Evolutionary Games

15

In this chapter we go deeper into evolutionary game theory. The concepts of evolutionary stable strategy and replicator dynamics, introduced in Chap. 8, are further explored. It may be helpful to study Chap. 8 first, although the present chapter is largely self-contained.

In Sect. 15.1 we briefly review symmetric two-player games. Section 15.2 discusses evolutionary stable strategies and Sect. 15.3 replicator dynamics.

15.1 Symmetric Two-Player Games

Much of evolutionary game theory is concerned with symmetric two-player games. A (finite) symmetric two-player game is a pair of $m \times m$ payoff matrices (A, B) such that $B = A^T$, i.e., $B = (b_{ij})_{i,j=1}^m$ is the transpose of $A = (a_{ij})_{i,j=1}^m$. In other words, for all $i, j \in \{1, 2, \ldots, m\}$, we have $b_{ij} = a_{ji}$.

In such a game we are particularly interested in symmetric (pure and mixed strategy) Nash equilibria. A Nash equilibrium (σ_1, σ_2) is *symmetric* if $\sigma_1 = \sigma_2$. We denote by $NE(A, A^T)$ the set of all Nash equilibria of (A, A^T) and by

$$NE(A) = \{\mathbf{x} \in \Delta^m \mid (\mathbf{x}, \mathbf{x}) \in NE(A, A^T)\}$$

the set of all strategies that occur in a symmetric Nash equilibrium. By a standard application of the Kakutani Fixed Point Theorem we prove that this set is nonempty.

Proposition 15.1 *For any $m \times m$-matrix A, $NE(A) \neq \emptyset$.*

Proof For each $\mathbf{x} \in \Delta^m$, viewed as a strategy of player 2 in (A, A^T), let $\beta_1(\mathbf{x})$ be the set of best replies of player 1 in (A, A^T). Then the correspondence $\mathbf{x} \mapsto \beta_1(\mathbf{x})$ is upper semi-continuous and convex-valued (check this), so that by the Kakutani Fixed Point Theorem 22.11 there is an $\mathbf{x}^* \in \Delta^m$ with $\mathbf{x}^* \in \beta_1(\mathbf{x}^*)$. Since player

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2's payoff matrix is the transpose of A, it also follows that $\mathbf{x}^* \in \beta_2(\mathbf{x}^*)$. Hence, $(\mathbf{x}^*, \mathbf{x}^*) \in NE(A, A^T)$, so $\mathbf{x}^* \in NE(A)$. ■

15.1.1 Symmetric 2 × 2-Games

For later reference it is convenient to have a classification of symmetric 2×2-games with respect to their symmetric Nash equilibria. Such a game is described by the payoff matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

For the purpose of Nash equilibrium analysis, we may consider without loss of generality the matrix

$$A' = \begin{pmatrix} a_{11} - a_{21} & a_{12} - a_{12} \\ a_{21} - a_{21} & a_{22} - a_{12} \end{pmatrix} = \begin{pmatrix} a_{11} - a_{21} & 0 \\ 0 & a_{22} - a_{12} \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},$$

where $a_1 := a_{11} - a_{21}$ and $a_2 := a_{22} - a_{12}$. Indeed, it is straightforward to verify that $(\mathbf{p}, \mathbf{q}) \in \Delta^2 \times \Delta^2$ is a Nash equilibrium of (A, A^T) if and only if it is a Nash equilibrium of (A', A'^T).

For a generic matrix A, implying $a_1, a_2 \neq 0$, there are essentially three different cases:

(1) $a_1 < 0, a_2 > 0$. In this case, $NE(A') = \{\mathbf{e}^2\}$, i.e., each player playing the second strategy is the unique symmetric Nash equilibrium.
(2) $a_1, a_2 > 0$. In this case, $NE(A') = \{\mathbf{e}^1, \mathbf{e}^2, \hat{\mathbf{x}}\}$, where $\hat{\mathbf{x}} = (a_2/(a_1+a_2), a_1/(a_1+a_2))$.
(3) $a_1, a_2 < 0$. In this case, $NE(A') = \{\hat{\mathbf{x}}\}$ with $\hat{\mathbf{x}}$ as in (2).

15.2 Evolutionary Stability

15.2.1 Evolutionary Stable Strategies

In evolutionary game theory the interpretation of a symmetric two-person game is that players in a possibly large population randomly meet in pairs. Let such a game be described by A, then a mixed strategy $\mathbf{x} \in \Delta^m$ is interpreted as a vector of population shares: for each k, x_k is the share of the population that 'plays' pure strategy k. Such a strategy is called *evolutionary stable* if it performs better against a small 'mutation' than that mutation performs against itself. Formally, we have the following definition.

Definition 15.2 A strategy $\mathbf{x} \in \Delta^m$ is an *evolutionary stable strategy* (*ESS*) in A if for every strategy $\mathbf{y} \in \Delta^m$, $\mathbf{y} \neq \mathbf{x}$, there exists some $0 < \varepsilon_{\mathbf{y}} < 1$ such that for all $0 < \varepsilon < \varepsilon_{\mathbf{y}}$ we have

$$\mathbf{x}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) > \mathbf{y}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}). \tag{15.1}$$

The set of all ESS is denoted by $ESS(A)$. □

Again, the interpretation of an ESS $\mathbf{x}$ is as follows. Consider any small *mutation* $\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}$ of $\mathbf{x}$. Condition (15.1) then says that against such a small mutation, the original strategy $\mathbf{x}$ is better than the *mutant* strategy $\mathbf{y}$. In other words, if the population $\mathbf{x}$ is *invaded* by a small part of the *mutant* population $\mathbf{y}$, then $\mathbf{x}$ survives since it fares better against this small mutation than the mutant $\mathbf{y}$ itself does.

Evolutionary stable strategies can be characterized as follows. In fact, this characterization was used as the definition of an evolutionary stable strategy in Chap. 8 (Definition 8.4).

Theorem 15.3 *Let A be a symmetric $m \times m$ game. Then*

$$ESS(A) = \{\mathbf{x} \in NE(A) \mid \forall \mathbf{y} \in \Delta^m, \mathbf{y} \neq \mathbf{x} \, [\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x} \Rightarrow \mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}]\} .$$

This theorem follows from Propositions 15.4 and 15.5.

Proposition 15.4 *Let A be an $m \times m$-matrix and let $\mathbf{x} \in ESS(A)$. Then $\mathbf{x} \in NE(A)$.*

Proof Let $\mathbf{y} \in \Delta^m$, then it is sufficient to show $\mathbf{x}A\mathbf{x} \geq \mathbf{y}A\mathbf{x}$. Let $\varepsilon_{\mathbf{y}}$ be as in Definition 15.2, then

$$\mathbf{x}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) > \mathbf{y}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x})$$

for all $0 < \varepsilon < \varepsilon_{\mathbf{y}}$ by (15.1). By letting ε go to zero, this implies $\mathbf{x}A\mathbf{x} \geq \mathbf{y}A\mathbf{x}$. ■

Proposition 15.5 *Let A be an $m \times m$-matrix. If $\mathbf{x} \in ESS(A)$, then, for all $\mathbf{y} \in \Delta^m$ with $\mathbf{y} \neq \mathbf{x}$ we have:*

$$\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x} \Rightarrow \mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y} . \tag{15.2}$$

Conversely, if $\mathbf{x} \in NE(A)$ and (15.2) holds, then $\mathbf{x} \in ESS(A)$.

Proof Let $\mathbf{x} \in ESS(A)$. Let $\mathbf{y} \in \Delta^m$ with $\mathbf{y} \neq \mathbf{x}$ and $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$. Suppose that $\mathbf{y}A\mathbf{y} \geq \mathbf{x}A\mathbf{y}$. Then, for any $\varepsilon \in [0,1]$, $\mathbf{y}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) \geq \mathbf{x}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x})$, contradicting (15.1).

Conversely, let $\mathbf{x} \in NE(A)$ and let (15.2) hold for $\mathbf{x}$. If $\mathbf{x}A\mathbf{x} > \mathbf{y}A\mathbf{x}$, then also $\mathbf{x}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) > \mathbf{y}A(\varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x})$ for small enough ε. If $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$, then $\mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}$, hence (15.1) holds for any $\varepsilon \in (0,1]$. ■

Clearly, Theorem 15.3 follows from the preceding propositions.

From Problems 15.1–15.3 it follows that the concept of ESS does not 'solve' the prisoners' dilemma nor the 'coordination problem'. Also, an ESS may be completely mixed (Hawk-Dove), or fail to exist (Rock-Paper-Scissors).

15.2.2 The Structure of the Set $ESS(A)$

Let $\mathbf{x}$ be an ESS and let $\mathbf{y} \in \Delta^m$, $\mathbf{y} \neq \mathbf{x}$ such that the carrier[1] of $\mathbf{y}$ is contained in the carrier of $\mathbf{x}$, i.e., $C(\mathbf{y}) \subseteq C(\mathbf{x})$. Since $\mathbf{x} \in NE(A)$ by Theorem 15.3, this implies $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$ and hence, again by Theorem 15.3, $\mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}$. We have established:

Proposition 15.6 *If* $\mathbf{x} \in ESS(A)$ *and* $\mathbf{y} \in \Delta^m$ *with* $\mathbf{y} \neq \mathbf{x}$ *and* $C(\mathbf{y}) \subseteq C(\mathbf{x})$*, then* $\mathbf{y} \notin NE(A)$.

This implies the following corollary (check!):

Corollary 15.7 *The set* $ESS(A)$ *is finite. If* $\mathbf{x} \in ESS(A)$ *is completely mixed, then* $ESS(A) = \{\mathbf{x}\}$.

15.2.3 Relations with Other Refinements

If $\mathbf{x} \in NE(A)$ is weakly dominated by $\mathbf{y} \in \Delta^m$, then $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$ and $\mathbf{y}A\mathbf{y} \geq \mathbf{x}A\mathbf{y}$; so by Theorem 15.3, $\mathbf{x} \notin ESS(A)$. Therefore, if $\mathbf{x} \in ESS(A)$, then $(\mathbf{x}, \mathbf{x})$ is an undominated equilibrium and hence perfect by Theorem 13.23. It can even be shown that $(\mathbf{x}, \mathbf{x})$ is proper. The next proposition summarizes these facts.

Proposition 15.8 *If* $\mathbf{x} \in ESS(A)$*, then* $(\mathbf{x}, \mathbf{x}) \in NE(A, A^T)$ *is undominated, perfect and proper.*

The unique (symmetric) equilibrium in the Rock-Paper-Scissors game in Problem 15.3 is proper (why?), but the associated equilibrium strategy is not ESS, so the converse of Proposition 15.8 does not hold.

15.2.4 Other Characterizations of ESS

15.2.4.1 Uniform Invasion Barriers

The number $\varepsilon_{\mathbf{y}}$ in the definition of an ESS can be interpreted as an 'invasion barrier': if the share of the mutant strategy $\mathbf{y}$ is smaller than $\varepsilon_{\mathbf{y}}$, then the 'incumbent' strategy $\mathbf{x}$ fares better against the mutated population than the mutant $\mathbf{y}$ itself does, so that

[1]Recall—see Chap. 13—that the carrier of $\mathbf{y}$, $C(\mathbf{y})$, is the set $\{i \in \{1, \ldots, m\} \mid y_i > 0\}$.

the mutant strategy becomes extinct. In a large but finite population, it would not make sense if this invasion barrier could become arbitrarily small since then the 'mutant' population would sometimes have to consist of less than one individual to guarantee survival of the strategy $\mathbf{x}$ under consideration. This gives rise to the following definition.

Definition 15.9 A strategy $\mathbf{x} \in \Delta^m$ has a *uniform invasion barrier* if there exists an $\bar{\varepsilon} \in (0, 1)$ such that (15.1) holds for all strategies $\mathbf{y} \neq \mathbf{x}$ and every $\varepsilon \in (0, \bar{\varepsilon})$. □

It turns out that possessing a uniform invasion barrier characterizes an evolutionary stable strategy.

Proposition 15.10 *For each* $\mathbf{x} \in \Delta^m$*,* $\mathbf{x} \in ESS(A)$ *if and only if* $\mathbf{x}$ *has a uniform invasion barrier.*

Proof Let $\mathbf{x} \in \Delta^m$. If $\mathbf{x}$ has a uniform invasion barrier $\bar{\varepsilon}$, then clearly $\mathbf{x}$ is an ESS by choosing, in (15.1), $\varepsilon_{\mathbf{y}} = \bar{\varepsilon}$ for each $\mathbf{y} \in \Delta^m$.

Conversely, let $\mathbf{x}$ be an ESS. Define the function $b : \Delta^m \setminus \{\mathbf{x}\} \to [0, 1]$ by

$$b(\mathbf{y}) = \sup\{\delta \in [0, 1] \mid \forall \varepsilon \in (0, \delta)\ [(\mathbf{x} - \mathbf{y})A(\varepsilon \mathbf{y} + (1 - \varepsilon)\mathbf{x}) > 0]\}$$

for all $\mathbf{y} \in \Delta^m \setminus \{\mathbf{x}\}$. We first consider the function b on the compact set $Z = \{\mathbf{z} \in \Delta^m \mid z_i = 0 \text{ for some } i \in C(\mathbf{x})\}$. Consider $\mathbf{y} \in Z$. Since $\mathbf{x}$ is an ESS, we have that $(\mathbf{x} - \mathbf{y})A(\varepsilon \mathbf{y} + (1 - \varepsilon)\mathbf{x})$ is positive for small positive values of ε. Since this expression depends linearly on ε, this implies that there can be at most one value of ε, which we denote by $\varepsilon'_{\mathbf{y}}$, such that $(\mathbf{x} - \mathbf{y})A(\varepsilon'_{\mathbf{y}}\mathbf{y} + (1 - \varepsilon'_{\mathbf{y}})\mathbf{x}) = 0$. If $\varepsilon'_{\mathbf{y}} \in (0, 1)$, then $(\mathbf{x} - \mathbf{y})A(\varepsilon'_{\mathbf{y}}\mathbf{y} + (1 - \varepsilon'_{\mathbf{y}})\mathbf{x}) = 0$ implies that $(\mathbf{x} - \mathbf{y})A(\mathbf{x} - \mathbf{y}) \neq 0$. To see this, suppose that $(\mathbf{x} - \mathbf{y})A(\mathbf{x} - \mathbf{y}) = 0$. Then

$$0 = (\mathbf{x} - \mathbf{y})A(\varepsilon'_{\mathbf{y}}\mathbf{y} + (1 - \varepsilon'_{\mathbf{y}})\mathbf{x}) = (\mathbf{x} - \mathbf{y})A\mathbf{x}\ ,$$

hence also $(\mathbf{x} - \mathbf{y})A\mathbf{y} = 0$ and, thus, $(\mathbf{x} - \mathbf{y})A(\varepsilon \mathbf{y} + (1 - \varepsilon)\mathbf{x}) = 0$ for all ε, a contradiction. Hence, in that case, $b(\mathbf{y}) = \varepsilon'_{\mathbf{y}} = (\mathbf{x} - \mathbf{y})A\mathbf{x}/(\mathbf{x} - \mathbf{y})A(\mathbf{x} - \mathbf{y})$; otherwise, $b(\mathbf{y}) = 1$. Clearly, b is a continuous function. Since b is positive and Z is compact, $\min_{\mathbf{y} \in Z} b(\mathbf{y}) > 0$. Hence, $\mathbf{x}$ has a uniform invasion barrier, namely this minimum value, on the set Z.

Now suppose that $\mathbf{y} \in \Delta^m$, $\mathbf{y} \neq \mathbf{x}$. We claim that there is a $\lambda \in (0, 1]$ such that $\mathbf{y} = \lambda \mathbf{z} + (1 - \lambda)\mathbf{x}$ for some $\mathbf{z} \in Z$. To see this, first note that we can take $\lambda = 1$ and $\mathbf{z} = \mathbf{y}$ if $\mathbf{y} \in Z$. If $\mathbf{y} \notin Z$ then consider, for each $\mu \geq 0$, the point $\mathbf{z}(\mu) = (1 - \mu)\mathbf{x} + \mu \mathbf{y}$, and let $\hat{\mu} \geq 1$ be the largest value of μ such that $\mathbf{z}(\mu) \in \Delta^m$. Then there is a coordinate $i \in \{1, \ldots, m\}$ with $z_i(\hat{\mu}) = 0$, $z_i(\mu) > 0$ for all $\mu < \hat{\mu}$, and $z_i(\mu) < 0$ for all $\mu > \hat{\mu}$. Clearly, this implies $x_i > y_i$, hence $i \in C(\mathbf{x})$, and thus $\mathbf{z}(\hat{\mu}) \in Z$. Then, for $\mathbf{z} = \mathbf{z}(\hat{\mu})$ and $\lambda = 1/\hat{\mu}$, we have $\mathbf{y} = \lambda \mathbf{z} + (1 - \lambda)\mathbf{x}$.

By straightforward computation we have

$$(\mathbf{x}-\mathbf{y})A(\varepsilon\mathbf{y}+(1-\varepsilon)\mathbf{x}) = \lambda(\mathbf{x}-\mathbf{z})A(\varepsilon\lambda\mathbf{z}+(1-\varepsilon\lambda)\mathbf{x})$$

for each $\varepsilon > 0$, so that $b(\mathbf{y}) = \min\{b(\mathbf{z})/\lambda, 1\} \geq b(\mathbf{z})$.

We conclude that $\bar{\varepsilon} = \min_{\mathbf{y}\in Z} b(\mathbf{y})$ is a uniform invasion barrier for $\mathbf{x}$. ■

15.2.4.2 Local Superiority

By Theorem 15.3, a completely mixed ESS earns a higher payoff against any mutant than such a mutant earns against itself. This global superiority property can be generalized to the following local version.

Definition 15.11 The strategy $\mathbf{x} \in \Delta^m$ is *locally superior* if it has an open neighborhood U such that $\mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}$ for all $\mathbf{y} \in U \setminus \{\mathbf{x}\}$. □

The local superiority condition provides another characterization of ESS.

Proposition 15.12 *For each $\mathbf{x} \in \Delta^m$, $\mathbf{x} \in ESS(A)$ if and only if $\mathbf{x}$ is locally superior.*

Proof Let $\mathbf{x} \in \Delta^m$.

First suppose that $\mathbf{x}$ is locally superior, and let U be as in Definition 15.11. Let $\mathbf{z} \in \Delta^m \setminus \{\mathbf{x}\}$ and define for each $0 < \varepsilon < 1$ the point $\mathbf{w}(\varepsilon)$ by $\mathbf{w}(\varepsilon) = \varepsilon\mathbf{z} + (1-\varepsilon)\mathbf{x}$. Then there is $\varepsilon_{\mathbf{z}} > 0$ such that $\mathbf{w}(\varepsilon) \in U$, hence $\mathbf{x}A\mathbf{w}(\varepsilon) > \mathbf{w}(\varepsilon)A\mathbf{w}(\varepsilon)$, for all $\varepsilon \in (0, \varepsilon_{\mathbf{z}})$. By slight rewriting, this implies $\mathbf{x}A\mathbf{w}(\varepsilon) > \mathbf{z}A\mathbf{w}(\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_{\mathbf{z}})$. In particular, we have

$$\varepsilon\mathbf{x}A\mathbf{z} + (1-\varepsilon)\mathbf{x}A\mathbf{x} > \varepsilon\mathbf{z}A\mathbf{z} + (1-\varepsilon)\mathbf{z}A\mathbf{x}$$

for all $\varepsilon \in (0, \varepsilon_{\mathbf{z}})$, hence $\mathbf{x}A\mathbf{x} \geq \mathbf{z}A\mathbf{x}$. So $\mathbf{x} \in NE(A)$. Suppose now that $\mathbf{z}A\mathbf{x} = \mathbf{x}A\mathbf{x}$. Then, for $\varepsilon \in (0, \varepsilon_{\mathbf{z}})$,

$$\begin{aligned}
\varepsilon\mathbf{x}A\mathbf{z} &= \mathbf{x}A\mathbf{w}(\varepsilon) - (1-\varepsilon)\mathbf{x}A\mathbf{x} \\
&> \mathbf{z}A\mathbf{w}(\varepsilon) - (1-\varepsilon)\mathbf{x}A\mathbf{x} \\
&= \varepsilon\mathbf{z}A\mathbf{z} + (1-\varepsilon)\mathbf{z}A\mathbf{x} - (1-\varepsilon)\mathbf{x}A\mathbf{x} \\
&= \varepsilon\mathbf{z}A\mathbf{z}\,,
\end{aligned}$$

so that $\mathbf{x}$ is an ESS.

Conversely, let $\mathbf{x}$ be an ESS with uniform invasion barrier (cf. Proposition 15.10) $\bar{\varepsilon} \in (0, 1)$, and let Z be as in the proof of Proposition 15.10. Let

$$V = \{\mathbf{y} \in \Delta^m \mid \mathbf{y} = \varepsilon\mathbf{z} + (1-\varepsilon)\mathbf{x} \text{ for some } \mathbf{z} \in Z \text{ and } \varepsilon \in [0, \bar{\varepsilon})\}.$$

Since Z is closed and $\mathbf{x} \notin Z$, there is an open neighborhood U of $\mathbf{x}$ such that $U \cap \Delta^m \subseteq V$. Suppose that $\mathbf{y} \neq \mathbf{x}$, $\mathbf{y} \in U \cap \Delta^m$. Then $\mathbf{y} \in V$, and by Proposition 15.10, $\mathbf{z}A\mathbf{y} = \mathbf{z}A(\varepsilon\mathbf{z} + (1-\varepsilon)\mathbf{x}) < \mathbf{x}A(\varepsilon\mathbf{z} + (1-\varepsilon)\mathbf{x}) = \mathbf{x}A\mathbf{y}$, with $\mathbf{z}$ as in the definition of V. This implies $\mathbf{y}A\mathbf{y} = \varepsilon\mathbf{z}A\mathbf{y} + (1-\varepsilon)\mathbf{x}A\mathbf{y} < \mathbf{x}A\mathbf{y}$. ■

15.2.4.3 Local Strict Efficiency

Consider the special case of a symmetric game (A, B) with $A^T = A$, hence A is itself symmetric and $B = A$. Call such a game *doubly symmetric*.

Definition 15.13 A strategy $\mathbf{x} \in \Delta^m$ is *locally strictly efficient* if it has an open neighborhood U such that $\mathbf{x}A\mathbf{x} > \mathbf{y}A\mathbf{y}$ for all $\mathbf{y} \in U \setminus \{\mathbf{x}\}$. □

For doubly symmetric games, local strict efficiency characterizes ESS.

Proposition 15.14 *Let $A = A^T$. Then $\mathbf{x} \in ESS(A)$ if and only if $\mathbf{x}$ is locally strictly efficient.*

Proof Let $\mathbf{x} \in \Delta^m$. For any $\mathbf{y} \neq \mathbf{x}$ and $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$, we have

$$\mathbf{y}A\mathbf{y} = \mathbf{x}A\mathbf{x} - 2\mathbf{x}A\mathbf{z} - 2\mathbf{z}A\mathbf{x} + 4\mathbf{z}A\mathbf{z} \; .$$

Hence, using the symmetry of A,

$$\mathbf{x}A\mathbf{x} - \mathbf{y}A\mathbf{y} = 4\left[\mathbf{x}A\mathbf{z} - \mathbf{z}A\mathbf{z}\right] \; .$$

If $\mathbf{x}$ is locally strictly efficient, then this identity implies that $\mathbf{x}$ is locally superior, and conversely. By Proposition 15.12, it follows that $\mathbf{x}$ is an ESS if and only if $\mathbf{x}$ is locally strictly efficient. ■

15.3 Replicator Dynamics and ESS

The concept of an evolutionary stable strategy is based on the idea of *mutation*. Incorporation of the evolutionary concept of *selection* calls for a more explicitly dynamic approach.

15.3.1 Replicator Dynamics

As before, consider a symmetric game described by the $m \times m$ matrix A. A mixed strategy $\mathbf{x} \in \Delta^m$ can be interpreted as a vector of population shares (a *state*) over the pure strategies, evolving over time. To express time dependence, we write $\mathbf{x} = \mathbf{x}(t)$. For each pure strategy i, the expected payoff of playing i when the population is in state $\mathbf{x}$ is equal to $\mathbf{e}^i A\mathbf{x}$, hence the average population payoff is equal to $\sum_{i=1}^m x_i \mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}$. In the *replicator dynamics* it is assumed that population shares develop according to the differential equation

$$\dot{x}_i = dx_i(t)/dt = \left[\mathbf{e}^i A\mathbf{x} - \mathbf{x}A\mathbf{x}\right] x_i \tag{15.3}$$

for each pure strategy $i = 1, 2, \ldots, m$, where dependence on t is (partly) suppressed from the notation. In other words, the share of the population playing strategy i changes with rate proportional to the difference between the expected payoff of i (individual fitness) and the average population payoff (average fitness).

To study the replicator dynamics in (15.3) one needs to apply the theory of differential equations and dynamical systems. For a first analysis we can restrict attention to a few basic concepts and facts.

For each *initial state* $\mathbf{x}(0) = \mathbf{x}^0 \in \Delta^m$, the system (15.3) induces a *solution* or *trajectory* $\xi(t, \mathbf{x}^0)$ in Δ^m. Call state $\mathbf{x}$ a *stationary point* of the dynamics (15.3) if $\dot{\mathbf{x}} = (\dot{x}_1, \ldots, \dot{x}_m) = (0, \ldots, 0)$. If $m = 2$ then $\dot{x}_1 = 0$ or $\dot{x}_2 = 0$ is sufficient for $\mathbf{x}$ to be a stationary point, since (15.3) implies the natural condition $\sum_{i=1}^m \dot{x}_i = 0$. Note that any $\mathbf{e}^i$ is a stationary point—this is a more or less artificial property of the replicator dynamics. A state $\mathbf{x}$ is *Lyapunov stable* if every open neighborhood B of $\mathbf{x}$ contains an open neighborhood B^0 of $\mathbf{x}$ such that $\xi(t, \mathbf{x}^0) \in B$ for all $\mathbf{x}^0 \in B^0$ and $t \geq 0$. A state $\mathbf{x}$ *asymptotically stable* if it is Lyapunov stable and it has an open neighborhood B^* such that $\lim_{t\to\infty} \xi(t, \mathbf{x}^0) = \mathbf{x}$ for all $\mathbf{x}^0 \in B^*$. It is easy to see that Lyapunov stability implies stationarity.

Before studying the replicator dynamics in more detail, we state the following useful fact without proof—see the Notes to this chapter for a reference. By Δ_0^m we denote the (relative) interior of the set Δ^m, i.e., $\Delta_0^m = \{\mathbf{x} \in \Delta^m \mid \mathbf{x} > \mathbf{0}\}$ is the set of completely mixed strategies or states.

Proposition 15.15 *The replicator dynamics (15.3) has a unique solution $\xi(t, \mathbf{x}^0)$, $t \in \mathbb{R}$, through any initial state $\mathbf{x}^0 \in \Delta^m$. The solution mapping $\xi : \mathbb{R} \times \Delta^m \to \Delta^m$ is continuous, and continuously differentiable with respect to time. Both Δ_0^m and $\Delta^m \setminus \Delta_0^m$ are invariant, that is, $\xi(t, \mathbf{x}^0) \in \Delta_0^m$ for all t whenever $\mathbf{x}^0 \in \Delta_0^m$ and $\xi(t, \mathbf{x}^0) \in \Delta^m \setminus \Delta_0^m$ for all t whenever $\mathbf{x}^0 \in \Delta^m \setminus \Delta_0^m$.*

15.3.2 Symmetric 2 × 2 Games

In order to analyze the replicator dynamics for symmetric 2×2 games corresponding to A, we can without loss of generality restrict attention again to the normalized game

$$A' = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} .$$

Now (15.3) reduces to

$$\dot{x}_1 = [a_1 x_1 - a_2 x_2] x_1 x_2 \tag{15.4}$$

(and $\dot{x}_2 = -\dot{x}_1$). For case (1) in Sect. 15.1.1, $a_1 < 0$ and $a_2 > 0$, the stationary points of the dynamics are $\mathbf{x} = \mathbf{e}^1$ and $\mathbf{x} = \mathbf{e}^2$. For all other $\mathbf{x}$, $\dot{x}_1 < 0$, which implies that the system then converges to $\mathbf{e}^2$, the unique ESS. Hence, the (unique) ESS is also the (unique) asymptotically stable state.

From the answers to Problems 15.5 and 15.6 we have:

Proposition 15.16 *Let A be a generic 2×2 matrix and let $\mathbf{x} \in \Delta^2$. Then $\mathbf{x} \in ESS(A)$ if and only if $\mathbf{x}$ is an asymptotically stable state of the replicator dynamics.*

Note that this proposition implies Proposition 8.5(b). Part (a) of Proposition 8.5 follows from Problem 15.2.

15.3.3 Dominated Strategies

Does the replicator dynamics discard of dominated strategies? One answer to this question is provided by the following proposition, which states that if we start from a completely mixed strategy eventually all strictly dominated pure strategies vanish, i.e., their population shares converge to zero.

Proposition 15.17 *Let $\mathbf{x}^0 \in \Delta^m$ be completely mixed and let pure strategy i be strictly dominated. Then* $\lim_{t\to\infty} \xi_i(t, \mathbf{x}^0) = 0$.

Proof Let i be strictly dominated by $\mathbf{y} \in \Delta^m$ and let

$$\varepsilon = \min_{\mathbf{x}\in\Delta^m} \mathbf{y}A\mathbf{x} - \mathbf{e}^i A\mathbf{x} .$$

By continuity of the expected payoff function and compactness of Δ^m, $\varepsilon > 0$. Define $v_i : \Delta_0^m \to \mathbb{R}$ by $v_i(\mathbf{x}) = \ln x_i - \sum_{j=1}^m y_j \ln(x_j)$. The function v_i is differentiable, with time derivative at any point $\mathbf{x} = \xi(t, \mathbf{x}^0)$ equal to

$$\begin{aligned}
\dot{v}_i(\mathbf{x}) &= \left[\frac{dv_i(\xi(t,\mathbf{x}^0))}{dt}\right]_{\xi(t,\mathbf{x}^0)=\mathbf{x}} \\
&= \sum_{j=1}^m \frac{\partial v_i(\mathbf{x})}{\partial x_j}\dot{x}_j \\
&= \frac{\dot{x}_i}{x_i} - \sum_{j=1}^m \frac{y_j \dot{x}_j}{x_j} \\
&= (\mathbf{e}^i - \mathbf{x})A\mathbf{x} - \sum_{j=1}^m y_j(\mathbf{e}^j - \mathbf{x})A\mathbf{x} \\
&= (\mathbf{e}^i - \mathbf{y})A\mathbf{x} \le -\varepsilon < 0 .
\end{aligned}$$

(Cf. Proposition 15.15.) Hence, $v_i(\xi(t, \mathbf{x}^0))$ decreases to minus infinity as $t \to \infty$. This implies $\ln(\xi(t, \mathbf{x}^0))$ decreases to minus infinity, so that $\xi_i(t, \mathbf{x}^0) \to 0$. ■

Proposition 15.17 remains true for pure strategies i that are iteratively strictly dominated. For weakly dominated strategies several things may happen, see Problem 15.7.

15.3.4 Nash Equilibrium Strategies

Consider again the finite symmetric two-player game with payoff matrix A. What is the relation between the replicator dynamics and Nash equilibrium strategies? The answer is given by the following proposition, where $ST(A)$ denotes the set of stationary states, hence (check!):

$$ST(A) = \{\mathbf{x} \in \Delta^m \mid \forall i \in C(\mathbf{x})\ [\mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}]\} . \qquad (15.5)$$

Proposition 15.18 *For any finite symmetric two-player game with payoff matrix A we have:*

(a) $\{\mathbf{e}^1, \ldots, \mathbf{e}^m\} \cup NE(A) \subseteq ST(A)$,
(b) $ST(A) \cap \Delta_0^m = NE(A) \cap \Delta_0^m$,
(c) $ST(A) \cap \Delta_0^m$ *is a convex set and if* $\mathbf{z} \in \Delta^m$ *is a linear combination of states in this set, then* $\mathbf{z} \in NE(A)$.

Proof It is straightforward from (15.3) that $\mathbf{e}^i \in ST(A)$ for every pure strategy i. If $\mathbf{x} \in NE(A)$, then every $i \in C(\mathbf{x})$ is a pure best reply, hence $\mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}$. Hence, $\mathbf{x} \in ST(A)$. This proves (a). As to (b), part (a) implies that $ST(A) \cap \Delta_0^m \supseteq NE(A) \cap \Delta_0^m$. Further, $\mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}$ for every $\mathbf{x} \in ST(A) \cap \Delta_0^m$ and every $i \in \Delta^m$, so that $\mathbf{x} \in NE(A) \cap \Delta_0^m$. This proves (b).

It remains to prove the last claim. Let $\mathbf{x}$ and $\mathbf{y}$ be completely mixed stationary points, and let $\alpha, \beta \in \mathbb{R}$ and $\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y} \in \Delta^m$. For any pure strategy i we have

$$\mathbf{e}^i A\mathbf{z} = \alpha\mathbf{e}^i A\mathbf{x} + \beta\mathbf{e}^i A\mathbf{y} = \alpha\mathbf{x}A\mathbf{x} + \beta\mathbf{y}A\mathbf{y}$$

since $\mathbf{x}, \mathbf{y} \in ST(A) \cap \Delta_0^m$. This implies that actually $\mathbf{e}^i A\mathbf{z} = \mathbf{z}A\mathbf{z}$ for all pure strategies i, hence $\mathbf{z}$ is stationary. If $\mathbf{z}$ is completely mixed, then we are done by part (b). Otherwise, $\mathbf{z}$ is a boundary point of $ST(A) \cap \Delta_0^m$ and hence of $NE(A) \cap \Delta_0^m$, so $\mathbf{z} \in NE(A)$ since $NE(A)$ is a closed set. Finally, since Δ^m is convex and $\mathbf{z} \in ST(A) \cap \Delta_0^m$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, $ST(A) \cap \Delta_0^m$ is a convex set. ■

Proposition 15.18 implies that every (symmetric) Nash equilibrium is stationary. The weakest form of dynamical stability, Lyapunov stability, leads to a refinement of Nash equilibrium, as the next result shows.

Proposition 15.19 *Let* $\mathbf{x} \in \Delta^m$ *be a Lyapunov stable stationary state. Then* $\mathbf{x} \in NE(A)$.

Proof Suppose $\mathbf{x} \notin NE(A)$. Then $\mathbf{e}^i A\mathbf{x} - \mathbf{x}A\mathbf{x} > 0$ for some $i \in N$. By continuity, there is a $\delta > 0$ and an open neighborhood U of $\mathbf{x}$ such that $\mathbf{e}^i A\mathbf{y} - \mathbf{y}A\mathbf{y} \geq \delta$ for all $\mathbf{y} \in U \cap \Delta^m$. Let $\mathbf{x}^0 \in U \cap \Delta^m$ with $x_i^0 > 0$. Then $\xi_i(t, \mathbf{x}^0) \geq x_i^0 \exp(\delta t)$ for all $t \geq 0$; this follows from the fact that the system $\dot{\eta} = \delta\eta$ has solution $\eta(t) = \eta(0)\exp(\delta t)$. So $\xi_i(t, \mathbf{x}^0)$ increases exponentially from any $\mathbf{x}^0 \in U \cap \Delta_0^m$ with $x_i^0 > 0$. This contradicts Lyapunov stability of $\mathbf{x}$. ■

The final result in this subsection says that if a trajectory of the replicator dynamics starts from an interior (completely mixed) state and converges, then the limit state is a Nash equilibrium strategy.

Proposition 15.20 *Let* $\mathbf{x}^0 \in \Delta_0^m$ *and* $\mathbf{x} \in \Delta^m$ *such that* $\mathbf{x} = \lim_{t\to\infty} \xi(t, \mathbf{x}^0)$. *Then* $\mathbf{x} \in NE(A)$.

Proof Suppose that $\mathbf{x} \notin NE(A)$. Then there is a pure strategy i and an $\varepsilon > 0$ such that $\mathbf{e}^i A\mathbf{x} - \mathbf{x}A\mathbf{x} = \varepsilon$. Hence, there is a $T \in \mathbb{R}$ such that $\mathbf{e}^i A\xi(t, \mathbf{x}^0) - \xi(t, \mathbf{x}^0)A\xi(t, \mathbf{x}^0) > \varepsilon/2$ for all $t \geq T$. By (15.3), $\dot{x}_i > x_i\,\varepsilon/2$ for all $t \geq T$, and hence (as in the proof of Proposition 15.19) $\xi_i(t, \mathbf{x}^0) > \xi_i(T, \mathbf{x}^0)\exp(\varepsilon(t - T)/2)$ for all $t \geq T$. Since $\xi_i(T, \mathbf{x}^0) > 0$, this implies $\xi_i(t, \mathbf{x}^0) \to \infty$ as $t \to \infty$, a contradiction. ■

15.3.5 Perfect Equilibrium Strategies

In the preceding subsection we have seen that Lyapunov stability implies Nash equilibrium. What are the implications of asymptotic stability?

First, asymptotic stability implies Lyapunov stability and therefore also Nash equilibrium. Since Nash equilibrium implies stationarity, however, it must be the case that an asymptotically stable Nash equilibrium strategy is *isolated*, meaning that it has an open neighborhood in which there are no other Nash equilibrium strategies. If not, there would be arbitrarily close stationary states, which conflicts with asymptotic stability.

Second, asymptotic stability also implies perfection.

Proposition 15.21 *Let* $\mathbf{x} \in \Delta^m$ *be asymptotically stable. Then* $\mathbf{x} \in NE(A)$ *and* $\mathbf{x}$ *is isolated. Moreover,* $(\mathbf{x}, \mathbf{x})$ *is a perfect equilibrium in* (A, A^T).

Proof We still have to prove that $(\mathbf{x}, \mathbf{x})$ is a perfect equilibrium in (A, A^T). Suppose not. Then $\mathbf{x}$ is weakly dominated by some $\mathbf{y} \in \Delta^m \setminus \{\mathbf{x}\}$, see Theorem 13.23. Hence

$\mathbf{y}A\mathbf{z} \geq \mathbf{x}A\mathbf{z}$ for all $\mathbf{z} \in \Delta^m$. Define $v : \Delta^m \to \mathbb{R}$ by

$$v(\mathbf{z}) = \sum_{i \in C(\mathbf{z})} (y_i - x_i) \ln(z_i)$$

for all $\mathbf{z} \in \Delta^m$. Similarly as in the proof of Proposition 15.17, we obtain that v is nondecreasing along all interior solution trajectories of (15.3), i.e., at any $\mathbf{z} \in \Delta_0^m$ (so $\mathbf{z}$ completely mixed),

$$\dot{v}(\mathbf{z}) = \sum_{i \in C(\mathbf{z})} (y_i - x_i) \frac{\dot{z}_i}{z_i} = \sum_{i=1}^{m} (y_i - x_i)[\mathbf{e}^i A\mathbf{z} - \mathbf{z}A\mathbf{z}] = (\mathbf{y} - \mathbf{x})A\mathbf{z} \geq 0 \, .$$

Since $\mathbf{x}$ is asymptotically stable, it has an open neighborhood U such that $\xi(t, \mathbf{x}^0) \to \mathbf{x}$ for all $\mathbf{x}^0 \in U \cap \Delta^m$. By nondecreasingness of v along all interior solution trajectories this implies $v(\mathbf{x}) \geq v(\mathbf{z})$ for all $\mathbf{z} \in U \cap \Delta_0^m$. We will construct, however, a $\mathbf{z}$ in $U \cap \Delta_0^m$ with $v(\mathbf{z}) > v(\mathbf{x})$. This is a contradiction and, hence, $\mathbf{x}$ must be perfect.

To construct such a $\mathbf{z}$, define for $\delta \in (0, 1)$, $\mathbf{w} \in \Delta_0^m$, and $\varepsilon \in [0, 1]$,

$$\mathbf{z} = (1 - \varepsilon)[(1 - \delta)\mathbf{x} + \delta\mathbf{y}] + \varepsilon\mathbf{w} \, .$$

For ε sufficiently small, we have $y_i > x_i \Rightarrow z_i > x_i$ and $y_i < x_i \Rightarrow z_i < x_i$. Moreover, $\mathbf{z} \in \Delta_0^m$ and

$$\begin{aligned} v(\mathbf{z}) - v(\mathbf{x}) &= \sum_{i=1}^{m} (y_i - x_i) \ln(z_i) - \sum_{i \in C(\mathbf{x})} (y_i - x_i) \ln(x_i) \\ &= \sum_{i \in C(\mathbf{x})} (y_i - x_i)[\ln(z_i) - \ln(x_i)] + \sum_{i \notin C(\mathbf{x})} y_i \ln(z_i) \, . \end{aligned}$$

Note that the first term after the second inequality sign is positive. We will show that the second term is zero, which completes the proof of the proposition, since then $v(\mathbf{z}) > v(\mathbf{x})$. To show that $\sum_{i \notin C(\mathbf{x})} y_i \ln(z_i) = 0$, it is sufficient to show that $C(\mathbf{y}) \subseteq C(\mathbf{x})$. Suppose that $j \in C(\mathbf{y})$ and $j \notin C(\mathbf{x})$. By asymptotic stability of $\mathbf{x}$, $\xi(t, \mathbf{x}^0) \to \mathbf{x}$ for all $\mathbf{x}^0 \in U \cap \Delta_0^m$. Write

$$\begin{aligned} v(\xi(t, \mathbf{x}^0)) = & \sum_{i : \xi_i(t, \mathbf{x}^0) > 0, i \in C(\mathbf{x})} (y_i - x_i) \ln(\xi_i(t, \mathbf{x}^0)) \\ & + \sum_{i : \xi_i(t, \mathbf{x}^0) > 0, i \notin C(\mathbf{x})} y_i \ln(\xi_i(t, \mathbf{x}^0)) \, . \end{aligned}$$

The first term after the inequality sign is bounded by some constant γ, since $\xi_i(t, \mathbf{x}^0) \to x_i > 0$. The second term converges to $-\infty$ since $\xi_j(t, \mathbf{x}^0) \to x_j = 0$ and $y_j > 0$. But $v(\xi(t, \mathbf{x}^0)) \to -\infty$ contradicts the nondecreasingness of v along the trajectory $\xi(t, \mathbf{x}^0)$. ■

15.4 Problems

15.1. *Computing ESS in* 2×2 *Games (1)*
Compute $ESS(A)$ for the following payoff matrices A.

(a) $A = \begin{pmatrix} 4 & 0 \\ 5 & 3 \end{pmatrix}$ (Prisoners' Dilemma)
(b) $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ (Coordination game)
(c) $A = \begin{pmatrix} -1 & 4 \\ 0 & 2 \end{pmatrix}$ (Hawk-Dove game)

15.2. *Computing ESS in* 2×2 *Games (2)*
Compute $ESS(A')$ for each of the cases (1), (2), and (3) in Sect. 15.1.1. Compare with your answers to Problem 15.1.

15.3. *Rock-Paper-Scissors (1)*
Show that the Rock-Paper-Scissors game

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

has no ESS.

15.4. *Uniform Invasion Barriers*
Find the maximal value of the uniform invasion barrier for the ESS's in each of the cases (1), (2), and (3) in Sect. 15.1.1.

15.5. *Replicator Dynamics in Normalized Game (1)*
Show that A and A' (see Sect. 15.3.2) result in the same replicator dynamics.

15.6. *Replicator Dynamics in Normalized Game (2)*

(a) Simplify the dynamics (15.4) for case (1) in Sect. 15.1.1 by substituting $x_2 = 1 - x_1$ and plot $\dot{x}_1$ as a function of $x_1 \in [0, 1]$.
(b) Carry out this analysis also for cases (2) and (3). What is your conclusion?

15.7. *Weakly Dominated Strategies and Replicator Dynamics*

(a) Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Investigate the trajectories of the replicator dynamics.

(b) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Investigate the trajectories of the replicator dynamics.
[Cf. Proposition 15.17.]

15.8. *Stationary Points and Nash Equilibria (1)*
Consider the two-person symmetric game with payoff matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

(a) Compute $NE(A)$.
(b) Compute $ST(A)$.

15.9. *Stationary Points and Nash Equilibria (2)*
Consider the two-person symmetric game with payoff matrix

$$A = \begin{pmatrix} 3 & 3 & 1 \\ 4 & 4 & 0 \\ 0 & 2 & 4 \end{pmatrix}.$$

(a) Compute $NE(A)$.
(b) Compute $ST(A)$.
(c) Which stationary states are asymptotically stable, which are Lyapunov stable, and which are not Lyapunov stable?
(d) Show directly from the definition of local superiority that the strategy $x = (1/2, 0, 1/2)$ is not locally superior. [Hint: consider the strategy $z = (1/2, \lambda, 1/2 - \lambda)$ for $0 < \lambda < 1/2$.]

15.10. *Lyapunov Stable States in* 2×2 *Games*
Consider the normalized two-player symmetric 2×2 game A'. Compute the Lyapunov stable states for cases (1), (2), and (3).

15.11. *Nash Equilibrium and Lyapunov Stability*
Consider the symmetric game with payoff matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} .$$

Compute $NE(A)$. Show that the unique element in this set is not Lyapunov stable.

15.12. *Rock-Paper-Scissors (2)*
Consider the generalized Rock-Paper-Scissors game with payoff matrix

$$A = \begin{pmatrix} 1 & 2+a & 0 \\ 0 & 1 & 2+a \\ 2+a & 0 & 1 \end{pmatrix}$$

where $a \in \mathbb{R}$.

(a) Write down the three equations of the replicator dynamics.
(b) Define $h(\mathbf{x}) = \ln(x_1 x_2 x_3)$ for $\mathbf{x}$ positive and show that $\dot{h}(\mathbf{x}) = 3 + a - 3\mathbf{x}A\mathbf{x}$.
(c) Show that the average payoff is equal to

$$\mathbf{x}A\mathbf{x} = 1 + \frac{a}{2}(1 - ||\mathbf{x}||^2)$$

for each $\mathbf{x} \in \Delta^3$, where $||\mathbf{x}||$ is the Euclidean norm of $\mathbf{x}$. Conclude that $\dot{h}(\mathbf{x}) = \frac{a}{2}\left(3||\mathbf{x}||^2 - 1\right)$.
(d) Show that $\dot{h}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 0$ and $\dot{h}(\mathbf{x})$ has the same sign as a for other $\mathbf{x} \in \Delta^m$.
(e) Show that the unique Nash equilibrium in this game is (i) asymptotically stable for $a > 0$; (ii) Lyapunov but not asymptotically stable for $a = 0$; (iii) not Lyapunov stable for $a < 0$.

15.5 Notes

This chapter is based mainly on Weibull (1995). The concept of an evolutionary stable strategy was introduced by Maynard Smith and Price (1973). See Selten (1980, 1983) for early applications of evolutionary stable strategies in game theory.

For properness of $(\mathbf{x}, \mathbf{x})$, where $\mathbf{x}$ is an ESS (Proposition 15.8), see van Damme (1991).

Uniform invasion barriers were introduced by Vickers and Cannings (1987), and local superiority by Hofbauer et al. (1979). Proposition 15.14 is due to Hofbauer and Sigmund (1988).

Replicator dynamics were introduced by Taylor and Jonker (1978). Proposition 15.17 remains true for pure strategies *i* that are iteratively strictly dominated: see Samuelson and Zhang (1992). For Proposition 15.21, in particular the result that asymptotic stability implies perfection, see Bomze (1986). For more on weakly dominated strategies see Weibull (1995), p. 83 ff. See Sect. 3.3.2 in the same book for more on the relation between Lyapunov stability and Nash equilibrium. For a proof of Proposition 15.15, see Weibull (1995), Proposition 3.20.

A general reference to the theory of differential equations and dynamical systems is Hirsch and Smale (1974).

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