21 Bargaining Problems

The game-theoretic literature on bargaining can be divided in two strands: the cooperative and the noncooperative approach. Here, the focus is on the cooperative approach, which was initiated by Nash (1950) and which is axiomatic in nature; see Sect. 10.1 for a first discussion. A seminal article on noncooperative bargaining is Rubinstein (1982). The basic idea of that paper is briefly repeated below, but see Sect. 6.7 for a more elaborate discussion. We conclude the chapter with a few remarks on games with nontransferable utility (NTU-games).

21.1 The Bargaining Problem

A 2-*person bargaining problem* is a pair $(S, \mathbf{d})$ where S is a compact convex nonempty subset of $\mathbb{R}^2$ and $\mathbf{d}$ is an element of S such that $\mathbf{x} > \mathbf{d}$ for some $\mathbf{x} \in S$. The elements of S are called *outcomes* and $\mathbf{d}$ is the *disagreement outcome*. The interpretation of such a problem $(S, \mathbf{d})$ is as follows. Two bargainers, 1 and 2, have to agree on some outcome $\mathbf{x} \in S$, yielding utility x_i to bargainer i. If they fail to reach such an agreement, they end up with the disagreement utilities $\mathbf{d} = (d_1, d_2)$. B denotes the family of all 2-person bargaining problems.

A (*bargaining*) *solution* is a map $F : B \to \mathbb{R}^2$ such that $F(S, \mathbf{d}) \in S$ for all $(S, \mathbf{d}) \in B$. Nash (1950) proposed to characterize such a solution by requiring it to satisfy certain axioms. More precisely, he proposed the following axioms.[1]

Weak Pareto Optimality (WPO): $F(S, \mathbf{d}) \in W(S)$ for all $(S, \mathbf{d}) \in B$, where $W(S) := \{\mathbf{x} \in S \mid \forall \mathbf{y} \in \mathbb{R}^2 : \mathbf{y} > \mathbf{x} \Rightarrow \mathbf{y} \notin S\}$ is the *weakly Pareto optimal subset of* S.[2]

[1] See Fig. 10.2 for an illustration of these axioms. In Sect. 10.1 the stronger Pareto Optimality is imposed instead of Weak Pareto Optimality. In the diagram—panel (a)—that does not make a difference.

[2] The notation $\mathbf{y} > \mathbf{x}$ means $y_i > x_i$ for $i = 1, 2$.

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Symmetry (SYM): $F_1(S,\mathbf{d}) = F_2(S,\mathbf{d})$ for all $(S,\mathbf{d}) \in B$ that are *symmetric*, i.e., $d_1 = d_2$ and $S = \{(x_2,x_1) \in \mathbb{R}^2 \mid (x_1,x_2) \in S\}$.

Scale Covariance (SC): $F(\mathbf{a}S+\mathbf{b},\mathbf{ad}+\mathbf{b}) = \mathbf{a}F(S,\mathbf{d})+\mathbf{b}$ for all $(S,\mathbf{d}) \in B$, where $\mathbf{b} \in \mathbb{R}^2$, $\mathbf{a} \in \mathbb{R}^2_{++}$, $\mathbf{ax} := (a_1x_1, a_2x_2)$ for all $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{a}S := \{\mathbf{ax} \mid \mathbf{x} \in S\}$, and $\mathbf{a}S + \mathbf{b} := \{\mathbf{ax}+\mathbf{b} \mid \mathbf{x} \in S\}$.[3]

Independence of Irrelevant Alternatives (IIA): $F(S,\mathbf{d}) = F(T,\mathbf{e})$ for all $(S,\mathbf{d})$, $(T,\mathbf{e}) \in B$ with $\mathbf{d} = \mathbf{e}$, $S \subseteq T$, and $F(T,\mathbf{e}) \in S$.

Weak Pareto Optimality says that it should not be possible for both bargainers to gain with respect to the solution outcome. If a game is symmetric, then there is no way to distinguish between the bargainers, and a solution should not do that either: that is what Symmetry requires. Scale Covariance requires the solution to be covariant under positive affine transformations: the underlying motivation is that the *utility functions* of the bargainers are usually assumed to be of the von Neumann–Morgenstern type, which implies that they are representations of preferences unique only up to positive affine transformations (details are omitted here). Independence of Irrelevant Alternatives requires the solution outcome not to change when the set of possible outcomes shrinks, the original solution outcome still remaining feasible.

The *Nash* (*bargaining*) *solution* $N : B \to \mathbb{R}^2$ is defined as follows. For every $(S,\mathbf{d}) \in B$,

$$N(S,\mathbf{d}) = \operatorname{argmax}\{(x_1-d_1)(x_2-d_2) \mid \mathbf{x} \in S,\ \mathbf{x} \geq \mathbf{d}\}\ .$$

That the Nash bargaining solution is well defined, follows from Problem 21.3.

The following theorem shows that the four conditions above characterize the Nash bargaining solution.

Theorem 21.1 *Let $F : B \to \mathbb{R}^2$ be a bargaining solution. Then the following two statements are equivalent:*

(a) $F = N$.
(b) *F satisfies WPO, SYM, SC, IIA.*

Proof The implication (a)⇒(b) is the subject of Problem 21.4. For the implication (b)⇒(a), assume F satisfies WPO, SYM, SC, and IIA. Let $(S,\mathbf{d}) \in B$, and $\mathbf{z} := N(S,\mathbf{d})$. Note that $\mathbf{z} > \mathbf{d}$. Let $T := \{((z_1-d_1)^{-1}, (z_2-d_2)^{-1})(\mathbf{x}-\mathbf{d}) \mid \mathbf{x} \in S\}$. By SC,

$$F(T,\mathbf{0}) = \left(\frac{F_1(S,\mathbf{d})}{z_1-d_1}, \frac{F_2(S,\mathbf{d})}{z_2-d_2}\right) - \left(\frac{d_1}{z_1-d_1}, \frac{d_2}{z_2-d_2}\right) \tag{21.1}$$

[3] $\mathbb{R}^2_{++} = \{\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2 \mid x_1,x_2 > 0\}$.

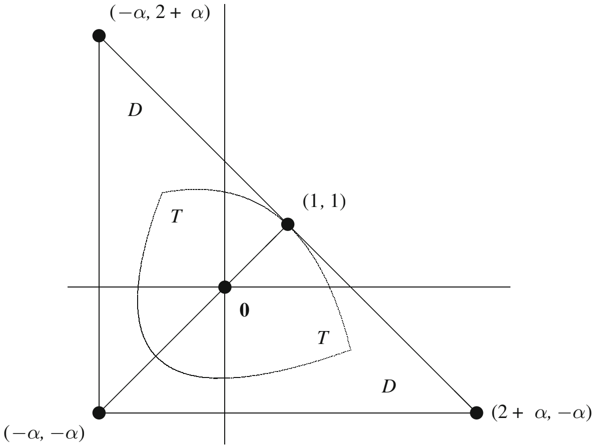

Fig. 21.1 Proof of Theorem 21.1

and

$$N(T, \mathbf{0}) = \left(\frac{z_1 - d_1}{z_1 - d_1}, \frac{z_2 - d_2}{z_2 - d_2} \right) = (1, 1) \ . \tag{21.2}$$

Hence, in order to prove $F(S, \mathbf{d}) = N(S, \mathbf{d})$, it is, in view of (21.1) and (21.2), sufficient to show that $F(T, \mathbf{0}) = (1, 1)$. By (21.2) and Problem 21.5, there is a supporting line of T at $(1, 1)$ with slope -1. So the equation of this supporting line is $x_1 + x_2 = 2$. Choose $\alpha > 0$ so large that $T \subseteq D := \text{conv}\{(-\alpha, -\alpha), (-\alpha, 2 + \alpha), (2 + \alpha, -\alpha)\}$. Cf. Fig. 21.1.

Then $(D, \mathbf{0}) \in B$, $(D, \mathbf{0})$ is symmetric, and $W(D) = \text{conv}\{(-\alpha, 2 + \alpha), (2 + \alpha, -\alpha)\}$. Hence by SYM and WPO of F:

$$F(D, 0) = (1, 1) \ . \tag{21.3}$$

Since $T \subseteq D$ and $(1, 1) \in T$, we have by IIA and (21.3): $F(T, 0) = (1, 1)$. This completes the proof. ∎

21.2 The Raiffa–Kalai–Smorodinsky Solution

Kalai and Smorodinsky (1975) replaced Nash's IIA (the most controversial axiom in Theorem 21.1) by the following condition. For a problem $(S, \mathbf{d}) \in B$,

$$u(S, \mathbf{d}) := (\max\{x_1 \mid \mathbf{x} \in S,\ \mathbf{x} \geq \mathbf{d}\}, \max\{x_2 \mid \mathbf{x} \in S,\ \mathbf{x} \geq \mathbf{d}\})$$

is called the *utopia point of* $(S, \mathbf{d})$.

Individual Monotonicity (IM): $F_j(S, \mathbf{d}) \leq F_j(T, \mathbf{e})$ for all $(S, \mathbf{d})$, $(T, \mathbf{e}) \in B$ and $i, j \in \{1, 2\}$ with $i \neq j$, $\mathbf{d} = \mathbf{e}$, $S \subseteq T$, and $u_i(S, \mathbf{d}) = u_i(T, \mathbf{e})$.

The *Raiffa–Kalai–Smorodinsky solution* $R : B \to \mathbb{R}^2$ is defined as follows. For every $(S, \mathbf{d}) \in B$, $R(S, \mathbf{d})$ is the point of intersection of $W(S)$ with the straight line joining $\mathbf{d}$ and $u(S, d)$.

The following theorem presents a characterization of the Raiffa–Kalai–Smorodinsky solution. In order to understand the proof it is recommended to draw pictures, just as in the proof of the characterization of the Nash bargaining solution.

Theorem 21.2 *Let $F : B \to \mathbb{R}^2$ be a bargaining solution. Then the following two statements are equivalent:*

(a) $F = R$.
(b) *F satisfies WPO, SYM, SC, and IM* .

Proof The implication (a)⇒(b) is the subject of Problem 21.7. For the converse implication, assume F has the four properties stated. Let $(S, \mathbf{d}) \in B$ and let $T := \{\mathbf{a}\mathbf{x} + \mathbf{b} \mid \mathbf{x} \in S\}$ with $\mathbf{a} := ((u_1(S, \mathbf{d}) - d_1)^{-1}, (u_2(S, \mathbf{d}) - d_2)^{-1})$, $\mathbf{b} := -\mathbf{a}\mathbf{d}$. By SC of R and F, $R(T, \mathbf{0}) = \mathbf{a}R(S, \mathbf{d}) + \mathbf{b}$ and $F(T, \mathbf{0}) = \mathbf{a}F(S, \mathbf{d}) + \mathbf{b}$. Hence, for $F(S, \mathbf{d}) = R(S, \mathbf{d})$, it is sufficient to prove that $R(T, \mathbf{0}) = F(T, \mathbf{0})$.

Since $u(T, \mathbf{0}) = (1, 1)$, $R(T, \mathbf{0})$ is the point of $W(T)$ with equal coordinates, so $R_1(T, \mathbf{0}) = R_2(T, \mathbf{0})$. If $R(T, \mathbf{0}) = (1, 1) = u(T, \mathbf{0})$, then let $L := \text{conv}\{(0, 0), (1, 1)\}$. Then by WPO, $F(L, \mathbf{0}) = (1, 1)$, so by IM, $F(T, \mathbf{0}) \geq F(L, \mathbf{0})$, hence $F(T, \mathbf{0}) = F(L, \mathbf{0}) = R(T, \mathbf{0})$.

Next assume $R(T, \mathbf{0}) < (1, 1)$. Let $\tilde{T} := \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{y} \leq \mathbf{x} \leq \mathbf{z} \text{ for some } \mathbf{y}, \mathbf{z} \in T\}$. Clearly $T \subseteq \tilde{T}$ and $u(\tilde{T}, 0) = u(T, \mathbf{0}) = (1, 1)$ so by IM:

$$F(\tilde{T}, \mathbf{0}) \geq F(T, \mathbf{0}) \ , \tag{21.4}$$

and further, since $R(T, \mathbf{0}) \in W(T)$ and $R_1(\tilde{T}, \mathbf{0}) = R_2(\tilde{T}, \mathbf{0})$,

$$R(\tilde{T}, \mathbf{0}) = R(T, \mathbf{0}) \ . \tag{21.5}$$

Let $V := \text{conv}\{\mathbf{0}, R(T, \mathbf{0}), (1, 0), (0, 1)\}$. By WPO and SYM, $F(V, \mathbf{0}) = R(T, \mathbf{0})$. By $V \subseteq \tilde{T}$, $u(V, \mathbf{0}) = u(\tilde{T}, \mathbf{0}) = (1, 1)$, and IM, we have $F(\tilde{T}, \mathbf{0}) \geq F(V, \mathbf{0}) = R(T, \mathbf{0})$, hence $F(\tilde{T}, \mathbf{0}) = R(T, \mathbf{0})$. Combined with (21.4), this implies $R(T, \mathbf{0}) \geq F(T, \mathbf{0})$, hence $R(T, \mathbf{0}) = F(T, \mathbf{0})$ by WPO and the fact $R(T, \mathbf{0}) < (1, 1)$. This completes the proof. ■

21.3 The Egalitarian Solution

Consider the following two properties for a bargaining solution F.

Pareto Optimality (PO): $F(S, \mathbf{d}) \in P(S)$ for all $(S, \mathbf{d}) \in B$, where $P(S) := \{\mathbf{x} \in S \mid \forall \mathbf{y} \in S : \mathbf{y} \geq \mathbf{x} \Rightarrow \mathbf{y} = \mathbf{x}\}$ is the *Pareto optimal subset of* S.[4]

Monotonicity (MON): $F(S, \mathbf{d}) \leq F(T, \mathbf{e})$ for all $(S, \mathbf{d})$, $(T, \mathbf{e}) \in B$ with $S \subseteq T$ and $\mathbf{d} = \mathbf{e}$.

Clearly, $P(S) \subseteq W(S)$ for every $(S, \mathbf{d}) \in B$, and Pareto optimality is a stronger requirement than Weak Pareto Optimality. The Nash and Raiffa–Kalai–Smorodinsky solutions are Pareto optimal, and therefore WPO can be replaced by PO in Theorems 21.1 and 21.2. Monotonicity is much stronger than Individual Monotonicity or Restricted Monotonicity (see Problem 21.8 for the definition of the last axiom) and in fact it is inconsistent with Weak Pareto Optimality. (See Problem 21.10.)

Call a problem $(S, \mathbf{d}) \in B$ *comprehensive* if $\mathbf{z} \leq \mathbf{y} \leq \mathbf{x}$ implies $\mathbf{y} \in S$ for all $\mathbf{z}$, $\mathbf{x} \in S$, $\mathbf{y} \in \mathbb{R}^2$. By B^c we denote the subclass of comprehensive problems.

The *egalitarian solution* $E : B^c \to \mathbb{R}^2$ assigns to each problem $(S, \mathbf{d}) \in B^c$ the point $E(S, \mathbf{d}) \in W(S)$ with $E_1(S, \mathbf{d}) - d_1 = E_2(S, \mathbf{d}) - d_2$.

The following axiom is a weakening of Scale Covariance.

Translation Covariance (TC): $F(S + \mathbf{e}, \mathbf{d} + \mathbf{e}) = F(S, \mathbf{d}) + \mathbf{e}$ for all $(S, \mathbf{d}) \in B^c$ and all $\mathbf{e} \in \mathbb{R}^2$.

The following theorem gives a characterization of the egalitarian solution based on Monotonicity.

Theorem 21.3 *Let $F : B^c \to \mathbb{R}^2$ be a bargaining solution. Then the following two statements are equivalent:*

(a) $F = E$.
(b) *F satisfies WPO, MON, SYM, and TC.*

Proof The implication (a)⇒(b) is the subject of Problem 21.11. For the converse implication, let $(S, \mathbf{d}) \in B^c$. We want to show $F(S, \mathbf{d}) = E(S, \mathbf{d})$.
In view of TC of F and E, we may assume $\mathbf{d} = \mathbf{0}$. Let $V := \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{0} \leq \mathbf{x} \leq E(S, \mathbf{0})\}$. Clearly, $(V, \mathbf{0}) \in B^c$ is a symmetric problem, so $F(V, \mathbf{0}) = E(S, \mathbf{0})$ by SYM and WPO of F. By MON,

$$F(S, \mathbf{0}) \geq F(V, \mathbf{0}) = E(S, \mathbf{0}) \ . \tag{21.6}$$

[4]The notation $\mathbf{y} \geq \mathbf{x}$ means $y_i \geq x_i$ for $i = 1, 2$.

If $E(S, \mathbf{0}) \in P(S)$, then (21.6) implies $F(S, \mathbf{0}) = E(S, \mathbf{0})$, so we are done. Now suppose $E(S, \mathbf{0}) \in W(S) \setminus P(S)$. Without loss of generality, assume $E_1(S, \mathbf{0}) = u_1(S, \mathbf{0})$, i.e., $E_1(S, \mathbf{0}) = \max\{x_1 \mid \mathbf{x} \in S, \mathbf{x} \geq \mathbf{0}\}$. Hence, $E_1(S, \mathbf{0}) = F_1(S, \mathbf{0})$ by (21.6).

Suppose $F_2(S, \mathbf{0}) > E_2(S, \mathbf{0})$. The proof will be finished by contradiction. Let $\alpha > 0$ with $E_2(S, \mathbf{0}) < \alpha < F_2(S, \mathbf{0})$. Let $T := \text{conv}(S \cup \{(\alpha, 0), (\alpha, \alpha)\})$. Then $(T, \mathbf{0}) \in B^c$ and $E(T, \mathbf{0}) = (\alpha, \alpha) \in P(T)$, so $F(T, \mathbf{0}) = (\alpha, \alpha)$ by our earlier argument [see the line below (21.6)]. On the other hand, by MON, $F_2(T, \mathbf{0}) \geq F_2(S, \mathbf{0}) > \alpha$, a contradiction. ■

An alternative characterization of the egalitarian solution can be obtained by considering the following axioms. For a bargaining problem $(S, \mathbf{d})$, denote $S_{\mathbf{d}} = \{\mathbf{x} \in S \mid \mathbf{x} > \mathbf{d}\}$.

Super-Additivity (SA): $F(S + T, \mathbf{d} + \mathbf{e}) \geq F(S, \mathbf{d}) + F(T, \mathbf{e})$ for all $(S, \mathbf{d}), (T, \mathbf{e}) \in B^c$. Here, $S + T := \{\mathbf{x} + \mathbf{y} \mid \mathbf{x} \in S, \mathbf{y} \in T\}$.

Independence of Non-Individually Rational Alternatives (INIR): $F(S, \mathbf{d}) = F(S_{\mathbf{d}}, \mathbf{d})$ for all $(S, \mathbf{d}) \in B^c$.

INIR is a stronger version of the following well-known axiom.

Individual Rationality (IR): $F(S, \mathbf{d}) \geq \mathbf{d}$ for all $(S, \mathbf{d}) \in B^c$.

Observe, indeed, that INIR implies IR.

Theorem 21.4 *Let $F : B^c \to \mathbb{R}^2$ be a bargaining solution. Then the following two statements are equivalent:*

(a) *$F = E$.*
(b) *F satisfies WPO, SA, SYM, INIR, and TC.*

Proof (a)⇒(b) follows from Theorem 21.3 and Problem 21.12. For the converse implication, let $(S, \mathbf{d}) \in B^c$. We wish to show $F(S, \mathbf{d}) = E(S, \mathbf{d})$. In view of TC of F and E we may assume $\mathbf{d} = \mathbf{0}$. For every $1 > \varepsilon > 0$ let $V^\varepsilon := \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{0} \leq \mathbf{x} \leq (1 - \varepsilon)E(S, \mathbf{0})\}$. Then $(V^\varepsilon, \mathbf{0}) \in B^c$ and $F(V^\varepsilon, \mathbf{0}) = E(V^\varepsilon, \mathbf{0}) = (1 - \varepsilon)E(S, \mathbf{0})$ by WPO and SYM of F and E. Define $W^\varepsilon = \{\mathbf{x} - (1 - \varepsilon)E(S, \mathbf{0}) \mid \mathbf{x} \in S\}$ and $S^\varepsilon = W^\varepsilon + V^\varepsilon$ for every $1 > \varepsilon > 0$. Then $S_{\mathbf{0}} = S^\varepsilon_{\mathbf{0}}$ and by SA we have

$$F(S^\varepsilon, \mathbf{0}) \geq (1 - \varepsilon)E(S, \mathbf{0}) + F(W^\varepsilon, \mathbf{0}) \ ,$$

hence by INIR

$$F(S, \mathbf{0}) \geq (1 - \varepsilon)E(S, \mathbf{0}) + F(W^\varepsilon, \mathbf{0}) \qquad (21.7)$$

for all $1 > \varepsilon > 0$. Letting ε decrease to 0, we obtain by (21.7) and IR (which is implied by INIR):

$$F(S, \mathbf{0}) \geq E(S, \mathbf{0}) \ . \tag{21.8}$$

If $E(S, \mathbf{0}) \in P(S)$, then (21.8) implies $F(S, \mathbf{0}) = E(S, \mathbf{0})$ and we are done. Otherwise, suppose without loss of generality that $E_1(S, \mathbf{0}) = \max\{x_1 \mid \mathbf{x} \in S, \mathbf{x} \geq \mathbf{0}\}$. Let $\mathbf{z}$ be the point of $P(S)$ with $z_1 = E_1(S, \mathbf{0})$, hence $\alpha := E_2(S, \mathbf{0}) - z_2 < 0$ since, by assumption, $E(S, \mathbf{0}) \notin P(S)$. For $\varepsilon > 0$, let $R^\varepsilon := \operatorname{conv}\{(0, \varepsilon), (0, \alpha), (\varepsilon, \alpha)\}$. Then $(R^\varepsilon, \mathbf{0}) \in B^c$. Further, let $T^\varepsilon := S + R^\varepsilon$. By construction, $E(T^\varepsilon, \mathbf{0}) \in P(T^\varepsilon)$, hence, as before, $F(T^\varepsilon, \mathbf{0}) = E(T^\varepsilon, \mathbf{0})$. If ε approaches 0, $F(T^\varepsilon, \mathbf{0})$ converges to $E(S, \mathbf{0})$ and by SA and IR, $F(T^\varepsilon, \mathbf{0}) \geq F(S, \mathbf{0})$. So $E(S, \mathbf{0}) \geq F(S, \mathbf{0})$. Combined with (21.8), this implies $F(S, \mathbf{0}) = E(S, \mathbf{0})$. ■

21.4 Noncooperative Bargaining

A different approach to bargaining is obtained by studying it a as strategic process. In this section we discuss the basics of the model of Rubinstein (1982) in an informal manner. See also Sect. 6.7 for a more elaborate treatment.

Point of departure is a bargaining problem $(S, \mathbf{d}) \in B$. Assume $\mathbf{d} = \mathbf{0}$ and write S instead of $(S, \mathbf{d})$. Suppose bargaining takes place over time, at moments $t = 0, 1, 2, \ldots$ At even moments, player 1 makes some proposal $\mathbf{x} = (x_1, x_2) \in P(S)$ and player 2 accepts or rejects it. At odd moments, player 2 makes some proposal $\mathbf{x} = (x_1, x_2) \in P(S)$ and player 1 accepts or rejects it. The game ends as soon as a proposal is accepted. If a proposal $\mathbf{x} = (x_1, x_2)$ is accepted at time t, then the players receive payoffs $(\delta^t x_1, \delta^t x_2)$. Here $0 < \delta < 1$ is a so called discount factor; it reflects impatience of the players, for instance because of foregone interest payments ('shrinking cake'). If no proposal is ever accepted, then the game ends with the disagreement payoffs of $(0, 0)$.

Suppose player 1 has in mind to make some proposal $\mathbf{y} = (y_1, y_2) \in P(S)$, and that player 2 has in mind to make some proposal $\mathbf{z} = (z_1, z_2) \in P(S)$. So player 1 offers to player 2 the amount y_2. Player 2 expects to get z_2 if he rejects $\mathbf{y}$, but he will get z_2 one round later. So player 1's proposal y will be rejected by player 2 if $y_2 < \delta z_2$; on the other hand, there is no need to offer strictly more than δz_2. This leads to the equation

$$y_2 = \delta z_2 \ . \tag{21.9}$$

By reversing in this argument the roles of players 1 and 2 one obtains

$$z_1 = \delta y_1 \ . \tag{21.10}$$

These two equations define unique points $\mathbf{y}$ and $\mathbf{z}$ in $P(S)$. The result of the Rubinstein bargaining approach is that player 1 starts by offering $\mathbf{y}$, player 2 accepts, and the game ends with the payoffs $\mathbf{y} = (y_1, y_2)$.

This description is informal. Formally, one defines a dynamic noncooperative game and looks for the (in this case) subgame perfect Nash equilibria of this game. It can be shown that all such equilibria result in the payoffs $\mathbf{y}$ (or in $\mathbf{z}$ if player 2 would start instead of player 1).

The surprising fact is that, although at first sight the Rubinstein approach is quite different from the axiomatic approach by Nash (Theorem 21.1) the resulting outcomes turn out to be closely related. From Eqs. (21.9) and (21.10) one derives easily that $y_1 y_2 = z_1 z_2$, i.e., the points $\mathbf{y}$ and $\mathbf{z}$ are on the same level curve of the function $\mathbf{x} = (x_1, x_2) \mapsto x_1 x_2$, which appears in the definition of the Nash bargaining solution. Moreover, if the discount factor δ approaches 1, the points $\mathbf{y}$ and $\mathbf{z}$ converge to one another on the curve $P(S)$, and hence to the Nash bargaining solution outcome. In words, as the players become more patient, the outcome of the Rubinstein model converges to the Nash bargaining solution outcome.

21.5 Games with Nontransferable Utility

Both TU-games and bargaining problems are special cases of NTU-games, games with nontransferable utility. In an NTU-game, the possibilities from cooperation for each coalition are described by a set, rather than a single number. For a TU-game (N, v) those sets can be defined as

$$V(S) = \{\mathbf{x} \in \mathbb{R}^S \mid x(S) \le v(S)\}$$

for every coalition S. For a two-person bargaining problem $(S, \mathbf{d})$ the set of feasible payoffs is S for the grand coalition $\{1, 2\}$ and $(-\infty, d_i]$ for each player i.

The core can be extended to NTU-games (for bargaining problems it is just the part of the Pareto optimal set weakly dominating the disagreement outcome). Also the balancedness concept can be extended; the main result here is that balanced games have a nonempty core, but the converse is not true. For further remarks and references see the Notes section.

21.6 Problems

21.1. *Anonymity and Symmetry*
Call a two-person bargaining solution *anonymous* if $F_1(S', \mathbf{d}') = F_2(S, \mathbf{d})$ and $F_2(S', \mathbf{d}') = F_1(S, \mathbf{d})$ whenever $(S, \mathbf{d}), (S', \mathbf{d}') \in B$ with $S' = \{(x_2, x_1) \in \mathbb{R}^2 \mid (x_1, x_2) \in S\}$ and $(d'_1, d'_2) = (d_2, d_1)$. Prove that Anonymity implies Symmetry but not vice versa.

21.2. *Revealed Preference*
Let $B_0 = \{(S, \mathbf{d}) \in B \mid \mathbf{d} = (0, 0)\}$. For $(S, \mathbf{d}) \in B_0$ write S instead of $(S, \mathbf{0})$. Let $\succeq$ be a binary relation on $\mathbb{R}^2$ and $F : B_0 \to \mathbb{R}^2$ a solution. Say that $\succeq$ *represents* F if for every $S \in B_0$:

$$\{F(S)\} = \{\mathbf{x} \in S \mid \mathbf{x} \succeq \mathbf{y} \text{ for every } \mathbf{y} \in S\} ,$$

i.e., if F uniquely maximizes $\succeq$ on S. Prove: F satisfies IIA if and only if F can be represented by a binary relation $\succeq$.

21.3. *The Nash Solution Is Well-Defined*
Show that N is well defined, i.e., that the function $(x_1 - d_1)(x_2 - d_2)$ takes its maximum on $\{\mathbf{x} \in S \mid \mathbf{x} \geq \mathbf{d}\}$ at a unique point.

21.4. $(a) \Rightarrow (b)$ *in Theorem 21.1*
Show that N satisfies the properties WPO, SYM, SC, and IIA.

21.5. *Geometric Characterization of the Nash Bargaining Solution*
Show that, for every $(S, \mathbf{d}) \in B$, $N(S, \mathbf{d}) = \mathbf{z} > \mathbf{d}$ if and only if there is a supporting line of S at $\mathbf{z}$ with slope the negative of the slope of the straight line through $\mathbf{d}$ and $\mathbf{z}$.

21.6. *Strong Individual Rationality*
Call a solution F *strongly individually rational* (SIR) if $F(S, \mathbf{d}) > \mathbf{d}$ for all $(S, \mathbf{d}) \in B$. The *disagreement* solution D is defined by $D(S, \mathbf{d}) := \mathbf{d}$ for every $(S, \mathbf{d}) \in B$. Show that the following two statements for a solution F are equivalent:

(a) $F = N$ or $F = D$.
(b) F satisfies IR, SYM, SC, and IIA.

Derive from this that N is the unique solution with the properties SIR, SYM, SC, and IIA. (Hint: For the implication (b)$\Rightarrow$(a), show that, for every $(S, \mathbf{d}) \in B$, either $F(S, \mathbf{d}) = \mathbf{d}$ or $F(S, \mathbf{d}) \in W(S)$. Also show that, if $F(S, \mathbf{d}) = \mathbf{d}$ for *some* $(S, \mathbf{d}) \in B$, then $F(S, \mathbf{d}) = \mathbf{d}$ for *all* $(S, \mathbf{d}) \in B$.)

21.7. $(a) \Rightarrow (b)$ *in Theorem 21.2*
Show that the Raiffa–Kalai–Smorodinsky solution has the properties WPO, SYM, SC, and IM.

21.8. *Restricted Monotonicity*
Call a solution $F : B \to \mathbb{R}^2$ *restrictedly monotonic* (RM) if $F(S, \mathbf{d}) \leq F(T, \mathbf{e})$ whenever $(S, \mathbf{d}), (T, \mathbf{e}) \in B$, $\mathbf{d} = \mathbf{e}$, $S \subseteq T$, $u(S, \mathbf{d}) = u(T, \mathbf{e})$.

(a) Prove that IM implies RM.
(b) Show that RM does not imply IM.

21.9. *Global Individual Monotonicity*
For a problem $(S, \mathbf{d}) \in B$, $g(S) := (\max\{x_1 \mid \mathbf{x} \in S\}, \max\{x_2 \mid \mathbf{x} \in S\})$ is called the *global utopia point of* S. *Global Individual Monotonicity* (GIM) is defined in the same way as IM, with the condition "$u_i(S, \mathbf{d}) = u_i(T, \mathbf{e})$" replaced by: $g_i(S) = g_i(T)$. The solution $G : B \to \mathbb{R}^2$ assigns to each $(S, \mathbf{d}) \in B$ the point of intersection of $W(S)$ with the straight line joining $\mathbf{d}$ and $g(S)$. Show that G is the unique solution with the properties WPO, SYM, SC, and GIM.

21.10. *Monotonicity and (Weak) Pareto Optimality*

(a) Show that there is no solution satisfying MON and WPO.
(b) Show that, on the subclass B_0 introduced in Problem 21.2, there is no solution satisfying MON and PO. Can you find a solution on this class with the properties MON and WPO?

21.11. *The Egalitarian Solution (1)*

(a) Show that E satisfies MON, SYM, and WPO (on B^c).
(b) Show that E is translation covariant on B^c.

21.12. *The Egalitarian Solution (2)*
Show that the egalitarian solution is super-additive.

21.13. *Independence of Axioms*
In the characterization Theorems 21.1–21.4, show that none of the axioms used can be dispensed with.

21.14. *Nash and Rubinstein*
Suppose two players (bargainers) bargain over the division of one unit of a perfectly divisible good. Player 1 has utility function $u_1(\alpha) = \alpha$ and player 2 has utility function $u_2(\alpha) = 1 - (1 - \alpha)^2$ for amounts $\alpha \in [0, 1]$ of the good. If they do not reach an agreement on the division of the good they both receive nothing.

(a) Determine the set of feasible utility pairs. Make a picture.
(b) Determine the Nash bargaining solution outcome, in terms of utilities as well as of the physical distribution of the good.
(c) Suppose the players' utilities are discounted by a factor $\delta \in (0, 1)$. Calculate the Rubinstein bargaining outcome.
(d) Determine the limit of the Rubinstein bargaining outcome, for δ approaching 1, in two ways: by using the result of (b) and by using the result of (c).

21.7 Notes

The axiomatic study of bargaining problems was initiated by Nash (1950). For comprehensive surveys see Peters (1992) or Thomson (1994). Theorem 21.1 is due to Nash (1950).

The Raiffa–Kalai–Smorodinsky solution was introduced by Raiffa (1953) and axiomatized by Kalai and Smorodinsky (1975). Theorem 21.2 is a modified version of this characterization.

For the analysis of the noncooperative bargaining model in Sect. 21.4 see Rubinstein (1982) or Sutton (1986). For an elaborate discussion of noncooperative bargaining models see Muthoo (1999). The observation about the relation with the Nash bargaining solution is due to Binmore et al. (1986).

The fact that balanced NTU-games have a nonempty core is due to Scarf (1976). A complete characterization of NTU games with nonempty core based on a kind of local balancedness condition is provided by Predtetchinski and Herings (2004).

Most other solution concepts for NTU-games—in particular the Harsanyi (1963) and Shapley (1969) NTU-values, and the consistent value of Hart and Mas-Collel (1996)—extend the Nash bargaining solution as well as the Shapley value for TU-games. An exception are the monotonic solutions of Kalai and Samet (1985), which extend the egalitarian solution of the bargaining problem. See de Clippel et al. (2004) for an overview of various axiomatic characterizations of values for NTU-games, and see Peters (2003) for an overview of NTU-games in general. An extensive textbook treatment can be found in Peleg and Sudhölter (2003, Part II).

Most (though not all) results of this chapter on bargaining can be extended to the n-person case without too much difficulty. This is not true for the Rubinstein approach, the extension of which is not obvious. One possibility is presented by Hart and Mas-Collel (1996).

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