10 Cooperative Game Models

The common features of a *cooperative game model*—such as the model of a game with transferable utility in Chap. 9—include: the abstraction from a detailed description of the strategic possibilities of a player; instead, a detailed description of what players and coalitions can attain in terms of outcomes or utilities; solution concepts based on strategic considerations and/or considerations of fairness, equity, efficiency, etc.; if possible, an axiomatic characterization of such solution concepts. For instance, one can argue that the core for TU-games is based on strategic considerations whereas the Shapley value is based on a combination of efficiency and symmetry or fairness with respect to contributions. The latter is made precise by an axiomatic characterization as in Problem 9.17.

In this chapter a few other cooperative game models are discussed: bargaining problems in Sect. 10.1, exchange economies in Sect. 10.2, matching problems in Sect. 10.3, and house exchange in Sect. 10.4.

10.1 Bargaining Problems

An example of a bargaining problem is the division problem in Sect. 1.3.5. A noncooperative, strategic approach to such a bargaining problem can be found in Sect. 6.7, see also Problems 6.16 and 6.17. In this section we treat the bargaining problem from a cooperative, axiomatic perspective. Surprisingly, there is a close relation between this approach and the strategic approach, as we will see. In Sect. 10.1.1 we discuss the Nash bargaining solution and in Sect. 10.1.2 its relation with the Rubinstein bargaining procedure of Sect. 6.7.

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10.1.1 The Nash Bargaining Solution

We start with the definition of a two-person bargaining problem.[1]

Definition 10.1 A two-person *bargaining problem* is a pair (S, d), where

(*i*) $S \subseteq \mathbb{R}^2$ is a convex, closed and bounded set,[2]
(*ii*) $\mathbf{d} = (d_1, d_2) \in S$ such that there is some point $\mathbf{x} = (x_1, x_2) \in S$ with $x_1 > d_1$ and $x_2 > d_2$.

S is the *feasible set* and $\mathbf{d}$ is the *disagreement point*. □

The interpretation of a bargaining problem $(S, \mathbf{d})$ is as follows. The two players bargain over the feasible outcomes in S. If they reach an agreement $\mathbf{x} = (x_1, x_2) \in S$, then player 1 receives utility x_1 and player 2 receives utility x_2. If they do not reach an agreement, then the game ends in the disagreement point $\mathbf{d}$, yielding utility d_1 to player 1 and d_2 to player 2. This is an interpretation, and the actual bargaining procedure is not spelled out.

For the example in Sect. 1.3.5, the feasible set and the disagreement point are given by

$$S = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 \leq x_1, x_2 \leq 1,\ x_2 \leq \sqrt{1 - x_1}\},\ d_1 = d_2 = 0\ .$$

See also Fig. 1.7. In general, a bargaining problem may look as in Fig. 10.1. The set of all such bargaining problems is denoted by $\mathcal{B}$.

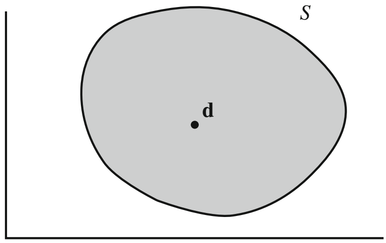

Fig. 10.1 A two-person bargaining problem

[1] We restrict attention here to two-person bargaining problems. For n-person bargaining problems and, more generally, NTU-games, see the Notes section at the end of the chapter and Chap. 21.

[2] A subset of $\mathbb{R}^k$ is convex if with each pair of points in the set also the line segment connecting these points is in the set. A set is closed if it contains its boundary or, equivalently, if for every sequence of points in the set that converges to a point that limit point is also in the set. It is bounded if there is a number $M > 0$ such that $|x_i| \leq M$ for all points $\mathbf{x}$ in the set and all coordinates i.

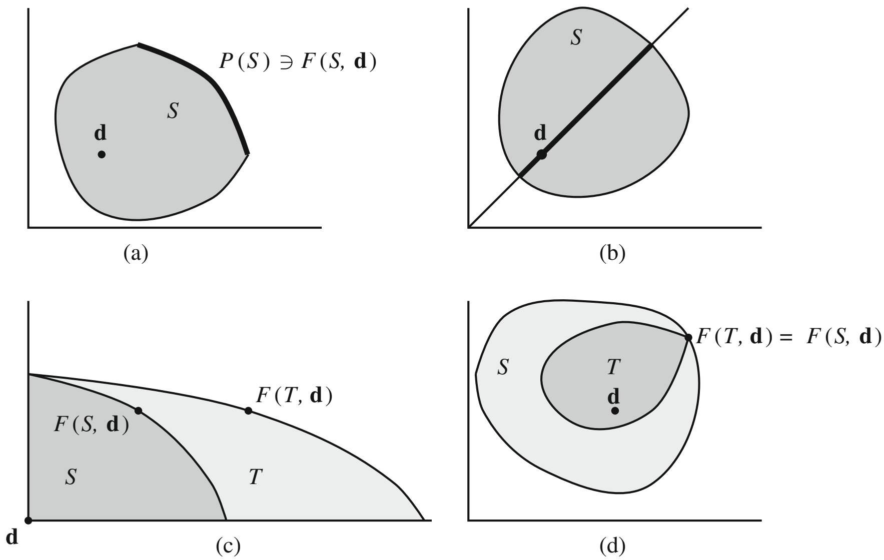

Fig. 10.2 Illustration of the four conditions ('axioms') determining the Nash bargaining solution—cf. Theorem 10.2. In (**a**) the Pareto optimal subset of S is the *thick black curve*. The bargaining problem $(S, \mathbf{d})$ in (**b**) is symmetric, and symmetry of F means that F should assign a point on the *thick black line* segment. In (**c**), which illustrates scale covariance, we took $\mathbf{d}$ to be the origin, and T results from S by multiplying all first coordinates by 2: then scale covariance implies that $F_1(T, \mathbf{d}) = 2F_1(S, \mathbf{d})$. The independence of irrelevant alternatives axiom is illustrated in (**d**)

We consider the following question: for any given bargaining problem $(S, \mathbf{d})$, what is a good compromise? We answer this question by looking for a map $F : \mathcal{B} \to \mathbb{R}^2$ which assigns a feasible point to every bargaining problem, i.e., satisfies $F(S, \mathbf{d}) \in S$ for every $(S, \mathbf{d}) \in \mathcal{B}$. Such a map is called a (two-person) *bargaining solution*. According to Nash (1950), a bargaining solution should satisfy four conditions, namely: Pareto optimality, symmetry, scale covariance, and independence of irrelevant alternatives. We discuss each of these conditions in detail. The conditions are illustrated in Fig. 10.2a–d.

For a bargaining problem $(S, \mathbf{d}) \in \mathcal{B}$, the Pareto optimal points of S are those where the utility of no player can be increased without decreasing the utility of the other player. Formally,

$$P(S) = \{\mathbf{x} \in S \mid \text{for all } \mathbf{y} \in S \text{ with } y_1 \geq x_1, y_2 \geq x_2, \text{ we have } \mathbf{y} = \mathbf{x}\}$$

is the *Pareto optimal* (sub)set of S. The bargaining solution F is *Pareto optimal* if $F(S, \mathbf{d}) \in P(S)$ for all $(S, \mathbf{d}) \in \mathcal{B}$. Hence, a Pareto optimal bargaining solution assigns a Pareto optimal point to each bargaining problem. See Fig. 10.2a for an illustration.

A bargaining problem $(S, \mathbf{d}) \in \mathcal{B}$ is *symmetric* if $d_1 = d_2$ and if S is symmetric with respect to the 45°-line through $\mathbf{d}$, i.e., if

$$S = \{(x_2, x_1) \in \mathbb{R}^2 \mid (x_1, x_2) \in S\} \ .$$

In a symmetric bargaining problem there is no way to distinguish between the players other than by the arbitrary choice of axes. A bargaining solution is *symmetric* if $F_1(S, \mathbf{d}) = F_2(S, \mathbf{d})$ for each symmetric bargaining problem $(S, \mathbf{d}) \in \mathcal{B}$. Hence, a symmetric bargaining solution assigns the same utility to each player in a symmetric bargaining problem. See Fig. 10.2b.

Observe that, for a symmetric bargaining problem $(S, \mathbf{d})$, Pareto optimality and symmetry of F would completely determine the solution point $F(S, \mathbf{d})$, since there is a unique symmetric Pareto optimal point in S.

The condition of scale covariance says that a bargaining solution should not depend on the choice of the origin or on a positive multiplicative factor in the utilities. For instance, in the wine division problem in Sect. 1.3.5, it should not matter if the utility functions were $\bar{u}_1(\alpha) = a_1\alpha + b_1$ and $\bar{u}_2(\alpha) = a_2\sqrt{\alpha} + b_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $a_1, a_2 > 0$. Saying that this should not matter means that the final outcome of the bargaining problem, the division of the wine, should not depend on this. One can think of $\bar{u}_1, \bar{u}_2$ expressing the same preferences about wine as u_1, u_2 in different units.[3] Formally, a bargaining solution F is *scale covariant* if for all $(S, \mathbf{d}) \in \mathcal{B}$ and all $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $a_1, a_2 > 0$ we have:

$$F\left(\{(a_1x_1 + b_1, a_2x_2 + b_2) \in \mathbb{R}^2 \mid (x_1, x_2) \in S\}, (a_1d_1 + b_1, a_2d_2 + b_2)\right)$$
$$= (a_1F_1(S, \mathbf{d}) + b_1, a_2F_2(S, \mathbf{d}) + b_2) \ .$$

For a simple case, this condition is illustrated in Fig. 10.2c.

The final condition is regarded as the most controversial one. Consider a bargaining problem $(S, \mathbf{d})$ with solution outcome $\mathbf{z} = F(S, \mathbf{d}) \in S$. In a sense, $\mathbf{z}$ can be regarded as the best compromise in S according to F. Now consider a smaller bargaining problem $(T, \mathbf{d})$ with $T \subseteq S$ and $\mathbf{z} \in T$. Since $\mathbf{z}$ was the best compromise in S, it is should certainly be regarded as the best compromise in T: $\mathbf{z}$ is available in T and every point of T is also available in S. Thus, we should conclude that $F(T, \mathbf{d}) = \mathbf{z} = F(S, \mathbf{d})$. As a less abstract example, suppose that in the wine division problem the wine is split fifty-fifty, with utilities $(1/2, \sqrt{1/2})$. Suppose now that no player wants to drink more than $3/4$ liter of wine: more wine does not increase utility. In that case, the new feasible set is

$$T = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 3/4,\ 0 \leq x_2 \leq \sqrt{3/4},\ x_2 \leq \sqrt{1 - x_1}\} \ .$$

[3]The usual assumption is that the utility functions are expected utility functions, which uniquely represent preferences up to choice of origin and scale.

According to the argument above, the wine should still be split fifty-fifty: $T \subseteq S$ and $(1/2, \sqrt{1/2}) \in T$. This may seem reasonable but it is not hard to change the example in such a way that the argument is, at the least, debatable. For instance,suppose that player 1 still wants to drink as much as possible but player 2 does not want to drink more than 1/2 L. In that case, the feasible set becomes

$$T' = \{\mathbf{x} \in \mathbb{R}^2 \mid 0 \le x_1 \le 1,\ 0 \le x_2 \le \sqrt{1/2},\ x_2 \le \sqrt{1 - x_1}\}\ ,$$

and we would still split the wine fifty-fifty. In this case player 2 would obtain his maximal feasible utility, and $(1/2, \sqrt{1/2})$ no longer seems a reasonable compromise since only player 1 makes a concession.

Formally, a bargaining solution F is *independent of irrelevant alternatives* if for all $(S, \mathbf{d}), (T, \mathbf{d}) \in \mathcal{B}$ with $T \subseteq S$ and $F(S, \mathbf{d}) \in T$, we have $F(T, \mathbf{d}) = F(S, \mathbf{d})$. See Fig. 10.2d for an illustration.

The theorem below says that these four conditions determine a unique bargaining solution F^{Nash}, defined as follows. For $(S, \mathbf{d}) \in \mathcal{B}$, $F^{\text{Nash}}(S, \mathbf{d})$ is equal to the unique point $\mathbf{z} \in S$ with $z_i \ge d_i$ for $i = 1, 2$ and such that

$$(z_1 - d_1)(z_2 - d_2) \ge (x_1 - d_1)(x_2 - d_2) \text{ for all } \mathbf{x} \in S \text{ with } x_i \ge d_i,\ i = 1, 2\ .$$

The solution F^{Nash} is called the *Nash bargaining solution*. The result of Nash is as follows.

Theorem 10.2 *The Nash bargaining solution F^{Nash} is the unique bargaining solution which is Pareto optimal, symmetric, scale covariant, and independent of irrelevant alternatives.*

For a proof of this theorem and the fact that F^{Nash} is well defined—i.e., the point $\mathbf{z}$ above exists and is unique—see Chap. 21.

Example 10.3 In this example we illustrate the role of the conditions in Theorem 10.2. In fact, we show how the proof of this theorem (cf. Chap. 21) works in an example. Consider the bargaining problem $(S, \mathbf{d})$, where $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 2,\ 0 \le x_2 \le 4 - x_1^2\}$ and $\mathbf{d} = (0, 1)$. The Nash bargaining solution outcome is obtained by solving the problem $\max_{0 \le x_1 \le 2}\ x_1(3 - x_1^2)$, which yields the point $(1, 3)$. Alternatively, consider the bargaining problem $(S', \mathbf{d}')$, obtained by subtracting 1 from the second coordinates of the points in S, including $\mathbf{d}$, yielding

$$S' = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 2,\ -1 \le x_2 \le 3 - x_1^2\},\ \mathbf{d}' = (0, 0)\ .$$

Next, consider the bargaining problem $(S'', \mathbf{d}'')$, obtained from $(S', \mathbf{d}')$ by dividing all second coordinates by 2, yielding

$$S'' = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 2,\ -\frac{1}{2} \le x_2 \le \frac{3}{2} - \frac{1}{2}x_1^2\},\ \mathbf{d}'' = (0, 0)\ .$$

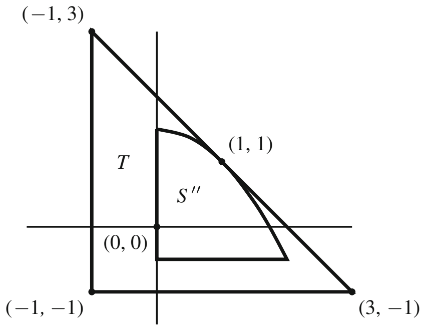

Fig. 10.3 Illustrating Example 10.3

The Pareto optimal boundary of S'' is described by the function $f(x_1) = \frac{3}{2} - \frac{1}{2}x_1^2$ for $0 \leq x_1 \leq 2$. At $x_1 = 1$ the derivative of this function is equal to -1, so that the straight line through the point $(1, 1)$ with slope -1 is tangential to S'', i.e., the set S'' is below this line. Now consider the bargaining problem $(T, (0, 0))$ with T the triangle and inside with vertices $(-1, -1)$, $(3, -1)$, and $(-1, 3)$; see Fig. 10.3. The bargaining problem $(T, (0, 0))$ is symmetric, so that by symmetry and Pareto optimality of the Nash bargaining solution, the outcome is the point $(1, 1)$. This point is also in S'', and moreover, S'' is a subset of T, so that by independence of irrelevant alternatives the point $(1, 1)$ is also the Nash bargaining solution outcome of $(S'', (0, 0))$. By scale covariance, this implies that the Nash bargaining solution outcome of $(S', (0, 0))$ is the point $(1, 2)$. Again by scale covariance, we obtain that the Nash bargaining solution outcome of $(S, (0, 1))$ is the point $(1, 3)$. We have reached this result by using only the properties of the Nash bargaining solution in Theorem 10.2, and not the formula. Observe that the result is in accordance with what we established by direct computation. □

10.1.2 Relation with the Rubinstein Bargaining Procedure

In the Rubinstein bargaining procedure the players make alternating offers. See Sect. 6.7.2 for a detailed discussion of this noncooperative game, and Problem 6.17(d) for the application to the wine division problem of Sect. 1.3.5. Here, we use this example to illustrate the relation with the Nash bargaining solution.

The Nash bargaining solution assigns to this bargaining problem the point $\mathbf{z} = (2/3, \sqrt{1/3})$. This means that player 1 obtains 2/3 of the wine and player 2 obtains 1/3. According to the Rubinstein infinite horizon bargaining game with discount factor $0 < \delta < 1$ the players make proposals $\mathbf{x} = (x_1, x_2) \in P(S)$ and $\mathbf{y} = (y_1, y_2) \in P(S)$ such that

$$x_2 = \delta y_2, \ y_1 = \delta x_1 \ , \tag{10.1}$$

and these proposals are accepted in (subgame perfect) equilibrium. From (10.1) we derive

$$\sqrt{1 - x_1} = x_2 = \delta y_2 = \delta\sqrt{1 - y_1} = \delta\sqrt{1 - \delta x_1}$$

hence $1 - x_1 = \delta^2(1 - \delta x_1)$, which implies

$$x_1 = \frac{1 - \delta^2}{1 - \delta^3} = \frac{1 + \delta}{1 + \delta + \delta^2} .$$

If we let δ increase to 1, we obtain

$$\lim_{\delta \to 1} \frac{1 + \delta}{1 + \delta + \delta^2} = \frac{2}{3} ,$$

which coincides with the payoff to player 1 according to the Nash bargaining solution. This is not a coincidence. It follows, in general, from (10.1) that

$$(x_1 - d_1)(x_2 - d_2) = x_1 x_2 = (y_1/\delta)(\delta y_2) = y_1 y_2 = (y_1 - d_1)(y_2 - d_2)$$

hence the Rubinstein proposals **x** and **y** have the same 'Nash product'. For our wine division example, the proposals **x** and **y** for $\delta = 0.5$ are represented in Fig. 10.4.

As δ increases to 1, this level curve shifts up until it passes through the point $\mathbf{z} = F^{\text{Nash}}(S, d)$, since this point maximizes the product $x_1 x_2$ on the set S: see again Fig. 10.4.

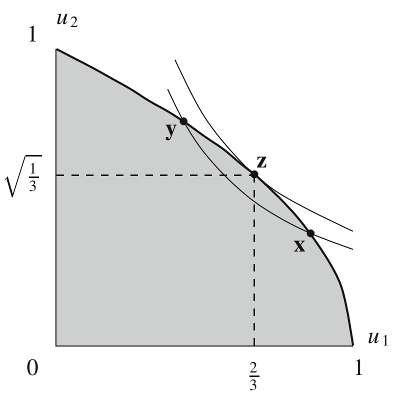

Fig. 10.4 The wine division problem. The disagreement point is the origin, and **z** is the Nash bargaining solution outcome. The points **x** and **y** are the proposals of players 1 and 2, respectively, in the subgame perfect equilibrium of the Rubinstein bargaining game for $\delta = 0.5$

We conclude that the subgame perfect equilibrium payoffs of the infinite horizon Rubinstein bargaining game converge to the Nash bargaining solution outcome as the discount factor δ approaches 1.

Example 10.4 In the bargaining game $(S', (0,0))$ of Example 10.3 the Rubinstein proposals are determined by the equations in (10.1) together with $x_2 = 3 - x_1^2$ and $y_2 = 3 - y_1^2$, yielding

$$3 - x_1^2 = x_2 = \delta y_2 = \delta(3 - y_1^2) = \delta(3 - \delta^2 x_1^2)$$

hence

$$x_1 = \sqrt{\frac{3(1-\delta)}{1-\delta^3}} = \sqrt{\frac{3}{1+\delta+\delta^2}}\,.$$

For δ approaching 1 we obtain

$$\lim_{\delta \to 1} \sqrt{\frac{3}{1+\delta+\delta^2}} = 1$$

which is the Nash bargaining solution payoff to player 1. □

10.2 Exchange Economies

In an exchange economy with n agents and k goods, each agent is endowed with a bundle of goods. Each agent has preferences over different bundles of goods, expressed by some utility function over these bundles. By exchanging goods among each other, it is in general possible to increase the utilities of all agents. One way to arrange this exchange is to introduce prices. For given prices the endowment of each agent represents the agent's income, which can be spent on buying a bundle of the goods that maximizes the agent's utility. If prices are such that the market for each good clears—total demand is equal to total endowment—while each agent maximizes utility, then the prices are in equilibrium: such an equilibrium is called Walrasian or competitive equilibrium. Alternatively, reallocations of the goods can be considered which are in the *core* of the exchange economy. A reallocation of the total endowment is in the core of the exchange economy if no coalition of agents can improve the utilities of its members by, instead, reallocating the total endowment of its own members among each other. It is well known that a competitive equilibrium allocation is an example of a core allocation.

This section is a first acquaintance with exchange economies. Attention is restricted to exchange economies with two agents and two goods. We work out an example of such an economy. Some variations are considered in Problem 10.4.

There are two agents, A and B, and two goods, 1 and 2. Agent A has an endowment $\mathbf{e}^A = (e_1^A, e_2^A) \in \mathbb{R}_+^2$ of the goods, and a utility function $u^A : \mathbb{R}_+^2 \to \mathbb{R}$, representing the preferences of A over bundles of goods.[4] Similarly, agent B has an endowment $\mathbf{e}^B = (e_1^B, e_2^B) \in \mathbb{R}_+^2$ of the goods, and a utility function $u^B : \mathbb{R}_+^2 \to \mathbb{R}$. (We use superscripts to denote the agents and subscripts to denote the goods.) This is a complete description of the exchange economy.

For our example we take $\mathbf{e}^A = (2, 3)$, $\mathbf{e}^B = (4, 1)$, $u^A(x_1, x_2) = x_1^2 x_2$ and $u^B(x_1, x_2) = x_1 x_2^2$. Hence, the total endowment in the economy is $\mathbf{e} = (6, 4)$, and the purpose of the exchange is to reallocate this bundle of goods such that both agents are better off.

Let $\mathbf{p} = (p_1, p_2)$ be a vector of positive prices of the goods. Given these prices, both agents want to maximize their utilities. Agent A has an income of $p_1 e_1^A + p_2 e_2^A$, i.e., the monetary value of his endowment. Then agent A solves the maximization problem

$$\begin{aligned} &\text{maximize } u^A(x_1, x_2) \\ &\text{subject to } p_1 x_1 + p_2 x_2 = p_1 e_1^A + p_2 e_2^A,\ x_1, x_2 \geq 0\ . \end{aligned} \tag{10.2}$$

The income constraint is called the *budget equation*. The solution of this maximization problem is a bundle $\mathbf{x}^A(\mathbf{p}) = (x_1^A(\mathbf{p}), x_2^A(\mathbf{p}))$, called agent A's *demand function*. Maximization problem (10.2) is called the *consumer problem* (of agent A). Similarly, agent B's consumer problem is:

$$\begin{aligned} &\text{maximize } u^B(x_1, x_2) \\ &\text{subject to } p_1 x_1 + p_2 x_2 = p_1 e_1^B + p_2 e_2^B,\ x_1, x_2 \geq 0\ . \end{aligned} \tag{10.3}$$

For our example, (10.2) becomes

$$\begin{aligned} &\text{maximize } x_1^2 x_2 \\ &\text{subject to } p_1 x_1 + p_2 x_2 = 2p_1 + 3p_2,\ x_1, x_2 \geq 0\ , \end{aligned}$$

which can be solved by using Lagrange's method or by substitution. By using the latter method the problem reduces to

$$\text{maximize } x_1^2\ ((2p_1 + 3p_2 - p_1 x_1)/p_2)$$

subject to $x_1 \geq 0$ and $2p_1 + 3p_2 - p_1 x_1 \geq 0$. Setting the derivative with respect to x_1 equal to 0 yields

$$2x_1 \left(\frac{2p_1 + 3p_2 - p_1 x_1}{p_2} \right) - x_1^2 \left(\frac{p_1}{p_2} \right) = 0\ ,$$

[4] $\mathbb{R}_+^2 := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \mid x_1,\ x_2 \geq 0\}$.

which after some simplifications yields the demand function $x_1 = x_1^A(\mathbf{p}) = (4p_1 + 6p_2)/3p_1$. By using the budget equation, $x_2^A(\mathbf{p}) = (2p_1 + 3p_2)/3p_2$. Similarly, solving (10.3) for our example yields $x_1^B(\mathbf{p}) = (4p_1 + p_2)/3p_1$ and $x_2^B(\mathbf{p}) = (8p_1 + 2p_2)/3p_2$ (check this!).

The prices $\mathbf{p}$ are *Walrasian equilibrium* prices if the markets for both goods clear. For the general model, this means that $x_1^A(\mathbf{p}) + x_1^B(\mathbf{p}) = e_1^A + e_1^B$ and $x_2^A(\mathbf{p}) + x_2^B(\mathbf{p}) = e_2^A + e_2^B$. For the example, this means

$$(4p_1 + 6p_2)/3p_1 + (4p_1 + p_2)/3p_1 = 6 \text{ and}$$

$$(2p_1 + 3p_2)/3p_2 + (8p_1 + 2p_2)/3p_2 = 4 \,.$$

Both equations result in the same condition, namely $10p_1 - 7p_2 = 0$. That there is only one equation left is no coincidence, since prices are only relative, as is easily seen from the budget equations. In fact, the prices represent the rate of exchange between the two goods, and are meaningful even if money does not exist in the economy. Thus, $\mathbf{p} = (7, 10)$ (or any positive multiple thereof) are the equilibrium prices in this exchange economy. The associated equilibrium demands are $\mathbf{x}^A(7, 10) = (88/21, 22/15)$ and $\mathbf{x}^B(7, 10) = (38/21, 38/15)$.

We now turn to the *core* of an exchange economy. A reallocation of the total endowments is in the core if no coalition can improve upon it. This definition is in the spirit as the corresponding definition for TU-games (Definition 9.2). In a two-person exchange economy, there are only three coalitions (excluding the empty coalition), namely $\{A\}$, $\{B\}$, and $\{A, B\}$. Consider an allocation $(\mathbf{x}^A, \mathbf{x}^B)$ with $x_1^A + x_1^B = e_1^A + e_1^B$ and $x_2^A + x_2^B = e_2^A + e_2^B$. To prevent that agents A or B can improve upon $(\mathbf{x}^A, \mathbf{x}^B)$ we need that

$$u^A(\mathbf{x}^A) \geq u^A(\mathbf{e}^A),\ u^B(\mathbf{x}^B) \geq u^B(\mathbf{e}^B) \,, \tag{10.4}$$

which are the *individual rationality* constraints. To prevent that the grand coalition $\{A, B\}$ can improve upon $(\mathbf{x}^A, \mathbf{x}^B)$ we need that

For *no* $(\mathbf{y}^A, \mathbf{y}^B)$ with $y_1^A + y_1^B = e_1^A + e_1^B$ and $y_2^A + y_2^B = e_2^A + e_2^B$ we have: $u^A(\mathbf{y}^A) \geq u^A(\mathbf{x}^A)$ and $u^B(\mathbf{y}^B) \geq u^B(\mathbf{x}^B)$ with at least one inequality strict. (10.5)

In words, (10.5) says that there should be no other reallocation of the total endowments such that no agent is worse off and at least one agent is strictly better off. This is the *efficiency* or *Pareto optimality* constraint.

We apply (10.4) and (10.5) to our example. The individual rationality constraints are

$$(x_1^A)^2 x_2^A \geq 12,\ x_1^B (x_2^B)^2 \geq 4 \,.$$

The Pareto optimal allocations, satisfying (10.5), can be computed as follows. Fix the utility level of one of the agents, say B, and maximize the utility of A subject to the utility level of B being fixed. By varying the fixed utility level of B we find all Pareto optimal allocations. In the example, we solve the following maximization problem for $c \in \mathbb{R}$:

$$\begin{aligned}&\text{maximize } (x_1^A)^2 x_2^A\\&\text{subject to } x_1^A + x_1^B = 6,\ x_2^A + x_2^B = 4,\ x_1^B (x_2^B)^2 = c,\ x_1^A, x_2^A, x_1^B, x_2^B \geq 0\ .\end{aligned}$$

By substitution this problem reduces to

$$\begin{aligned}&\text{maximize } (x_1^A)^2 x_2^A\\&\text{subject to } (6 - x_1^A)(4 - x_2^A)^2 = c,\ x_1^A, x_2^A \geq 0\ .\end{aligned}$$

The associated Lagrange function is $(x_1^A)^2 x_2^A - \lambda[(6 - x_1^A)(4 - x_2^A)^2 - c]$ and the first-order conditions are

$$2x_1^A x_A^2 + \lambda(4 - x_2^A)^2 = 0,\ (x_1^A)^2 + 2\lambda(6 - x_1^A)(4 - x_2^A) = 0\ .$$

Extracting λ from both equations and simplifying yields

$$x_2^A = \frac{4x_1^A}{24 - 3x_1^A}\ .$$

Thus, for any value of x_1^A between 0 and 6 this equation returns the corresponding value of x_2^A, resulting in a Pareto optimal allocation with $x_1^B = 6 - x_1^A$ and $x_2^B = 4 - x_2^A$.

It is straightforward to check by substitution that the Walrasian equilibrium allocation $\mathbf{x}^A(7, 10) = (88/21, 22/15)$ and $\mathbf{x}^B(7, 10) = (38/21, 38/15)$ found above, is Pareto optimal. This is no coincidence: the *First Welfare Theorem* states that in an exchange economy such as the one under consideration, a Walrasian equilibrium allocation is Pareto optimal.

Combining the individual rationality constraint for agent A with the Pareto optimality constraint yields $4(x_1^A)^3/(24 - 3x_1^A) \geq 12$, which holds for x_1^A larger than approximately 3.45. For agent B, similarly, the individual rationality and Pareto optimality constraints imply

$$(6 - x_1^A)\left(\frac{96 - 16x_1^A}{24 - 3x_1^A}\right)^2 \geq 4\ ,$$

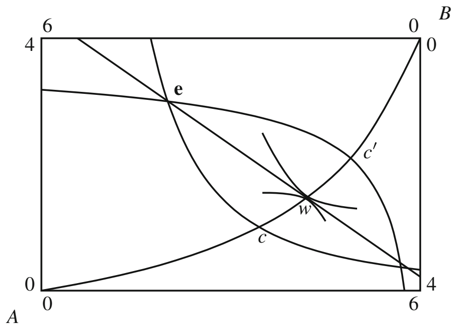

Fig. 10.5 The contract curve is the curve through c and c'. The point c is the point of intersection of the contract curve and the indifference curve of agent A through the endowment point **e**. The point c' is the point of intersection of the contract curve and the indifference curve of agent B through the endowment point **e**. The core consists of the allocations on the contract curve between c and c'. The *straight line* ('budget line') through **e** is the graph of the budget equation for A at the equilibrium prices, i.e., $7x_1 + 10x_2 = 44$, and its point of intersection with the contract curve, w, is the Walrasian equilibrium allocation. At this point the indifference curves of the two agents are both tangential to the budget line

which holds for x_1^A smaller than approximately 4.88. Hence, the core of the exchange economy in the example is, approximately, the set

$$\{(x_1^A, x_2^A, x_1^B, x_2^B) \in \mathbb{R}^4 \mid 3.45 \le x_1^A \le 4.88,$$
$$x_2^A = \frac{4x_1^A}{24-3x_1^A},\ x_1^B = 6 - x_1^A,\ x_2^B = 4 - x_2^A\}\ .$$

Clearly, the Walrasian equilibrium allocation is in the core, since $3.45 \le 88/21 \le 4.88$, and also this holds more generally. Thus, decentralization of the reallocation process through prices leads to an allocation that is in the core.

For an exchange economy with two agents and two goods a very useful pictorial device is the *Edgeworth box*, see Fig. 10.5. The Edgeworth box consists of all possible reallocations of the two goods. The origin for agent A is the South West corner and the origin for agent B the North East corner. In the diagram, the indifference curves of the agents through the endowment point are plotted, as well as the *contract curve*, i.e., the set of Pareto optimal allocations. The core is the subset of the contract curve between the indifference curves of the agents through the endowment point.

10.3 Matching Problems

In a matching problem there is a group of agents that have to form couples. Examples are: students who have to be coupled with schools; hospitals that have to be coupled with doctors; workers who have to be coupled with firms; men who

Table 10.1 A matching problem

m_1	m_2	m_3	w_1	w_2	w_3
w_2	w_1	w_1	m_1	m_2	m_1
w_1	w_2	w_2	m_3	m_1	m_3
	w_3		m_2	m_3	m_2

have to be coupled with women; etc. In this section we consider so-called one-to-one matching problems.

The agents are divided in two equally large (finite and nonempty) sets, denoted M and W. Each agent in M has a strict preference over those agents in W which he prefers over staying single. Similarly, each agent in W has a strict preference over those agents in M which she prefers over staying single. In such a *matching problem*, a *matching* assigns to each agent in M at most one agent in W, and vice versa; thus, no two agents in M are assigned the same agent in W, and vice versa.

Such matching problems are also called *marriage problems*, and the agents of M and W are called *men* and *women*, respectively. While the problem may indeed refer to the 'marriage market', this terminology is of course adopted for convenience. Other examples are matching tasks and people, or matching rooms and people.

As an example, consider the matching problem in Table 10.1. The set of men is $M = \{m_1, m_2, m_3\}$ and the set of women is $W = \{w_1, w_2, w_3\}$. The columns in the table represent the preferences. For instance m_1 prefers w_2 over w_1 over staying single, but prefers staying single over w_3. An example of a matching in this particular matching problem is (m_1, w_1), (m_3, w_2), m_2, w_3, meaning that m_1 is married to w_1 and m_3 to w_2, while m_2 and w_3 stay single.[5] Observe that this matching does not seem very 'stable': m_1 and w_2 would prefer to be married to each other instead of to their partners in the given matching. Moreover, m_2 and w_3 would prefer to be married to each other instead of being single. Also, for instance, any matching in which m_1 would be married to w_3 would not be plausible, since m_1 would prefer to stay single.

The obvious way to formalize these considerations is to require that a matching should be in the core of the matching problem. A matching is in the *core* of there is no subgroup (coalition) of men and/or women who can do better by marrying (or staying single) among each other. For a matching to be in the core, the following two requirements are certainly necessary:

(c1) each person prefers his/her partner over being single;
(c2) if $m \in M$ and $w \in W$ are not matched to each other, then it is *not* the case that both m prefers w over his current partner if m is married or over being single if m is not married; and w prefers m over her current partner if w is married or over being single if w is not married.

[5]Check that there are 34 possible different matchings for this problem.

Obviously, if (c1) were violated then the person in question could improve by divorcing and becoming single; if (c2) were violated then m and w would both be better off by marrying each other. A matching satisfying (c1) and (c2) is called *stable*. Hence, any matching in the core must be stable. Interestingly, the converse is also true: any stable matching is in the core. To see this, suppose there is a matching outside the core and satisfying (c1) and (c2). Then there is a coalition of agents each of whom can improve by marrying or staying single within that coalition. If a member of the coalition improves by becoming single, then (c1) is violated. If two coalition members improve by marrying each other, then (c2) is violated. This contradiction establishes the claim that stable matchings must be in the core. Thus, the core of a matching problem is the set of all stable matchings.

How can stable matchings be computed? A convenient procedure is the *deferred acceptance procedure*, developed by Gale and Shapley. In this procedure, the members of one of the two parties propose and the members of the other party accept or reject proposals. Suppose men propose. In the first round, each man proposes to his favorite woman (or stays single if he prefers that) and each woman, if proposed to at least once, chooses her favorite man among those who have proposed to her (which may mean staying single). This way, a number of couples may form, and the involved men and women are called 'engaged'. In the second round, the rejected men propose to their second-best woman (or stay single); then each woman again picks here favorite among the men who proposed to her including possibly the man to whom she is currently engaged. The procedure continues until all proposals are accepted. Then all currently engaged couples marry and a matching is established.

It is not hard to verify that this matching is stable. A man who stays single was rejected by all women he preferred over staying single and therefore can find no woman who prefers him over her husband or over being single. A woman who stays single was never proposed to by any man whom she prefers over staying single. Consider, finally, an $m \in M$ and a $w \in W$ who are married but not to each other. If m prefers w over his current wife, then w must have rejected him for a better partner somewhere in the procedure. If w prefers m over her current husband, then m has never proposed to her and, thus, prefers his wife over her.

Of course, the deferred acceptance procedure can also be applied with women as proposers, resulting in a stable matching that is different in general.

Table 10.2 shows how the deferred acceptance procedure with the men proposing works, applied to the matching problem in Table 10.1.

There may be other stable matchings than those found by applying the deferred acceptance procedure with the men and with the women proposing. It can be shown

Table 10.2 The deferred acceptance procedure applied to the matching problem of Table 10.1

	Stage 1	Stage 2	Stage 3	Stage 4
m_1	$\to w_2$	rejected	$\to w_1$	
m_2	$\to w_1$ rejected	$\to w_2$		
m_3	$\to w_1$		rejected	$\to w_2$ rejected

The resulting matching is (m_1, w_1), (m_2, w_2), m_3, w_3

that the former procedure—with the men proposing—results in a stable matching that is optimal, among all stable matchings, from the point of view of the men, whereas the latter procedure—with the women proposing—produces the stable matching optimal from the point of view of the women. See also Problems 10.6 and 10.7.

10.4 House Exchange

In a house exchange problem each one of finitely many agents owns a house, and has a preference over all houses. The purpose of the exchange is to make the agents better off. A house exchange problem is an exchange economy with as many goods as there are agents, and where each agent is endowed with one unit of a different, indivisible good.

Formally, the set of agents is $N = \{1, \ldots, n\}$, and each agent $i \in N$ owns house h_i, and has a strict preference of the set of all (n) houses. In a *core* allocation, each agent obtains exactly one house, and there is no coalition that can make each of its members strictly better off by exchanging their *initially owned* houses among themselves.[6]

As an example, consider the house exchange problem in Table 10.3. In this problem there are six possible different allocations of the houses. Table 10.4 lists these allocations and also which coalitions could improve by exchanging their own houses.

Especially for larger problems, the 'brute force' analysis as in Table 10.4 is rather cumbersome. A different and more convenient way is to use the top trading cycle procedure. In a given house exchange problem a *top trading cycle* is a sequence $i_1, i_2, \ldots, i_k$ of agents, with $k \geq 1$, such that the favorite house of agent i_1 is house h_{i_2}, the favorite house of agent i_2 is house h_{i_3}, …, and the favorite house of agent i_k is house h_{i_1}. If $k = 1$, then this simply means that agent i_1 already owns his favorite house. In the top trading cycle procedure, we look for a top trading cycle, assign houses within the cycle, and next the involved agents and their houses leave the scene. Then we repeat the procedure for the remaining agents, etc., until no agent is left.

Table 10.3 A house exchange problem with three agents. For instance, agent 1 prefers the house of agent 3 over the house of agent 2 over his own house

Agent 1	Agent 2	Agent 3
h_3	h_1	h_2
h_2	h_2	h_3
h_1	h_3	h_1

[6]Hence, by definition players in coalitions can only possibly improve by exchanging their initially owned houses, not the houses they acquired after the exchange has taken place.

Table 10.4 Analysis of the house exchange problem of Table 10.3. There are two core allocations

Agent 1	Agent 2	Agent 3	Improving coalition(s)
h_1	h_2	h_3	$\{1,2\}$, $\{1,2,3\}$
h_1	h_3	h_2	$\{2\}$, $\{1,2\}$
h_2	h_1	h_3	None: core allocation
h_2	h_3	h_1	$\{2\}$, $\{3\}$, $\{2,3\}$, $\{1,2,3\}$
h_3	h_1	h_2	None: core allocation
h_3	h_2	h_1	$\{3\}$

In the example in Table 10.3 there is only one top trading cycle, namely $1, 3, 2$, resulting in the allocation $1 : h_3$, $3 : h_2$, $2 : h_1$, a core allocation: in fact, each agent obtains his top house. In general, it is true that *for strict preferences the top trading cycle procedure results in a core allocation.* The reader should check the validity of this claim (Problem 10.8).

What about the other core allocation found in Table 10.4? In this allocation, the grand coalition could *weakly improve*: by the allocation $1 : h_3$, $3 : h_2$, $2 : h_1$ agents 1 and 3 would be strictly better off, while agent 2 would not be worse off. We define the *strong core* as consisting of those allocations on which no coalition could even weakly improve, that is, make all its members at least as well off and at least one member strictly better off. In the example, only the allocation $1 : h_3$, $3 : h_2$, $2 : h_1$ is in the strong core. In general, one can show that *the strong core of a house exchange problem with strict preferences consists of the unique allocation produced by the top trading cycle procedure.*

10.5 Problems

10.1. *A Division Problem (1)*

Suppose two players (bargainers) bargain over the division of one unit of a perfectly divisible good. Player 1 has utility function $u_1(\alpha) = \alpha$ and player 2 has utility function $u_2(\beta) = 1 - (1-\beta)^2$ for amounts $\alpha, \beta \in [0,1]$ of the good.

(a) Determine the set of feasible utility pairs. Make a picture.
(b) Determine the Nash bargaining solution outcome, in terms of utilities as well as of the distribution of the good.
(c) Suppose the players' utilities are discounted by a factor $\delta \in [0,1)$. Calculate the Rubinstein bargaining outcome, i.e., the subgame perfect equilibrium outcome of the infinite horizon alternating offers bargaining game.
(d) Determine the limit of the Rubinstein bargaining outcome, for δ approaching 1, in two ways: by using the result of (b) and by using the result of (c).

10.2. *A Division Problem (2)*
Suppose that two players (bargainers) bargain over the division of one unit of a perfectly divisible good. Assume that player 1 has utility function $u(\alpha)$ $(0 \leq \alpha \leq 1)$ and player 2 has utility function $v(\alpha) = 2u(\alpha)$ $(0 \leq \alpha \leq 1)$.

Determine the distribution of the good according to the Nash bargaining solution. Can you say something about the resulting utilities? (Hint: use the relevant properties of the Nash bargaining solution.)

10.3. *A Division Problem (3)*
Suppose that two players (bargainers) bargain over the division of two units of a perfectly divisible good. Assume that player 1 has a utility function $u(\alpha) = \frac{\alpha}{2}$ $(0 \leq \alpha \leq 2)$ and player 2 has utility function $v(\alpha) = \sqrt[3]{\alpha}$ $(0 \leq \alpha \leq 2)$.

(a) Determine the distribution of the good according to the Rubinstein bargaining procedure, for any discount factor $0 < \delta < 1$.
(b) Use the result to determine the Nash bargaining solution distribution.
(c) Suppose player 1's utility function changes to $w(\alpha) = \alpha$ for $0 \leq \alpha \leq 1.6$ and $w(\alpha) = 1.6$ for $1.6 \leq \alpha \leq 2$. Determine the Nash bargaining solution outcome, both in utilities and in distribution, for this new situation.

10.4. *An Exchange Economy*
Consider an exchange economy with two agents A and B and two goods. The agents are endowed with initial bundles $\mathbf{e}^A = (3, 1)$ and $\mathbf{e}^B = (1, 3)$. Their preferences are represented by the utility functions $u^A(x_1, x_2) = \ln(x_1 + 1) + \ln(x_2 + 2)$ and $u^B(x_1, x_2) = 3\ln(x_1 + 1) + \ln(x_2 + 1)$.

(a) Compute the demand functions of the agents.
(b) Compute Walrasian equilibrium prices and the equilibrium allocation.
(c) Compute the contract curve and the core. Sketch the Edgeworth box.
(d) Show that the Walrasian equilibrium allocation is in the core.
(e) How would you set up a two-person bargaining problem associated with this economy? Would it make sense to consider the Nash bargaining solution in order to compute an allocation? Why or why not?

10.5. *The Matching Problem of Table 10.1 Continued*

(a) Apply the deferred acceptance procedure to the matching problem of Table 10.1 with the women proposing.
(b) Are there any other stable matchings in this example?

10.6. *Another Matching Problem*
Consider the matching problem with three men, three women, and preferences as in Table 10.5.

Table 10.5 The matching problem of Problem 10.6

m_1	m_2	m_3	w_1	w_2	w_3
w_1	w_1	w_1	m_1	m_1	m_1
w_2	w_2	w_3	m_2	m_3	m_2
w_3	w_3	w_2	m_3	m_2	m_3

Table 10.6 The matching problem of Problem 10.7

m_1	m_2	m_3	w_1	w_2	w_3
w_2	w_1	w_1	m_1	m_3	m_1
w_1	w_3	w_2	m_3	m_1	m_3
w_3	w_2	w_3	m_2	m_2	m_2

(a) Compute the two matchings produced by the deferred acceptance procedure with the men and with the women proposing.
(b) Are there any other stable matchings?
(c) Verify the claim made in the text about the optimality of the matchings in (a) for the men and the women, respectively.

10.7. *Yet Another Matching Problem: Strategic Behavior*
Consider the matching problem with three men, three women, and preferences as in Table 10.6.

(a) Compute the two matchings produced by the deferred acceptance procedure with the men and with the women proposing.
(b) Are there any other stable matchings?

Now consider the following noncooperative game. The players are w_1, w_2, and w_3. The strategy set of a player is simply the set of all possible preferences over the men. (Thus, each player has 16 different strategies.) The outcomes of the game are the matchings produced by the deferred acceptance procedure with the men proposing, assuming that each man uses his true preference given in Table 10.6.

(c) Show that the following preferences form a Nash equilibrium: w_2 and w_3 use their true preferences, as given in Table 10.6; w_1 uses the preference (m_1, m_2, m_3). Conclude that sometimes it may pay off to lie about one's true preference. (Hint: in a Nash equilibrium, no player can gain by deviating.)

10.8. *Core Property of Top Trading Cycle Procedure*
Show that for strict preferences the top trading cycle results in a core allocation.

10.9. *House Exchange with Identical Preferences*
Consider the n-agent house exchange problem where all agents have identical strict preferences. Find the house allocation(s) in the core.

Table 10.7 The house exchange problem of Problem 10.10

Player 1	Player 2	Player 3	Player 4
h_3	h_4	h_1	h_3
h_2	h_1	h_4	h_2
h_4	h_2	h_3	h_1
h_1	h_3	h_2	h_4

10.10. *A House Exchange Problem*
Consider the house exchange problem with four agents in Table 10.7. Compute all core allocations and all strong core allocations.

10.11. *Cooperative Oligopoly*
Consider the Cournot oligopoly game with n firms with different costs $c_1, c_2, \ldots, c_n$. (This is the game of Problem 6.2 with heterogenous costs.) As before, each firm i offers $q_i \geq 0$, and the price-demand function is $p = \max\{0, a - \sum_{i=1}^{n} q_i\}$, where $0 < c_i < a$ for all i.

(a) Show that the reaction function of player i is

$$q_i = \max\{0, \frac{a - c_i - \sum_{j \neq i} q_i}{2}\} .$$

(b) Show that the unique Nash equilibrium of the game is $\mathbf{q}^* = (q_1^*, \ldots, q_n^*)$ with

$$q_i^* = \frac{a - nc_i + \sum_{j \neq i} c_j}{n+1} ,$$

for each i, assuming that this quantity is positive.
(c) Derive that the corresponding profits are

$$\frac{(a - nc_i + \sum_{j \neq i} c_j)^2}{(n+1)^2}$$

for each player i.

Let the firms now be the players in a cooperative TU-game with player set $N = \{1, 2, \ldots, n\}$, and consider a coalition $S \subseteq N$. What is the total profit that S can make on its own? This depends on the assumptions that we make on the behavior of the players outside S. Very pessimistically, one could solve the problem

$$\max_{q_i : i \in S} \min_{q_j : j \notin S} \sum_{i \in S} P_i(q_1, \ldots, q_n) ,$$

which is the profit that S can guarantee independent of the players outside S. This view is very pessimistic because it presumes maximal resistance of the outside players, even if this means that these outside players hurt themselves. In the present case it is not hard to see that this results in zero profit for S.

Two alternative scenarios are: S plays a Cournot-Nash equilibrium in the $(n - |S| + 1)$-player oligopoly game against the outside firms as separate firms, or S plays a Cournot-Nash equilibrium in the duopoly game against $N \setminus S$.

In the first case we in fact have an oligopoly game with costs c_j for every player $j \notin S$ and with cost $c_S := \min\{c_i : i \in S\}$ for 'player' (coalition) S.

(d) By using the results of (a)–(c) show that coalition S obtains a profit of

$$v_1(S) = \frac{[a - (n - |S| + 1)c_S + \sum_{j \notin S} c_j]^2}{(n - |S| + 2)^2}$$

in this scenario. Thus, this scenario results in a cooperative TU-game (N, v_1).

(e) Assume $n = 3$, $a = 7$, and $c_i = i$ for $i = 1, 2, 3$. Compute the core, the Shapley value, and the nucleolus for the TU-game (N, v_1).

(f) Show that in the second scenario, coalition S obtains a profit of

$$v_2(S) = \frac{(a - 2c_S + c_{N-S})^2}{9},$$

resulting in a cooperative game (N, v_2).

(g) Assume $n = 3$, $a = 7$, and $c_i = i$ for $i = 1, 2, 3$. Compute the core, the Shapley value, and the nucleolus for the TU-game (N, v_2).

10.6 Notes

The Nash bargaining model and solution were proposed and characterized in Nash (1950). Critique on the independence of irrelevant alternatives axiom was formalized in Kalai and Smorodinsky (1975), see Chap. 21.

For a proof of the First Welfare Theorem, see for instance Jehle and Reny (2001). There one can also find a proof of the fact that the Walrasian equilibrium results in a core allocation.

The deferred acceptance procedure for matching problems was first proposed in Gale and Shapley (1962). Our introduction to matching problems and house exchange is largely based on Osborne (2004). Also Problems 10.5–10.7 are from that source. Problem 10.10 is from Moulin (1995).

Two-person bargaining problems and TU-games (Chap. 9) are both special cases of the general model of *cooperative games without transferable utility*, so-called NTU-games. In an NTU-game, a *set* of feasible utility vectors $V(T)$ is assigned to each coalition T of players. For a TU-game (N, v) and a coalition T, this set takes the special form $V(T) = \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i \in T} x_i \leq v(T)\}$, i.e., a coalition T can attain any vector of utilities such that the sum of the utilities for the players in T does not exceed the worth of the coalition. In a two-player bargaining problem $(S, \mathbf{d})$, one can set $V(\{1,2\}) = S$ and $V(\{i\}) = \{\alpha \in \mathbb{R} \mid \alpha \leq d_i\}$ for $i = 1, 2$. See also Chap. 21.

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