13 Finite Games

This chapter builds on Chap. 3, where we studied finite two person games—bimatrix games. (Re)reading Chap. 3 may serve as a good preparation for the present chapter, which offers a more rigorous treatment of finite games, i.e., games with finitely many players—often two—who have finitely many pure strategies over which they can randomize. We only discuss games with complete information. In the terminology of Chap. 5, each player has only one type.

In Sect. 13.1 a proof of Nash's existence theorem is provided. Section 13.2 goes deeper into bimatrix games. In Sect. 13.3 the notion of iterated dominance is studied, and its relation with rationalizability indicated. Sections 13.4–13.6 present some basics about refinements of Nash equilibrium. Section 13.7 is on correlated equilibrium in bimatrix games, and Sect. 13.8 concludes with an axiomatic characterization of Nash equilibrium based on a reduced game (consistency) condition.

13.1 Existence of Nash Equilibrium

We start with a general definition of a finite game. Matrix and bimatrix games are special cases.

A *finite game* is a $2n+1$-tuple

$$G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n) ,$$

where

- $N = \{1, \ldots, n\}$, with $n \in \mathbb{N}$, is the set of *players*;
- for every $i \in N$, S_i is the finite *pure strategy set* of player i;
- for every $i \in N$, $u_i : S = S_1 \times \ldots \times S_n \to \mathbb{R}$ is the *payoff function* of player i; i.e., for every pure strategy combination $(s_1, \ldots, s_n) \in S$ where $s_1 \in S_1, \ldots, s_n \in S_n$, $u_i(s_1, \ldots, s_n) \in \mathbb{R}$ is player i's payoff.

H. Peters, *Game Theory*, Springer Texts in Business and Economics,
DOI 10.1007/978-3-662-46950-7_13

This definition is identical to the definition of an n-person game in Chap. 6, except that the pure strategy sets are now finite. The elements of S_i are the pure strategies of player i. A (mixed) *strategy* of player i is a probability distribution over S_i. The set of (mixed) strategies of player i is denoted by $\Delta(S_i)$. Observe that, whenever we talk about a strategy, we mean a mixed strategy (which may of course be pure).

Let $(\sigma_1, \ldots, \sigma_n) \in \Delta(S_1) \times \ldots \times \Delta(S_n)$ be a strategy combination. Player i's *payoff* from this strategy combination is defined to be his expected payoff. With some abuse of notation this is also denoted by $u_i(\sigma_1, \ldots, \sigma_n)$. Formally,

$$u_i(\sigma_1, \ldots, \sigma_n) = \sum_{(s_1,\ldots,s_n)\in S} \left(\prod_{j\in N} \sigma_j(s_j) \right) u_i(s_1, \ldots, s_n) \ .$$

For a strategy combination σ and a player $i \in N$ we denote by (σ_i', σ_{-i}) the strategy combination in which player i plays $\sigma_i' \in \Delta(S_i)$ and each player $j \neq i$ plays σ_j.

A *best reply* of player i to the strategy combination σ_{-i} of the other players is a strategy $\sigma_i \in \Delta(S_i)$ such that $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for all $\sigma_i' \in \Delta(S_i)$.

A *Nash equilibrium* of G is a strategy combination $\sigma^* \in \prod_{i\in N} \Delta(S_i)$ such that for each player i, σ_i^* is a best reply to σ_{-i}^*.

As in Chaps. 3 and 6, β_i denotes player i's *best reply correspondence*. That is, $\beta_i : \prod_{j\in N,\, j\neq i} \Delta(S_j) \to \Delta(S_i)$ assigns to each strategy combination of the other players the set of all best replies of player i.

Theorem 13.1 (Existence of Nash Equilibrium) *Every finite game* $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ *has a Nash equilibrium.*

The proof of this theorem below is based on the Kakutani Fixed Point Theorem (Sect. 22.5).

Proof of Theorem 13.1 Consider the correspondence

$$\beta : \prod_{i\in N} \Delta(S_i) \to \prod_{i\in N} \Delta(S_i),\ (\sigma_1, \ldots, \sigma_n) \mapsto \prod_{i\in N} \beta_i(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n) \ .$$

This correspondence is convex-valued and upper semi-continuous (Problem 13.2). By the Kakutani Fixed Point Theorem (Theorem 22.11) it has a fixed point σ^*. By definition of β, any fixed point is a Nash equilibrium of G. ■

An alternative proof is obtained by using the Brouwer Fixed Point Theorem (Theorem 22.10). See Problem 13.1.

13.2 Bimatrix Games

Two-person finite games—bimatrix games—were studied in Chap. 3. Here we present some extensions. In Sect. 13.2.1 we give some formal relations between pure and mixed strategies in a Nash equilibrium. In Sect. 13.2.2 we extend the graphical method for computing Nash equilibria (cf. Sect. 3.2.2). In Sect. 13.2.3 a general mathematical programming method is described by which equilibria of bimatrix games can be found. Section 13.2.4 reconsiders matrix games as a special kind of bimatrix games. Section 13.2.5 is about Zermelo's theorem on the game of chess.

13.2.1 Pure and Mixed Strategies in Nash Equilibrium

Let (A, B) be an $m \times n$ bimatrix game (Definition 3.1). The first lemma implies that to determine whether a strategy pair is a Nash equilibrium it is sufficient to compare the expected payoff of a (mixed) strategy with the payoffs of pure strategies.

Lemma 13.2 *Let* $\mathbf{p} \in \Delta^m$ *and* $\mathbf{q} \in \Delta^n$. *Then* $\mathbf{p} \in \beta_1(\mathbf{q})$ *if and only if* $\mathbf{p}A\mathbf{q} \geq \mathbf{e}^i A\mathbf{q}$ *for all* $i = 1, \ldots, m$; *and* $\mathbf{q} \in \beta_2(\mathbf{p})$ *if and only if* $\mathbf{p}B\mathbf{q} \geq \mathbf{p}B\mathbf{e}^j$ *for all* $j = 1, \ldots, n$.

Proof Problem 13.3. ■

The next lemma says that a player always has a pure best reply against any strategy of the opponent.

Lemma 13.3 *Let* $\mathbf{p} \in \Delta^m$ *and* $\mathbf{q} \in \Delta^n$. *Then there is an* $i \in \{1, \ldots, m\}$ *with* $\mathbf{e}^i \in \beta_1(\mathbf{q})$ *and a* $j \in \{1, \ldots, n\}$ *with* $\mathbf{e}^j \in \beta_2(\mathbf{p})$.

Proof Problem 13.4. ■

In light of these lemmas it makes sense to introduce the pure best reply correspondences.

Definition 13.4 Let (A, B) be an $m \times n$ bimatrix game and let $\mathbf{p} \in \Delta^m$ and $\mathbf{q} \in \Delta^n$. Then

$$PB_1(\mathbf{q}) = \{i \in \{1, \ldots, m\} \mid \mathbf{e}^i A\mathbf{q} = \max_k \mathbf{e}^k A\mathbf{q}\}$$

is the set of *pure best replies* of player 1 to $\mathbf{q}$ and

$$PB_2(\mathbf{p}) = \{j \in \{1, \ldots, n\} \mid \mathbf{p}B\mathbf{e}^j = \max_k \mathbf{p}B\mathbf{e}^k\}$$

is the set of *pure best replies* of player 2 to $\mathbf{p}$. □

Observe that, with some abuse of notation, the pure best replies in this definition are labelled by the row and column numbers.

The *carrier* $C(\mathbf{p})$ of a mixed strategy $\mathbf{p} \in \Delta^k$, where $k \in \mathbb{N}$, is the set of coordinates that are positive, i.e.,

$$C(\mathbf{p}) = \{i \in \{1, \dots, k\} \mid p_i > 0\} .$$

The next lemma formalizes the observation used already in Chap. 3, namely that in a best reply a player puts positive probability only on those pure strategies that maximize his expected payoff (cf. Problem 3.8).

Lemma 13.5 *Let (A, B) be an $m \times n$ bimatrix game, $\mathbf{p} \in \Delta^m$ and $\mathbf{q} \in \Delta^n$. Then*

$$\mathbf{p} \in \beta_1(\mathbf{q}) \Leftrightarrow C(\mathbf{p}) \subseteq PB_1(\mathbf{q})$$

and

$$\mathbf{q} \in \beta_2(\mathbf{p}) \Leftrightarrow C(\mathbf{q}) \subseteq PB_2(\mathbf{p}) .$$

Proof We only show the first equivalence.

First let $\mathbf{p} \in \beta_1(\mathbf{q})$, and assume $i \in C(\mathbf{p})$ and, contrary to what we want to prove, that $\mathbf{e}^i A\mathbf{q} < \max_k \mathbf{e}^k A\mathbf{q}$. Then

$$\mathbf{p}A\mathbf{q} = \max_k \mathbf{e}^k A\mathbf{q} = \sum_{k'=1}^{m} p_{k'} \max_k \mathbf{e}^k A\mathbf{q} > \sum_{k=1}^{m} p_k \, \mathbf{e}^k A\mathbf{q} = \mathbf{p}A\mathbf{q} ,$$

where the first equality follows from Lemma 13.3. This is a contradiction, hence $\mathbf{e}^i A\mathbf{q} = \max_k \mathbf{e}^k A\mathbf{q}$ and $i \in PB_1(\mathbf{q})$.

Next, assume that $C(\mathbf{p}) \subseteq PB_1(\mathbf{q})$. Then

$$\mathbf{p}A\mathbf{q} = \sum_{i=1}^{m} p_i \, \mathbf{e}^i A\mathbf{q} = \sum_{i \in C(\mathbf{p})} p_i \, \mathbf{e}^i A\mathbf{q} = \sum_{i \in C(\mathbf{p})} p_i \max_k \mathbf{e}^k A\mathbf{q} = \max_k \mathbf{e}^k A\mathbf{q} .$$

So $\mathbf{p}A\mathbf{q} \geq \mathbf{e}^i A\mathbf{q}$ for all $i = 1, \dots, m$, which by Lemma 13.2 implies $\mathbf{p} \in \beta_1(\mathbf{q})$. ∎

The following corollary is an immediate consequence of Lemma 13.5. It is, in principle, helpful to find Nash equilibria or to determine whether a given strategy combination is a Nash equilibrium. See Example 13.7.

Corollary 13.6 *A strategy pair $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium in a bimatrix game (A, B) if and only if $C(\mathbf{p}) \subseteq PB_1(\mathbf{q})$ and $C(\mathbf{q}) \subseteq PB_2(\mathbf{p})$.*

Example 13.7 Consider the bimatrix game

$$(A, B) = \begin{pmatrix} 1,1 & 0,1 & 0,1 & 0,1 \\ 1,1 & 1,1 & 0,1 & 0,1 \\ 1,1 & 1,1 & 1,1 & 0,1 \\ 1,1 & 1,1 & 1,1 & 1,1 \end{pmatrix}$$

and the strategies $\mathbf{p} = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\mathbf{q} = (\frac{1}{2}, \frac{1}{2}, 0, 0)$. Since

$$A\mathbf{q} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{p}B = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} ,$$

$PB_1(\mathbf{q}) = \{2, 3, 4\}$ and $PB_2(\mathbf{p}) = \{1, 2, 3, 4\}$. Since $C(\mathbf{p}) = \{2, 3, 4\}$ and $C(\mathbf{q}) = \{1, 2\}$, we have $C(\mathbf{p}) \subseteq PB_1(\mathbf{q})$ and $C(\mathbf{q}) \subseteq PB_2(\mathbf{p})$. So Corollary 13.6 implies that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium.

13.2.2 Extension of the Graphical Method

In Sect. 3.2.2 we learnt how to solve 2×2 bimatrix games graphically. We now extend this method to 2×3 and 3×2 games. For larger games it becomes impractical or impossible to use this graphical method.

As an example consider the 2×3 bimatrix game

$$(A, B) = \begin{pmatrix} 2,1 & 1,0 & 1,1 \\ 2,0 & 1,1 & 0,0 \end{pmatrix} .$$

The Nash equilibria of this game are elements of the set $\Delta^2 \times \Delta^3$ of all possible strategy combinations. This set can be represented as in Fig. 13.1.

Here player 2 chooses a point in the triangle with vertices $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$, while player 1 chooses a point of the horizontal line segment with vertices $\mathbf{e}_1$ and $\mathbf{e}_2$.

In order to determine the best replies of player 1 note that

$$A\mathbf{q} = \begin{pmatrix} 2q_1 + q_2 + q_3 \\ 2q_1 + q_2 \end{pmatrix} .$$

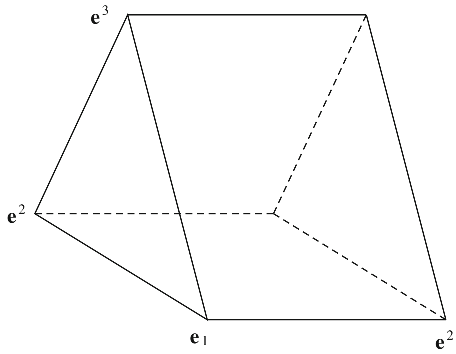

Fig. 13.1 The set $\Delta^2 \times \Delta^3$

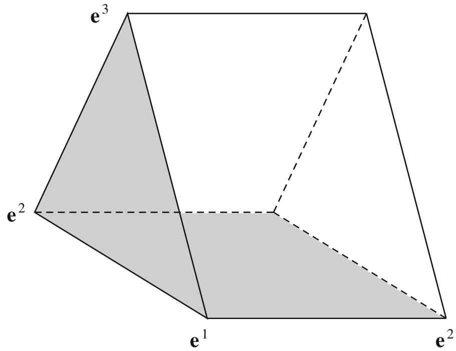

Fig. 13.2 The best reply correspondence of player 1 (*shaded*)

As $\mathbf{e}_1 A\mathbf{q} = \mathbf{e}_2 A\mathbf{q} \Leftrightarrow q_3 = 0$, it follows that

$$\beta_1(\mathbf{q}) = \begin{cases} \{\mathbf{e}^1\} & \text{if } q_3 > 0 \\ \Delta^2 & \text{if } q_3 = 0 \ . \end{cases}$$

This yields the best reply correspondence represented in Fig. 13.2.

Similarly,

$$\mathbf{p}B = \begin{pmatrix} p_1 & p_2 & p_1 \end{pmatrix}$$

implies

$$\beta_2(\mathbf{p}) = \begin{cases} \{\mathbf{e}^2\} & \text{if } p_1 < p_2 \\ \Delta^3 & \text{if } p_1 = p_2 \\ \{\mathbf{q} \in \Delta^3 \mid q_2 = 0\} & \text{if } p_1 > p_2. \end{cases}$$

This yields the best reply correspondence represented in Fig. 13.3.

Figure 13.4 represents the intersection of the two best reply correspondences and, thus, the set of Nash equilibria.

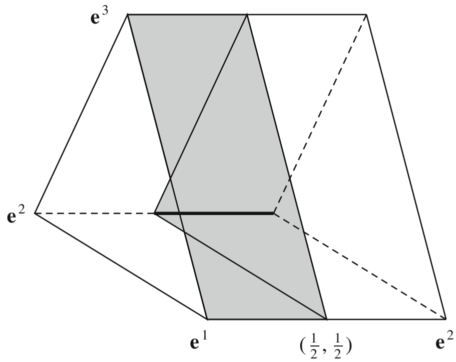

Fig. 13.3 The best reply correspondence of player 2 (*shaded/thick*)

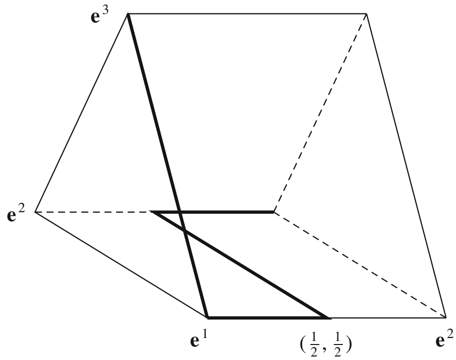

Fig. 13.4 The set of Nash equilibria: $\{((1,0),\mathbf{q}) \mid q_2 = 0\} \cup \{(\mathbf{p},(1,0,0)) \mid 1 \geq p_1 \geq \frac{1}{2}\} \cup \{((\frac{1}{2},\frac{1}{2}),\mathbf{q}) \mid q_3 = 0\} \cup \{(\mathbf{p},(0,1,0)) \mid \frac{1}{2} \geq p_1 \geq 0\}$

13.2.3 A Mathematical Programming Approach

In Sect. 12.2 we have seen that matrix games can be solved by linear programming. Nash equilibria of an $m \times n$ bimatrix game (A,B) can be found by considering the following *quadratic* programming problem:

$$\max_{\mathbf{p}\in\Delta^m,\ \mathbf{q}\in\Delta^n,\ a,b\in\mathbb{R}} f(\mathbf{p},\mathbf{q},a,b) := \mathbf{p}A\mathbf{q} + \mathbf{p}B\mathbf{q} - a - b$$

$$\text{subject to} \quad \begin{aligned} \mathbf{e}^i A\mathbf{q} &\leq a \quad \text{for all } i = 1,2,\ldots,m \\ \mathbf{p}B\mathbf{e}^j &\leq b \quad \text{for all } j = 1,2,\ldots,n\,. \end{aligned} \tag{13.1}$$

Theorem 13.8 *The following two statements are equivalent:*

(1) $(\mathbf{p},\mathbf{q},a,b)$ *is a solution of* (13.1)
(2) $(\mathbf{p},\mathbf{q})$ *is a Nash equilibrium of* (A,B), $a = \mathbf{p}A\mathbf{q}$, $b = \mathbf{p}B\mathbf{q}$.

Proof Problem 13.9. ■

If (A, B) is a zero-sum game, i.e., if $B = -A$, then (13.1) reduces to

$$\begin{aligned} & \max_{\mathbf{p}\in\Delta^m,\ \mathbf{q}\in\Delta^n,\ a,b\in\mathbb{R}} \quad -a-b \\ & \text{subject to} \quad \mathbf{e}^i A\mathbf{q} \le a \ \text{ for all } i = 1,2,\ldots,m \\ & \qquad\qquad\quad -\mathbf{p}A\mathbf{e}^j \le b \ \text{ for all } j = 1,2,\ldots,n\ . \end{aligned} \tag{13.2}$$

Program (13.2) can be split up into two independent programs

$$\begin{aligned} & \max_{\mathbf{q}\in\Delta^n,\ a\in\mathbb{R}} \quad -a \\ & \text{subject to} \quad \mathbf{e}^i A\mathbf{q} \le a \ \text{ for all } i = 1,2,\ldots,m \end{aligned} \tag{13.3}$$

and

$$\begin{aligned} & \min_{\mathbf{p}\in\Delta^m,\ b\in\mathbb{R}} \quad b \\ & \text{subject to} \quad \mathbf{p}A\mathbf{e}^j \ge -b \ \text{ for all } j = 1,2,\ldots,n\ . \end{aligned} \tag{13.4}$$

One can check that these problems are equivalent to the LP and its dual for matrix games in Sect. 12.2, see Problem 13.10.

13.2.4 Matrix Games

Since matrix games are also bimatrix games, everything that we know about bimatrix games is also true for matrix games. In fact, the Minimax Theorem (Theorem 12.3) can be derived directly from the existence theorem for Nash equilibrium (Theorem 13.1). Moreover, each Nash equilibrium in a matrix game consists of a pair of optimal (maximin and minimax) strategies, and each such pair is a Nash equilibrium. As a consequence, in a matrix game, Nash equilibrium strategies are exchangeable—there is no coordination problem, and all Nash equilibria result in the same payoffs.

All these facts are collected in the following theorem. For terminology concerning matrix games see Chap. 12. The 'new' contribution of this theorem is part (2), part (1) is just added to provide an alternative proof of the Minimax Theorem.

Theorem 13.9 *Let A be an $m \times n$ matrix game. Then:*

(1) $v_1(A) = v_2(A)$.
(2) *A pair $(\mathbf{p}^*, \mathbf{q}^*) \in \Delta^m \times \Delta^n$ is a Nash equilibrium of $(A, -A)$ if and only if $\mathbf{p}^*$ is an optimal strategy for player 1 in A and $\mathbf{q}^*$ is an optimal strategy for player 2 in A.*

Proof

(1) In view of Lemma 12.2 it is sufficient to prove that $v_1(A) \geq v_2(A)$. Choose $(\mathbf{p}^*, \mathbf{q}^*) \in \Delta^m \times \Delta^n$ to be a Nash equilibrium of $(A, -A)$—this is possible by Theorem 13.1. Then

$$\mathbf{p}A\mathbf{q}^* \leq \mathbf{p}^*A\mathbf{q}^* \leq \mathbf{p}^*A\mathbf{q} \text{ for all } \mathbf{p} \in \Delta^m, \mathbf{q} \in \Delta^n .$$

This implies $\max_{\mathbf{p}} \mathbf{p}A\mathbf{q}^* \leq \mathbf{p}^*A\mathbf{q}$ for all $\mathbf{q}$, hence $v_2(A) = \min_{\mathbf{q}'} \max_{\mathbf{p}} \mathbf{p}A\mathbf{q}' \leq \mathbf{p}^*A\mathbf{q}$ for all $\mathbf{q}$. So

$$v_2(A) \leq \min_{\mathbf{q}} \mathbf{p}^*A\mathbf{q} \leq \max_{\mathbf{p}} \min_{\mathbf{q}} \mathbf{p}A\mathbf{q} = v_1(A) .$$

(2) First, suppose that $(\mathbf{p}^*, \mathbf{q}^*) \in \Delta^m \times \Delta^n$ is a Nash equilibrium of $(A, -A)$. Then

$$\mathbf{p}^*A\mathbf{q}^* = \max_{\mathbf{p}} \mathbf{p}A\mathbf{q}^* = v_2(\mathbf{q}^*) \geq \min_{\mathbf{q}} v_2(\mathbf{q}) = v_2(A) = v(A) .$$

If $\mathbf{p}^*$ were not optimal, then $\mathbf{p}^*A\mathbf{q} < v(A)$ for some $\mathbf{q} \in \Delta^n$, so $\mathbf{p}^*A\mathbf{q}^* \leq \mathbf{p}^*A\mathbf{q} < v(A)$, a contradiction. Similarly, $\mathbf{q}^*$ must be optimal.

Conversely, suppose that $\mathbf{p}^*$ and $\mathbf{q}^*$ are optimal strategies. Since $\mathbf{p}A\mathbf{q}^* \leq v(A)$ for all $\mathbf{p} \in \Delta^m$ and $\mathbf{p}^*A\mathbf{q} \geq v(A)$ for all $\mathbf{q} \in \Delta^n$, it follows that $\mathbf{p}^*$ and $\mathbf{q}^*$ are mutual best replies and, thus, $(\mathbf{p}^*, \mathbf{q}^*)$ is a Nash equilibrium in $(A, -A)$. ■

13.2.5 The Game of Chess: Zermelo's Theorem

One of the earliest formal results in game theory is Zermelo's Theorem on the game of chess. In this subsection we provide a simple proof of this theorem, based on Theorem 13.9.

The game of chess is a classical example of a zero-sum game. There are three possible outcomes: a win for White, a win for Black, and a draw. Identifying player 1 with White and player 2 with Black, we can associate with these outcomes the payoffs $(1, -1)$, $(-1, 1)$, and $(0, 0)$, respectively. In order to guarantee that the (extensive form) game stops after finitely many moves, we assume the following stopping rule: if the same configuration on the chess board has occurred more than twice, the game ends in a draw. Since there are only finitely many configurations on the chess board, the game must stop after finitely many moves. Note that the chess game is a finite extensive form game of perfect information and therefore it has a Nash equilibrium in pure strategies—see Sect. 4.3. To be precise, this is a pure strategy Nash equilibrium in the associated matrix game, where mixed strategies are allowed as well.

Theorem 13.10 (Zermelo's Theorem) *In the game of chess, either White has a pure strategy that guarantees a win, or Black has a pure strategy that guarantees a win, or both players have pure strategies that guarantee at least a draw.*

Proof Let $A = (a_{ij})$ denote the associated matrix game, and let row i^* and column j^* constitute a pure strategy Nash equilibrium. We distinguish three cases.

Case 1. $a_{i^*j^*} = 1$, i.e., White wins. By Theorem 13.9, $v(A) = 1$. Hence, White has a pure strategy that guarantees a win, namely to play row i^*.

Case 2. $a_{i^*j^*} = -1$, i.e., Black wins. By Theorem 13.9, $v(A) = -1$. Hence, Black has a pure strategy that guarantees a win, namely to play column j^*.

Case 3. $a_{i^*j^*} = 0$, i.e., the game ends in a draw. By Theorem 13.9, $v(A) = 0$. Hence, both White and Black can guarantee at least a draw by playing row i^* and column j^*, respectively. ■

13.3 Iterated Dominance and Best Reply

A pure strategy of a player in a finite game is strictly dominated if there is another (mixed or pure) strategy that yields always—whatever the other players do—a strictly higher payoff. Such a strategy is not played in a Nash equilibrium, and can therefore be eliminated. In the smaller game there may be another pure strategy of the same or of another player that is strictly dominated and again may be eliminated. This way a game may be reduced to a smaller game for which it is easier to compute the Nash equilibria. If the procedure results in a unique surviving strategy combination then the game is called *dominance solvable*, but this is a rare exception.

We applied these ideas before, in Chaps. 2 and 3. In this section we show, formally, that by this procedure of iterated elimination of strictly dominated strategies no Nash equilibria of the original game are lost, and no Nash equilibria are added.

For iterated elimination of weakly dominated strategies the situation is different: Nash equilibria may be lost, and the final result may depend on the order of elimination. See Problem 3.6.

We start with repeating the definition of a strictly dominated strategy for an arbitrary finite game.

Definition 13.11 Let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a finite game, $i \in N$, $s_i \in S_i$. Strategy s_i is *strictly dominated by strategy* $\sigma_i \in \Delta(S_i)$ if $u_i(\sigma_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i})$ for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$. Strategy $s_i \in S_i$ is *strictly dominated* if it is strictly dominated by some strategy $\sigma_i \in \Delta(S_i)$. □

The fact that iterated elimination of strictly dominated strategies does not essentially change the set of Nash equilibria of a game is a straightforward consequence of the following lemma.

Lemma 13.12 *Let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a finite game, $i \in N$, and let $s_i \in S_i$ be strictly dominated. Let $G' = (N, S_1, \ldots, S_{i-1}, S_i \setminus \{s_i\}, S_{i+1}, \ldots, S_n, u'_1, \ldots, u'_n)$ be the game arising from G be eliminating s_i from S_i and restricting the payoff functions accordingly. Then:*

(1) *If σ is a Nash equilibrium in G, then $\sigma_i(s_i) = 0$ (where $\sigma_i(s_i)$ is the probability assigned by σ_i to pure strategy $s_i \in S_i$) and σ' is a Nash equilibrium in G', where $\sigma'_j = \sigma_j$ for each $j \in N \setminus \{i\}$ and σ'_i is the restriction of σ_i to $S_i \setminus \{s_i\}$.*
(2) *If σ' is a Nash equilibrium in G', then σ is a Nash equilibrium in G, where $\sigma_j = \sigma'_j$ for each $j \in N \setminus \{i\}$ and $\sigma_i(t_i) = \sigma'_i(t_i)$ for all $t_i \in S_i \setminus \{s_i\}$.*

Proof

(1) Let σ be a Nash equilibrium in G, and let $\tau_i \in \Delta(S_i)$ strictly dominate s_i. If $\sigma_i(s_i) > 0$, then

$$u_i(\hat{\sigma}_i + \sigma_i(s_i)\tau_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}) ,$$

where $\hat{\sigma}_i : S_i \to \mathbb{R}$ is defined by $\hat{\sigma}_i(t_i) = \sigma_i(t_i)$ for all $t_i \in S_i \setminus \{s_i\}$ and $\hat{\sigma}_i(s_i) = 0$. This contradicts the assumption that σ is a Nash equilibrium in G. Therefore, $\sigma_i(s_i) = 0$. With σ' and G' as above, we have

$$u'_i(\sigma'_1, \ldots, \sigma'_{i-1}, \tau'_i, \sigma'_{i+1}, \ldots, \sigma'_n) = u_i(\sigma_1, \ldots, \sigma_{i-1}, \bar{\tau}'_i, \sigma_{i+1}, \ldots, \sigma_n) ,$$

for every $\tau'_i \in \Delta(S_i \setminus \{s_i\})$, where $\bar{\tau}'_i \in \Delta(S_i)$ assigns 0 to s_i and is equal to τ'_i otherwise. From this it follows that σ'_i is still a best reply to σ'_{-i}. It is straightforward that also for each $j \neq i$, σ'_j is still a best reply to σ'_{-j}. Hence σ' is a Nash equilibrium in G'.
(2) Let σ' and σ be as in (2) of the lemma. Obviously, for every player $j \neq i$, σ_j is still a best reply in σ since $\sigma_i(s_i) = 0$, i.e., player i puts zero probability on the new pure strategy s_i. For player i, σ_i is certainly a best reply among all strategies that put zero probability on s_i. But then, σ_i is a best reply among all strategies, since strategies that put nonzero probability on s_i can never be best replies by the first argument in the proof of (1). Hence, σ is a Nash equilibrium in G. ■

Obviously, a strictly dominated pure strategy is not only never played in a Nash equilibrium, but, a fortiori, is never (part of) a best reply. Formally, we say that a pure strategy s_i of player i in the finite game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ is *never a best reply* if for all $(\sigma_j)_{j \neq i}$ and all $\sigma_i \in \beta_i((\sigma_j)_{j \neq i})$, we have $\sigma_i(s_i) = 0$. The following result shows that for two-player games also the converse holds.

Theorem 13.13 *In a finite two-person game every pure strategy that is never a best reply, is strictly dominated.*

Proof Let (A, B) be an $m \times n$ bimatrix game and suppose without loss of generality that pure strategy $\mathbf{e}^1 \in \Delta^m$ of player 1 is never a best reply. Let $\mathbf{b} = (-1, -1, \ldots, -1) \in \mathbb{R}^n$.

Let $\tilde{A}$ be the $(m-1) \times n$ matrix with i-th row equal to the first row of A minus the $i+1$-th row of A, i.e., $\tilde{a}_{ij} = a_{1j} - a_{i+1,j}$ for every $i = 1, \ldots, m-1$ and $j = 1, \ldots, n$. Thus,

$$\tilde{A} = \begin{pmatrix} a_{11} - a_{21} & \cdots & a_{1n} - a_{2n} \\ a_{11} - a_{31} & \cdots & a_{1n} - a_{3n} \\ \vdots & \ddots & \vdots \\ a_{11} - a_{n1} & \cdots & a_{1n} - a_{nn} \end{pmatrix} .$$

The assumption that the pure strategy $\mathbf{e}^1$ of player 1 is never a best reply is equivalent to the statement that the system

$$\tilde{A}\mathbf{q} = \begin{pmatrix} \mathbf{e}^1 A\mathbf{q} - \mathbf{e}^2 A\mathbf{q} \\ \vdots \\ \mathbf{e}^1 A\mathbf{q} - \mathbf{e}^n A\mathbf{q} \end{pmatrix} \geq \mathbf{0}, \ \mathbf{q} \in \Delta^n$$

has no solution. This, in turn, is equivalent to the statement that the system

$$\tilde{A}\mathbf{q} \geq \mathbf{0}, \ \mathbf{q} \geq \mathbf{0}, \ \mathbf{q} \cdot \mathbf{b} < 0$$

has no solution. This means that the system in (2) of Lemma 22.7 (with $\tilde{A}$ instead of A there) has no solution. Hence, this lemma implies that the system

$$\mathbf{x} \in \mathbb{R}^{m-1}, \ \mathbf{x}\tilde{A} \leq \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}$$

has a solution. By definition of $\mathbf{b}$ and $\tilde{A}$ we have for such a solution $\mathbf{x} = (x_2, \ldots, x_m)$:

$$\mathbf{x} \geq \mathbf{0} \ \text{ and } \ \sum_{i=2}^{m} x_i \, \mathbf{e}^i A \geq \sum_{i=2}^{m} x_i \, \mathbf{e}^1 A + (1, \ldots, 1) .$$

This implies that $\mathbf{x} \neq \mathbf{0}$ and therefore that $\mathbf{e}^1$ is strictly dominated by the strategy

$$\left(0, x_2 / \sum_{i=2}^{m} x_i, \ldots, x_m / \sum_{i=2}^{m} x_i\right) \in \Delta^m .$$

Hence, $\mathbf{e}^1$ is strictly dominated. ■

For games with more than two players Theorem 13.13 does not hold, see Problem 13.13 for a counterexample.

The concept of 'never a best reply' is closely related to the concept of *rationalizability*. Roughly, rationalizable strategies are strategies that survive a process of iterated elimination of strategies that are never a best reply. Just like the strategies surviving iterated elimination of strictly dominated strategies, rationalizable strategies constitute a set-valued solution concept. The above theorem implies that for two-player games the two solution concepts coincide. In general, the set of rationalizable strategies is a subset of the set of strategies that survive iterated elimination of strictly dominated strategies.

The implicit assumption justifying iterated elimination of strategies that are dominated or never a best reply is quite demanding. Not only should a player believe that some other player will not play a such a strategy, but he should also believe that the other player believes that he (the first player) believes this and, in turn, will not use such a strategy in the reduced game, and so on and so forth. See the notes at the end of the chapter for references.

13.4 Perfect Equilibrium

Since a game may have many, quite different Nash equilibria, the literature has focused since a long time on *refinements* of Nash equilibrium. We have seen examples of this in extensive form games, such as subgame perfect equilibrium and perfect Bayesian equilibrium (Chaps. 4 and 5). One of the earliest and best known refinements of Nash equilibrium in strategic form games is the concept of 'trembling hand perfection'. This refinement excludes Nash equilibria that are not robust against 'trembles' in the players' strategies.

Formally, let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a finite game and let μ be an *error function*, assigning a number $\mu_{ih} \in (0, 1)$ to every $i \in N$ and $h \in S_i$, such that $\sum_{h \in S_i} \mu_{ih} < 1$ for every player i. The number μ_{ih} is the minimum probability with which player i is going to play pure strategy h, perhaps by 'mistake' ('trembling hand'). Let, for each $i \in N$, $\Delta(S_i, \mu) = \{\sigma_i \in \Delta(S_i) \mid \sigma_i(h) \geq \mu_{ih} \text{ for all } h \in S_i\}$, and let $G(\mu)$ denote the game derived from G by assuming that each player i may only choose strategies from $\Delta(S_i, \mu)$. The game $G(\mu)$ is called the *μ-perturbed game*. Denote the set of Nash equilibria of G by $NE(G)$ and of $G(\mu)$ by $NE(G(\mu))$.

Lemma 13.14 *For every error function μ, $NE(G(\mu)) \neq \emptyset$.*

Proof Analogous to the proof of Theorem 13.1. ■

A perfect equilibrium is a strategy combination that is the limit of *some* sequence of Nash equilibria of perturbed games. Formally:

Definition 13.15 A strategy combination σ is a *perfect equilibrium* if there is a sequence $G(\mu^t)$, $t \in \mathbb{N}$, of perturbed games with $\mu^t \to \mathbf{0}$ for $t \to \infty$ and a sequence of Nash equilibria $\sigma^t \in G(\mu^t)$ such that $\sigma^t \to \sigma$ for $t \to \infty$. □

As will follow from Theorem 13.17 below, a perfect equilibrium is a Nash equilibrium. So the expressions *perfect equilibrium* and *perfect Nash equilibrium* are equivalent.

Call a strategy combination σ in G *completely mixed* if $\sigma_i(h) > 0$ for all $i \in N$ and $h \in S_i$.

Lemma 13.16 *A completely mixed Nash equilibrium of G is a perfect equilibrium.*

Proof Problem 13.14. ■

Also if a game has no completely mixed Nash equilibrium, it still has a perfect equilibrium.

Theorem 13.17 *Every finite game* $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ *has a perfect equilibrium. Every perfect equilibrium is a Nash equilibrium.*

Proof Take any sequence $(G(\mu^t))_{\mu^t \to \mathbf{0}}$ of perturbed games and $\sigma^t \in NE(G(\mu^t))$ for each $t \in \mathbb{N}$. Since $\prod_{i \in N} \Delta(S_i)$ is a compact set we may assume without loss of generality that the sequence $(\sigma^t)_{t \in \mathbb{N}}$ converges to some $\sigma \in \prod_{i \in N} \Delta(S_i)$. Then σ is perfect. It is easy to verify that $\sigma \in NE(G)$. ■

Example 13.18 Consider the bimatrix game

$$G = \begin{array}{c} \\ U \\ M \\ D \end{array} \begin{array}{c} \begin{array}{ccc} L & C & R \end{array} \\ \begin{pmatrix} 1,1 & 0,0 & 2,0 \\ 1,2 & 1,2 & 1,1 \\ 0,0 & 1,1 & 1,1 \end{pmatrix} \end{array}.$$

(Cf. Problem 3.6.) The set of Nash equilibria in this game is the union of the following sets (Problem 13.8):

(i) $\{((p, 0, 1-p), (0, \frac{1}{2}, \frac{1}{2})) \mid 0 \le p \le \frac{1}{2}\}$,
(ii) $\{((0, 0, 1), (0, q, 1-q)) \mid \frac{1}{2} \le q \le 1\}$,
(iii) $\{((p, 1-p, 0), (1, 0, 0)) \mid 0 \le p \le 1\}$,
(iv) $\{((0, 1, 0), (q, 1-q, 0)) \mid 0 \le q \le 1\}$,
(v) $\{((0, p, 1-p), (0, 1, 0)) \mid 0 \le p \le 1\}$.

We consider these collections one by one. The Nash equilibria in the first collection are not perfect, for the following reason. In any perturbed game, player 1 plays each pure strategy with positive probability. As a consequence, for player 2, R always gives a strictly lower expected payoff than C: if player 1 plays $\mathbf{p} = (p_1, p_2, p_3) > \mathbf{0}$, then the payoff from C is $2p_2 + p_3$, and hence strictly larger than $p_2 + p_3$, which is the payoff from R. Thus, any best reply of player 2 in a perturbed game puts minimal probability on R, so that the strategy $(0, \frac{1}{2}, \frac{1}{2})$ can never be the limit of strategies of player 2 in Nash equilibria of perturbed games. In fact, this argument preludes on Theorem 13.21, which states that weakly dominated strategies (the third column in this case) are never played with positive probability in a perfect equilibrium. For a similar reason, the equilibria in the second collection are not perfect: D gives player 1 a strictly lower payoff than M if player 2 plays each column with positive probability, as is the case in a perturbed game. Hence, player 1's strategy $(0, 0, 1)$ cannot occur as the limit of Nash equilibrium strategies in perturbed games.

Now consider the collection in (iii). We will show that all Nash equilibria in this collection are perfect. For every $\varepsilon > 0$ let $G(\varepsilon)$ be the perturbed game with $\mu_{ij} = \varepsilon$ for each player $i = 1, 2$ and each row/column $j = 1, 2, 3$. First, suppose $p = 1$ and consider the strategy combination $((1 - 2\varepsilon, \varepsilon, \varepsilon), (1 - 2\varepsilon, \varepsilon, \varepsilon))$ in $G(\varepsilon)$.[1] Given the strategy of player 2, the three pure strategies (rows) of player 1 result in the payoffs 1, 1, and 2ε, respectively, so that player 1's strategy $(1 - 2\varepsilon, \varepsilon, \varepsilon)$ is a best reply: the inferior row D is only played with the minimally required probability. Similarly, given this strategy of player 1, the three pure strategies (columns) of player 2 yield 1, 3ε, and 2ε, so that player 2's strategy $(1 - 2\varepsilon, \varepsilon, \varepsilon)$ is a best reply: it puts only the minimally required probability on the inferior columns C and R. Hence, $((1 - 2\varepsilon, \varepsilon, \varepsilon), (1 - 2\varepsilon, \varepsilon, \varepsilon))$ is a Nash equilibrium in $G(\varepsilon)$. In fact, with these arguments we are preluding on Lemma 13.19, in particular part (2). Since $((1 - 2\varepsilon, \varepsilon, \varepsilon), (1 - 2\varepsilon, \varepsilon, \varepsilon)) \to ((1, 0, 0), (1, 0, 0))$ for $\varepsilon \to 0$, we conclude that $((1, 0, 0), (1, 0, 0))$ is a perfect equilibrium.

For the other cases in (iii) the arguments are similar. For $p = 0$, take the strategy combination $((\varepsilon, 1 - 2\varepsilon, \varepsilon), (1 - 2\varepsilon, \varepsilon, \varepsilon))$ in $G(\varepsilon)$. Given the strategy of player 2 the three rows yield 1, 1, and 2ε as before, so that player 1's strategy is a best reply. Given the strategy of player 1 the three columns now yield $2 - 3\varepsilon$, $2 - 3\varepsilon$, and $1 - \varepsilon$, so that player 2 still has a best reply, putting only the minimally required probability on the inferior R. Thus, $((\varepsilon, 1 - 2\varepsilon, \varepsilon), (1 - 2\varepsilon, \varepsilon, \varepsilon))$ is a Nash equilibrium in $G(\varepsilon)$, converging to $((0, 1, 0), (1, 0, 0))$ as ε goes to zero. Hence, $((0, 1, 0), (1, 0, 0))$ is a perfect equilibrium. Finally, for $0 < p < 1$, consider the strategy combination $((p - \frac{\varepsilon}{2}, 1 - p - \frac{\varepsilon}{2}, \varepsilon), (1 - 2\varepsilon, \varepsilon, \varepsilon))$ in $G(\varepsilon)$. Player 1 still plays a best reply, as before. For player 2, the three columns yield $2 - p - \frac{3\varepsilon}{2}$, $2 - 2p$, and $1 - p + \frac{\varepsilon}{2}$. Since player 2's strategy puts only the minimally required probability on C and R, it is a best reply. Thus, the strategy combination $((p - \frac{\varepsilon}{2}, 1 - p - \frac{\varepsilon}{2}, \varepsilon), (1 - 2\varepsilon, \varepsilon, \varepsilon))$ is a Nash equilibrium in $G(\varepsilon)$, implying that its limit $((p, 1 - p, 0), (1, 0, 0))$ is perfect.

Also the Nash equilibria in (iv) are perfect: the arguments are analogous to those for the collection in (iii) (Problem 13.8). In collection (v), finally, if $1 - p > 0$, then the Nash equilibrium cannot be perfect since D always gives a strictly lower payoff than M in any perturbed game, and therefore should be played with minimal probability, hence zero in the limit. The Nash equilibrium $((0, 1, 0), (0, 1, 0))$ is also an element of the collection in (iv), and thus perfect. □

[1] Whenever needed we assume that ε is sufficiently small, which is without loss of generality since we consider the limit for $\varepsilon \to 0$. To comply with our definitions we may take values $\varepsilon = \varepsilon^t = 1/t$, $t \in \mathbb{N}$, but this is not essential.

The following lemma formalizes some of the arguments already used in Example 13.18. It relates the Nash equilibria of a perturbed game to pure best replies in such a game. The first part says that, in a Nash equilibrium of a perturbed game, if a player puts more than the minimally required probability on some pure strategy, then that pure strategy must be a best reply. The second part says that, in some strategy combination, if all players only put more than the minimally required probabilities on pure best replies, then that combination must be a Nash equilibrium.

Lemma 13.19 *Let $G(\mu)$ be a perturbed game and let σ a strategy combination in $G(\mu)$.*

(1) *If $\sigma \in NE(G(\mu))$, $i \in N$, $h \in S_i$, and $\sigma_i(h) > \mu_{ih}$, then $h \in \beta_i(\sigma_{-i})$.*
(2) *If for all $i \in N$ and $h \in S_i$, $\sigma_i(h) > \mu_{ih}$ implies $h \in \beta_i(\sigma_{-i})$, then $\sigma \in NE(G(\mu))$.*

Proof

(1) Let $\sigma \in NE(G(\mu))$, $i \in N$, $h \in S_i$, and $\sigma_i(h) > \mu_{ih}$. Suppose, contrary to what we wish to prove, that $h \notin \beta_i(\sigma_{-i})$. Take $h' \in S_i$ with $h' \in \beta_i(\sigma_{-i})$. (Such an h' exists by an argument similar to the proof of Lemma 13.3.) Consider the strategy σ_i' defined by $\sigma_i'(h) = \mu_{ih}$, $\sigma_i'(h') = \sigma_i(h') + \sigma_i(h) - \mu_{ih}$, and $\sigma_i'(k) = \sigma_i(k)$ for all $k \in S_i \setminus \{h, h'\}$. Then $\sigma_i' \in \Delta(S_i, \mu)$ and $u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma)$, contradicting the assumption $\sigma \in NE(G(\mu))$.
(2) Let $i \in N$. The condition in (2) implies that, if $h \in S_i$ and $h \notin \beta_i(\sigma_{-i})$, then $\sigma_i(h) = \mu_{ih}$. This implies that σ_i is a best reply to $(\sigma_j)_{j \neq i}$. Thus, $\sigma \in NE(G(\mu))$. ■

Below we present two characterizations of perfect Nash equilibrium that both avoid sequences of perturbed games. The first one is based on the notion of ε-perfect equilibrium, defined as follows. Let $\varepsilon > 0$. A strategy combination $\sigma \in \prod_{i \in N} \Delta(S_i)$ is an *ε-perfect equilibrium* of G if it is completely mixed and $\sigma_i(h) \leq \varepsilon$ for all $i \in N$ and all $h \in S_i$ with $h \notin \beta_i(\sigma_{-i})$.

An ε-perfect equilibrium of G need not be a Nash equilibrium of G, but it puts probabilities of at most ε on pure strategies that are not best replies.

The announced characterizations are collected in the following theorem. The theorem says that a perfect equilibrium is a limit of ε-perfect equilibria. Also, a perfect equilibrium is a limit of completely mixed strategy combinations such that the strategy of each player in the perfect equilibrium under consideration is a best reply to the strategy combinations of the other players in those completely mixed strategy combinations.

Theorem 13.20 *Let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ and $\sigma \in \prod_{i \in N} \Delta(S_i)$. The following statements are equivalent:*

(1) *σ is a perfect equilibrium of G;*
(2) *σ is a limit of a sequence of ε-perfect equilibria σ^{ε} of G for $\varepsilon \to 0$;*

(3) *σ is a limit of a sequence of completely mixed strategy combinations σ^ε for $\varepsilon \to 0$, where $\sigma_i \in \beta_i(\sigma^\varepsilon_{-i})$ for each $i \in N$ and each σ^ε in this sequence.*

Proof (1) $\Rightarrow$ (2): Take a sequence of perturbed games $G(\mu^t)$, $t \in \mathbb{N}$ with $\mu^t \to \mathbf{0}$ and a sequence $\sigma^t \in NE(G(\mu^t))$ with $\sigma^t \to \sigma$. For each t define $\varepsilon^t \in \mathbb{R}$ by $\varepsilon^t = \max\{\mu^t_{ih} \mid i \in N,\ h \in S_i\}$. Then, by Lemma 13.19(1), σ^t is an ε^t-perfect equilibrium for every t. So (2) follows.
(2) $\Rightarrow$ (3): Take a sequence of ε-perfect equilibria σ^ε as in (2) converging to σ for $\varepsilon \to 0$. Let $i \in N$. By the definition of ε-perfect equilibrium, if $\sigma_i(h) > 0$ for some $h \in S_i$, then for ε sufficiently small we have $h \in \beta^\varepsilon_i(\sigma_{-i})$. This implies $\sigma_i \in \beta_i(\sigma^\varepsilon_{-i})$, and (3) follows.
(3) $\Rightarrow$ (1): Let σ^{ε^t}, $t \in \mathbb{N}$, be a sequence as in (3) with $\varepsilon^t \to 0$ and $\sigma^{\varepsilon^t} \to \sigma$ as $t \to \infty$. For each $t \in \mathbb{N}$, $i \in N$ and $h \in S_i$ define $\mu^t_{ih} = \sigma^{\varepsilon^t}_i(h)$ if $\sigma_i(h) = 0$ and $\mu^t_{ih} = \varepsilon^t$ otherwise. Then, for t sufficiently large, μ^t is an error function, $G(\mu^t)$ is a perturbed game, and σ^{ε^t} is a strategy combination in $G(\mu^t)$. By Lemma 13.19(2), $\sigma^{\varepsilon^t} \in NE(G(\mu^t))$. So σ is a perfect Nash equilibrium of G. ■

There is a close relation between the concept of domination and the concept of perfection. We first extend the concept of (weak) domination to mixed strategies. In the game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$, call a strategy $\sigma_i \in \Delta(S_i)$ *(weakly) dominated by* $\sigma'_i \in \Delta(S_i)$ if $u_i(\sigma_i, \sigma_{-i}) \le u_i(\sigma'_i, \sigma_{-i})$ for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$, with at least one inequality strict. (Observe that it is actually sufficient to check this for combinations $s_{-i} \in \prod_{j \neq i} S_i$.) Call σ_i *undominated* if there is no σ'_i by which it is dominated, and call a strategy combination σ *undominated* if σ_i is undominated for every $i \in N$. We now have:

Theorem 13.21 *Every perfect Nash equilibrium in G is undominated.*

Proof Let σ be a perfect Nash equilibrium and suppose that (say) σ_1 is dominated. Then there is a $\sigma'_1 \in \Delta(S_1)$ such that $u_1(\sigma_1, s_{-1}) \le u_1(\sigma'_1, s_{-1})$ for all $s_{-1} \in \prod_{i=2}^n S_i$, with at least one inequality strict. Take a sequence $(\sigma^t)_{t \in \mathbb{N}}$ of strategy combinations as in (3) of Theorem 13.20, converging to σ. Then, since every σ^t is completely mixed, we have $u_1(\sigma_1, \sigma^t_{-1}) < u_1(\sigma'_1, \sigma^t_{-1})$ for every t. This contradicts the fact that σ_1 is a best reply to σ^t_{-1}. ■

The converse of Theorem 13.21 is only true for two-person games. For a counterexample involving three players, see Problem 13.15.

For proving the converse of the theorem for bimatrix games, we use the following auxiliary lemma. In this lemma, for a matrix game $\tilde{A}$, $C_2(\tilde{A})$ denotes the set of all columns of $\tilde{A}$ that are in the carrier of some optimal strategy of player 2 in $\tilde{A}$.

Lemma 13.22 *Let $G = (A, B)$ be an $m \times n$ bimatrix game and let $\mathbf{p} \in \Delta^m$. Define the $m \times n$ matrix $\tilde{A} = (\tilde{a}_{ij})$ by $\tilde{a}_{ij} = a_{ij} - \mathbf{p}A\mathbf{e}^j$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Then $\mathbf{p}$ is undominated in G if and only if $v(\tilde{A}) = 0$ and $C_2(\tilde{A}) = \{1, \ldots, n\}$.*

Proof First note that $\mathbf{p}\tilde{A} = \mathbf{0}$ and therefore $v(\tilde{A}) \geq 0$.

For the if-direction, suppose that $\mathbf{p}$ is dominated in G, say by $\mathbf{p}'$. Then $\mathbf{p}'A \gneqq \mathbf{p}A$, hence $\mathbf{p}'\tilde{A} \gneqq \mathbf{0}$. Therefore, if $v(\tilde{A}) = 0$, then $\mathbf{p}'$ is an optimal strategy in $\tilde{A}$; take a column j with $\mathbf{p}'\tilde{A}\mathbf{e}^j > 0$, then we have $j \notin C_2(\tilde{A})$ and, thus, $C_2(\tilde{A}) \neq \{1, \ldots, n\}$. This proves the if-direction.

For the only-if direction suppose that $\mathbf{p}$ is undominated in G. Suppose we had $v(\tilde{A}) > 0$. Then take an optimal strategy $\tilde{\mathbf{p}}$ of player 1 in $\tilde{A}$, so that $\tilde{\mathbf{p}}\tilde{A} > \mathbf{0}$, hence $\tilde{\mathbf{p}}A > \mathbf{p}A$, a contradiction. Thus, $v(\tilde{A}) = 0$. Suppose there is a column j that is not an element of $C_2(\tilde{A})$. By Problem 12.7 there must be an optimal strategy $\mathbf{p}'$ of player 1 in $\tilde{A}$ such that $\mathbf{p}'\tilde{A}\mathbf{e}^j > 0$, so that $\mathbf{p}'\tilde{A} \gneqq \mathbf{0}$, hence $\mathbf{p}'A \gneqq \mathbf{p}A$. So $\mathbf{p}'$ dominates $\mathbf{p}$ in G, a contradiction. This proves the only-if direction. ■

Theorem 13.23 *Let $G = (A, B)$ be a bimatrix game, and let $(\mathbf{p}, \mathbf{q})$ be an undominated Nash equilibrium. Then $(\mathbf{p}, \mathbf{q})$ is perfect.*

Proof Let $\tilde{A}$ as in Lemma 13.22, then $\mathbf{p}$ is an optimal strategy for player 1 in $\tilde{A}$ since $\mathbf{p}\tilde{A} = \mathbf{0}$ and $v(\tilde{A}) = 0$. By Lemma 13.22 we can find a completely mixed optimal strategy $\mathbf{q}'$ for player 2 in $\tilde{A}$. So $\mathbf{p}$ is a best reply to $\mathbf{q}'$ in $\tilde{A}$, i.e., $\mathbf{p}\tilde{A}\mathbf{q}' \geq \tilde{\mathbf{p}}\tilde{A}\mathbf{q}'$ for all $\tilde{\mathbf{p}}$, and thus $\tilde{\mathbf{p}}A\mathbf{q}' - \mathbf{p}A\mathbf{q}' \leq 0$ for all $\tilde{\mathbf{p}}$. So $\mathbf{p}$ is also a best reply to $\mathbf{q}'$ in G. For $1 > \varepsilon > 0$ define $\mathbf{q}^\varepsilon = (1 - \varepsilon)\mathbf{q} + \varepsilon\mathbf{q}'$. Then $\mathbf{q}^\varepsilon$ is completely mixed, $\mathbf{p}$ is a best reply to $\mathbf{q}^\varepsilon$, and $\mathbf{q}^\varepsilon \to \mathbf{q}$ for $\varepsilon \to 0$. In the same way we can construct a sequence $\mathbf{p}^\varepsilon$ with analogous properties, converging to $\mathbf{p}$. Then implication (3) $\Rightarrow$ (1) in Theorem 13.20 implies that $(\mathbf{p}, \mathbf{q})$ is perfect. ■

Example 13.24 Consider again the game G from Example 13.18. Row D is weakly dominated by M, and R by C. There are no other weakly dominated strategies in this game. Thus, as established earlier, but now using Theorems 13.21 and 13.23, the set of perfect equilibria is the set $\{((p, 1-p, 0), (1, 0, 0)) \mid 0 \leq p \leq 1\} \cup \{((0, 1, 0), (q, 1-q, 0)) \mid 0 \leq q \leq 1\}$. □

The following example shows an advantage but at the same time a drawback of perfect equilibrium: a perfect equilibrium may be payoff-dominated by another Nash equilibrium.

Example 13.25 Consider the bimatrix game

$$\begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 1, 1 & 10, 0 \\ 0, 10 & 10, 10 \end{pmatrix} \end{array},$$

which has two Nash equilibria, both pure, namely (U, L) and (D, R). Only (U, L) is perfect, as can be seen by direct inspection or by applying Theorems 13.21 and 13.23. At the equilibrium (D, R), each player has an incentive to deviate to the other pure strategy since the opponent may deviate by mistake. This equilibrium is excluded by perfection. On the other hand, the unique perfect equilibrium (U, L) is payoff-dominated by the equilibrium (D, R). □

Another drawback of perfect equilibrium is the fact that adding strictly dominated strategies may result in adding perfect Nash equilibria, as the following example shows.

Example 13.26 In the game

$$\begin{array}{c} \\ U \\ M \\ D \end{array} \begin{array}{c} \begin{array}{ccc} L & C & R \end{array} \\ \begin{pmatrix} 1,1 & 0,0 & -1,-2 \\ 0,0 & 0,0 & 0,-2 \\ -2,-1 & -2,0 & -2,-2 \end{pmatrix} \end{array},$$

there are two perfect Nash equilibria, namely (U, L) and (M, C). If we reduce the game by deleting the strictly dominated pure strategies D and R, the only perfect equilibrium that remains is (U, L). □

This motivated the introduction of a further refinement called *proper Nash equilibrium*. See the next section.

13.5 Proper Equilibrium

A perfect equilibrium is required to be robust only against *some* 'trembles' and, moreover, there are no further conditions on these trembles. We now propose the additional restriction that trembles be less probable if they are more 'costly'.

Given some $\varepsilon > 0$, call a strategy combination σ in the game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ an *ε-proper equilibrium* if σ is completely mixed and for all $i \in N$ and $h, k \in S_i$ we have

$$u_i(h, \sigma_{-i}) < u_i(k, \sigma_{-i}) \Rightarrow \sigma_i(h) \leq \varepsilon \sigma_i(k) \ .$$

Observe that an ε-proper equilibrium does not have to be a Nash equilibrium.

Definition 13.27 A strategy combination σ in G is *proper* if, for some sequence $\varepsilon^t \to 0$, $t \in \mathbb{N}$, there exist ε^t-proper equilibria $\sigma(\varepsilon^t)$ such that $\sigma(\varepsilon^t) \to \sigma$. □

Since, in a proper strategy combination σ, a pure strategy h of a player i that is not a best reply to σ_{-i} is played with probability 0, it follows that a proper strategy combination is a Nash equilibrium. Moreover, since it is straightforward by the definitions that an ε-proper equilibrium is also an ε-perfect equilibrium, it follows from Theorem 13.20 that a proper equilibrium is perfect. Hence, properness is a refinement of perfection.

Remark 13.28 By replacing the word 'proper' by 'perfect' in Definition 13.27, we obtain an alternative definition of perfect equilibrium. This follows from Theorem 13.20. □

Example 13.26 shows that properness is a strict refinement of perfection: the Nash equilibrium (M, C) is perfect but not proper. To see this, for $\varepsilon^t > 0$ with $\varepsilon^t \to 0$ as $t \to \infty$, let $(\mathbf{p}(\varepsilon^t), \mathbf{q}(\varepsilon^t))$ be a converging sequence of ε^t-proper equilibria. Then for each t we must have $q_3(\varepsilon^t) \leq \varepsilon^t q_1(\varepsilon^t)$. In turn, this implies that U is the unique best reply to $\mathbf{q}(\varepsilon^t)$, hence $p_2(\varepsilon^t) \leq \varepsilon^t p_1(\varepsilon^t)$ for each t. But then $\mathbf{p}(\varepsilon^t)$ cannot converge to $(0, 1, 0)$.

A proper Nash equilibrium always exists:

Theorem 13.29 *Let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a finite game. Then G has a proper Nash equilibrium.*

Proof It is sufficient to show that for $\varepsilon > 0$ close to 0 there exists an ε-proper equilibrium of G. Let $0 < \varepsilon < 1$ and define the error function μ by $\mu_{ik} = \varepsilon^{|S_i|}/|S_i|$ for all $i \in N$ and $k \in S_i$. For every $i \in N$ and $\sigma \in \prod_{j \in N} \Delta(S_j, \mu)$ define

$$F_i(\sigma) = \{\tau_i \in \Delta(S_i, \mu) \mid \forall k, l \in S_i \ [u_i(k, \sigma_{-i}) < u_i(l, \sigma_{-i}) \Rightarrow \tau_i(k) \leq \varepsilon \tau_i(l)]\} .$$

Then $F_i(\sigma) \neq \emptyset$, as can be seen as follows. Define

$$v_i(\sigma, k) = |\{l \in S_i \mid u_i(k, \sigma_{-i}) < u_i(l, \sigma_{-i})\}| ,$$

and define $\tau_i(k) = \varepsilon^{v_i(\sigma,k)} / \sum_{l \in S_i} \varepsilon^{v_i(\sigma,l)}$, for all $k \in S_i$. Then $\tau_i \in F_i(\sigma)$. Consider the correspondence

$$F : \prod_{j \in N} \Delta(S_j, \mu) \to \prod_{j \in N} \Delta(S_j, \mu), \quad \sigma \mapsto \prod_{i \in N} F_i(\sigma) .$$

Then F satisfies the conditions of the Kakutani Fixed Point Theorem (Theorem 22.11)—see Problem 13.16. Hence, F has a fixed point, and each fixed point of F is an ε-proper equilibrium of G. ■

In spite of the original motivation for introducing properness, this concept suffers from the same deficit as perfect equilibrium: adding strictly dominated strategies may enlarge the set of proper Nash equilibria. See Problem 13.17 for an example of this.

Example 13.30 We consider again the game G from Example 13.18:

$$G = \begin{matrix} & L & C & R \\ U \\ M \\ D \end{matrix}\begin{pmatrix} 1,1 & 0,0 & 2,0 \\ 1,2 & 1,2 & 1,1 \\ 0,0 & 1,1 & 1,1 \end{pmatrix}.$$

The perfect equilibria are:

(a) $\{((p, 1-p, 0), (1, 0, 0)) \mid 0 \le p \le 1\}$,

(b) $\{((0, 1, 0), (q, 1-q, 0)) \mid 0 \le q \le 1\}$.

First consider the equilibria in (a). Which ones are proper? In any ε-proper equilibrium $(\mathbf{p}^\varepsilon, \mathbf{q}^\varepsilon)$, player 1 plays a completely mixed strategy. This implies that C gives player 2 a higher payoff than R, so that $q_3^\varepsilon \le \varepsilon q_2^\varepsilon$. For player 1, the payoff of U is $q_1^\varepsilon + 2q_3^\varepsilon$ and the payoff of M is $q_1^\varepsilon + q_2^\varepsilon + q_3^\varepsilon > q_1^\varepsilon + 2q_3^\varepsilon$ for ε small, so that we obtain $p_1^\varepsilon \le \varepsilon p_2^\varepsilon$. Hence, for $p > 0$ the equilibria in (a) are not proper. The equilibrium $((0, 1, 0), (1, 0, 0))$ is proper: it is the limit of the ε-proper equilibria $((\varepsilon, 1, \varepsilon^2)/\eta, (1, \varepsilon, \varepsilon^2)/\eta)$, where $\eta = 1 + \varepsilon + \varepsilon^2$.

Next consider the equilibria in (b). As before, in any ε-proper equilibrium $(\mathbf{p}^\varepsilon, \mathbf{q}^\varepsilon)$ we must have $q_3^\varepsilon \le \varepsilon q_2^\varepsilon$. This implies that for player 1, the payoff from U is higher than the payoff from D, so that we must have $p_3^\varepsilon \le \varepsilon p_1^\varepsilon$. In turn, this implies that for player 2 the payoff from L is higher than the payoff from C, so that $q_2^\varepsilon \le \varepsilon q_1^\varepsilon$. Thus, the only candidate in (b) for a proper equilibrium is $((0, 1, 0), (1, 0, 0))$, and we have already established that this combination is proper indeed. Hence, it is the unique proper equilibrium of G. □

13.6 Strictly Perfect Equilibrium

Another refinement of perfect equilibrium is obtained by requiring robustness of a Nash equilibrium with respect to *all* 'trembles'. This results in the concept of strictly perfect equilibrium.

Definition 13.31 A strategy combination σ in the game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ is a *strictly perfect equilibrium* if, for every sequence $\{G(\mu^t)\}$, $t \in \mathbb{N}$, of perturbed games with $\mu^t \to 0$ for $t \to \infty$, there exist profiles $\sigma^t \in NE(G(\mu^t))$ such that $\sigma^t \to \sigma$.

Clearly, a strictly perfect equilibrium is a perfect equilibrium. For some further observations concerning strictly perfect equilibrium see Problem 13.18.

A clear drawback of strictly perfect equilibrium is the fact that it does not have to exist, as the following (continued) example shows.

Example 13.32 We consider again the game G from Example 13.18:

$$G = \begin{matrix} U \\ M \\ D \end{matrix} \overset{\begin{matrix} L & \quad C & \quad R \end{matrix}}{\begin{pmatrix} 1,1 & 0,0 & 2,0 \\ 1,2 & 1,2 & 1,1 \\ 0,0 & 1,1 & 1,1 \end{pmatrix}},$$

with perfect equilibria:

(a) $\{((p, 1-p, 0), (1, 0, 0)) \mid 0 \leq p \leq 1\}$,

(b) $\{((0, 1, 0), (q, 1-q, 0)) \mid 0 \leq q \leq 1\}$.

Consider the equilibria in (a) and suppose $p > 0$. Let $G(\mu)$ be the perturbed game as in Sect. 13.4. We may assume that, since $p > 0$, L must yield player 2 a higher payoff than C in a Nash equilibrium of $G(\mu)$. In turn this implies that player 2's equilibrium strategy in any Nash equilibrium of $G(\mu)$ is equal to $(1 - \mu_{22} - \mu_{23}, \mu_{22}, \mu_{23})$ (we already saw that player 2 must always put minimal probability on R). Now choose $\mu_{23} < \mu_{22}$, then the payoff $1 - \mu_{22} + \mu_{23}$ to player 1 from playing U is smaller than the payoff 1 from M. This implies that player 1 plays M in the limit. Thus, a perfect equilibrium of the form $((p, 1-p, 0), (1, 0, 0))$ with $p > 0$ is not strictly perfect.

Next, consider the equilibrium $((0, 1, 0), (1, 0, 0))$, which is both in (a) and in (b). Assume again that $\mu_{23} < \mu_{22}$. If player 2 plays (q_1, q_2, q_3) in $G(\mu)$, then $q_3 = \mu_{23}$ and the payoff to player 1 from U is $q_1 + 2\mu_{23}$, whereas the payoff from M is $q_1 + q_2 + \mu_{23}$. Since $q_2 \geq \mu_{22} > \mu_{23}$, this implies that M yields player 1 a higher payoff than U. In turn, this implies that player 1's strategy in any Nash equilibrium of $G(\mu)$ is equal to $(\mu_{11}, 1 - \mu_{11} - \mu_{13}, \mu_{13})$ (recall that player 1 puts minimal probability on D). Now suppose that $\mu_{11} < \mu_{13}$, then the payoff to player 2 from C is higher than the payoff from L, so that player 2 must play C in the limit. Thus, also the perfect equilibrium $((0, 1, 0), (1, 0, 0))$ with $p > 0$ is not strictly perfect.

For the equilibria in (b) with $q < 1$, the argument is almost identical to the one in the preceding paragraph. In the end, choose $\mu_{11} > \mu_{13}$, then the payoff to player 2 from L is higher than the payoff from C, so that player 2 must play L in the limit. Hence, no equilibrium of the form $\{((0, 1, 0), (q, 1-q, 0))$ with $q < 1$ is strictly perfect. We conclude that this game has no strictly perfect equilibria. □

13.7 Correlated Equilibrium

In the preceding sections we studied several refinements of Nash equilibrium. In this section the set of Nash equilibria is extended in a way to become clear below. It is, however, not the intention to enlarge the set of Nash equilibria but rather to enable the players to reach better payoffs by allowing some communication device. This will result in the concept of *correlated equilibrium*. Attention in this section is restricted to bimatrix games.

In order to fix ideas, consider the situation where two car drivers approach a road crossing. Each driver has two pure strategies: 'stop' (s) or 'cross' (c). The preferences for the resulting combinations are as expressed by the following table:

$$(A,B) \;=\; \begin{matrix} & c & s \\ c & -10,-10 & 5,0 \\ s & 0,5 & -1,-1 \end{matrix} \;.$$

This bimatrix game has two asymmetric and seemingly unfair pure Nash equilibria, and one symmetric mixed Nash equilibrium $((3/8,5/8),(3/8,5/8))$, resulting in an expected payoff of $-5/8$ for both, and therefore also not quite satisfying.

Now suppose that traffic lights are installed that indicate c ('green') or s ('red') according to the probabilities in the following table:

$$\begin{matrix} & c & s \\ c & 0.00 & 0.55 \\ s & 0.40 & 0.05 \end{matrix} \;.$$

For example, with probability 0.55 (55 % of the time) the light is green for driver 1 and red for driver 2. Assume that the players (drivers) are not forced to obey the traffic lights but know the probabilities as given in the table. We argue that it is in each player's own interest to obey the lights if the other player does so.

If the light is green for player 1 then player 1 knows with certainty that the light is red for player 2. So if player 2 obeys the lights and stops, it is indeed optimal for player 1 to cross. If the light is red for player 1, then the conditional probability that player 2 crosses (if he obeys the lights) is equal to $0.4/0.45 \approx 0.89$ and the conditional probability that player 2 stops is $0.05/0.45 \approx 0.11$. So if player 1 stops, his expected payoff is $0.89 \cdot 0 + 0.11 \cdot -1 = -0.11$, and if he crosses his expected payoff is $0.89 \cdot -10 + 0.11 \cdot 5 = -8.35$. Clearly, it is optimal for player 1 to obey the light and stop.

For player 2 the argument is similar. If the light is green for player 2 then he knows with certainty that player 1 has red light. Thus, if player 1 obeys the red light and stops, it is optimal for player 2 to cross. If player 2 has red light then he knows that the conditional probabilities are $0.55/0.60 \approx 0.92$ and $0.05/0.60 \approx 0.08$ that player 1 has green light and red light, respectively. For player 2, assuming that player 1 obeys the traffic lights, stopping yields an expected payoff of $0.08 \cdot -1$ and crossing an expected payoff of $0.92 \cdot -10 + 0.08 \cdot 5$: clearly, stopping is optimal.

Thus, we can indeed talk of an equilibrium: such an equilibrium is called a *correlated equilibrium*. Note that there is no mixed strategy combination in the game (A,B) that induces these probabilities. If $\mathbf{p} = (p_1,p_2)$ and $\mathbf{q} = (q_1,q_2)$ are mixed strategies of players 1 and 2, respectively, then we would need $p_1 \neq 0$ since $p_1q_2 = 0.55$ and we would need $q_1 \neq 0$ since $p_2q_1 = 0.40$, but then $p_1q_1 \neq 0$, hence the combination (c,c) would have positive probability. In terms of the situation in the example, this particular equilibrium cannot be reached without traffic lights serving as a communication device between the players. The overall expected payoffs of the players are $0.55 \cdot 5 + 0.05 \cdot -1 = 2.7$ for player 1 and $0.40 \cdot 5 + 0.05 \cdot -1 = 1.95$ for player 2, which is considerably better for both than the payoffs in the mixed Nash equilibrium.

In general, let (A, B) be an $m \times n$ bimatrix game. A *correlated strategy* is an $m \times n$ matrix $P = (p_{ij})$ with $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1$ and $p_{ij} \geq 0$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n$. A correlated strategy P can be thought of as a communication device: the pair (i, j) is chosen with probability p_{ij}, and if that happens, player 1 receives the signal i and player 2 the signal j. Suppose player 2 obeys the signal. If player 1 receives signal i and indeed plays i, his expected payoff is

$$\sum_{j=1}^{n} p_{ij} a_{ij} \Bigg/ \sum_{j=1}^{n} p_{ij} \ ,$$

and if he plays row k instead, his expected payoff is

$$\sum_{j=1}^{n} p_{ij} a_{kj} \Bigg/ \sum_{j=1}^{n} p_{ij} \ .$$

So to keep player 1 from disobeying the received signal, we should have

$$\sum_{j=1}^{n} (a_{ij} - a_{kj}) p_{ij} \geq 0 \text{ for all } i, k = 1, \ldots, m \ . \tag{13.5}$$

The analogous condition for player 2 is:

$$\sum_{i=1}^{m} (b_{ij} - b_{il}) p_{ij} \geq 0 \text{ for all } j, l = 1, \ldots, n \ . \tag{13.6}$$

Definition 13.33 A *correlated equilibrium* in the bimatrix game (A, B) is a correlated strategy $P = (p_{ij})$ satisfying (13.5) and (13.6). □

For the two-driver example conditions (13.5) and (13.6) result in four inequalities, which are not difficult to solve (Problem 13.19). In general, any Nash equilibrium of a bimatrix game results in a correlated equilibrium (Problem 13.20), so existence of a correlated equilibrium is not an issue.

The set of correlated equilibria is convex (Problem 13.21), so the convex hull of all payoff pairs corresponding to the Nash equilibria of a bimatrix game consists of payoff pairs attainable in correlated equilibria. Problem 13.22 presents an example of a game in which some payoff pairs can be reached in correlated equilibria but not as convex combinations of payoff pairs of Nash equilibria.

In general, correlated equilibria can be computed using linear programming. Specifically, let (A, B) be an $m \times n$ bimatrix game. We associate with (A, B) an $mn \times (m(m-1) + n(n-1))$ matrix C as follows. For each pair (i, j) of a row and a column in (A, B) we have a row in C, and for each pair (h, k) of two different rows in (A, B) or two different columns in (A, B) we have a column in C. We define

$$c_{(i,j)(h,k)} = \begin{cases} a_{ij} - a_{kj} & \text{if } i = h \in \{1, \ldots, m\} \text{ and } k \in \{1, \ldots, m\} \\ b_{ij} - b_{ik} & \text{if } j = h \in \{1, \ldots, n\} \text{ and } k \in \{1, \ldots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

Let $P = (p_{ij})$ be a correlated strategy in (A, B). Then P can be seen as a vector $\mathbf{p} \in \mathbb{R}^{mn}$. Let $c_{(h,k)}$ be a column in C. If h and k are rows of (A, B) we have

$$\mathbf{p} \cdot c_{(h,k)} = \sum_{j=1}^{n} p_{hj}(a_{hj} - a_{kj})$$

and if h and k are columns of (A, B) we have

$$\mathbf{p} \cdot c_{(h,k)} = \sum_{i=1}^{m} p_{ih}(b_{ih} - b_{ik}) \ .$$

Hence, by (13.5) and (13.6), P is a correlated equilibrium of (A, B) if and only if $\mathbf{p}C \geq \mathbf{0}$. If we consider C as a matrix game and $\mathbf{p}$ as a strategy of player 1 in C—not to be confused with player 1 in (A, B)—then the existence of a correlated equilibrium implies that $v(C)$, the value of the matrix game C, is nonnegative. In particular, this implies that any optimal strategy of player 1 in C is a correlated equilibrium in (A, B). If $v(C) = 0$ then any correlated equilibrium in (A, B) is an optimal strategy of player 1 in C, but if $v(C) > 0$ then there may be correlated equilibria in (A, B) that are not optimal strategies for player 1 in C—they may only guarantee zero. The latter is the case in the two-drivers example (Problem 13.23).

Matrix games can be solved by linear programming, see Sect. 12.2. We conclude with an example in which the described technique is applied.

Example 13.34 Consider the bimatrix game

$$(A, B) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{pmatrix} 3, 1 & 2, 5 & 6, 0 \\ 1, 4 & 3, 3 & 2, 6 \end{pmatrix} \end{array} .$$

The associated matrix game C is as follows.

$$\begin{array}{c} (1,1') \\ (1,2') \\ (1,3') \\ (2,1') \\ (2,2') \\ (2,3') \end{array} \overset{\begin{array}{cccccccc} (1,2) & (2,1) & (1',2') & (1',3') & (2',1') & (2',3') & (3',1') & (3',2') \end{array}}{\begin{pmatrix} 2 & 0 & -4 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 4 & 5 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & -1 & -5 \\ 0 & -2 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -3 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 & 2 & 3 \end{pmatrix}}.$$

It can be checked that this game has value 0, by using the optimal strategies

$$P = \begin{array}{c} \\ 1 \\ 2 \end{array} \overset{\begin{array}{ccc} 1' & 2' & 3' \end{array}}{\begin{pmatrix} 0 & 0.3 & 0.075 \\ 0 & 0.5 & 0.125 \end{pmatrix}}$$

for player 1 in C and $(0, 0, 1/2, 1/2, 0, 0, 0, 0)$ for player 2 in C. The optimal strategy for player 1 in C is unique, and since $v(C) = 0$ this implies that the game (A, B) has a unique correlated equilibrium. Consequently, this must correspond to the unique Nash equilibrium of the game. Indeed, $((3/8, 5/8), (0, 4/5, 1/5))$ is the unique Nash equilibrium of (A, B) and it results in the probabilities given by P. □

13.8 A Characterization of Nash Equilibrium

The concept of Nash equilibrium requires strong behavioral assumptions about the players. Each player should be able to guess what other players will do, assume that other players know this and make similar conjectures, and so on, and all this should be in equilibrium. The basic difficulty is that Nash equilibrium is a circular concept: a player plays a best reply against the conjectured strategies of the opponents but, in turn, this best reply should be conjectured by the opponents and they should play best replies as well. Not surprisingly, theories of repeated play or learning or, more generally, dynamic models that aim to explain how players in a game come to play a Nash equilibrium, have in common that they change the strategic decision into a collection of single-player decision problems.

In this section we review a different approach, which is axiomatic in nature. The Nash equilibrium concept is viewed as a solution concept: a correspondence which assigns to any finite game a set of strategy combinations. One of the conditions (axioms) put on this correspondence is a condition of consistency with respect to changes in the number of players: if a player leaves the game, leaving his strategy as an input behind, then the other players should not want to change their strategies. This is certainly true for Nash equilibrium, and it can be imposed as a condition on a solution correspondence. By assuming that players in single-player games—hence, in ‘simple’ maximization problems—behave rationally, and by adding a converse consistency condition, it follows that the solution correspondence must be the Nash equilibrium correspondence. We proceed with a formal treatment of this axiomatic characterization.

Let Γ be a collection of finite games of the form $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$. (It is implicit that also the set of players N may vary in Γ.) A *solution* on Γ is a function φ that assigns to each $G \in \Gamma$ a set of strategy combinations $\varphi(G) \subseteq \prod_{i \in N} \Delta(S_i)$. A particular solution is the Nash correspondence NE, assigning to each $G \in \Gamma$ the set $NE(G)$ of all Nash equilibria of G.

Definition 13.35 The solution φ satisfies *one-person rationality* (OPR) if

$$\varphi(G) = \{\sigma_i \in \Delta(S_i) \mid u_i(\sigma_i) \geq u_i(\tau_i) \text{ for all } \tau_i \in \Delta(S_i)\}$$

for every one-person game $G = (\{i\}, S_i, u_i)$ in Γ. □

The interpretation of OPR is clear and needs no further comment.

Let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a game, $\emptyset \neq M \subseteq N$, and let σ be a strategy combination in G. The *reduced game* of G with respect to M and σ is the game $G^{M,\sigma} = (M, (S_i)_{i \in M}, (u_i^{\sigma})_{i \in M})$, where $u_i^{\sigma}(\tau) = u_i(\tau, \sigma_{N \setminus M})$ for all $\tau \in \prod_{j \in M} \Delta(S_j)$.[2] The interpretation of such a reduced game is straightforward: if the players of $N \setminus M$ leave the game, leaving their strategy combination $\sigma_{N \setminus M}$ behind, then the remaining players are faced with the game $G^{M,\sigma}$. Alternatively, if it is common knowledge among the players in M that the players outside M play according to σ, then they are faced with the game $G^{M,\sigma}$. Call a collection of games Γ *closed* if it is closed under taking reduced games.

Definition 13.36 Let Γ be a closed collection of games and let φ be a solution on Γ. Then φ is *consistent* (CONS) if for every game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$, every $\emptyset \neq M \subseteq N$, and every strategy combination $\sigma \in \varphi(G)$, we have $\sigma_M \in \varphi(G^{M,\sigma})$. □

The interpretation of consistency is as follows. If the players outside M have left the game while leaving the strategy combination $\sigma_{N \setminus M}$ behind, then there should be no need for the remaining players to revise their strategies.

The consequence of imposing OPR and CONS on a solution correspondence is that it can contain only Nash equilibria:

Proposition 13.37 *Let Γ be a closed collection of games and let φ be a solution on Γ satisfying OPR and CONS. Then $\varphi(G) \subseteq NE(G)$ for every $G \in \Gamma$.*

Proof Let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n) \in \Gamma$ and $\sigma \in \varphi(G)$. By CONS, $\sigma_i \in \varphi(G^{\{i\},\sigma})$ for every $i \in N$. By OPR, $u_i^{\sigma}(\sigma_i) \geq u_i^{\sigma}(\tau_i)$ for every $\tau_i \in \Delta(S_i)$ and $i \in N$.

[2] For a subset $T \subseteq N$, we denote $(\sigma_j)_{j \in T}$ by σ_T.

Hence

$$u_i(\sigma_i, \sigma_{N\setminus\{i\}}) \geq u_i(\tau_i, \sigma_{N\setminus\{i\}}) \text{ for every } \tau_i \in \Delta(S_i) \text{ and } i \in N.$$

Thus, $\sigma \in NE(G)$. ■

Proposition 13.37 says that NE is the maximal solution (with respect to set-inclusion) satisfying OPR and CONS. (It it trivial to see that NE satisfies these conditions.) To derive a similar minimal set-inclusion result we use another condition.

Let Γ be a closed collection of games and let φ be a solution on Γ. For a game $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n) \in \Gamma$ with $|N| \geq 2$ denote

$$\tilde{\varphi}(G) = \{\sigma \in \prod_{i \in N} \Delta(S_i) \mid \text{ for all } \emptyset \neq M \subsetneq N, \sigma_M \in \varphi(G^{M,\sigma})\}\ .$$

Definition 13.38 A solution φ on a closed collection of games satisfies *converse consistency* (COCONS) if for every game G with at least two players, $\tilde{\varphi}(G) \subseteq \varphi(G)$. □

Converse consistency says that strategy combinations of which the restrictions belong to the solution in smaller reduced games, should also belong to the solution of the game itself. Note that consistency can be defined by the converse inclusion $\varphi(G) \subseteq \tilde{\varphi}(G)$ for every $G \in \Gamma$, which explains the expression 'converse consistency'. Obviously, the Nash equilibrium correspondence satisfies COCONS.

Proposition 13.39 *Let Γ be a closed collection of games and let φ be a solution on Γ satisfying OPR and COCONS. Then $\varphi(G) \supseteq NE(G)$ for every $G \in \Gamma$.*

Proof The proof is by induction on the number of players. For one-person games the inclusion follows (with equality) from OPR. Assume that $NE(G) \subseteq \varphi(G)$ for all t-person games in Γ, where $t \leq k$ and $k \geq 1$. Let G_0 be a $k+1$-person game in Γ. Note that $NE(G_0) \subseteq \widetilde{NE}(G_0)$ by CONS of NE. By the induction hypothesis, $\widetilde{NE}(G_0) \subseteq \tilde{\varphi}(G_0)$ and by COCONS, $\tilde{\varphi}(G_0) \subseteq \varphi(G_0)$. Thus, $NE(G_0) \subseteq \varphi(G_0)$. ■

Corollary 13.40 *Let Γ be a closed collection of games. The Nash equilibrium correspondence is the unique solution on Γ satisfying OPR, CONS, and COCONS.*

It can be shown that the axioms in Corollary 13.40 are independent (Problem 13.26).

In general, the consistency approach fails when applied to refinements of Nash equilibrium. For instance, Problem 13.27 shows that the correspondence of perfect equilibria is not consistent.

13.9 Problems

13.1. *Existence of Nash Equilibrium Using Brouwer*
Let $G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a finite game, as defined in Sect. 13.1. Define the function $f : \prod_{i \in N} \Delta(S_i) \to \prod_{i \in N} \Delta(S_i)$ by

$$f_{i,s_i}(\sigma) = \frac{\sigma_i(s_i) + \max\{0, u_i(s_i, \sigma_{-i}) - u_i(\sigma)\}}{1 + \sum_{s_i' \in S_i} \max\{0, u_i(s_i', \sigma_{-i}) - u_i(\sigma)\}}$$

for all $i \in N$ and $s_i \in S_i$, where $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$.

(a) Show that f is well-defined, i.e., that $f(\sigma) \in \prod_{i \in N} \Delta(S_i)$ for every $\sigma \in \prod_{i \in N} \Delta(S_i)$.
(b) Argue that f has a fixed point, i.e., there is $\sigma^* \in \prod_{i \in N} \Delta(S_i)$ with $f(\sigma^*) = \sigma^*$, by using the Brouwer Fixed Point Theorem (Theorem 22.10).
(c) Show that $\sigma^* \in \prod_{i \in N} \Delta(S_i)$ is a fixed point of f if and only if it is a Nash equilibrium of G.

13.2. *Existence of Nash Equilibrium Using Kakutani*
Prove that the correspondence β in the proof of Theorem 13.1 is upper semi-continuous and convex-valued. Also check that every fixed point of β is a Nash equilibrium of G.

13.3. *Lemma 13.2*
Prove Lemma 13.2.

13.4. *Lemma 13.3*
Prove Lemma 13.3.

13.5. *Dominated Strategies*
Let (A, B) be an $m \times n$ bimatrix game. Suppose there exists a $\mathbf{q} \in \Delta^n$ such that $q_n = 0$ and $B\mathbf{q} > B\mathbf{e}^n$ (i.e., there exists a mixture of the first $n-1$ columns of B that is strictly better than playing the n-th column).

(a) Prove that $q_n^* = 0$ for every Nash equilibrium $(\mathbf{p}^*, \mathbf{q}^*)$.
Let (A', B') be the bimatrix game obtained from (A, B) be deleting the last column.
(b) Prove that $(\mathbf{p}^*, \mathbf{q}')$ is a Nash equilibrium of (A', B') if and only if $(\mathbf{p}^*, \mathbf{q}^*)$ is a Nash equilibrium of (A, B), where $\mathbf{q}'$ is the strategy obtained from $\mathbf{q}^*$ by deleting the last coordinate.

13.6. *A* 3×3 *Bimatrix Game*
Consider the 3×3 bimatrix game

$$(A,B) = \begin{pmatrix} 0,4 & 4,0 & 5,3 \\ 4,0 & 0,4 & 5,3 \\ 3,5 & 3,5 & 6,6 \end{pmatrix}.$$

Let $(\mathbf{p},\mathbf{q})$ be a Nash equilibrium in (A,B).

(a) Prove that $\{1,2\} \not\subseteq C(\mathbf{p})$.
(b) Prove that $C(\mathbf{p}) \neq \{2,3\}$.
(c) Find all Nash equilibria of this game.

13.7. *A* 3×2 *Bimatrix Game*
Use the graphical method to compute the Nash equilibria of the bimatrix game

$$(A,B) = \begin{pmatrix} 0,0 & 2,1 \\ 2,2 & 0,2 \\ 2,2 & 0,2 \end{pmatrix}.$$

13.8. *The Nash Equilibria in Example 13.18*

(a) Compute the Nash equilibria of the game in Example 13.18.
(b) Show that the Nash equilibria in the set $\{((0,1,0),(q,1-q,0)) \mid 0 \leq q \leq 1\}$ are perfect by using the definition of perfection.

13.9. *Proof of Theorem 13.8*
Prove Theorem 13.8.

13.10. *Matrix Games*
Show that the pair of linear programs (13.3) and (13.4) is equivalent to the LP and its dual in Sect. 12.2 for solving matrix games.

13.11. *Tic-Tac-Toe*
The two-player game of Tic-Tac-Toe is played on a 3×3 board. Player 1 starts by putting a cross on one of the nine fields. Next, player 2 puts a circle on one of the eight remaining fields. Then player 1 puts a cross on one of the remaining seven fields, etc. If player 1 achieves three crosses or player 2 achieves three circles in a row (either vertically or horizontally or diagonally) then that player wins. If this does not happen and the board is full, then the game ends in a draw.

(a) Design a pure maximin strategy for player 1. Show that this maximin strategy guarantees at least a draw to him.
(b) Show that player 1 cannot guarantee a win.
(c) What is the value of this game?

13.12. *Iterated Elimination in a Three-Player Game*
Solve the following three-player game, where player 1 chooses rows, player 2 columns, and player 3 one of the two games L and R:

$$L: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 14,24,32 & 8,30,27 \\ 30,16,24 & 13,12,50 \end{pmatrix} \end{array} \quad R: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 16,24,30 & 30,16,24 \\ 30,23,14 & 14,24,32 \end{pmatrix} \end{array}.$$

13.13. *Never a Best Reply and Domination*
In the following game player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices. The diagram gives the payoffs of player 3. Show that Y is never a best reply for player 3, and that Y is not strictly (and not even weakly) dominated.

$$V: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \quad W: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 0 & 9 \\ 9 & 0 \end{pmatrix} \end{array}$$

$$X: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & 9 \end{pmatrix} \end{array} \quad Y: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \end{array}$$

13.14. *Completely Mixed Nash Equilibria Are Perfect*
Prove Lemma 13.16.

13.15. *A 3-Player Game with an Undominated But Not Perfect Equilibrium*
Consider the following 3-player game, where player 1 chooses rows, player 2 columns, and player 3 matrices:

$$L: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 1,1,1 & 1,0,1 \\ 1,1,1 & 0,0,1 \end{pmatrix} \end{array} \quad R: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 1,1,0 & 0,0,0 \\ 0,1,0 & 1,0,0 \end{pmatrix} \end{array}.$$

(a) Show that (U, l, L) is the only perfect Nash equilibrium of this game.
(b) Show that (D, l, L) is an undominated Nash equilibrium.

13.16. *Existence of Proper Equilibrium*
Prove that the correspondence F in the proof of Theorem 13.29 satisfies the conditions of the Kakutani Fixed Point Theorem.

13.17. *Strictly Dominated Strategies and Proper Equilibrium*
Consider the 3-person game

$$L: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 1,1,1 & 0,0,1 \\ 0,0,1 & 0,0,1 \end{pmatrix} \end{array} \quad R: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} l & r \end{array} \\ \begin{pmatrix} 0,0,0 & 0,0,0 \\ 0,0,0 & 1,1,0 \end{pmatrix} \end{array},$$

where player 1 chooses rows, player 2 chooses columns, and player 3 chooses matrices.

(a) First assume that player 3 is a dummy and has only one strategy, namely L. Compute the perfect and proper Nash equilibrium or equilibria of the game.
(b) Now suppose that player 3 has two pure strategies. Compute the perfect and proper Nash equilibrium or equilibria of the game. Conclude that adding a strictly dominated strategy (namely, R) has resulted in an additional proper equilibrium.

13.18. *Strictly Perfect Equilibrium*

(a) Show that a completely mixed Nash equilibrium in a finite game G is strictly perfect.
(b) Show that a strict Nash equilibrium in a game G is strictly perfect. (A Nash equilibrium is *strict* if any unilateral deviation of a player leads to a strictly lower payoff for that player.)
(c) Compute all Nash equilibria, perfect equilibria, proper equilibria, and strictly perfect equilibria in the following game, where $\alpha, \beta > 0$. (Conclude that strictly perfect equilibria may fail to exist.)

$$(A,B) = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{ccc} L & M & R \end{array} \\ \begin{pmatrix} 0,\beta & \alpha,0 & 0,0 \\ 0,\beta & 0,0 & \alpha,0 \end{pmatrix} \end{array}.$$

13.19. *Correlated Equilibria in the Two-Driver Example (1)*
Compute all correlated equilibria in the game

$$(A,B) = \begin{array}{c} \\ c \\ s \end{array} \begin{array}{c} \begin{array}{cc} c & s \end{array} \\ \begin{pmatrix} -10,-10 & 5,0 \\ 0,5 & -1,-1 \end{pmatrix} \end{array},$$

by using the definition of correlated equilibrium.

13.20. *Nash Equilibria Are Correlated*
Let $(\mathbf{p}, \mathbf{q})$ be a Nash equilibrium in the $m \times n$ bimatrix game (A, B). Let $P = (p_{ij})$ be the $m \times n$ matrix defined by $p_{ij} = p_i q_j$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Show that P is a correlated equilibrium.

13.21. *The Set of Correlated Equilibria Is Convex*
Show that the set of correlated equilibria in a bimatrix game (A, B) is convex.

13.22. *Correlated vs. Nash Equilibrium*
Consider the bimatrix game

$$(A, B) = \begin{pmatrix} 6,6 & 2,7 \\ 7,2 & 0,0 \end{pmatrix}$$

and the correlated strategy

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix} .$$

(a) Compute all Nash equilibria of (A, B).
(b) Show that P is a correlated equilibrium and that the associated payoffs fall outside the convex hull of the payoff pairs associated with the Nash equilibria of (A, B).

13.23. *Correlated Equilibria in the Two-Driver Example (2)*
Consider again the game of Problem 13.19 and set up the associated matrix C as in Sect. 13.7. Show that the value of the matrix game C is equal to 3, and that player 1 in C has a unique optimal strategy. (Hence, this method gives one particular correlated equilibrium.)

13.24. *Finding Correlated Equilibria*
Compute (the) correlated equilibria in the following game directly, and by using the associated matrix game.

$$(A, B) = \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} \begin{matrix} 1' & \quad 2' \end{matrix} \\ \begin{pmatrix} 5,2 & 1,3 \\ 2,3 & 4,1 \end{pmatrix} \end{matrix} .$$

13.25. *Nash, Perfect, Proper, Strictly Perfect, and Correlated Equilibria*
Consider the bimatrix game $(A, B) = \begin{pmatrix} 0,6 & 0,4 & 6,0 \\ 4,0 & 0,0 & 4,0 \\ 6,0 & 0,4 & 0,6 \end{pmatrix}$.
Let $(\mathbf{p}, \mathbf{q}) \in \Delta^3 \times \Delta^3$ be a Nash equilibrium in (A, B).

(a) Show that, if $p_3 = 0$, then $\mathbf{p} = (0, 1, 0)$.
(b) Show that, if $p_1 > 0$ and $p_3 > 0$, then $\mathbf{q} = (0, 1, 0)$.
(c) Show that in each Nash equilibrium at least one player has payoff 0.
(d) Compute all Nash equilibria of (A, B).

(e) Which Nash equilibria are perfect?
(f) Which Nash equilibria are proper? Strictly perfect?

Consider the correlated strategy $P = \begin{pmatrix} 0 & \alpha & 0 \\ \beta & 0 & \gamma \\ 0 & \delta & 0 \end{pmatrix}$,
with $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \beta + \gamma + \delta = 1$.

(g) Under which conditions is P a correlated equilibrium?
(h) Find a correlated equilibrium with payoff 3 for player 1 and 1 for player 2.

13.26. *Independence of the Axioms in Corollary 13.40*
Show that the three conditions in Corollary 13.40 are independent: for each pair of conditions, exhibit a solution that satisfies these two conditions but not the third one.

13.27. *Inconsistency of Perfect Equilibria*
Consider the 3-person game G_0

$$D: \begin{matrix} & L & R \\ T & 1,1,1 & 1,0,1 \\ B & 1,1,1 & 0,0,1 \end{matrix} \qquad U: \begin{matrix} & L & R \\ T & 0,1,0 & 0,0,0 \\ B & 1,1,0 & 0,0,0 \end{matrix}$$

where player 1 chooses rows, player 2 columns, and player 3 matrices. Let Γ consist of this game and all its reduced games. Use this collection to show that the perfect Nash equilibrium correspondence is not consistent.

13.10 Notes

The result that every finite game has a Nash equilibrium in mixed strategies is due to Nash (1951). For the quadratic programming problem in Sect. 13.2.3, see Mangasarian and Stone (1964). Lemke and Howson (1964) provide an algorithm to find at least one Nash equilibrium of a bimatrix game. See von Stengel (2002) for an overview.

For the first proof of Theorem 13.10 see Zermelo (1913). Theorem 13.13 is due to Pearce (1984). For the concept of rationalizability, see Bernheim (1984) and Pearce (1984). For a study of the assumptions underlying the procedure of iterated elimination of strictly dominated strategies see Tan and Werlang (1988) or Perea (2001). Perea (2012) presents a recent overview on epistemic game theory.

The concept of perfect equilibrium (trembling hand perfection) is due to Selten (1975). The notions of ε-perfect and (ε-)proper equilibrium were introduced in Myerson (1978). Theorem 13.20 is based on Selten (1975) and Myerson (1978) and appears as Theorem 2.2.5 in van Damme (1991). The notion of strictly perfect equilibrium was introduced in Okada (1981).

Correlated equilibria were introduced in Aumann (1974). Our treatment of the topic closely follows the presentation in Owen (1995); Example 13.34 corresponds to Example VIII.4.4 in that book.

For theories of strategic learning, see Young (2004). Section 13.8 is based on Peleg and Tijs (1996); the presentation there is for more general games. See Norde et al. (1996) for a study of the notion of consistency for refinements of Nash equilibrium.

Problem 13.12 is from Watson (2002). Problem 13.13 is from Fudenberg and Tirole (1991). Problems 13.15 and 13.17 are based on van Damme (1991). Problem 13.22 is based on Aumann (1974).

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