = \frac{1}{768}(\text{in})^4$$

$$k_1 = \frac{3 \times 30 \times 10^6}{(12.5)^3 \times 768} = 60 \text{ lb/in}$$

and

$$\frac{1}{k_e} = \frac{1}{60} + \frac{1}{100}$$
$$k_e = 37.5 \text{ lb/in}$$

The natural frequency for this system is then given by Eq. (1.16a) as

$$\omega = \sqrt{k_e/m} \quad (\text{m} = \text{W/g and g} = 386\,\text{in/sec}^2)$$
$$\omega = \sqrt{37.5 \times 386/50.0}$$
$$\omega = 17.01\,\text{rad/sec}$$

or using Eq. (1.22)

$$f = 2.71\,\text{cps} \quad \text{(Ans)}$$

1.9 Amplitude of Motion

Let us now examine in more detail Eq. (1.20), the solution describing the free vibratory motion of the undamped oscillator. A simple trigonometric transformation may show us that we can rewrite this equation in the equivalent forms, namely

$$u = C \sin(\omega t + \alpha) \tag{1.23}$$

or

$$u = C \cos(\omega t - \beta) \tag{1.24}$$

where

$$C = \sqrt{u_0^2 + (v_0/\omega)^2} \tag{1.25}$$

$$\tan\alpha = \frac{u_0}{v_0/\omega} \tag{1.26}$$

and

$$\tan\beta = \frac{v_0/\omega}{u_0} \tag{1.27}$$

The simplest way to obtain Eq. (1.23) or Eq. (1.24) is to multiply and divide Eq. (1.20) by the factor C defined in Eq. (1.25) and to define α (or β) by Eq. (1.26) [or Eq. (1.27)]. Thus

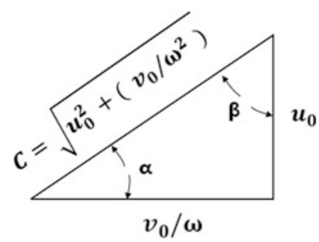

Fig. 1.9 Definition of angle α or angle β

$$u = C\left(\frac{u_0}{C}\cos\omega t + \frac{v_0/\omega}{C}\sin\omega t\right) \tag{1.28}$$

With the assistance of Fig. 1.9, we recognize that

$$\sin\alpha = \frac{u_0}{C} \tag{1.29}$$

and

$$\cos\alpha = \frac{v_0/\omega}{C} \tag{1.30}$$

The substitution of Eqs. (1.29) and (1.30) into Eq. (1.28) gives

$$u = C(\sin\alpha\cos\omega t + \cos\alpha\sin\omega t) \tag{1.31}$$

The expression within the parentheses of Eq. (1.31) is identical to *sin(ωt + α),* which yields Eq. (1.23). Similarly, the reader should verify without difficulty, the form of solution given by Eq. (1.24).

The value of C in Eq. (1.23) (or Eq. (1.24)) is referred to as the amplitude of motion and the angle α (or β) as the phase angle. The solution for the motion of the simple oscillator is shown graphically in Fig. 1.10.

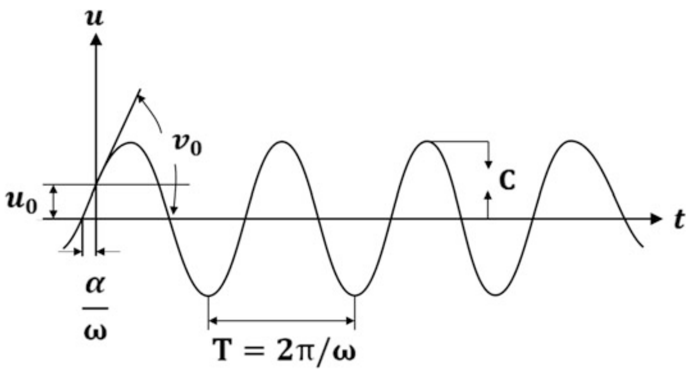

Fig. 1.10 Undamped free-vibration response

Illustrative Example 1.3

Consider the steel frame shown in Fig. 1.11a having a rigid horizontal member to which a horizontal dynamic force is applied. As part of the overall structural design it is required to determine the natural frequency of this structure. Two assumptions are made:

1. The masses of the columns are neglected.
2. The horizontal members are sufficiently rigid to prevent rotation at the tops of the columns.

These assumptions are not mandatory for the solution of the problem, but they serve to simplify the analysis. Under these conditions, the frame may be modeled by the spring-mass system shown in Fig. 1.11b.

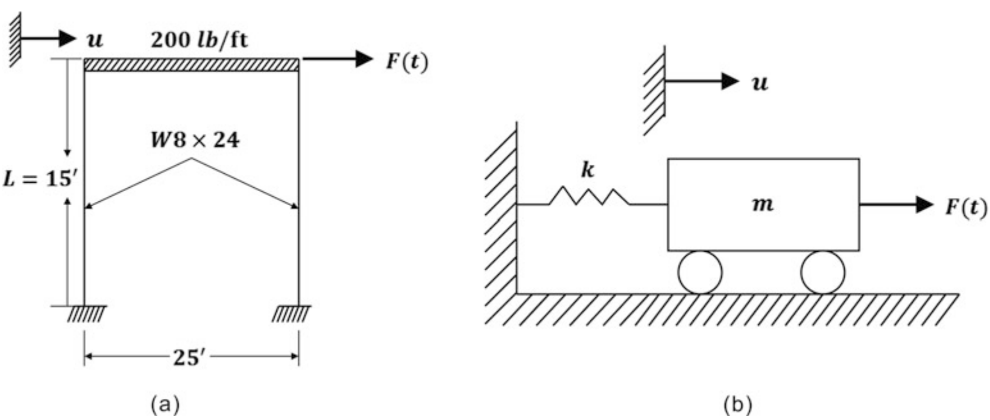

Fig. 1.11 One-degree-of-freedom frame and corresponding analytical model for Illustrative Example 1.3

Solution:
The parameters of this model may be computed as follows:

$$W = 200 \times 25 = 5000\,\text{lb}$$
$$I = 82.5\,\text{in}^4$$
$$E = 30 \times 10^6\,\text{psi}$$
$$k = \frac{12E(2I)}{L^3} = \frac{12 \times 30 \times 10^6 \times 165}{(15 \times 12)^3}$$
$$k = 10,185\,\text{lb/in} \qquad \text{(Ans)}$$

Note: A unit displacement of the top of a fixed column requires a force equal to $12EI/L^3$.
Therefore, the natural frequency from Eqs. (1.16b) and (1.22) is

$$f = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} = \frac{1}{2\pi}\sqrt{\frac{10,185 \times 386}{5000}}$$
$$f = 4.46\,\text{cps} \qquad \text{(Ans)}$$

Illustrative Example 1.4
The elevated water tower tank with a capacity for 5000 gallons of water shown in Fig. 1.12a has a natural period in lateral vibration of 1.0 sec when empty. When the tank is full of water, its period lengthens to 2.2 sec. Determine the lateral stiffness k of the tower and the weight W of the tank. Neglect the mass of the supporting columns (one gallon of water weighs approximately 8.34 lb).

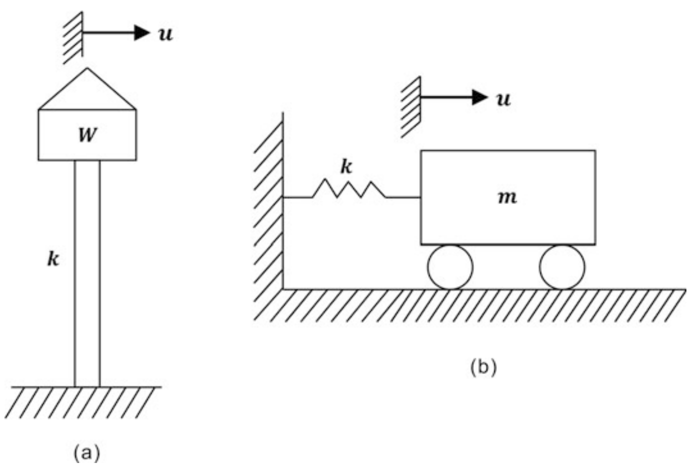

Fig. 1.12 (**a**) Water tower tank of Illustrative Example 1.4. (**b**) Analytical model

Solution:
In its lateral motion, the water tower is modeled by the simple oscillator shown in Fig. 1.12b in which k is the lateral stiffness of the tower and m is the vibrating mass of the tank.

(a) Natural frequency ω_E (tank empty):

$$\omega_E = \frac{2\pi}{T_E} = \frac{2\pi}{1.0} = \sqrt{\frac{kg}{W}} \tag{a}$$

(b) Natural frequency ω_F (tank full of water)

Weight of water W_w:

$$W_w = 5000 \times 8.34 = 41,700\,\text{lb}$$

$$\omega_F = \frac{2\pi}{T_F} = \frac{2\pi}{1.0} = \sqrt{\frac{kg}{W + 41,700}} \tag{b}$$

Squaring Eqs. (a) and (b) and dividing correspondingly the left and right sides of these equations, results in

$$\frac{(2.2)^2}{(1.0)^2} = \frac{W + 41,700}{W}$$

and solving for W

$$W = 10,860 \text{ lb} \tag{Ans}$$

Substituting in Eq. (a), $W = 10{,}860$ lb and $g = 386$ in/sec^2, yields

$$\frac{2\pi}{1.00} = \sqrt{\frac{k386}{10,860}}$$

and

$$k = 1110 \text{ lb/in} \tag{Ans}$$

Illustrative Example 1.5

The steel frame shown in Fig. 1.13a is fixed at the base and has a rigid top W that weighs 1000 lb. Experimentally, it has been found that its natural period in lateral vibration is equal to 1/10 of a second. It is required to shorten or lengthen its period by 20% by adding weight or strengthening the columns. Determine needed additional weight or additional stiffness (neglect the weight of the columns).

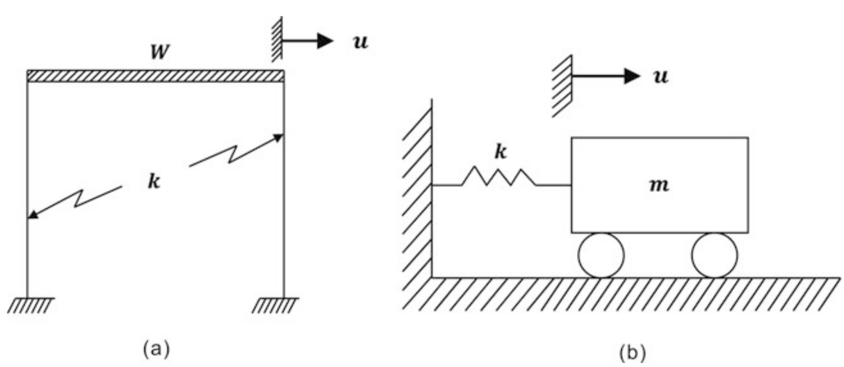

Fig. 1.13 (**a**) Frame of Illustrative Example 1.5. (**b**) Analytical model

Solution:
The frame is modeled by the spring-mass system shown in Fig. 1.13b. Its stiffness is calculated from

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{kg}{W}}$$

as

$$\frac{2\pi}{0.1} = \sqrt{\frac{kg}{1000}} \quad \left(g = 386\ \text{in}/\sec^2\right)$$

or

$$k = 10,228\ \text{lb/in}$$

(a) Lengthen the period to $T_L = 1.2 \times 0.10 = 0.12$ sec by adding weight ΔW:

$$\omega = \frac{2\pi}{0.12} = \sqrt{\frac{10,228 \times 386}{1000 + \Delta W}}$$

Solve for ΔW:

$$\Delta W = 440\ \text{lb} \qquad \text{(Ans)}$$

(b) Shorten the period to $T_s = 0.8 \times 0.1 = 0.08$ sec by strengthening columns in Δk:

$$\omega = \frac{2\pi}{0.08} = \sqrt{\frac{(10,228 + \Delta K)(386)}{1000}}$$

Solve for Δk:

$$\Delta k = 5753\ \text{lb/in} \qquad \text{(Ans)}$$

1.10 Response of SDF Using MATLAB Program

Plot the displacement as a function of time, u(t) ranging from 0 to 5 sec.
Given:

- Initial conditions: $u_0 = 1$ in. and $\dot{u}_0 = 0.2$ in./sec
- Natural period: $T = 0.5$ sec (Fig. 1.14).

```
clear all
clc

%%%%-GIVEN VALUES-%%%%
%%% Set Initial Conditions

u0=1;                                            %Initial Displacement
v0=2;                                            %Initial Velocity

%%%Define period and frequency

T=0.5;                                           %Natural Period
omega=2*pi/T;                                    %Natural Frequency

%%%%-ESTIMATION-%%%%
%%% Generate time stamp equally between 0 to 5 sec with a total of 500 %%% data

t=linspace(0,5,500);

%%%Calculate the displacement response

A = u0;                                                  % Eq. 19a
B = v0/omega;                                            % Eq. 19b

u= A*cos(omega*t)+B*sin(omega*t);                        % Eq. 20

C= sqrt(A^2+B^2);                                        %Maximum amplitude (Eq. 1.25)
udot = -A*omega*sin(omega*t)+B*omega*cos(omega*t);       %Velocity


%%%Plot the response curve

plot(t, u);
title ('Response');
xlabel ('Time (sec)');                                   %Label the x-axis of the plot
ylabel ('Displacement (in.)');                           %Label the y-axis of the plot
```

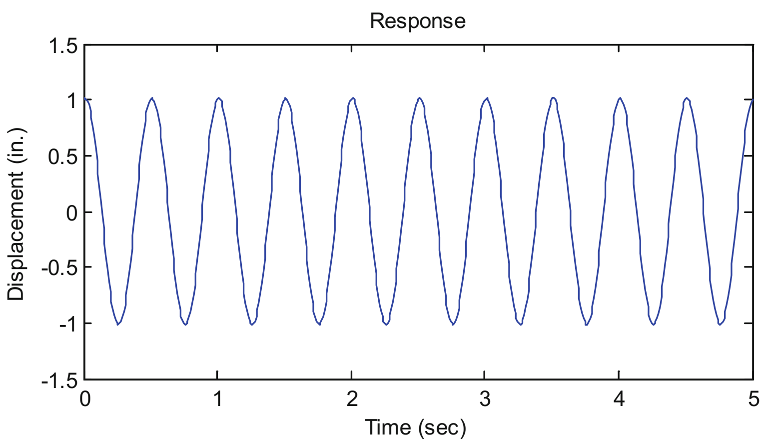

Fig. 1.14 Response of SDF using MATLAB

1.11 Summary

Several basic concepts were introduced in this chapter:

1. The analytical or mathematical model of a structure is an idealized representation for its analysis.
2. The number of degrees of freedom of a structural system is equal to the number of independent coordinates necessary to describe its position.
3. The free body diagram (FBD) for dynamic equilibrium (to allow application of D'Alembert's Principle) is a diagram of the system isolated from all other bodies, showing all the external forces on the system, including the inertial force.
4. The stiffness or spring constant of a linear system is the force necessary to produce a unit displacement.
5. The differential equation of the undamped simple oscillator in free motion is

$$m\ddot{u} + ku = 0$$

and its general solution is

$$u = A\cos\omega t + B\sin\omega t$$

where A and B are constants of integration determined from initial conditions of the displacement u_0 and of the velocity v_0:

$$A = u_0$$

$$B = v_0/\omega$$

$$\omega = \sqrt{k/m} \text{ is the natural frequency in rad/ sec}$$

$$f = \frac{\omega}{2\pi} \text{ is the natural frequency in cps}$$

$$T = \frac{1}{f} \text{ is the natural period in seconds}$$

6. The equation of motion may be written in the alternate forms:

$$u = C\sin(\omega t + \alpha)$$

or

$$u = C\cos(\omega t - \beta)$$

where

$$C = \sqrt{u_0^2 + (v_0/\omega)^2}$$

and

$$\tan\alpha = \frac{u_0}{v_0/\omega}$$

$$\tan\beta = \frac{v_0/\omega}{u_0}$$

1.12 Problems

Problem 1.1
Determine the natural period for the system in Fig. P1.1. Assume that the beam and springs supporting the weight W are massless.

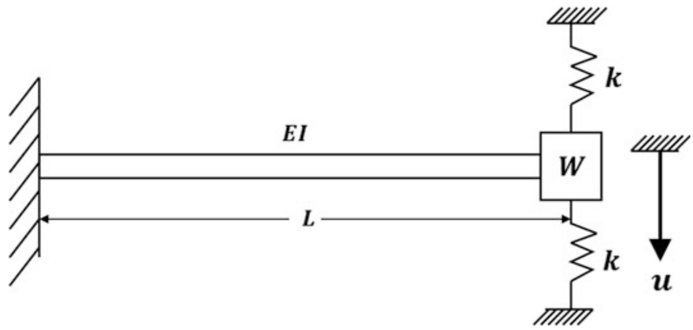

Fig. P1.1

Problem 1.2
The following numerical values are given in Problem 1.1: $L = 100$ in. $EI = 10^8$ (lb.in^2), $W = 3000$ lb, $k = 2000$ lb/in. If the weight W has an initial displacement of $u_0 = 1.0$ in and an initial velocity of $u_0 = 20$ in/sec, determine the displacement and the velocity 1 sec later.

Problem 1.3
Determine the natural frequency for horizontal motion of the steel frame in Fig. P1.3. Assume the horizontal girder to be infinitely rigid and neglect the mass of the columns.

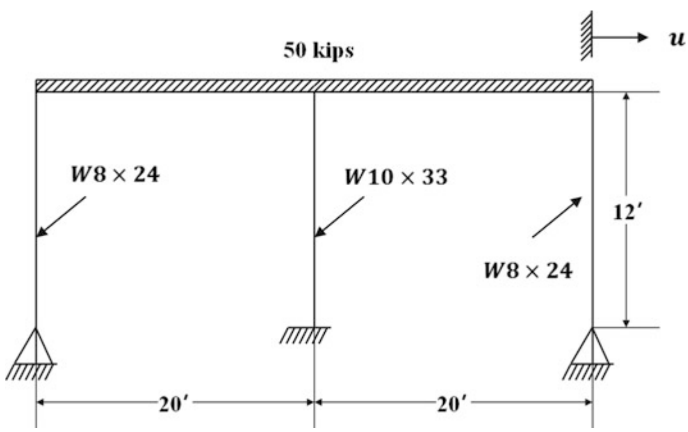

Fig. P1.3

Problem 1.4
Calculate the natural frequency in the horizontal mode of the steel frame in Fig. P1.4 for the following cases:

(a) The horizontal member is assumed to be rigid.
(b) The horizontal member is flexible and made of steel sections-- W 8 × 24.

Hint: *When the girder stiffness needs to be considered to determine the effective stiffness of column fixed on the ground, the following formula is useful.*

$$k_e = k_c(left) + k_c(right)$$
$$= \frac{24E_cI_c}{h^3}\frac{(1+6\gamma)}{(4+6\gamma)}$$

where,

$$\gamma = \frac{I_g/L}{I_c/h}$$

I_g and L are the moment of inertia and span length for girder.

I_c and h are the moment of inertia and height of column.

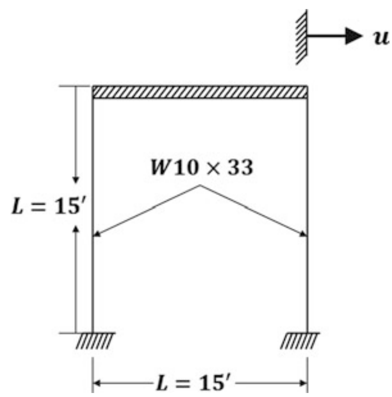

Fig. P1.4

Problem 1.5

Determine the natural frequency of the fixed beam in Fig. P1.5 carrying a concentrated weight *W at* its center. Neglect the mass of the beam.

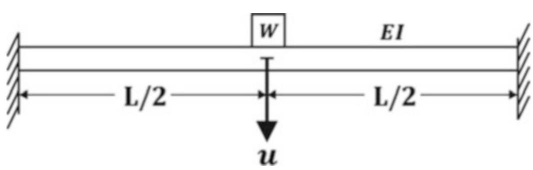

Fig. P1.5

Problem 1.6

The numerical values for Problem 1.5 are given as: $L = 120$ in. $EI = 10^9$ (lb.in^2), $W = 5000$ lb, with initial conditions $u_0 = 0.5$ in and $v_0 = 15$ in/sec. Determine the displacement, velocity, and acceleration of *W at* $t = 2$ sec later. Plot the responses (i.e., displacement, velocity, and acceleration) using MATLAB and determine the maximum amplitude.

Problem 1.7

Consider the simple pendulum of weight W illustrated in Fig. P1.7. If the cord length is L, determine the motion of the pendulum. The initial angular displacement and initial angular velocity are θ_0 and $\dot{\theta}_0$, respectively (Assume the angle θ is small).

Note: A simple pendulum is a particle of concentrated weight that oscillates in a vertical arc and is supported by a weightless cord. The only forces acting are those of gravity and the cord tension (i.e., frictional resistance is neglected).

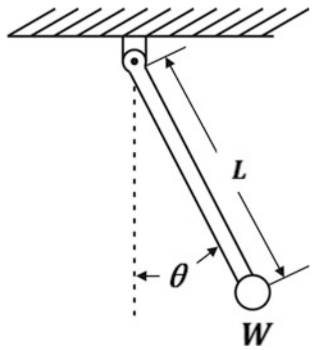

Fig. P1.7

Problem 1.8

A diver standing at the end of a diving board that cantilevers 2 ft oscillates at a frequency 2 cps. Determine the flexural rigidity *EI* of the diving board. The weight of the diver is 180 lb. (Neglect the mass of the diving board).

Problem 1.9

A bullet weighing 0.2 lb is fired at a speed of 100 ft/sec into a wooden block weighing $W = 50$ lb and supported by a spring of stiffness 300 lb/in (Fig. P1.9). Determine the displacement $u(t)$ and velocity $u(t)$ of the block after t sec.

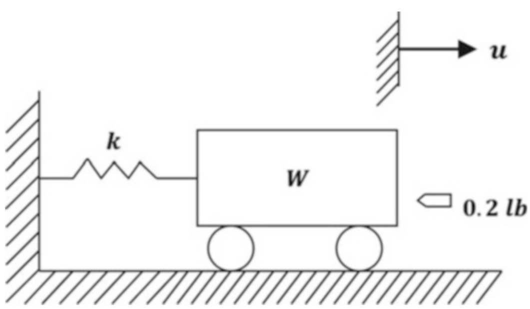

Fig. P1.9

Problem 1.10

An elevator weighing 500 lb is suspended from a spring having a stiffness of $k = 600$ lb/in. A weight of 300 lb is suspended through a cable to the elevator as shown schematically in Fig. P1.10. Determine the equation of motion of the elevator if the cable of the suspended weight suddenly breaks.

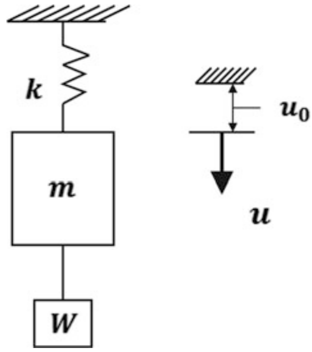

Fig. P1.10

Problem 1.11

Write the differential equation of motion for the inverted pendulum shown in Fig. P1.11 and determine its natural frequency. Assume small oscillations, and neglect the mass of the rod.

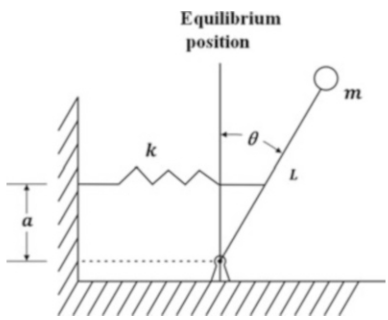

Fig. P1.11

Problem 1.12

Show that the natural frequency for the system of Problem 1.11 may be expressed as

$$f = f_0\sqrt{1 - \frac{W}{W_{cr}}}$$

where $W = mg$, W_{cr} is the critical buckling weight, and f_0 is the natural frequency neglecting the effect of gravity.

Problem 1.13

A vertical pole of length L and flexural rigidity EI carries a mass m at its top, as shown in Fig. P1.13. Neglecting the weight of the pole, derive the differential equation for small horizontal vibrations of the mass, and find the natural frequency. Assume that the effect of gravity is small and neglect nonlinear effects.

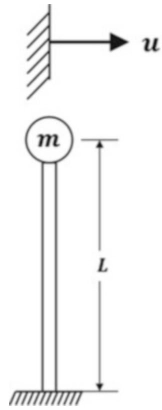

Fig. P1.13

Problem 1.14

Show that the natural frequency for the system in Problem 1.13 may be expressed as

$$f = f_0\sqrt{1 - \frac{W}{W_{cr}}}$$

where f_0 is the natural frequency calculated neglecting the effect of gravity and W_{cr} is the critical buckling weight.

Problem 1.15

Determine an expression for the natural frequency of the weight W in each of the cases shown in Fig. P1.15. The beams are uniform of cross-sectional moment of inertia I and modulus of elasticity E. Neglect the mass of the beams.

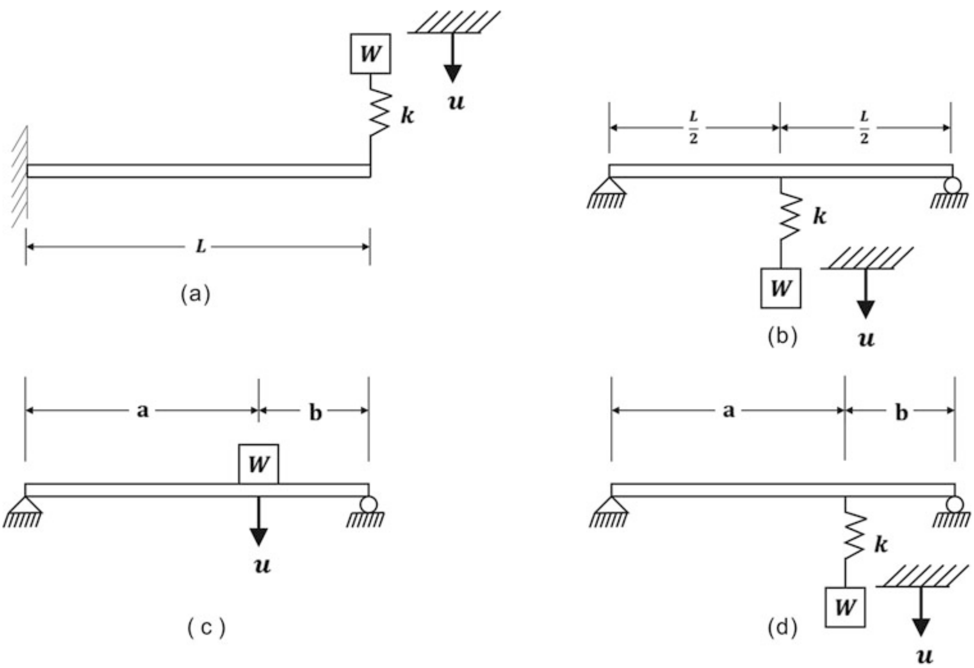

Fig. P1.15

Problem 1.16

A system is modeled by two freely vibrating masses m_1 and m_2 interconnected by a spring having a constant k as shown in Fig. P1.16. Determine for this system the differential equation of motion for the relative displacement $u_r = u_2 - u_1$ between the two masses. Also determine the corresponding natural frequency of the system.

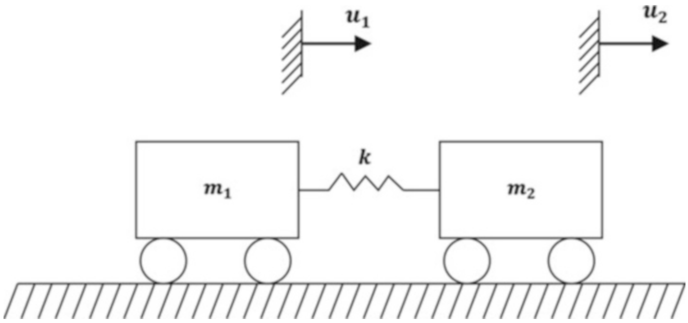

Fig. P1.16

Problem 1.17

Calculate the natural frequency for the vibration of the mass m shown in Fig. P1.17. Member AE is rigid with a hinge at C and a supporting spring of stiffness k at D. (Problem contributed by Professors Vladimir N. Alekhin and Alesksey A. Antipin of the Urals State Technical University, Russia.)

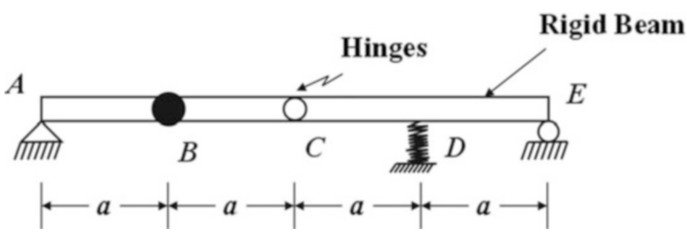

Fig. P1.17

Problem 1.18

Determine the natural frequency of vibration in the vertical direction for the rigid foundation (Fig. P1.18) transmitting a uniformly distributed pressure on the soil having a resultant force $Q = 2000$ kN. The area of the foot of the foundation is $A = 10\ \text{m}^2$. The coefficient of elastic compression of the soil is $k = 25{,}000\ \text{kN/m}^3$. (Problem contributed by Professors Vladimir N. Alekhin and Alesksey A. Antipin of the Urals State Technical University, Russia.)

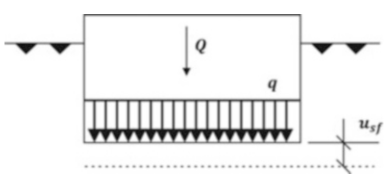

Fig. P1.18

Problem 1.19

Calculate the natural frequency of free vibration of the chimney on elastic foundation (Fig. P1.19), permitting the rotation of the structure as a rigid body about the horizontal axis *x-x*. The total weight of the structure is *W* with its center of gravity at a height *h* from the base of the foundation. The mass moment of inertia of the structure about the axis *x-x* is *I* and the rotational stiffness of the soil is *k* (resisting moment of the soil per unit rotation). (Problem contributed by Professors Vladimir N. Alekhin and Alesksey A. Antipin of the Urals State Technical University, Russia.)

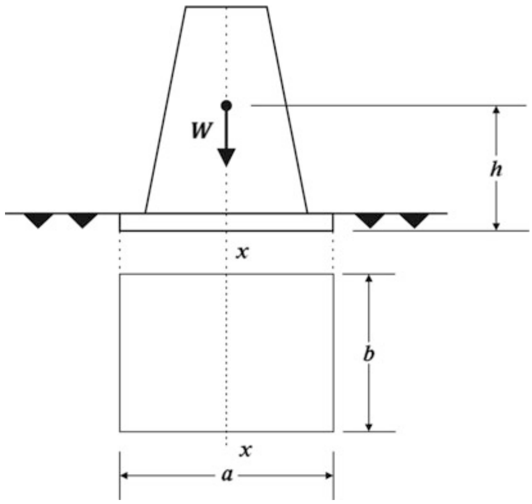

Fig. P1.19