

The implicit assumption in a cooperative game is that players can form coalitions and make binding agreements on how to distribute the proceeds of these coalitions. A cooperative game is more abstract than a noncooperative game in the sense that strategies are not explicitly modelled: rather, the game describes what each possible coalition can earn by cooperation. In a cooperative game with *transferable utility* it is assumed that the earnings of a coalition can be expressed by one number. One may think of this number as an amount of money, which can be distributed among the players in any conceivable way—including negative payments—if the coalition is actually formed. More generally, it is an amount of *utility* and the implicit assumption is that it makes sense to transfer this utility among the players—for instance, due to the presence of a medium like money, assuming that individual utilities can be expressed in monetary terms.

This chapter presents a first acquaintance with the theory of cooperative games with transferable utility. A few important solution concepts—the core, the Shapley value, and the nucleolus—are briefly discussed in Sects. 9.2–9.4. We start with examples and preliminaries in Sect. 9.1.

9.1 Examples and Preliminaries

In Chap. 1 we have seen several examples of cooperative games with transferable utility: the three cities game, a glove game, a permutation game, and a voting game. For the stories giving rise to these games the reader is referred to Sect. 1.3.4. Here we reconsider the resulting games.

In the three cities game, cooperation between cities leads to cost savings expressed in amounts of money, as in Table 9.1. In the first line of this table all possible coalitions are listed. It is important to note that the term ‘coalition’ is used for any subset of the set of players. So a coalition is not necessarily formed. The empty subset (empty coalition) has been added for convenience: it is assigned the number

Table 9.1 The three cities game

S	\emptyset	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	0	0	0	0	90	100	120	220

Table 9.2 A glove game

S	\emptyset	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	0	0	0	0	0	1	1	1

Table 9.3 A permutation game

S	\emptyset	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	0	2	5	4	14	18	9	24

0 by convention. The numbers in the second line of the table are called the ‘worths’ of the coalitions. For instance, coalition $S = \{1, 2\}$ has worth 90. In this particular example, 90 are the costs saved by cities 1 and 2 if they cooperate. It is assumed that this amount can be split between the two players (cities) if the coalition is actually formed: that is, player 1 may receive $x_1 \in \mathbb{R}$ and player 2 may receive $x_2 \in \mathbb{R}$ such that $x_1 + x_2 = 90$ or, more generally, $x_1 + x_2 \leq 90$.

In the glove game in Sect. 1.3.4 coalitions may make pairs of gloves. The game is described in Table 9.2. In this game the worth 1 of the ‘grand coalition’ $\{1, 2, 3\}$, for instance, means that this coalition can earn 1 by producing one pair of gloves. One can think of this number as expressing the monetary value of this pair of gloves. Alternatively, one can think of one pair of gloves having ‘utility’ equal to 1. Again, it is assumed that the players can split up this amount in any way they like. So a possible distribution of the worth of the grand coalition takes the form $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1 + x_2 + x_3 \leq 1$. For $i = 1, 2, 3$, the number x_i may represent the money that player i receives, or (if nonnegative) the percentage of time that player i is allowed to wear the gloves.

The permutation game (dentist game) of Sect. 1.3.4 is reproduced in Table 9.3. In this game, one could think of the worth of a coalition as expressing, for instance, savings of opportunity costs by having dentist appointments on certain days. What is important is that, again, these worths can be distributed in any way among the players of the coalitions.

For the voting game related to the UN Security Council, a table could be constructed as well, but this table would be huge: there are $2^{15} = 32,768$ possible coalitions (cf. Problem 9.1). Therefore, it is convenient to describe this game as follows. Let the permanent members be the players $1, \dots, 5$ and let the other members be the players $6, \dots, 15$. Denote by $N = \{1, 2, \dots, 15\}$ the grand coalition of all players and by $v(S)$ the worth of a coalition $S \subseteq N$. Then

$$v(S) := \begin{cases} 1 & \text{if } \{1, \dots, 5\} \subseteq S \text{ and } |S| \geq 9 \\ 0 & \text{otherwise} \end{cases}$$

where $|S|$ denotes the number of players in S . In this case the number 1 indicates that the coalition is ‘winning’ and the number 0 that the coalition is ‘losing’.

In analyzing games like this the resulting numbers—e.g., nonnegative numbers x_1, \dots, x_{15} summing to 1—are usually interpreted as power indices, expressing the power of a player in some way or another.

We summarize the concepts introduced informally in the preceding examples, formally in the following definition.

Definition 9.1 A *cooperative game with transferable utility* or *TU-game* is a pair (N, v) , where $N = \{1, \dots, n\}$ with $n \in \mathbb{N}$ is the set of *players*, and v is a function assigning to each *coalition* S , i.e., to each subset $S \subseteq N$ a real number $v(S)$, such that $v(\emptyset) = 0$. The function v is called the *characteristic function* and $v(S)$ is called the *worth* of S . The coalition N is called the *grand coalition*. A *payoff distribution* or *payoff vector* for coalition S is a vector of real numbers $(x_i)_{i \in S}$. \square

When analyzing a TU-game there are two important questions to answer: which coalitions are formed; and how are the worths of these coalitions distributed among their members? In this chapter we assume that the grand coalition is formed and we concentrate on the second question. This is less restrictive than it may seem at first sight, since coalition formation depends, naturally, on how the proceeds of a coalition are going to be distributed among its members. Thus, also if smaller coalitions are formed the distribution question has to be considered for these coalitions.

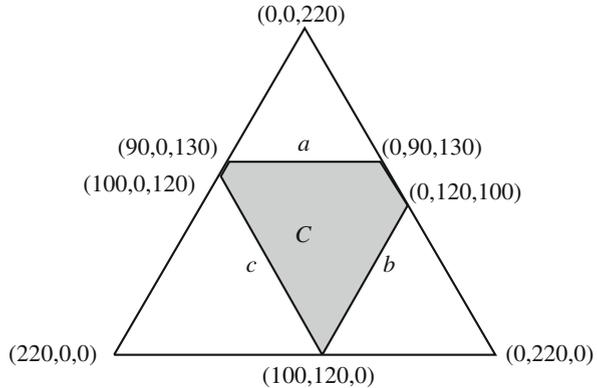
9.2 The Core

Consider the three cities game in Table 9.1, suppose that the grand coalition gets together, and suppose that there is a proposal $x_1 = 40, x_2 = 40$, and $x_3 = 140$ for distribution of the savings $v(N) = 220$ on the bargaining table. One can imagine, for instance, that player 3 made such a proposal. In that case, players 1 and 2 could protest successfully, since they can save $v(\{1, 2\}) = 90 > 80 = x_1 + x_2$ without player 3. We express this by saying that $\mathbf{x} = (x_1, x_2, x_3)$ is not in the “core” of this game. More generally, the *core* of the three cities game is the set of payoff distributions for $N = \{1, 2, 3\}$ such that the sum of the payoffs is equal to $v(N) = 220$ and each nonempty coalition S obtains at least its own worth. Thus, it is the set

$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0, \\ x_1 + x_2 \geq 90, x_1 + x_3 \geq 100, x_2 + x_3 \geq 120, \\ x_1 + x_2 + x_3 = 220\}.$$

To obtain a better idea of what this set looks like, we can make a diagram. Although C is a subset of \mathbb{R}^3 , the constraint $x_1 + x_2 + x_3 = 220$ makes that the set C is contained in a two-dimensional subset of \mathbb{R}^3 , i.e., the plane through the points $(220, 0, 0)$, $(0, 220, 0)$, and $(0, 0, 220)$. The triangle formed by these three points is represented in Fig. 9.1. By the constraints $x_i \geq 0$ for every $i = 1, 2, 3$, the set C must be a

Fig. 9.1 The set C (shaded) is the core of the three cities game. Line segment a corresponds to the constraint $x_1 + x_2 \geq 90$; it consists of the payoff vectors in the triangle with $x_1 + x_2 = 90$ or, equivalently, $x_3 = 130$. Line segment b corresponds to the constraint $x_1 + x_3 \geq 100$, and line segment c corresponds to the constraint $x_2 + x_3 \geq 120$



subset of this triangle. The set C is further restricted by the three constraints for the two-person coalitions: it is the shaded area in Fig. 9.1.

Hence, the core of the three cities game is the polygon and its inside with vertices $(100, 120, 0)$, $(0, 120, 100)$, $(0, 90, 130)$, $(90, 0, 130)$, and $(100, 0, 120)$.

We now give the formal definition of the core and of some other related concepts. We write $x(S) := \sum_{i \in S} x_i$ for a payoff distribution $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and a nonempty coalition $S \subseteq N = \{1, \dots, n\}$. Hence, $x(S)$ is what the members of the coalition S obtain together if the payoff vector is \mathbf{x} .

Definition 9.2 For a TU-game (N, v) , a payoff distribution $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ is

- *efficient* if $x(N) = v(N)$,
- *individually rational* if $x_i \geq v(\{i\})$ for all $i \in N$,
- *coalitionally rational* if $x(S) \geq v(S)$ for all nonempty coalitions S .

The *core* of (N, v) is the set

$$C(N, v) = \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } \emptyset \neq S \subseteq N\}.$$

Thus, the core of (N, v) is the set of all efficient and coalitionally rational payoff distributions. \square

The core of a game can be a large set, as in the three cities game; a small set, as in the glove game (see Problem 9.2); or it can be empty (see again Problem 9.2). In general, core elements can be computed by linear programming. For games with two or three players the core can be computed graphically, as we did for the three cities game. Sometimes, the core can be computed by using the special structure of the specific game under consideration.

We conclude this section with analyzing the core of the general glove game (cf. Problem 1.9).

Example 9.3 In the general glove game, there are $\ell > 0$ players who own a left-hand glove and $r > 0$ players who own a right-hand glove. In total there are n players, so $n = \ell + r$. The coalition N of all players is the grand coalition. The worth of a coalition S is equal to the number of pairs of gloves that the members of the coalition can make. Hence, it is equal to the *minimum* of two numbers: the number of left-hand glove owners in S and the number of right-hand glove owners in S . If we denote the set of all left-hand glove owners by L and the set of all right-hand glove owners by R , then we can also write this as

$$v(S) = \min\{|S \cap L|, |S \cap R|\}.$$

Here, $S \cap L$ is the intersection of S and L , i.e., the set of left-hand glove owners in S , and as before $|\cdot|$ denotes the number of elements in a set, in this case the number of players. What is the core of this game?

To answer this question we cannot just make a diagram as before. Even if we take specific numbers for ℓ and r , as soon as $\ell + r$ is larger than three we cannot make a picture. (This is not quite true: we could still make a picture for four players, but that would be a three-dimensional picture, which is not easy to draw.) Therefore, we have to argue in a different way.

Let us assume, first, that $\ell > r$. Then $v(N) = r$: we can use the above formula or simply observe that the grand coalition can make r pairs. Since each single player can make zero pairs, we also have $v(\{i\}) = 0$ for each player $i \in N$. Hence, for a payoff vector $\mathbf{x} = (x_1, \dots, x_n)$ to be in the core, we already need $x_1 + \dots + x_n = r$ and $x_i \geq 0$ for each $i \in N$. Now take a player $j \in L$, that is, j owns a left-hand glove. Consider the coalition S that consists of all right-hand glove owners and of at least r left-hand glove owners but not player j : this is possible since $\ell > r$. Then we still have $v(S) = r$, since S can still make r pairs, but this means that the members of S together should obtain at least r . In turn, this implies that they obtain exactly r , and thus player j obtains zero: $x_j = 0$. Since this argument holds for every arbitrary left-hand glove owner, every such player obtains zero. So far we have derived: all right-hand glove owners together obtain r , each one of them obtains at least zero, and every left-hand glove owner obtains zero.

Now let i be a right-hand glove owner and j a left-hand glove owner. Then $v(\{i, j\}) = 1$, hence $x_i + x_j \geq 1$. Since $x_j = 0$, as already established, we have $x_i \geq 1$. But this holds for every right-hand glove owner. Since there are r of them and together they obtain r , we must have that $x_i = 1$ for every right-hand glove owner. So we have found that \mathbf{x} is unique: it assigns 1 to right-hand glove owners and 0 to left-hand glove owners. Thus, we have found that *if* \mathbf{x} is in the core, *then* it can only be this specific payoff vector.

Conversely, this payoff vector is indeed in the core. All players together receive r , which is indeed equal to $v(N)$. An arbitrary coalition S receives an amount which is

equal to the number of right-hand glove owners in S : this is indeed at least equal to the worth of S , i.e., the number of glove pairs that S can make.

Altogether, in one formula, we have for the core of the general glove game with $\ell > r$:

$$C(v) = \{\mathbf{x}\} \text{ with } \mathbf{x} \in \mathbb{R}^n, x_i = 1 \text{ for } i \in R \text{ and } x_i = 0 \text{ for } i \in L .$$

Similarly we find for $r > \ell$:

$$C(v) = \{\mathbf{x}\} \text{ with } \mathbf{x} \in \mathbb{R}^n, x_i = 1 \text{ for } i \in L \text{ and } x_i = 0 \text{ for } i \in R .$$

The case $\ell = r$ is left as Problem 9.4. □

9.3 The Shapley Value

The Shapley value is a solution concept for TU-games that is quite different from the core. Whereas the core is a (possibly empty) *set*, the Shapley value assigns a *unique payoff distribution* for the grand coalition to every TU-game. The Shapley value is not so much based on strategic considerations but, rather, assigns to each player his “average marginal contribution” in the game. For three-player games we already explained the definition of the Shapley value in Chap. 1. Here, we repeat and extend this definition.

Consider again the three cities game of Table 9.1. Imagine a setting where the players enter a bargaining room one by one, and upon entering each player demands and obtains what he contributes to the worth of the coalition present in the room. Suppose that player 1 enters first, player 2 enters next, and player 3 enters last. Player 1 enters an empty room and can take his “marginal contribution” $v(\{1\}) - v(\emptyset) = 0 - 0 = 0$. When player 2 enters, player 1 is already present, and player 2 obtains his marginal contribution $v(\{1, 2\}) - v(\{1\}) = 90 - 0 = 90$. When, finally, player 3 enters, then the coalition $\{1, 2\}$ is already present. So player 3 obtains his marginal contribution $v(\{1, 2, 3\}) - v(\{1, 2\}) = 220 - 90 = 130$. Hence, this procedure results in the payoff distribution $(0, 90, 130)$, which is called a *marginal vector*. Of course, this payoff distribution does not seem fair since it depends on the order in which the players enter the room, and this order is arbitrary: there are five other possible orders. The Shapley value takes the marginal vectors of all six orders into consideration, and assigns to a TU-game their average. See Table 9.4.

For an arbitrary TU-game (N, v) with player set $N = \{1, \dots, n\}$ the Shapley value can be computed in the same way. There are $n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1 = n!$ possible orders of the players. First compute the marginal vectors corresponding to these $n!$ different orders, and then take the average—that is, sum all marginal vectors and divide the result by $n!$. If the number of players is large, then this is a huge task. In the UN security council voting game of Sect. 9.1, for instance, this would mean

Table 9.4 Computation of the Shapley value for the three cities game. The Shapley value is obtained by dividing the totals of the marginal contributions by 6

Order of entry	1	2	3
1, 2, 3	0	90	130
1, 3, 2	0	120	100
2, 1, 3	90	0	130
2, 3, 1	100	0	120
3, 1, 2	100	120	0
3, 2, 1	100	120	0
Total	390	450	480
Shapley value	65	75	80

computing $15! > 13 \times 10^{11}$ marginal vectors. Fortunately, there is a more clever way to compute the total marginal contribution of a player.

For instance, let (N, v) be a TU-game with ten players. Consider player 7 and the coalition $\{3, 5, 9\}$. The marginal contribution $v(\{3, 5, 9, 7\}) - v(\{3, 5, 9\})$ accruing to player 7 occurs in more than one marginal vector. In how many marginal vectors does it occur? To compute this, note that first players 3, 5, and 9 must enter, and this can happen in $3!$ different orders. Then player 7 enters. Finally, the other six players enter, and this can happen in $6!$ different orders. Therefore, the total number of marginal vectors in which player 7 obtains the marginal contribution $v(\{3, 5, 9, 7\}) - v(\{3, 5, 9\})$ is equal to $3! \times 6!$. By counting in this way the number of computations is greatly reduced.

We now repeat this argument for an arbitrary TU-game (N, v) , an arbitrary player $i \in N$, and an arbitrary coalition S that does not contain player i . By the same argument as in the preceding paragraph, the total number of marginal vectors in which player i receives the marginal contribution $v(S \cup \{i\}) - v(S)$ is equal to the number of different orders in which the players of S can enter first, $|S|!$, multiplied by the number of different orders in which the players not in $S \cup \{i\}$ can enter after player i , which is $(n - |S| - 1)!$. Hence, the total contribution obtained by player i by entering after the coalition S is equal to $|S|!(n - |S| - 1)! [v(S \cup \{i\}) - v(S)]$. The Shapley value for player i is then obtained by summing over all coalitions S not containing player i , and dividing by $n!$. In fact, we use this alternative computation as the definition of the Shapley value.

Definition 9.4 The *Shapley value* of a TU-game (N, v) is denoted by $\Phi(N, v)$. Its i -th coordinate, i.e., the Shapley value payoff to player $i \in N$, is given by

$$\Phi_i(N, v) = \sum_{S \subseteq N: i \notin S} \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)] .$$

□

Especially for larger TU-games it is easier to work with the formula in Definition 9.4 than to use the definition based on marginal vectors. For some purposes, however, it is easier to use the latter definition (Problem 9.7).

The Shapley value of the three cities game is an element of the core of that game (check this). In general, however, this does not have to be the case even if the core is nonempty (Problem 9.8).

Example 9.5 Consider the general glove game of Example 9.3. We assume $\ell = 4$ and $r = 2$. The worth $v(N)$ of the grand coalition is equal to 2. First note that, for reasons of symmetry, in the Shapley value all left-hand glove owners receive the same payoff and also all right-hand glove owners receive the same payoff. (This is intuitive, but can also be made more precise: see Problem 9.17 or Chap. 16.) Since the total payoff is 2, this means that it is sufficient to compute the Shapley value of either one left-hand glove owner or one right-hand glove owner: if, say, the payoff of a left-hand glove owner in the Shapley value is α , then every left-hand glove owner receives α and every right-hand glove owner receives $(2 - 4\alpha)/2$. Let us compute α .

Suppose i is a left-hand glove owner. Then i makes a contribution of 1 to any coalition with strictly less left-hand players than right-hand players. To all other coalitions i 's contribution is zero, so we do not have to take those into consideration. Take a coalition S with k right-hand glove owners and j left-hand glove owners, such that player i is not in S and such that $j < k$, where k is equal to 1 or 2. Then indeed $v(S \cup \{i\}) - v(S) = 1$, since $v(S) = j$ and $v(S \cup \{i\}) = j + 1$. There are $\binom{2}{k} \cdot \binom{3}{j}$ coalitions with k right-hand glove owners and j left-hand glove owners.¹ Since $|S| = k + j$, the weight of S as in the formula in Definition 9.4 is $(k + j)!(5 - k - j)!$. Hence, summing over all possible values of k and j we obtain

$$\Phi_i(N, v) = \alpha = \sum_{k=1}^2 \sum_{j=0}^{k-1} \frac{(k + j)!(5 - k - j)!}{6!} \binom{2}{k} \cdot \binom{3}{j}.$$

This is readily computed and yields $\alpha = 2/15$. Hence, every left-hand glove owner receives $2/15$ in the Shapley value, which implies that for the two right-hand glove owners $2 - 4 \cdot 2/15 = 22/15$ is left. This implies

$$\Phi_i(N, v) = \begin{cases} 2/15 & \text{if } i \text{ is a left-hand glove owner} \\ 11/15 & \text{if } i \text{ is a right-hand glove owner.} \end{cases}$$

Observe that the Shapley value of this game is *not* in the core (see Example 9.3).

¹In general, $\binom{p}{q} = \frac{p!}{q!(p-q)!}$ is the number of ways in which we can choose a set of q elements from a set of p elements.

The same arguments can be used to compute the Shapley value for any number of left-hand and right-hand glove owners. If $\ell = r$, then for reasons of symmetry, all players receive the same payoff in the Shapley value, namely $1/2$. \square

The definition of the Shapley value as assigning to each player in a game his average marginal contribution, can be regarded as a justification of this solution concept by itself. In the literature there are, moreover, a number of axiomatic characterizations of the Shapley value. In an axiomatic characterization one proceeds as follows. Consider an arbitrary map, which (like the Shapley value) assigns to each game with player set N a payoff vector. Next, define “reasonable” properties or *axioms* for this map. Such axioms limit the possible maps (i.e., solution concepts), and if the axioms are strong enough, they admit only one solution concept. This so-called *axiomatic approach* is common in cooperative game theory. Problem 9.17 preludes to this. For details, see Chap. 17.

9.4 The Nucleolus

The last concept we discuss in this introduction to TU-games is the *nucleolus*. Like the Shapley value it assigns a unique payoff distribution to a game. An advantage compared to the Shapley value is that the nucleolus assigns a payoff distribution in the core of a game, provided the core is nonempty. The nucleolus is defined for games which possess the following property. A TU-game (N, v) is *essential* if $v(N) \geq \sum_{i \in N} v(\{i\})$. For an essential game there are payoff distributions for the grand coalition that are both efficient and individually rational. Such payoff distributions are called *imputations*. The set

$$I(N, v) = \{\mathbf{x} \in \mathbb{R}^N \mid x(N) = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N\}$$

is called the *imputation set* of the TU-game (N, v) . Hence, a game (N, v) is essential if and only if $I(N, v) \neq \emptyset$ (check!).

Let (N, v) be an essential TU-game, let $\mathbf{x} \in I(N, v)$, and let S be a nonempty coalition unequal to N . The *excess of S at \mathbf{x}* , denoted by $e(S, \mathbf{x})$, is defined by

$$e(S, \mathbf{x}) = v(S) - x(S) .$$

The excess of a coalition S at a payoff distribution \mathbf{x} is, thus, the difference between what S can acquire on its own and what it receives in total from \mathbf{x} . The excess can be seen as a measure of the dissatisfaction of the coalition S with the imputation \mathbf{x} : the smaller the total payoff of S at \mathbf{x} , the larger $e(S, \mathbf{x})$. In particular, if this excess is positive then S obtains less than its own worth.

In words, the *nucleolus* of an essential TU-game (N, v) is defined as follows.

1. First, find all imputations for which the maximal excess among all coalitions (not equal to N or the empty set) is as small as possible. If there is a unique such imputation, then that is the nucleolus.
2. If not, then determine those coalitions for which the maximal excess found in (1) cannot be decreased any further. Then continue with the remaining coalitions and among the imputations found in (1) find those imputations for which the maximal excess among these remaining coalitions is as small as possible. If there is a unique such imputation, then that is the nucleolus.
3. If not, then determine those coalitions for which the maximal excess found in (2) cannot be decreased any further. Then continue with the remaining coalitions and among the imputations found in (2) find those imputations for which the maximal excess among these remaining coalitions is as small as possible. If there is a unique such imputation, then that is the nucleolus.
4. Etc.

Thus, the idea behind the nucleolus is to make the largest dissatisfaction as small as possible. If there is more than one possibility to do this, then we also make the second largest dissatisfaction as small as possible, and so on, until a unique distribution is reached.

A formal definition of the nucleolus can be found in Chap. 19. Here we content ourselves with the given verbal description and some examples.

Our first illustration of this procedure is its application to the three cities game, reproduced in Table 9.5. The third line of the table gives the excesses at the imputation $(70, 70, 80)$. The choice of this particular imputation is arbitrary: we use it as a starting point to find the nucleolus. The largest excess at this imputation is -30 , namely for the coalition $\{2, 3\}$. Clearly, we can decrease this excess by giving players 2 and 3 more at the expense of player 1. Doing so implies that the excesses of $\{1, 2\}$ or of $\{1, 3\}$ or of both will increase. These excesses are equal to -50 . We can increase the payoffs of players 2 and 3 by $6\frac{2}{3}$ and decrease the payoff of player 1 by $2 \cdot 6\frac{2}{3} = 13\frac{1}{3}$, so that these three excesses become equal. Thus we obtain the imputation $(56\frac{2}{3}, 76\frac{2}{3}, 86\frac{2}{3})$, at which the excesses of the three two-player coalitions are all equal to $-43\frac{1}{3}$, and these are also the maximal excesses. Now first observe that at this imputation the maximal excess is as small as possible. This follows since

Table 9.5 Heuristic determination of the nucleolus of the three cities game

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	0	0	0	90	100	120	220
$e(S, (70, 70, 80))$	-70	-70	-80	-50	-50	-30	
$e(S, (56\frac{2}{3}, 76\frac{2}{3}, 86\frac{2}{3}))$	$-56\frac{2}{3}$	$-76\frac{2}{3}$	$-86\frac{2}{3}$	$-43\frac{1}{3}$	$-43\frac{1}{3}$	$-43\frac{1}{3}$	

the sum of the excesses of the three two-player coalitions at *any* imputation must be the same, namely equal to -130 , as follows from

$$\begin{aligned} e(\{1, 2\}, \mathbf{x}) + e(\{1, 3\}, \mathbf{x}) + e(\{2, 3\}, \mathbf{x}) &= v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\}) \\ &\quad - 2(x_1 + x_2 + x_3) \\ &= 310 - 2 \cdot 220 \\ &= -130 . \end{aligned}$$

This implies that none of these excesses can be decreased without increasing at least one other excess. Second, the imputation at which these three excesses are equal is unique, since the system

$$\begin{aligned} 90 - x_1 - x_2 &= 100 - x_1 - x_3 \\ 100 - x_1 - x_3 &= 120 - x_2 - x_3 \\ x_1 + x_2 + x_3 &= 220 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

has a unique solution—namely, indeed, $(56\frac{2}{3}, 76\frac{2}{3}, 86\frac{2}{3})$. So this imputation must be the nucleolus of the three cities game.

This example suggests that, at least for a three player TU-game, it is easy to find the nucleolus, namely simply by equating the excesses of the three two-player coalitions. Unfortunately, this is erroneous. It works if the worths of the two-player coalitions are large relative to the worths of the single player coalitions, but otherwise it may fail to result in the nucleolus. Consider the three-player TU-game in Table 9.6, which is identical to the three cities game except that now $v(\{1\}) = 20$. The third line of the table shows the excesses at $(56\frac{2}{3}, 76\frac{2}{3}, 86\frac{2}{3})$ in this TU-game. (This vector is still an imputation.) The maximal excess is now $-36\frac{2}{3}$ for the single-player coalition $\{1\}$. Clearly, $(56\frac{2}{3}, 76\frac{2}{3}, 86\frac{2}{3})$ is no longer the nucleolus: the excess of $\{1\}$ can be decreased by giving player 1 more at the expense of players 2 and/or 3. Suppose we equalize the excesses of $\{1\}$ and $\{2, 3\}$ by solving the equation $20 - x_1 = 120 - x_2 - x_3$. Together with $x_1 + x_2 + x_3 = 220$ this yields $x_1 = 60$ and $x_2 + x_3 = 160$. Trying the imputation $(60, 75, 85)$, obtained by taking away the same amount $1\frac{2}{3}$ from players 2 and 3, yields the excesses in the fourth line of

Table 9.6 Heuristic determination of the nucleolus in the three cities game with the worth of coalition $\{1\}$ changed to 20

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	20	0	0	90	100	120	220
$e(S, (56\frac{2}{3}, 76\frac{2}{3}, 86\frac{2}{3}))$	$-36\frac{2}{3}$	$-76\frac{2}{3}$	$-86\frac{2}{3}$	$-43\frac{1}{3}$	$-43\frac{1}{3}$	$-43\frac{1}{3}$	
$e(S, (60, 75, 85))$	-40	-75	-85	-45	-45	-40	

Table 9.6. We claim that $(60, 75, 85)$ is the nucleolus of this TU-game. The maximal excess is -40 , reached by the coalitions $\{1\}$ and $\{2, 3\}$, and this cannot be decreased: decreasing the excess for one of those two coalitions implies increasing the excess for the other coalition. Hence, x_1 has to be equal to 60 in the nucleolus. The second maximal excess is -45 , reached by the coalitions $\{1, 2\}$ and $\{1, 3\}$. Since x_1 has already been fixed at 60, a decrease in the excess for one of these two coalitions implies an increase of the excess for the other coalition. Hence, also x_2 and x_3 are fixed, at 75 and 85, respectively.

These two examples indicate that it may not be easy to compute the nucleolus. For three-player games the heuristic method above works well. In general, it can be computed by solving a series of linear programs. The following example illustrates this for the three-player games considered above.

Example 9.6 The maximal excess at the nucleolus of the three-cities game in Table 9.5 can be found by solving the following linear minimization problem.

Minimize α subject to

$$x_1 + x_2 + x_3 = 220$$

$$x_1, x_2, x_3 \geq 0$$

$$0 - x_1 \leq \alpha$$

$$0 - x_2 \leq \alpha$$

$$0 - x_3 \leq \alpha$$

$$90 - x_1 - x_2 \leq \alpha$$

$$100 - x_1 - x_3 \leq \alpha$$

$$120 - x_2 - x_3 \leq \alpha$$

Here, α is the maximal excess to be minimized. The first two constraints make sure that α is minimized over the set of imputations. The next three constraints are those for the excesses of the single-player coalitions, and the last three constraints are those for the excesses of the two-player coalitions: these six inequalities ensure that α will be the maximal excess. By using the efficiency constraint $x_1 + x_2 + x_3 = 220$ we can rewrite the problem into a minimization problem with three variables (α and for instance x_1, x_2) which could be solved graphically (by making a three-dimensional diagram); in general, however, we can use a computer program, based on for instance the simplex method, to solve the problem. In this case, from the preceding analysis we already know that the optimal solution of the problem is $\alpha = -43\frac{1}{3}$, attained at a unique point $(x_1, x_2, x_3) = (56\frac{2}{3}, 76\frac{2}{3}, 86\frac{2}{3})$.

For the modified three-cities game in Table 9.6 the linear program becomes:

Minimize α subject to

$$x_1 + x_2 + x_3 = 220$$

$$x_1, x_2, x_3 \geq 0$$

$$20 - x_1 \leq \alpha$$

$$0 - x_2 \leq \alpha$$

$$0 - x_3 \leq \alpha$$

$$90 - x_1 - x_2 \leq \alpha$$

$$100 - x_1 - x_3 \leq \alpha$$

$$120 - x_2 - x_3 \leq \alpha$$

From the preceding analysis the solution of this problem is $\alpha = -40$, and at all points at which this value is attained we have $x_1 = 60$ and, consequently, $x_2 + x_3 = 160$. To determine the second maximal excess we need to solve the following linear program:

Minimize α subject to

$$x_2 + x_3 = 160$$

$$x_2, x_3 \geq 0$$

$$-x_2 \leq \alpha$$

$$-x_3 \leq \alpha$$

$$30 - x_2 \leq \alpha$$

$$40 - x_3 \leq \alpha$$

The optimal solution to this problem is $\alpha = -45$, attained at $(x_2, x_3) = (75, 85)$ —this follows from the preceding analysis, or from solving the problem graphically after reducing it to a problem of two variables using the constraint $x_2 + x_3 = 160$. Thus, the nucleolus of the modified three-cities problem is $(60, 75, 85)$, as found earlier. \square

Although the nucleolus is not easy to compute it is an attractive solution for the following reasons. It assigns a unique imputation to every essential game and if a game has a nonempty core, the nucleolus assigns a core element.

Proposition 9.7 *Let (N, v) be a game with nonempty core. Then the nucleolus of (N, v) is a payoff distribution in the core.*

Table 9.7 The game in Example 9.8

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	0	0	0	10	0	20	20
$e(S, (0, 10, 10))$	0	-10	-10	0	-10	0	
$e(S, (0, 15, 5))$	0	-15	-5	-5	-5	0	

Proof Take any \mathbf{x} in the core of (N, v) . Then for every nonempty coalition $S \neq N$ we have $x(S) \geq v(S)$, hence $e(S, \mathbf{x}) = v(S) - x(S) \leq 0$. Suppose that \mathbf{z} is the nucleolus of (N, v) . Since the nucleolus minimizes the maximal excess over all imputations, and we already have an imputation, namely \mathbf{x} , at which all excesses and thus also the maximal excess are non-positive, we must have that the maximal excess at \mathbf{z} is non-positive and, hence, all excesses at \mathbf{z} are non-positive. This means that for every nonempty coalition $S \neq N$ we have $e(S, \mathbf{z}) = v(S) - z(S) \leq 0$, hence $z(S) \geq v(S)$. But this means that \mathbf{z} is in the core of (N, v) . ■

In view of Proposition 9.7, if the core of a game is known then this may be very helpful in finding the nucleolus since we can restrict consideration to the core. We illustrate this by the next examples.

Example 9.8 Consider the three-person game in Table 9.7. This game has a nonempty core, and player 1 obtains 0 in every core distribution. Starting with the core distribution $(0, 10, 10)$, we find that the maximal excess is equal to 0, reached for the coalitions $\{1\}$, $\{1, 2\}$, and $\{2, 3\}$. Clearly, this excess cannot be decreased any further since a decrease of the excess for $\{1\}$ implies an increase for $\{2, 3\}$, and conversely. Hence, if \mathbf{z} is the nucleolus, then $z_1 = 0$ and $z_2 + z_3 = 20$. Note, however, that the excess for $\{1, 2\}$ can be decreased by increasing the payoff for player 2 at the expense of player 3. For the payoff distribution $(0, 15, 5)$, we find that the second maximal excess is reached by $\{1, 2\}$, $\{1, 3\}$, and $\{3\}$. We cannot decrease the excess of $\{1, 2\}$ without increasing it for $\{1, 3\}$ and $\{3\}$, given that z_1 is already fixed at 0. Hence, we have obtained the nucleolus: $\mathbf{z} = (0, 15, 5)$. □

Example 9.9 Consider again the general glove game of Example 9.3. For $\ell > r$, the core of this game consists of the unique payoff distribution where the left-hand glove owners obtain 0 and the right-hand glove owners obtain 1. By Proposition 9.7, this is also the nucleolus of this game. □

We conclude with an example illustrating again the core, Shapley value and nucleolus of a game.

Example 9.10 Consider the following six-player cooperative game. The player set is $N = \{1, \dots, 6\}$. A coalition S has worth 1 in exactly two cases: either it contains player 1 and at least one other player, or it is the coalition $\{2, \dots, 6\}$. All other

coalitions have worth 0. (Such a game is called an *apex game*, player 1 is the *apex player* and the other players are called *minor players*.)

First observe that the core of this game is empty. To see this, suppose $\mathbf{x} = (x_1, \dots, x_6)$ is in the core. Then $x_1 + \dots + x_5 \geq v(\{1, \dots, 5\}) = 1$. Since $x_6 \geq v(\{6\}) = 0$ and $x_1 + \dots + x_6 = v(N) = 1$, we must have $x_1 + \dots + x_5 = 1$ and $x_6 = 0$. Similarly, one derives $x_2 = \dots = x_5 = 0$. Hence, $x_2 + \dots + x_6 = 0$, but then $1 = v(\{2, \dots, 6\}) > x_2 + \dots + x_6$, so that the core constraint for coalition $\{2, \dots, 6\}$ is violated. From this contradiction, we conclude that the core of this game is empty.

To find the nucleolus, for reasons of symmetry we may assume that it is of the form $(1 - 5\alpha, \alpha, \dots, \alpha)$, where $0 \leq \alpha \leq 1/5$ in order to make it an imputation. The excess of a coalition of the form $\{1, i\}$ for every $i \in \{2, \dots, 6\}$ is equal to $1 - (1 - 5\alpha) - \alpha = 4\alpha$, and the excess of the coalition $\{2, \dots, 6\}$ is $1 - 5\alpha$. Clearly, these will be the maximal excesses. By increasing α the excesses of the $\{1, i\}$ coalitions will increase and the excess of the coalition $\{2, \dots, 6\}$ will decrease; by decreasing α the effects will be opposite. Therefore, we find the nucleolus by equating these excesses. Setting $4\alpha = 1 - 5\alpha$ yields $\alpha = 1/9$, hence the nucleolus is $(4/9, 1/9, \dots, 1/9)$.

To find the Shapley value, we use Definition 9.4. (There are $6! = 720$ possible orderings of the six players, so listing all of them is an inefficient method to compute the Shapley value.) It is sufficient to compute the Shapley value for player 2 (for instance) since then we also know the Shapley value for the other players. Player 2 makes a nonzero contribution, equal to 1, in exactly two cases: either to $S = \{1\}$ or to $S = \{3, \dots, 6\}$. So we obtain

$$\Phi_2(N, v) = \frac{1!(6 - 1 - 1)!}{6!} \cdot 1 + \frac{4!(6 - 4 - 1)!}{6!} \cdot 1 = \frac{1}{15}.$$

Thus, the Shapley value of this game is

$$\Phi(N, v) = (1 - 5 \cdot \frac{1}{15}, \frac{1}{15}, \dots, \frac{1}{15}) = (\frac{2}{3}, \frac{1}{15}, \dots, \frac{1}{15}).$$

□

9.5 Problems

9.1. Number of Coalitions

Show that a set of $n \in \mathbb{N}$ elements has 2^n different subsets.

9.2. Computing the Core

- (a) Compute the core of the glove game of Table 9.2 by making a diagram.
- (b) Compute the core of the dentist game of Table 9.3 by making a diagram.
- (c) Compute the core of the UN security council voting game in Sect. 9.1.

9.3. The Core of a Two-Person Game

Consider the two-person game $(\{1, 2\}, v)$ given by $v(\{1\}) = a$, $v(\{2\}) = b$, and $v(\{1, 2\}) = c$, where $a, b, c \in \mathbb{R}$. Give a necessary and sufficient condition on a , b , and c for the core of $(\{1, 2\}, v)$ to be nonempty. Make a diagram and compute the core.

9.4. The Core of the General Glove Game

Compute the core of the general glove game in Example 9.3 for the case where the numbers of left-hand glove owners and right-hand glove owners are equal. Is the Shapley value (cf. Example 9.5) in the core?

9.5. A Condition for Nonemptiness of the Core of a Three-Person Game

Let $(\{1, 2, 3\}, v)$ be a three-person game which has a nonempty core. Show that $2v(\{1, 2, 3\}) \geq v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\})$. (Hint: Take a core element $\mathbf{x} = (x_1, x_2, x_3)$ and write down the core constraints.)

9.6. Non-monotonicity of the Core

Consider the following four-person game: $v(\{i\}) = 0$ for every $i = 1, \dots, 4$, $v(\{1, 2\}) = v(\{3, 4\}) = 0$, $v(S) = 1$ for all other two-person coalitions and for all three-person coalitions, and $v(N) = 2$.

- Show that $C(N, v) = \{(\alpha, \alpha, 1 - \alpha, 1 - \alpha) \in \mathbb{R}^4 \mid 0 \leq \alpha \leq 1\}$.
- Consider the game (N, v') equal to (N, v) except for $v'(\{1, 3, 4\}) = 2$. Show that the core of (N, v') consists of a single element. What about the payoff to player 1 if core elements in (N, v) and (N, v') are compared? Conclude that the core is not “monotonic” (consider player 1).

9.7. Efficiency of the Shapley Value

Let (N, v) be an arbitrary TU-game. Show that the Shapley value $\Phi(N, v)$ is efficient. [Hint: take an order i_1, i_2, \dots, i_n of the players and show that the sum of the coordinates of the corresponding marginal vector is equal to $v(N)$; use this to conclude that $\Phi(N, v)$ is efficient.]

9.8. Computing the Shapley Value

- Compute the Shapley value of the glove game of Table 9.2. Is it an element of the core?
- Compute the Shapley value of the dentist game of Table 9.3. Is it an element of the core?
- Compute the Shapley value of the UN security council voting game in Sect. 9.1. (Hint: observe the—more or less—obvious fact that the Shapley value assigns the same payoff to all permanent members and also to all nonpermanent members. Use the formula in Definition 9.4.) Is it an element of the core?

9.9. The Shapley Value and the Core

For every real number a the three-player TU-game v_a is given by: $v_a(\{i\}) = 0$ for $i = 1, 2, 3$, $v_a(\{1, 2\}) = 3$, $v_a(\{1, 3\}) = 2$, $v_a(\{2, 3\}) = 1$, $v_a(\{1, 2, 3\}) = a$.

- Determine the minimal value of a so that the TU-game v_a has a nonempty core.
- Calculate the Shapley value of v_a for $a = 6$.
- Determine the minimal value of a so that the Shapley value of v_a is a core distribution.

9.10. Shapley Value in a Two-Player Game

Let (N, v) be a two-player TU-game, i.e., $N = \{1, 2\}$. Compute the Shapley value (expressed in $v(\{1\})$, $v(\{2\})$, and $v(\{1, 2\})$), and show that it is in the core of the game provided the core is nonempty. Make a diagram.

9.11. Computing the Nucleolus

- Compute the nucleolus of the glove game of Table 9.2.
- Compute the nucleolus of the dentist game of Table 9.3.
- Compute the nucleolus of the UN security council voting game in Sect. 9.1. (Hint: use Proposition 9.7.)
- Compute the nucleolus of the games (N, v) and (N, v') in Problem 9.6.

9.12. Nucleolus of Two-Player Games

Let (N, v) be an essential two-player TU-game. Compute the nucleolus.

9.13. Computing the Core, the Shapley Value, and the Nucleolus

- Compute the Shapley value and the nucleolus in the three-player TU-game given by: $v(\{i\}) = 1$ for $i = 1, 2, 3$, $v(\{1, 2\}) = 2$, $v(\{1, 3\}) = 3$, $v(\{2, 3\}) = 4$, $v(\{1, 2, 3\}) = 6$. Is the Shapley value a core element in this game?
- Compute the core of this game. Make a picture.
- Suppose we increase $v(\{1\})$. What is the maximal value of $v(\{1\})$ such that the game still has a nonempty core?

9.14. Voting (1)

Suppose in Parliament there are four parties A, B, C, D with numbers of votes equal to 40, 30, 20, 10, respectively. To pass any law a two-third majority is needed.

- Formulate this situation as a four-person cooperative game where winning coalitions have worth 1 and losing coalitions worth 0. Determine the Shapley value of this game.
- Determine also the core and the nucleolus of this game.

9.15. Voting (2)

A voting committee consists of five members: members 1 and 2 belong to party I, while members 3, 4, and 5 belong to party II. In order to pass a decision at least a weak majority of each party (that is, at least 50% of the votes of each party) is required. A coalition that has a weak majority of both parties is called *winning*. We model this situation as a so-called *simple game*: winning coalitions obtain worth 1, all other coalitions worth 0.

- (a) Show that there are six winning coalitions of minimal size: list all of them.
- (b) Use your answer to (a) to give a concise description of the game (i.e., without listing all 32 coalitions).
- (c) Compute the Shapley value of this game. According to the Shapley value, which players (members) are most powerful?
- (d) Compute the nucleolus of this game. According to the nucleolus, which players (members) are most powerful?
- (e) Compute the core of this game.

9.16. Two Buyers and a Seller

Players 1 and 2 are buyers, while player 3 is a seller. The seller owns an object that is worth nothing to him, but has value 1 for buyer 1 and value 2 for buyer 2. These are the prices that the buyers are willing to pay to the seller in order to get the object.

- (a) Model this situation as a three-person TU-game, where the worth of each coalition is the maximal surplus it can create by a transaction between a buyer and the seller.
- (b) Compute the core of this game.
- (c) Compute the Shapley value of this game. Is it in the core?
- (d) Compute the nucleolus of this game.

9.17. Properties of the Shapley Value

The properties of the Shapley value described in (a)–(c) below are called symmetry, additivity, and dummy property, respectively. It can be shown (Chap. 17) that the Shapley value is the unique solution concept that assigns exactly one payoff vector to each TU-game and has these three properties together with efficiency (cf. Problem 9.7). In other words, a solution concept has these four properties if, and only if, it is the Shapley value. In this exercise you are asked to show the “easy” part of this statement, namely the if-part. (Hint: in each case, decide which of the two formulas for the Shapley value—the one in Definition 9.4 or the formula based on marginal vectors—is most convenient to use.)

- (a) Let (N, v) be a TU-game, and suppose players i and j are *symmetric* in this game, i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$ for all coalitions S which do not contain i and j . Show that i and j obtain the same payoff from the Shapley value.

- (b) Let (N, v) and (N, w) be two TU-games with the same player set N . Define the sum of these TU-games as the TU-game with player set N where the worth of each coalition S is given by $v(S) + w(S)$; denote this TU-game by $(N, v + w)$. Show that the Shapley value assigns to this sum TU-game the payoff vector which is the sum of the Shapley values of (N, v) and (N, w) .
- (c) Call player i a *dummy* in the TU-game (N, v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every coalition S to which player i does not belong. Show that the Shapley value assigns exactly the payoff $v(\{i\})$ to player i .

9.6 Notes

For a more advanced treatment of cooperative game theory, see Chaps. 16–20. Games with nonempty cores were characterized in Bondareva (1962) and Shapley (1967), see Chap. 16.

The Shapley value was introduced and axiomatically characterized in Shapley (1953)—see Chap. 17.

Imputations were first introduced by von Neumann and Morgenstern (1944/1947). They introduced cooperative games in order to cope with multi-person zero-sum games, which in general do not have a value.

The nucleolus was introduced in Schmeidler (1969). The nucleolus is similar in spirit to the main principle of distributive justice proposed in Rawls (1971), namely to maximize the lot of the worst off people in society.

The arguments used above to show that a particular imputation is indeed the nucleolus implicitly use a general property of the nucleolus called the Kohlberg criterion. See Chap. 19 for a detailed study of the nucleolus.

Problem 9.6 is taken from Moulin (1988).

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