18 Core, Shapley Value, and Weber Set

In Chap. 17 we have seen that the Shapley value of a game does not have to be in the core of the game, nor even an imputation (Problem 17.5). In this chapter we introduce a set-valued extension of the Shapley value, the Weber set, and show that it always contains the core (Sect. 18.1). Next, we study so-called convex games and show that these are exactly those games for which the core and the Weber set coincide. Hence, for such games the Shapley value is an attractive core selection (Sect. 18.2). Finally, we study random order values (Sect. 18.3), which fill out the Weber set, and the subset of weighted Shapley values, which still cover the core (Sect. 18.4).

18.1 The Weber Set

Let $(N, v) \in \mathcal{G}^N$. Recall the definition of a marginal vector from Sect. 17.1.

Definition 18.1 The *Weber set* of a game $(N, v) \in \mathcal{G}^N$ is the convex hull of its marginal vectors:

$$W(v) := \text{conv}\{m^\sigma(v) \mid \sigma \in \Pi(N)\} .$$

□

Example 18.2 Consider the three-person game $(\{1, 2, 3\}, v)$ defined by $v(12) = v(13) = 1$, $v(23) = -1$, $v(123) = 3$, and $v(i) = 0$ for every $i \in \{1, 2, 3\}$. The marginal vectors of this game, the core and the Weber set are given in Fig. 18.1. □

We show now that the core is always a subset of the Weber set.

Theorem 18.3 *Let* $(N, v) \in \mathcal{G}^N$. *Then* $C(v) \subseteq W(v)$.

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$(\sigma(1), \sigma(2), \sigma(3))$	m_1^σ	m_2^σ	m_3^σ
$(1, 2, 3)$	0	1	2
$(1, 3, 2)$	0	2	1
$(2, 1, 3)$	1	0	2
$(2, 3, 1)$	4	0	−1
$(3, 1, 2)$	1	2	0
$(3, 2, 1)$	4	−1	0

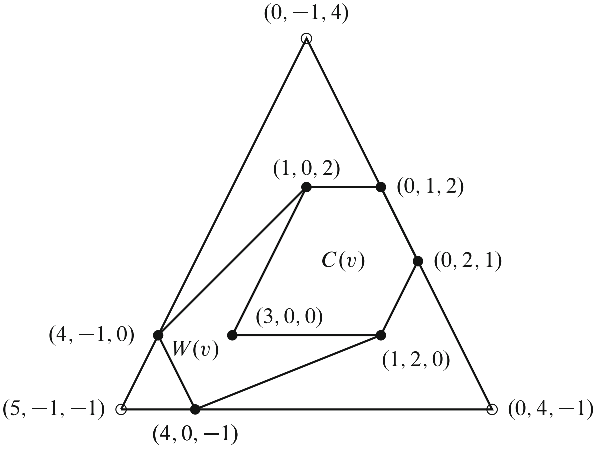

Fig. 18.1 Example 18.2. The core is the convex hull of the vectors $(3, 0, 0)$, $(1, 2, 0)$, $(0, 2, 1)$, $(0, 1, 2)$, and $(1, 0, 2)$. The Weber set is the convex hull of the six marginal vectors

Proof Suppose there is an $\mathbf{x} \in C(v) \setminus W(v)$. By a separation theorem (Theorem 22.1), there exists a vector $\mathbf{y} \in \mathbb{R}^N$ such that $\mathbf{w} \cdot \mathbf{y} > \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{w} \in W(v)$. In particular,

$$m^\sigma \cdot \mathbf{y} > \mathbf{x} \cdot \mathbf{y} \ \text{ for every } \sigma \in \Pi(N) \ . \tag{18.1}$$

Let $\pi \in \Pi(N)$ with $y_{\pi(1)} \geq y_{\pi(2)} \geq \ldots \geq y_{\pi(n)}$. Since $\mathbf{x} \in C(v)$,

$$\begin{aligned} m^\pi \cdot \mathbf{y} &= \sum_{i=1}^{n} y_{\pi(i)}(v(\pi(1), \pi(2), \ldots, \pi(i)) - v(\pi(1), \pi(2), \ldots, \pi(i-1))) \\ &= y_{\pi(n)} v(N) - y_{\pi(1)} v(\emptyset) + \sum_{i=1}^{n-1} (y_{\pi(i)} - y_{\pi(i+1)}) v(\pi(1), \pi(2) \ldots, \pi(i)) \\ &\leq y_{\pi(n)} \sum_{j=1}^{n} x_{\pi(j)} + \sum_{i=1}^{n-1} (y_{\pi(i)} - y_{\pi(i+1)}) \sum_{j=1}^{i} x_{\pi(j)} \end{aligned}$$

$$= \sum_{i=1}^{n} y_{\pi(i)} \sum_{j=1}^{i} x_{\pi(j)} - \sum_{i=2}^{n} y_{\pi(i)} \sum_{j=1}^{i-1} x_{\pi(j)}$$

$$= \sum_{i=1}^{n} y_{\pi(i)} x_{\pi(i)} = \mathbf{x} \cdot \mathbf{y}$$

contradicting (18.1). ■

18.2 Convex Games

For the coincidence of core and Weber set the following possible property of a game plays a crucial role.

Definition 18.4 A TU-game (N, v) is *convex* if the following condition holds for all $S, T \subseteq N$:

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \ . \qquad (18.2)$$

□

Observe that convexity of a game implies super-additivity [cf. (16.2)]: v is super-additive if (18.2) holds whenever S and T have empty intersection. The intuition is similar: larger coalitions have a relatively larger worth. This intuition is also apparent in the following condition:

$$\text{For all } i \in N \text{ and } S \subseteq T \subseteq N \setminus \{i\}\text{:} \ \ v(S \cup i) - v(S) \leq v(T \cup i) - v(T) \ . \qquad (18.3)$$

Lemma 18.5 *A game* $(N, v) \in \mathcal{G}^N$ *is convex if and only if it satisfies* (18.3).

Proof Let $v \in \mathcal{G}^N$. Obviously, (18.3) follows from (18.2) by taking, for S and T in (18.2), $S \cup i$ and T from (18.3), respectively.

In order to derive (18.2) from (18.3), first take $S_0 \subseteq T_0 \subseteq N$ and $R \subseteq N \setminus T_0$, say $R = \{i_1, \ldots, i_k\}$. By repeated application of (18.3) one obtains

$$v(S_0 \cup i_1) - v(S_0) \leq v(T_0 \cup i_1) - v(T_0)$$

$$v(S_0 \cup i_1 i_2) - v(S_0 \cup i_1) \leq v(T_0 \cup i_1 i_2) - v(T_0 \cup i_1)$$

$$\vdots$$

$$v(S_0 \cup i_1 \cdots i_k) - v(S_0 \cup i_1 \cdots i_{k-1}) \leq v(T_0 \cup i_1 \cdots i_k) - v(T_0 \cup i_1 \cdots i_{k-1}) \ .$$

Adding these inequalities yields

$$v(S_0 \cup R) - v(S_0) \leq v(T_0 \cup R) - v(T_0)$$

for all $R \subseteq N \setminus T_0$. Applying this inequality to arbitrary S, T by setting $S_0 = S \cap T$, $T_0 = T$, and $R = S \setminus T$, yields (18.2). ■

The importance of convexity of a game for the relation between the core and the Weber set follows from the following theorem.

Theorem 18.6 *Let $v \in \mathcal{G}^N$. Then v is convex if and only if $C(v) = W(v)$.*

Proof

(a) Suppose v is convex. For the 'only if' part it is, in view of Theorem 18.3 and convexity of the core, sufficient to prove that each marginal vector $m^\pi(v)$ of v is in the core. In order to show this, assume for notational simplicity that π is identity. Let $S \subseteq N$ be an arbitrary coalition, say $S = \{i_1, \ldots, i_s\}$ with $i_1 \leq \ldots \leq i_s$. Then, for $1 \leq k \leq s$, by (18.3):

$$v(i_1, \ldots, i_k) - v(i_1, \ldots, i_{k-1}) \leq v(1, 2, \ldots, i_k) - v(1, 2, \ldots, i_k - 1) = m^\pi_{i_k}(v) \; .$$

Summing these inequalities from $k = 1$ to $k = s$ yields

$$v(S) = v(i_1, \ldots, i_s) \leq \sum_{k=1}^{s} m^\pi_{i_k}(v) = \sum_{i \in S} m^\pi_i(v) \; ,$$

which shows $m^\pi(v) \in C(v)$.

(b) For the converse, suppose that all marginal vectors of v are in the core. Let $S, T \subseteq N$ be arbitrary. Order the players of N as follows:

$$N = \{\underbrace{i_1, \ldots, i_k}_{S \cap T}, \underbrace{i_{k+1}, \ldots, i_\ell}_{T \setminus S}, \underbrace{i_{\ell+1}, \ldots, i_s}_{S \setminus T}, \underbrace{i_{s+1}, \ldots, i_n}_{N \setminus (S \cup T)}\} \; .$$

This defines an ordering or permutation π, namely by $\pi(j) = i_j$ for all $j \in N$, with corresponding marginal vector $m(v) = m^\pi(v)$. Since $m(v) \in C(v)$,

$$\begin{aligned} v(S) &\leq \sum_{i \in S} m_i(v) \\ &= \sum_{j=1}^{k} m_{i_j}(v) + \sum_{j=\ell+1}^{s} m_{i_j}(v) \\ &= v(i_1, \ldots, i_k) + \end{aligned}$$

$$\begin{aligned}
&[v(i_1,\ldots,i_{\ell+1}) - v(i_1,\ldots,i_\ell)] + \\
&[v(i_1,\ldots,i_{\ell+2}) - v(i_1,\ldots,i_{\ell+1}] + \ldots \\
&[v(i_1,\ldots,i_s) - v(i_1,\ldots,i_{s-1}] \\
= {}& v(S \cap T) + v(S \cup T) - v(T) \ ,
\end{aligned}$$

which implies (18.2). So v is a convex game. ■

An immediate consequence of Theorem 18.6 and the definition of the Shapley value (Definition 17.1) is the following corollary.

Corollary 18.7 *Let $v \in \mathcal{G}^N$ be convex. Then $\Phi(v) \in C(v)$, i.e., the Shapley value is in the core.*

18.3 Random Order Values

A value $\psi : \mathcal{G}^N \to \mathbb{R}^N$ is called a *random order value* if there is a probability distribution p over the set of permutations $\Pi(N)$ of N such that

$$\psi(N, v) = \sum_{\pi \in \Pi(N)} p(\pi) m^\pi(N, v)$$

for every $(N, v) \in \mathcal{G}^N$. In that case, we denote ψ by Φ^p. Observe that Φ^p is the Shapley value Φ if $p(\pi) = 1/n!$ for every $\pi \in \Pi(N)$. Obviously,

$$W(v) = \{\mathbf{x} \in \mathbb{R}^N \mid \mathbf{x} = \Phi^p(v) \text{ for some } p\}.$$

Random order values satisfy the following two conditions.

Monotonicity (MON): $\psi(v) \geq \mathbf{0}$ for all monotonic games $v \in \mathcal{G}^N$. [The game v is *monotonic* if $S \subseteq T$ implies $v(S) \leq v(T)$ for all $S, T \subseteq N$.]

Linearity (LIN): $\psi(\alpha v + \beta w) = \alpha\psi(v) + \beta\psi(w)$ for all $v, w \in \mathcal{G}^N$, and $\alpha, \beta \in \mathbb{R}$ [where, for each S, $(\alpha v + \beta w)(S) = \alpha v(S) + \beta w(S)$].

Monotonicity says that in a monotonic game, where larger coalitions have higher worths, i.e., all marginal contributions are nonnegative, every player should receive a nonnegative payoff. Linearity is a strengthening of additivity. The main result in this section is the following characterization of random order values. (See Problem 18.7 for a strengthening of the 'only if' part of this theorem.)

Theorem 18.8 *Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a value. Then ψ satisfies LIN, DUM, EFF, and MON if and only if it is a random order value.*

The proof of Theorem 18.8 is based on a sequence of propositions and lemmas, which are of independent interest.

Proposition 18.9 *Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a linear value. Then there is a collection of constants $\{a^i_T \in \mathbb{R} \mid i \in N,\ \emptyset \neq T \subseteq N\}$ such that $\psi_i(v) = \sum_{\emptyset \neq T \subseteq N} a^i_T v(T)$ for every $v \in \mathcal{G}^N$ and $i \in N$.*

Proof Let $a^i_T := \psi_i(1_T)$ for all $i \in N$ and $\emptyset \neq T \subseteq N$ (cf. Problem 17.1). For every $v \in \mathcal{G}^N$ we have $v = \sum_{T \neq \emptyset} v(T) 1_T$. The desired conclusion follows from linearity of ψ. ■

Proposition 18.10 *Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a linear value satisfying DUM. Then there is a collection of constants $\{p^i_T \in \mathbb{R} \mid i \in N,\ T \subseteq N \setminus i\}$ with $\sum_{T \subseteq N \setminus i} p^i_T = 1$ for all $i \in N$, such that for every $v \in \mathcal{G}^N$ and every $i \in N$:*

$$\psi_i(v) = \sum_{T \subseteq N \setminus i} p^i_T [v(T \cup i) - v(T)] \, .$$

Proof Let $v \in \mathcal{G}^N$ and $i \in N$. By Proposition 18.9 there are a^i_T such that $\psi_i(v) = \sum_{\emptyset \neq T \subseteq N} a^i_T v(T)$. Then $0 = \psi_i(u_{N \setminus i}) = a^i_N + a^i_{N \setminus i}$, where the first equality follows from DUM. Assume now as induction hypothesis that $a^i_{T \cup i} + a^i_T = 0$ for all $T \subseteq N \setminus i$ with $|T| \geq k \geq 2$ (we have just established this for $k = n - 1$), and let $S \subseteq N \setminus i$ with $|S| = k - 1$. Then

$$\begin{aligned} 0 &= \psi_i(u_S) \\ &= \sum_{T : S \subseteq T} a^i_T \\ &= \sum_{T : S \subsetneq T \subseteq N \setminus i} (a^i_{T \cup i} + a^i_T) + a^i_{S \cup i} + a^i_S \\ &= a^i_{S \cup i} + a^i_S \, , \end{aligned}$$

where the last equality follows by induction and the first one by DUM. Hence, we have proved that $a^i_{T \cup i} + a^i_T = 0$ for all $T \subseteq N \setminus i$ with $0 < |T| \leq n - 1$. Now define, for all $i \in N$ and all such T, $p^i_T := a^i_{T \cup i} = -a^i_T$, and define $p^i_\emptyset := a^i_{\{i\}}$. Then for every $v \in \mathcal{G}^N$ and $i \in N$:

$$\begin{aligned} \psi_i(v) &= \sum_{\emptyset \neq T \subseteq N} a^i_T v(T) \\ &= \sum_{T :\ i \in T} a^i_T v(T) + \sum_{\emptyset \neq T \subseteq N \setminus i} a^i_T v(T) \end{aligned}$$

$$= \sum_{T:\, i \notin T} a^i_{T \cup i} v(T \cup i) - \sum_{T \subseteq N \setminus i} (-a^i_T) v(T)$$

$$= \sum_{T \subseteq N \setminus i} p^i_T [v(T \cup i) - v(T)] \,.$$

Finally, by DUM,

$$1 = u_{\{i\}}(i) = \psi_i(u_{\{i\}}) = \sum_{T \subseteq N \setminus i} p^i_T \,,$$

which completes the proof of the proposition. ■

Proposition 18.11 *Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a linear value satisfying DUM and MON. Then there is a collection of constants $\{p^i_T \in \mathbb{R} \mid i \in N,\ T \subseteq N \setminus i\}$ with $\sum_{T \subseteq N \setminus i} p^i_T = 1$ and $p^i_S \geq 0$ for all $S \subseteq N \setminus i$ and $i \in N$, such that for every $v \in \mathcal{G}^N$ and every $i \in N$:*

$$\psi_i(v) = \sum_{T \subseteq N \setminus i} p^i_T [v(T \cup i) - v(T)] \,.$$

Proof In view of Proposition 18.10 we only have to prove that the weights p^i_T are nonnegative. Let $i \in N$ and $T \subseteq N \setminus i$ and consider the game $\hat{u}_T$ assigning worth 1 to all strict supersets of T and 0 otherwise. Then $\psi_i(\hat{u}_T) = \sum_{S \subseteq N \setminus i} p^i_S [\hat{u}_T(S \cup i) - \hat{u}_T(S)]$ by Proposition 18.10, hence $\psi_i(\hat{u}_T) = p^i_T$. Since $\hat{u}_T$ is a monotonic game, this implies $p^i_T \geq 0$. ■

Lemma 18.12 *Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a value and $\{p^i_T \in \mathbb{R} \mid i \in N,\ T \subseteq N \setminus i\}$ be a collection of constants such that for every $v \in \mathcal{G}^N$ and every $i \in N$:*

$$\psi_i(v) = \sum_{T \subseteq N \setminus i} p^i_T [v(T \cup i) - v(T)] \,.$$

Then ψ is efficient if and only if $\sum_{i \in N} p^i_{N \setminus i} = 1$ and $\sum_{i \in T} p^i_{T \setminus i} = \sum_{j \in N \setminus T} p^j_T$ for all $\emptyset \neq T \neq N$.

Proof Let $v \in \mathcal{G}^N$. Then

$$\psi(v)(N) = \sum_{i \in N} \sum_{T \subseteq N \setminus i} p^i_T [v(T \cup i) - v(T)]$$

$$= \sum_{i \in N} \left(\sum_{T \subseteq N:\, i \in T} p^i_{T \setminus i} v(T) - \sum_{T \subseteq N:\, i \notin T} p^i_T v(T) \right)$$

$$= \sum_{T \subseteq N} v(T) \left(\sum_{i \in T} p^i_{T \setminus i} - \sum_{i \notin T} p^i_T \right)$$

$$= \sum_{T \subseteq N} v(T) \left(\sum_{i \in T} p^i_{T \setminus i} - \sum_{j \in N \setminus T} p^j_T \right) .$$

Clearly, this implies efficiency of ψ if the relations in the lemma hold. Conversely, suppose that ψ is efficient. Let $\emptyset \neq T \subseteq N$ and consider the games u_T and $\hat{u}_T$, where $\hat{u}$ was defined in the proof of Proposition 18.11. Then the preceding equation implies that

$$\psi(u_T)(N) - \psi(\hat{u}_T)(N) = \sum_{S \subseteq N} (u_T(S) - \hat{u}_T(S)) \left(\sum_{i \in S} p^i_{S \setminus i} - \sum_{j \in N \setminus S} p^j_S \right)$$

$$= \sum_{i \in T} p^i_{T \setminus i} - \sum_{j \in N \setminus T} p^j_T .$$

The relations in the lemma now follow by efficiency of ψ, since $u_T(N) - \hat{u}_T(N)$ is equal to 1 if $T = N$ and equal to 0 otherwise. ■

We are now sufficiently equipped to prove Theorem 18.8.

Proof of Theorem 18.8 We leave it to the reader to verify that random order values satisfy the four axioms in the theorem. Conversely, let ψ satisfy these four axioms. By Proposition 18.11 there is a collection of constants $\{p^i_T \in \mathbb{R} \mid i \in N,\ T \subseteq N \setminus i\}$ with $\sum_{T \subseteq N \setminus i} p^i_T = 1$ and $p^i_S \geq 0$ for all $S \subseteq N \setminus i$ and $i \in N$, such that for every $v \in \mathcal{G}^N$ and every $i \in N$:

$$\psi_i(v) = \sum_{T \subseteq N \setminus i} p^i_T [v(T \cup i) - v(T)] .$$

For all $i \in N$ and $T \subseteq N \setminus i$ define $A(T) := \sum_{j \in N \setminus T} p^j_T$ and $A(i;T) := p^i_T / A(T)$ if $A(T) \neq 0$, $A(i;T) := 0$ if $A(T) = 0$. For any permutation $\pi \in \Pi(N)$ write $\pi = (i_1, \ldots, i_n)$ (that is, $\pi(k) = i_k$ for all $k \in N$). Define

$$p(\pi) = p^{i_1}_{\emptyset} A(i_2; \{i_1\}) A(i_3; \{i_1, i_2\}) \cdots A(i_n; \{i_1, \ldots, i_{n-1}\}) .$$

Then we claim that

$$\sum_{\pi \in \Pi(N)} p(\pi) = \sum_{i \in N} p^i_{\emptyset} . \qquad (18.4)$$

In order to prove (18.4), note that when considering the sum $\sum_{\pi \in \Pi(N)} p(\pi)$ we may leave out any π with $p(\pi) = 0$. This means in particular that in what follows all expressions $A(\cdot;\cdot)$ are positive. Now

$$\begin{aligned}
\sum_{\pi \in \Pi(N)} p(\pi) &= \sum_{i_1 \in N} \sum_{i_2 \in N \setminus i_1} \cdots \sum_{i_{n-1} \in N \setminus i_1 \cdots i_{n-2}} p_{\emptyset}^{i_1} A(i_2; \{i_1\}) \cdot A(i_3; \{i_1, i_2\}) \cdot \\
&\quad \ldots \cdot A(i_{n-1}; \{i_1, \ldots, i_{n-2}\}) \cdot A(i_n; \{i_1, \ldots, i_{n-1}\}) \\
&= \sum_{i_1 \in N} \sum_{i_2 \in N \setminus i_1} \cdots \sum_{i_{n-2} \in N \setminus i_1 \cdots i_{n-3}} p_{\emptyset}^{i_1} A(i_2; \{i_1\}) \cdot A(i_3; \{i_1, i_2\}) \cdot \\
&\quad \ldots \cdot A(i_{n-2}; \{i_1, \ldots, i_{n-3}\}) \\
&\quad \cdot \left(\frac{p_{N \setminus i_1 \cdots i_{n-2}}^{\ell}}{p_{N \setminus i_1 \cdots i_{n-2}}^{\ell} + p_{N \setminus i_1 \cdots i_{n-2}}^{k}} + \frac{p_{N \setminus i_1 \cdots i_{n-2}}^{k}}{p_{N \setminus i_1 \cdots i_{n-2}}^{\ell} + p_{N \setminus i_1 \cdots i_{n-2}}^{k}} \right) \\
&= \sum_{i_1 \in N} \sum_{i_2 \in N \setminus i_1} \cdots \sum_{i_{n-2} \in N \setminus i_1 \cdots i_{n-3}} p_{\emptyset}^{i_1} A(i_2; \{i_1\}) \cdot A(i_3; \{i_1, i_2\}) \cdot \\
&\quad \ldots \cdot A(i_{n-2}; \{i_1, \ldots, i_{n-3}\}) \\
&= \sum_{i_1 \in N} \sum_{i_2 \in N \setminus i_1} \cdots \sum_{i_{n-3} \in N \setminus i_1 \cdots i_{n-4}} p_{\emptyset}^{i_1} A(i_2; \{i_1\}) \cdot A(i_3; \{i_1, i_2\}) \cdot \\
&\quad \ldots \cdot A(i_{n-3}; \{i_1, \ldots, i_{n-4}\}) \\
&= \ldots \\
&= \sum_{i_1 \in N} p_{\emptyset}^{i_1} ,
\end{aligned}$$

where after the first equality sign, $i_n \in N \setminus \{i_1, \ldots, i_{n-1}\}$, and $A(i_n; \{i_1, \ldots, i_{n-1}\}) = 1$ by definition; after the second equality sign $\ell, k \in N \setminus \{i_1, \ldots, i_{n-2}\}$ with $\ell \neq k$; the third equality sign follows since the sum involving ℓ and k is equal to 1; the remaining equality signs follow from repetition of this argument. This concludes the proof of (18.4).

We next claim that for every $0 \leq t \leq n-1$ we have

$$\sum_{T:|T|=t} \sum_{i \in N \setminus T} p_T^i = 1 . \tag{18.5}$$

To prove this, first let $t = n-1$. Then the sum on the left-hand side of (18.5) is equal to $\sum_{i \in N} p_{N \setminus i}^i$, which is equal to 1 by Lemma 18.12. Now as induction hypothesis assume that (18.5) holds for $t+1$. Then

$$\begin{aligned}\sum_{T:\ |T|=t} \ \sum_{i\in N\setminus T} p^i_T &= \sum_{i\in N} \ \sum_{T:\ |T|=t+1,\ i\in T} p^i_{T\setminus i} \\ &= \sum_{T:\ |T|=t+1} \ \sum_{i\in T} p^i_{T\setminus i} \\ &= \sum_{T:\ |T|=t+1} \ \sum_{i\in N\setminus T} p^i_T \\ &= 1\ ,\end{aligned}$$

where the second equality follows by Lemma 18.12 and the last equality by induction. This proves (18.5). In particular, for $t = 0$, we have $\sum_{i\in N} p^i_\emptyset = 1$. Together with (18.4) this implies that $p(\cdot)$ as defined above is a probability distribution on $\Pi(N)$.

In order to complete the proof of the theorem, it is sufficient to show that $\psi = \Phi^p$. For every game $v \in \mathcal{G}^N$ and $i \in N$ we can write

$$\Phi^p_i(v) = \sum_{\pi\in\Pi(N)} p(\pi)[v(P_\pi(i)\cup i) - v(P_\pi(i))]\ ,$$

where $P_\pi(i)$ denotes the set of predecessors of player i under the permutation π (cf. Sect. 17.1). Hence, it is sufficient to prove that for all $i \in N$ and $T \subseteq N\setminus i$ we have

$$p^i_T = \sum_{\pi\in\Pi(N):\ T=P_\pi(i)} p(\pi)\ . \tag{18.6}$$

In order to prove (18.6), first let $|T| = t$. By using a similar argument as for the proof of (18.4), we can write

$$\sum_{\pi:T=P_\pi(i)} p(\pi) = \sum_{i_1\in T}\ \sum_{i_2\in T\setminus i_1} \cdots \sum_{i_t\in T\setminus i_1 i_2\cdots i_{t-1}} p^{i_1}_\emptyset A(i_2;\{i_1\})\cdots A(i_t;T\setminus\{i_t\})\cdot A(i;T)\ .$$

Hence,

$$\begin{aligned}\sum_{\pi:T=P_\pi(i)} p(\pi) &= \frac{p^i_T}{\sum_{j\in N\setminus T} p^j_T} \\ &\quad \cdot \sum_{i_t\in T} \frac{p^{i_t}_{T\setminus i_t}}{\sum_{j\in(N\setminus T)\cup i_t} p^j_{T\setminus i_t}} \cdot \sum_{i_{t-1}\in T\setminus i_t} \frac{p^{i_{t-1}}_{T\setminus i_t i_{t-1}}}{\sum_{j\in(N\setminus T)\cup i_t i_{t-1}} p^j_{T\setminus i_t i_{t-1}}} \\ &\quad \cdot\cdots\cdot \sum_{i_1\in T\setminus i_t\cdots i_2} p^{i_1}_\emptyset\end{aligned}$$

$$= \frac{p_T^i}{\sum_{j\in T} p_{T\setminus j}^j}$$

$$\cdot \sum_{i_t \in T} \frac{p_{T\setminus i_t}^{i_t}}{\sum_{j\in T\setminus i_t} p_{(T\setminus i_t)\setminus j}^j} \cdot \sum_{i_{t-1}\in T\setminus i_t} \frac{p_{T\setminus i_t i_{t-1}}^{i_{t-1}}}{\sum_{j\in T\setminus i_t i_{t-1}} p_{(T\setminus i_t i_{t-1})\setminus j}^j}$$

$$\cdot \cdots \cdot \sum_{i_1 \in T\setminus i_t\cdots i_2} p_{\emptyset}^{i_1}$$

$$= p_T^i \,.$$

Here, the first equality sign follows from rearranging terms and substituting the expressions for $A(\cdot;\cdot)$; the second equality sign follows from Lemma 18.12; the final equality sign follows from reading the preceding expression from right to left, noting that the remaining sum in each enumerator cancels against the preceding denominator. ■

18.4 Weighted Shapley Values

The Shapley value is a random order value that distributes the dividend of each coalition equally among all the members of that coalition (see Theorem 17.7). In this sense, it treats players consistently over coalitions. This is not necessarily the case for every random order value. To be specific, consider Example 18.2. The payoff vector $(2\frac{1}{2}, -\frac{1}{2}, 1)$ is a point of the Weber set, namely the midpoint of the marginal vectors $(4, -1, 0)$ and $(1, 0, 2)$. Thus, it can be obtained uniquely as $\Phi^p(v)$, where the probability distribution p assigns weights $1/2$ to the permutations $(3, 2, 1)$ and $(2, 1, 3)$. In terms of dividends, the two marginal vectors can be written as

$$(\Delta_v(1) + \Delta_v(12) + \Delta_v(13) + \Delta_v(123), \Delta_v(2) + \Delta_v(23), \Delta_v(3))$$

for the permutation $(3, 2, 1)$ and

$$(\Delta_v(1) + \Delta_v(12), \Delta_v(2), \Delta_v(3) + \Delta_v(13) + \Delta_v(23) + \Delta_v(123))$$

for the permutation $(2, 1, 3)$. (Cf. Problem 18.1, where this is generalized.) Thus, $\Delta_v(123)$ is split equally between players 1 and 3, whereas $\Delta_v(23)$ is split equally between players 2 and 3. Hence, whereas player 2 has zero power compared to player 3 in distributing $\Delta_v(123)$, they have equal power in distributing $\Delta_v(23)$. In this respect, players 2 and 3 are not treated consistently by Φ^p.

In order to formalize the idea of consistent treatment, we first define positively weighted Shapley values. Let $\boldsymbol{\omega} \in \mathbb{R}^N$ with $\boldsymbol{\omega} > \mathbf{0}$. The *positively weighted Shapley* *value* $\Phi^{\boldsymbol{\omega}}$ is defined as the unique linear value which assigns to each unanimity game (N, u_S):

$$\Phi_i^{\boldsymbol{\omega}}(u_S) = \begin{cases} \omega_i/\omega(S) & \text{for } i \in S \\ 0 & \text{for } i \in N \setminus S \,. \end{cases} \tag{18.7}$$

We will show that these positively weighted Shapley values are random order values. Define independently distributed random variables X_i $(i \in N)$ on $[0,1]$ by their cumulative distribution functions $[0,1] \ni t \mapsto t^{\omega_i}$ (that is, $X_i \leq t$ with probability t^{ω_i}). Then, define the probability distribution $p^{\boldsymbol{\omega}}$ by

$$p^{\boldsymbol{\omega}}(\pi) = \int_0^1 \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} dt_1^{\omega_{i_1}} \ldots dt_{n-2}^{\omega_{i_{n-2}}} dt_{n-1}^{\omega_{i_{n-1}}} dt_n^{\omega_{i_n}} \tag{18.8}$$

for every permutation $\pi = (i_1, i_2, \ldots, i_n)$. That is, $p^{\boldsymbol{\omega}}(\pi)$ is defined as the probability that i_1 comes before i_2, i_2 before i_3, etc., evaluated according to the independent random variables X_i. It is straightforward to check that, indeed, p^{ω} is a probability distribution over the set of permutations. Then we have:

Theorem 18.13 *For every* $\boldsymbol{\omega} \in \mathbb{R}^N$ *with* $\boldsymbol{\omega} > \mathbf{0}$, $\Phi^{\boldsymbol{\omega}} = \Phi^{p^{\boldsymbol{\omega}}}$.

Proof Let S be a nonempty coalition and $i \in S$. Since $\Phi^{\boldsymbol{\omega}}$ and $\Phi^{p^{\boldsymbol{\omega}}}$ are linear, it is sufficient to prove that $\Phi_i^{p^{\boldsymbol{\omega}}}(u_S) = \omega_i/\omega(S)$. Note that

$$\Phi_i^{p^{\boldsymbol{\omega}}}(u_S) = \sum_{\pi \in \Pi(N)} p^{\omega}(\pi) m_i^{\pi}(u_S) = \sum_{\pi \in \Pi(N):\, S \setminus i \subseteq P_\pi(i)} p^{\boldsymbol{\omega}}(\pi) \,,$$

and the right-hand side of this identity is equal to

$$\int_0^1 \int_0^{t_i} dt^{\omega(S\setminus i)} dt_i^{\omega_i} \,.$$

Hence,

$$\Phi_i^{p^{\boldsymbol{\omega}}}(u_S) = \int_0^1 \int_0^{t_i} dt^{\omega(S\setminus i)} dt_i^{\omega_i} = \int_0^1 t_i^{\omega(S\setminus i)} dt_i^{\omega_i} = \int_0^1 \omega_i t^{\omega(S)-1} dt = \omega_i/\omega(S)$$

which concludes the proof of the theorem. ■

Next, we extend the concept of weighted Shapley value to include zero weights. Consider for instance, the three-person random order value that puts weight $1/2$ on the permutations $(1,2,3)$ and $(1,3,2)$. Then each of the marginal vectors

$$(\Delta_v(1), \Delta_v(2) + \Delta_v(12), \Delta_v(3) + \Delta_v(13) + \Delta_v(23) + \Delta_v(123))$$

and

$$(\Delta_v(1), \Delta_v(2) + \Delta_v(12) + \Delta_v(23) + \Delta_v(123), \Delta_v(3) + \Delta_v(13))$$

gets weight $1/2$. (Cf. again Problem 18.1.) The dividend $\Delta_v(12)$ goes to player 2, the dividend $\Delta_v(13)$ to player 3, and the dividends $\Delta_v(23)$ and $\Delta_v(123)$ are split equally between players 2 and 3. Thus, this random order value treats players consistently but we cannot just formalize this by giving player 1 weight 0 since player 1 does obtain $\Delta_v(1)$.

To accommodate this kind of random order values we introduce the concept of a weight system. A *weight system* w is an ordered partition $(S_1, \ldots, S_k)$ of N together with a vector $\boldsymbol{\omega} \in \mathbb{R}^N$ such that $\boldsymbol{\omega} > \mathbf{0}$. The *weighted Shapley value* Φ^w is defined as the unique linear value which assigns to each unanimity game $u_S \in \mathcal{G}^N$, $S \neq \emptyset$:

$$\Phi_i^w(u_S) = \begin{cases} \omega_i/\omega(S \cap S_m) & \text{for } i \in S \cap S_m \text{ and } m = \max\{h : S_h \cap S \neq \emptyset\} \\ 0 & \text{otherwise.} \end{cases} \tag{18.9}$$

Hence, S_h is more powerful as h is larger; for each coalition S we consider the subset of the most powerful players $S \cap S_m$, where m is the largest index h such that the intersection of S_h with S is nonempty, and they distribute the dividend of coalition S according to their (relative) weights $\omega_i/\omega(S \cap S_m)$. Clearly, for $k = 1$ we obtain a positively weighted Shapley value as defined above.

Weighted Shapley values are again random order values. For a weight system w with ordered partition $(S_1, \ldots, S_k)$ we only assign positive probability to those permutations in which all players of S_1 enter before all players of S_2, all players of S_2 enter before all players of S_3, etc. Given such a permutation we can assign probability $p_1(\pi)$ to the order induced by π on S_1 in the same way as we did above in Eq. (18.8); similarly, we assign probabilities $p_2(\pi), \ldots, p_k(\pi)$ to the orders induced on $S_2, \ldots, S_k$, respectively. Then we define

$$p^w(\pi) = \prod_{h=1}^{k} p_h(\pi)\ .$$

It can be shown again that $\Phi^w = \Phi^{p^w}$.

We conclude with an axiomatic characterization of weighted Shapley values. To this end we consider the following axiom for a value $\psi : \mathcal{G}^N \to \mathbb{R}^N$.

Partnership (PA): $\psi_i(\psi(u_T)(S)u_S) = \psi_i(u_T)$ for all $S \subseteq T \subseteq N$ and all $i \in S$.

Theorem 18.14 *Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a value. Then ψ satisfies LIN, DUM, EFF, MON, and PA, if and only if it is a weighted Shapley value.*

Proof Problem 18.8. ■

By Theorem 18.3 we know that the core of any game is included in the Weber set and, thus, in any game any core element corresponds to at least one random order value. The following theorem, included here without a proof, states that, in fact, the core is always covered by the set of weighted Shapley values.

Theorem 18.15 *Let $v \in \mathcal{G}^N$ and $\mathbf{x} \in C(v)$. Then there is a weight system w such that $\mathbf{x} = \Phi^w(v)$.*

18.5 Problems

18.1. *Marginal Vectors and Dividends*
Let $(N, v) \in \mathcal{G}^N$.

(a) Show that

$$v(S) = \sum_{T \subseteq S} \Delta_v(T) \tag{18.10}$$

where $\Delta_v(T)$ are the dividends defined in Sect. 17.1.
(b) Express each marginal vector m^π in terms of dividends.

18.2. *Convexity and Marginal Vectors*
Prove that a game (N, v) is convex if and only if for all $T \in 2^N \setminus \{\emptyset\}$:

$$v(T) = \min_{\pi \in \Pi(N)} \sum_{i \in T} m_i^\pi(v) .$$

18.3. *Strictly Convex Games*
Call a game (N, v) *strictly convex* if all inequalities in (18.3) hold strictly. Show that in a strictly convex game all marginal vectors are different.

18.4. *Sharing Profits*
Consider the following situation with $n+1$ players. Player 0 (the landlord) owns the land and players $1, 2, \ldots, n$ are n identical workers who own their labor only. The production $f : \{0, 1, \ldots, n\} \to \mathbb{R}$ describes how much is produced by the workers. Assume that f is nondecreasing and that $f(0) = 0$. We associate with this situation a TU-game that reflects the production possibilities of coalitions. Without agent 0 a coalition has zero worth, otherwise the worth depends on the number of workers.

More precisely,

$$v(S) := \begin{cases} 0 & \text{if } 0 \notin S \\ f(|S|-1) & \text{if } 0 \in S \end{cases}$$

for every coalition $S \subseteq \{0, 1, \ldots, n\}$.

(a) Compute the marginal vectors and the Shapley value of this game.
(b) Compute the core of this game.
(c) Give a necessary and sufficient condition on f such that the game is convex. [So in that case, the core and the Weber set coincide and the Shapley value is in the core.]

18.5. *Sharing Costs*
Suppose that n airlines share the cost of a runway. To serve the planes of company i, the length of the runway must be c_i, which is also the cost of a runway of that length. Assume $0 \leq c_1 \leq c_2 \leq \ldots \leq c_n$. The cost of coalition S is defined as $c_S = \max_{i \in S} c_i$ for every nonempty coalition S.

(a) Model this situation as a cost savings game (cf. the three communities game in Chap. 1).
(b) Show that the resulting game is convex, and compute the marginal vectors, the Shapley value, and the core.

18.6. *Independence of the Axioms in Theorem 18.8*
Show that the axioms in Theorem 18.8 are independent.

18.7. *Null-Player in Theorem 18.8*
Show that Theorem 18.8 still holds if DUM is replaced by NP.

18.8. *Characterization of Weighted Shapley Values*
Prove Theorem 18.14. Also show that the axioms are independent.

18.9. *Core and Weighted Shapley Values in Example 18.2*
In Example 18.2, determine for each $\mathbf{x} \in C(v)$ a weight system w such that $\mathbf{x} = \Phi^w(v)$.

18.6 Notes

The Weber set was introduced by Weber (1988), who also gave a proof of Theorem 18.3 (the core is always a subset of the Weber set); our proof is due to Derks (1992). Theorem 18.6 (coincidence of the core and Weber set of convex games) is from Shapley (1971) and Ichiishi (1981). The proof of Theorem 18.13

(positively weighted Shapley values are random order values) is from Owen (1972). Theorem 18.14 (axiomatic characterization of weighted Shapley values) is from Kalai and Samet (1987), see also Derks et al. (2000). Theorem 18.15 is due to Monderer et al. (1992).

Problem 18.5 is based on Littlechild and Owen (1974).

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