8
The Fate of the Universe

Will the universe continue to expand forever, or will it eventually halt and start to collapse? We shall see that this question has a rather simple answer that depends only on the average density of the universe, ρ.[1] The larger the density, the stronger is the force of gravity that slows down the expansion. If ρ is greater than a certain critical value, ρ_c, expansion will be followed by contraction, and the universe will end in a big crunch. Otherwise, the expansion will continue eternally, and the universe will grow colder and darker as the stars exhaust their nuclear fuel, and the galaxies get further and further apart. Our goal in this chapter is to calculate the critical density ρ_c. Then, in the following chapter, we will discuss the measured value of the average density ρ and compare it to ρ_c.

8.1 The Critical Density

It is a fortunate happenstance that one does not need to employ the full blown mathematical machinery of general relativity to determine the critical density—a Newtonian analysis will lead to the correct result and will offer useful insights along the way. So let us start by considering an expanding spherical region of radius *R*, which represents a portion of the expanding universe. We will imagine that our hypothetical sphere is uniformly sprinkled with galaxies, which we will call "particles". Now let us consider the

[1]In this chapter we assume that there is no cosmological constant in the universe. We will revisit the fate of the universe later when we discover evidence that the cosmological constant is not zero.

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motion of a "test" particle that lies on the boundary of the sphere. The gravitational effect of the rest of the sphere on this test particle is the same as if the sphere's mass, M, were concentrated at the center. Also, the distribution of matter outside the sphere has no effect on our test particle (or any other particle within the sphere), as discussed in Chap. 2 (Fig. 8.1).

As the sphere expands, the particle (and the rest of the sphere) will be slowed down by gravity, and will either come to a halt and collapse, or will keep expanding forever. So how do we determine the outcome? We use energy conservation (this is exactly the same principle that was used when we calculated the escape speed for a projectile in Chap. 2). The particle's energy is the sum of its kinetic and gravitational potential energy, and is given by

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R} = constant \tag{8.1}$$

The mass of the test particle is m, its velocity is v, M is the mass of the whole sphere, and R is the distance of the particle to the center of the sphere. The way the sphere behaves depends on whether the total energy is negative, positive or zero.

If the total energy is negative, the particle will stop and fall back inwards. Indeed, the whole sphere will collapse. To understand why this is the case, consider the two terms that contribute to the total energy. As the particle gets further and further away, the negative potential energy term gets smaller and smaller, while the kinetic energy term is always positive. Thus, in order for the total energy to be conserved, the particle has to stop and turn around

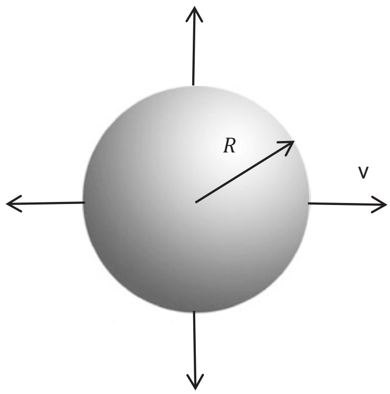

Fig. 8.1 An expanding sphere of mass *M* and radius *R* representing a portion of the universe

(if it got to infinity, it would have either zero or positive total energy depending on what the residual velocity would be at infinity). On the other hand, if the total energy is positive, then the expansion will continue, and the velocity will approach a constant value (can you determine what this value is in terms of E?). There is a third possibility—the energy could be zero,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{R} = 0 \tag{8.2}$$

This is called the critical case. The density in this case is called the critical density ρ_c.

The mass inside the sphere of radius R is related to the average mass density of the sphere via $M = \frac{4\pi}{3}R^3\rho$. Note, the sphere is expanding, so both the radius and average energy density are functions of time, while the mass remains constant. Also, from Hubble's law, the velocity of the particle is $v = HR$.

Inserting these expressions for velocity and mass into Eq. (8.2), we find $\frac{1}{2}H^2R^2 = G\frac{4\pi}{3}R^2\rho_c$, which may be rearranged to yield the expression for the critical density ρ_c in terms of the Hubble constant H and Newton's constant G:

$$\rho_c = \frac{3H^2}{8\pi G} \tag{8.3}$$

Note that ρ_c does not depend on the arbitrary radius R of the sphere. Note also that ρ_c is time-dependent because H is time-dependent.

Thus, using energy conservation, we have found that if the sphere (or the universe) has the critical density given by Eq. (8.3), expansion will continue forever, but at a speed that approaches zero (as the potential energy goes to zero, so too must the kinetic energy and hence the velocity). If the average density $\rho > \rho_c$, the expansion will halt, and will be followed by contraction and collapse. And if $\rho < \rho_c$, the universe will expand forever.

Using the current best estimate for the Hubble constant, $H_0 = 2.2 \times 10^{-18}\, s^{-1}$ we find $\rho_{c,0} \approx 10^{-26}\,\mathrm{kg/m^3}$—which corresponds to only about 6 protons per cubic meter.[2] This is all it takes to make the universe collapse! Now, if we measure the average density ρ_0, we should be able to forecast the ultimate fate of the universe. We will discuss more about ρ_0 in the next chapter, but before we get there, let us introduce a closely related parameter that is indispensable to cosmologists.

[2] The zero subscripts of H_0 and ρ_0 indicate the values of H and ρ measured at the present cosmic time.

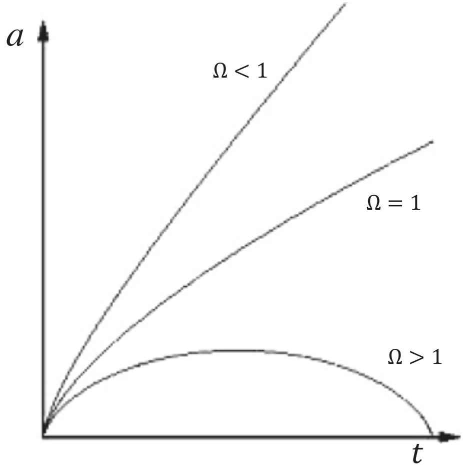

Fig. 8.2 Evolution of the scale factor (and thus the separation between any generic pair of galaxies) is determined by the density parameter. For $\Omega < 1$ galaxies approach constant recession speeds (different for different pairs of galaxies); when $\Omega = 1$ the recession speeds get smaller with time, approaching zero; and $\Omega > 1$ universes eventually contract

8.2 The Density Parameter

The density parameter is defined as the ratio of the actual (average) density to the critical density:

$$\Omega = \frac{\rho}{\rho_c} \tag{8.4}$$

We can recast the results of the previous sections of this chapter in terms of this parameter. If $\Omega > 1$, the universe eventually collapses; and if $\Omega \leq 1$, the universe expands forever (see Fig. 8.2).

Our calculation of the critical density has been performed in a Newtonian framework. Had we used general relativity, we would have found precisely the same relation between the universe's fate and the density parameter.[3] In

[3]This is not just a lucky coincidence. Newtonian gravity is a good approximation to GR when (1) the gravitational field is weak and (2) the velocities are small compared to the speed of light. Are these conditions satisfied in our calculation? When the radius of the sphere R is sufficiently small, they are. Since R is an arbitrary parameter in our calculation, we can choose it to be small enough to ensure that the Newtonian approximation is indeed valid.

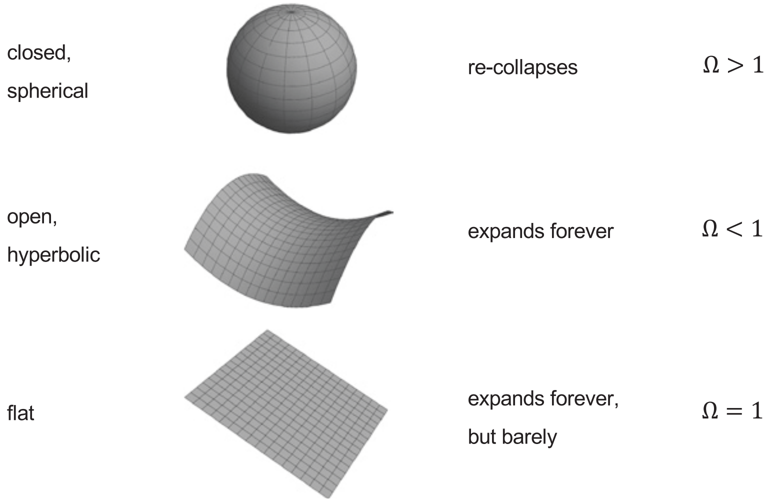

Fig. 8.3 Relation between the geometry of the universe, its fate, and the density parameter

addition, it turns out that the value of the density parameter also determines the geometry of the universe. This geometrical link can only be understood using general relativity. A closed Friedmann universe that we discussed in Chap. 5 has $\Omega > 1$; his open model has $\Omega < 1$; and a flat universe has $\Omega = 1$. The relation between the geometry of the universe, its fate, and the density parameter is summarized in Fig. 8.3.

Thus it seems as though we only have to measure Ω in order to determine the fate of the universe. However, things are not quite so straightforward—the analysis in this chapter makes use of certain assumptions regarding the contents of the universe; we will revisit the issue of our cosmic fate in the next chapter, when another important component of the universe will be introduced.

Summary

The fate of the universe is determined by its average density ρ. If ρ is larger than a certain critical value ρ_c, the force of gravity will slow the expansion down, until it halts; then the universe will contract and collapse to a big crunch. If $\rho < \rho_c$, the universe will continue expanding forever, with galaxies ultimately reaching constant recession velocities. A critical density universe with $\rho = \rho_c$ will also expand forever, but at an ever decreasing rate. Underlying these conclusions, there are certain assumptions about the contents of the universe; we revisit the issue of our cosmic fate in the next chapter.

The average density also determines the geometry of the universe: the universe is closed if $\rho > \rho_c$, open (hyperbolic) if $\rho < \rho_c$, and flat if $\rho = \rho_c$.

This relation between the average density and geometry holds regardless of the contents of the universe.

Questions

1. What properties of the universe determine whether or not it will expand forever?
2. In terms of M and R, what velocity does the test particle have in Eq. (8.2)? Can you interpret what this velocity means? If a test particle had less than this specific velocity, would it continue to move radially outwards, or would it fall back inwards?
3. Explain why gravitational potential energy is negative? Hint: Consider two objects falling towards one another from rest at a large initial distance, and think about energy conservation.
4. How is launching a projectile into space similar to the expansion of the universe?
5. At earlier cosmic times the Hubble parameter H was greater than it is now. Can you explain why?
6. Suppose astronomers living at an early epoch, when the Hubble parameter H had twice its present value H_0, set out to determine the fate of the universe. They want to measure the density of the universe and compare it to the critical density ρ_c. Is the value of ρ_c at that epoch the same as it is now? If not, how does it differ?
7. Why is Ω such an important parameter?
8. Does the density parameter Ω change with time? If Ω is greater than one at some moment, can it become less than one at a later time?
9. Do you have a philosophical preference for a universe that ends in fire (the Big Crunch) or ice (expanding eternally)?