Finite Two-Person Games

3

In this chapter we consider two-player games where each player chooses from finitely many pure strategies or randomizes among these strategies. In contrast to Chap. 2 it is no longer required that the sum of the players' payoffs is zero (or, equivalently, constant). This allows for a much larger class of games, including many games relevant for economic or other applications. Famous examples are the Prisoners' Dilemma and the Battle of the Sexes discussed in Sect. 1.3.2.

In Sect. 3.1 we introduce the model and the concept of Nash equilibrium. Section 3.2 shows how to compute Nash equilibria in pure strategies for arbitrary games, all Nash equilibria in games where both players have exactly two pure strategies, and how to use the concept of strict domination to facilitate computation of Nash equilibria and to compute equilibria also of larger games. The structure of this chapter thus parallels the structure of Chap. 2. For a deeper and more comprehensive analysis of finite two-person games see Chap. 13.

3.1 Basic Definitions and Theory

The data of a finite two-person game can be summarized by two matrices. Usually, these matrices are written as one matrix with two numbers at each position. Therefore, such games are often called 'bimatrix games'. The definition is as follows.

Definition 3.1 (Bimatrix Game) A *bimatrix game* is a pair of $m \times n$ matrices (A, B), where m is the number of rows and n the number of columns. □

The interpretation of a bimatrix game (A, B) is that, if player 1 (the row player) plays row i and player 2 (the column player) plays column j, then player 1 receives payoff a_{ij} and player 2 receives b_{ij}, where these numbers are the corresponding entries of A and B, respectively. Definitions and notations for pure and mixed strategies, strategy

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sets and expected payoffs are similar to those for matrix games, see Sect. 2.1, but for easy reference we repeat them here. A *(mixed) strategy* of player 1 is a probability distribution $\mathbf{p}$ over the rows of A and B, i.e., an element of the set

$$\Delta^m := \{\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} p_i = 1,\ p_i \geq 0 \text{ for all } i = 1, \ldots, m\} .$$

Similarly, a *(mixed) strategy* of player 2 is a probability distribution $\mathbf{q}$ over the columns of A and B, i.e., an element of the set

$$\Delta^n := \{\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} q_j = 1,\ q_j \geq 0 \text{ for all } j = 1, \ldots, n\} .$$

A strategy $\mathbf{p}$ of player 1 is called *pure* if there is a row i with $p_i = 1$. This strategy is also denoted by $\mathbf{e}^i$. Similarly, a strategy $\mathbf{q}$ of player 2 is called *pure* if there is a column j with $q_j = 1$. This strategy is also denoted by $\mathbf{e}^j$. If player 1 plays $\mathbf{p}$ and player 2 plays $\mathbf{q}$ then the payoff to player 1 is the expected payoff

$$\mathbf{p}A\mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij} ,$$

and the payoff to player 2 is the expected payoff

$$\mathbf{p}B\mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j b_{ij} .$$

As mentioned, the entries of A and B are usually grouped together in one (bi)matrix, by putting the pair a_{ij}, b_{ij} at position (i, j) of the matrix. Cf. the examples in Sect. 1.3.2.

Central to noncooperative game theory is the idea of best reply. It says that a rational selfish player should always maximize his (expected) payoff, given his knowledge of or conjecture about the strategies chosen by the other players.

Definition 3.2 (Best Reply) A strategy $\mathbf{p}$ of player 1 is a *best reply* to a strategy $\mathbf{q}$ of player 2 in an $m \times n$ bimatrix game (A, B) if

$$\mathbf{p}A\mathbf{q} \geq \mathbf{p}'A\mathbf{q} \text{ for all } \mathbf{p}' \in \Delta^m .$$

Similarly, $\mathbf{q}$ is a *best reply* of player 2 to $\mathbf{p}$ if

$$\mathbf{p}B\mathbf{q} \geq \mathbf{p}B\mathbf{q}' \text{ for all } \mathbf{q}' \in \Delta^n \ .$$

□

In a Nash equilibrium, each player's strategy is a best reply to the other player's strategy.

Definition 3.3 (Nash Equilibrium) A pair of strategies $(\mathbf{p}^*, \mathbf{q}^*)$ in a bimatrix game (A, B) is a *Nash equilibrium* if $\mathbf{p}^*$ is a best reply of player 1 to $\mathbf{q}^*$ and $\mathbf{q}^*$ is a best reply of player 2 to $\mathbf{p}^*$. A Nash equilibrium $(\mathbf{p}^*, \mathbf{q}^*)$ is called *pure* if both $\mathbf{p}^*$ and $\mathbf{q}^*$ are pure strategies. □

The concept of a Nash equilibrium can be extended to arbitrary games, including games with arbitrary numbers of players, strategy sets, and payoff functions. We will see many examples in later chapters.

Every bimatrix game has a Nash equilibrium: for a proof see Sect. 13.1. Generally speaking, the main concern with Nash equilibrium is not its existence but rather the opposite, namely its abundance, as well as its interpretation. In many games, there are many Nash equilibria, and then the questions of equilibrium selection and equilibrium refinement are relevant (cf. Chap. 13). With respect to interpretation, an old question is how the players would come to play a Nash equilibrium in reality. The definition of Nash equilibrium does not say anything about this.

For a Nash equilibrium in mixed strategies as in Definition 3.3, an additional question is what the meaning of such a mixed strategy is. Does it mean that the players actually randomize when playing the game? A different and common interpretation is that a mixed strategy of a player, say player 1, represents the belief, or conjecture, of the other player, player 2, about what player 1 will do. Thus, it embodies the 'strategic uncertainty' of the players in a game.

For now, we just leave these questions aside and take the definition of Nash equilibrium at face value. We show how to compute pure Nash equilibria in general, and all Nash equilibria in games where both players have two pure strategies. Just as in Chap. 2, we also consider the role of strict domination.

3.2 Finding Nash Equilibria

To find all Nash equilibria of an arbitrary bimatrix game is a difficult task. We refer to Sect. 13.2.3 for more discussion on this problem. Here we restrict ourselves to, first, the much easier problem of finding all Nash equilibria in pure strategies of an arbitrary bimatrix game and, second, to showing how to find all Nash equilibria in 2×2 games graphically. It is also possible to solve 2×3 and 3×2 games graphically, see Sect. 13.2.2. For larger games, graphical solutions are impractical or, indeed, impossible.

3.2.1 Pure Nash Equilibria

To find the pure Nash equilibria in a bimatrix game, one can first determine the pure best replies of player 2 to every pure strategy of player 1, and next determine the pure best replies of player 1 to every pure strategy of player 2. Those pairs of pure strategies that are mutual best replies are the pure Nash equilibria of the game. To illustrate this method, consider the bimatrix game

$$\begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} 2,2 & 4,0 & 1,1 & 3,2 \\ 0,3 & 1,5 & 4,4 & 3,4 \\ 2,0 & 2,1 & 5,1 & 1,0 \end{pmatrix} \end{array}.$$

First we determine the pure best replies of player 2 to every pure strategy of player 1, indicated by underlining the corresponding entries. This yields:

$$\begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} 2,\underline{2} & 4,0 & 1,1 & 3,\underline{2} \\ 0,3 & 1,\underline{5} & 4,4 & 3,4 \\ 2,0 & 2,\underline{1} & 5,\underline{1} & 1,0 \end{pmatrix} \end{array}.$$

Next, we determine the pure best replies of player 1 to every pure strategy of player 2, again indicated by underlining the corresponding entries. This yields:

$$\begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} \underline{2},2 & \underline{4},0 & 1,1 & \underline{3},2 \\ 0,3 & 1,5 & 4,4 & \underline{3},4 \\ \underline{2},0 & 2,1 & \underline{5},1 & 1,0 \end{pmatrix} \end{array}.$$

Putting the two results together yields:

$$\begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} \underline{2},\underline{2} & \underline{4},0 & 1,1 & \underline{3},\underline{2} \\ 0,3 & 1,\underline{5} & 4,4 & \underline{3},4 \\ \underline{2},0 & 2,\underline{1} & \underline{5},\underline{1} & 1,0 \end{pmatrix} \end{array}.$$

We conclude that the game has three Nash equilibria in pure strategies, namely (T, W), (T, Z), and (B, Y). In mixed strategy notation, these are the pairs $(\mathbf{e}^1, \mathbf{e}^1)$, $(\mathbf{e}^1, \mathbf{e}^4)$, and $(\mathbf{e}^3, \mathbf{e}^3)$. In more extensive notation: $((1,0,0), (1,0,0,0))$, $((1,0,0), (0,0,0,1))$, and $((0,0,1), (0,0,1,0))$.

Strictly speaking, one should also consider mixed best replies to a pure strategy in order to establish whether this pure strategy can occur in a Nash equilibrium, but it is not difficult to see that any mixed best reply is a combination of pure best replies and, thus, can never lead to a higher payoff. For instance, in the example above, any strategy of the form $(q, 0, 0, 1-q)$ played against T yields to player 2 a payoff of 2 $(= 2q + 2(1-q))$ and is therefore a best reply, but does not yield a payoff higher than W or Z. However, the reader can check that all strategy pairs of the form $(T, (q, 0, 0, 1-q))$ $(0 < q < 1)$ are also Nash equilibria of this game.

It is also clear from this example that a Nash equilibrium does not have to result in Pareto optimal payoffs: a pair of payoffs is *Pareto optimal* if there is no other pair of payoffs which are at least as high for both players and strictly higher for at least one player. The payoff pair $(4, 4)$, resulting from (M, Y), is better for both players than the equilibrium payoffs $(2, 2)$, resulting from (T, W). We know this phenomenon already from the Prisoners' Dilemma game in Sect. 1.3.2.

3.2.2 2 x 2 Games

We demonstrate the graphical solution method for 2×2 games by means of an example. Consider the bimatrix game

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 2,2 & 0,1 \\ 1,1 & 3,3 \end{pmatrix} \end{array} .$$

Observe that this game has two Nash equilibria in pure strategies, namely (T, L) and (B, R). To find all Nash equilibria we determine the best replies of both players.

First consider the strategy $(q, 1-q)$ of player 2. The unique best reply of player 1 to this strategy is T or, equivalently, $(1, 0)$, if the expected payoff from playing T is higher than the expected payoff from playing B, since then it is also higher than the expected payoff from playing any combination $(p, 1-p)$ of T and B. Hence, the best reply is T if

$$2q + 0(1-q) > 1q + 3(1-q) ,$$

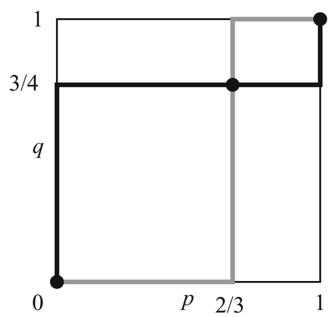
so if $q > \frac{3}{4}$. Similarly, we find that B is the unique best reply if $q < \frac{3}{4}$, and that T and B are both best replies if $q = \frac{3}{4}$. In the last case, since T and B yield the same payoff to player 1 against $(q, 1-q)$, it follows that any $(p, 1-p)$ is a best reply. Summarizing, if we denote the set of best replies of player 1 against $(q, 1-q)$ by $\beta_1(q, 1-q)$, we have

$$\beta_1(q, 1-q) = \begin{cases} \{(1,0)\} & \text{if } \frac{3}{4} < q \le 1 \\ \{(p, 1-p) \mid 0 \le p \le 1\} & \text{if } q = \frac{3}{4} \\ \{(0,1)\} & \text{if } 0 \le q < \frac{3}{4} \ . \end{cases} \tag{3.1}$$

By analogous arguments, we find that for a strategy $(p, 1-p)$ the best replies $\beta_2(p, 1-p)$ of player 2 are given by

$$\beta_2(p, 1-p) = \begin{cases} \{(1,0)\} & \text{if } \frac{2}{3} < p \leq 1 \\ \{(q, 1-q) \mid 0 \leq q \leq 1\} & \text{if } p = \frac{2}{3} \\ \{(0,1)\} & \text{if } 0 \leq p < \frac{2}{3}\,. \end{cases} \tag{3.2}$$

By definition, the Nash equilibria of the game are the strategy combinations $(\mathbf{p}^*, \mathbf{q}^*)$ such that $\mathbf{p}^* \in \beta_1(\mathbf{q}^*)$ and $\mathbf{q}^* \in \beta_2(\mathbf{p}^*)$, i.e., the points of intersection of the best reply functions in (3.1) and (3.2). A convenient way to find these points is by drawing the graphs of $\beta_1(q, 1-q)$ and $\beta_2(p, 1-p)$. We put p on the horizontal axis and q on the vertical axis and obtain the following diagram.



The solid black curve depicts the best replies of player 1 and the solid grey curve depicts the best replies of player 2. The solid circles indicate the three Nash equilibria of the game: $((1,0),(1,0))$, $((2/3,1/3),(3/4,1/4))$, and $((0,1),(0,1))$.

3.2.3 Strict Domination

The graphical method discussed in Sect. 3.2.2 is suited for 2×2 games. It can be extended to 2×3 and 3×2 games as well, see Sect. 13.2.2.

In general, for the purpose of finding Nash equilibria the size of a game can sometimes be reduced by iteratively eliminating strictly dominated strategies. We look for a strictly dominated (pure) strategy of a player, eliminate the associated row or column, and continue this procedure for the smaller game until there is no more strictly dominated strategy. It can be shown (see Sect. 13.3) that no pure strategy that is eliminated by this procedure is ever played with positive probability in a Nash equilibrium of the original game. Thus, no Nash equilibrium of the original game is eliminated. Also, no Nash equilibrium is added. It follows, in particular, that the order in which strictly dominated strategies are eliminated does not matter.

For completeness we first repeat the definition of strict domination, formulated for a bimatrix game, and then present an example.

Definition 3.4 (Strict Domination) Let (A, B) be an $m \times n$ bimatrix game and i a row. The pure strategy $\mathbf{e}^i$ is *strictly dominated* if there is a strategy $\mathbf{p} = (p_1, \ldots, p_m) \in \Delta^m$ such that $\mathbf{p}A\mathbf{e}^j > \mathbf{e}^i A\mathbf{e}^j$ for every $j = 1, \ldots, n$. Similarly, let j be a column. The pure strategy $\mathbf{e}^j$ is *strictly dominated* if there is a strategy $\mathbf{q} = (q_1, \ldots, q_n) \in \Delta^n$ such that $\mathbf{e}^i B\mathbf{q} > \mathbf{e}^i B\mathbf{e}^j$ for every $i = 1, \ldots, m$. □

We observe that Remark 2.6 is still valid: if $\mathbf{e}^i$ for player 1 is strictly dominated, then it is strictly dominated by some $\mathbf{p}$ with $p_i = 0$; and similar for player 2.

Consider the following bimatrix game

$$\begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} 2,2 & 2,1 & 2,2 & 0,0 \\ 1,0 & 4,1 & 2,4 & 1,5 \\ 0,4 & 3,1 & 3,0 & 3,3 \end{pmatrix} \end{array}.$$

Observe, first, that no pure strategy (row) of player 1 is strictly dominated by another pure strategy of player 1, and that no pure strategy (column) of player 2 is strictly dominated by another pure strategy of player 2. Consider the pure strategy X of player 2. Note that the payoffs in column X for player 2 are below the maximum of the payoffs in columns W and Y: $1 < \max\{2, 2\}$, $1 < \max\{0, 4\}$, and $1 < \max\{4, 0\}$. Therefore, we may try and see if X can be strictly dominated by a combination of W and Y, i.e., by a strategy of the form $(q, 0, 1-q, 0)$. For this we need: $1 < 2q + 2(1-q)$, $1 < 0q + 4(1-q)$, and $1 < 4q + 0(1-q)$. These three inequalities hold for all q with $\frac{1}{4} < q < \frac{3}{4}$. For instance, X is strictly dominated by $(\frac{1}{2}, 0, \frac{1}{2}, 0)$. So X can be eliminated, to obtain

$$\begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{ccc} W & Y & Z \end{array} \\ \begin{pmatrix} 2,2 & 2,2 & 0,0 \\ 1,0 & 2,4 & 1,5 \\ 0,4 & 3,0 & 3,3 \end{pmatrix} \end{array}.$$

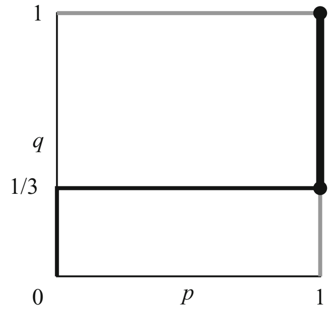
Next, in this reduced game, for player 1 pure strategy M is strictly dominated by any strategy of the form $(p, 0, 1-p)$ with $\frac{1}{2} < p < \frac{2}{3}$. So M can be eliminated to obtain

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{ccc} W & Y & Z \end{array} \\ \begin{pmatrix} 2,2 & 2,2 & 0,0 \\ 0,4 & 3,0 & 3,3 \end{pmatrix} \end{array}.$$

Here, finally, Z can be eliminated since it is strictly dominated by W, and we are left with the 2×2 game

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} W & Y \end{array} \\ \begin{pmatrix} 2,2 & 2,2 \\ 0,4 & 3,0 \end{pmatrix} \end{array}.$$

This game can be solved by the graphical method of Sect. 3.2.2. Doing so results in the diagram



The solid black (grey) curve depicts player 1's (2's) best replies. In this case, the curves overlap in infinitely many points, resulting in the set of Nash equilibria $\{((1,0),(q,1-q)) \mid 1/3 \le q \le 1\}$. In the original 3×4 game, the set of all Nash equilibria is therefore equal to

$$\{((1,0,0),(q,0,1-q,0)) \mid 1/3 \le q \le 1\} .$$

3.3 Problems

3.1. *Some Applications*
In each of the following situations, set up the corresponding bimatrix game and solve for all Nash equilibria.

(a) *Pure coordination*. Two firms (Smith and Brown) decide whether to design the computers they sell to use large or small floppy disks. Both players will sell more computers if their disk drives are compatible. If they both choose for large disks the payoffs will be 2 for each. If they both choose for small disks the payoffs will be 1 for each. If they choose different sizes the payoffs will be -1 for each.
(b) *The welfare game*. This game models a government that wishes to aid a pauper if he searches for work but not otherwise, and a pauper who searches for work only if he cannot depend on government aid, and who may not succeed in finding a job even if the tries. The payoffs are $3,2$ (for government, pauper) if the government aids and the pauper tries to work; $-1,1$ if the government does not aid and the pauper tries to work; $-1,3$ if the government aids and the pauper does not try to work; and $0,0$ in the remaining case. [These payoffs represent the preferences of the players, rather than monetary values. For instance, the government ranks the combination Aid/Search above the combination No

Aid/No Search, which in turn is ranked above the combinations Aid/No Search and No Aid/Search, between which the government is indifferent.]

(c) *Wage game*. Each of two firms has one job opening. Suppose that firm i ($i = 1, 2$) offers wage w_i, where $0 < \frac{1}{2}w_1 < w_2 < 2w_1$ and $w_1 \neq w_2$. Imagine that there are two workers, each of whom can apply to only one firm. The workers simultaneously decide whether to apply to firm 1 or firm 2. If only one worker applies to a given firm, that worker gets the job; if both workers apply to the same firm, each worker has probability $1/2$ of getting the job while the other worker remains unemployed and has a payoff of zero.

(d) *Marketing game*. Two firms sell a similar product. Each percent of market share yields a net payoff of 1. Without advertising both firms have 50 % of the market. The cost of advertising is equal to 10 but leads to an increase in market share of 20 % at the expense of the other firm. The firms make their advertising decisions simultaneously and independently. The total market for the product is of fixed size.

(e) *Voting game.* Two political parties, I and II, each have three votes that they can distribute over three party-candidates each. A committee is to be elected, consisting of three members. Each political party would like to see as many as possible of their own candidates elected in the committee. Of the total of six candidates, those three who have most of the votes will be elected; in case of ties, tied candidates are drawn with equal probabilities.

(f) *Voting game, revisited.* Consider the situation in (e) but now assume that each party is risk averse. For instance, each party strictly prefers to have one candidate for sure in the committee over a lottery in which it has zero or two candidates in the committee each with probability 50 %. Model this by each party having a payoff of $\sqrt{c}$ for a sure number $c \in \{0, 1, 2, 3\}$ of its candidates in the committee.

3.2. *Matrix Games*

(a) Since a matrix game is a special case of a bimatrix game, it may be analyzed by the graphical method of Sect. 3.2.2. Do this for the game in Problem 2.1(a). Compare your answer with what you found previously.

(b) Argue that a pair consisting of a maximin and a minimax strategy in a matrix game is a Nash equilibrium; and that any Nash equilibrium in a matrix game must be a pair consisting of a maximin and a minimax strategy. (You may give all your arguments in words.)

(c) Define a maximin strategy for player 1 in the bimatrix game (A, B) to be a maximin strategy in the matrix game A. Which definition is appropriate for player 2 in this respect? With these definitions, find examples showing that a Nash equilibrium in a bimatrix game does not have to consist of maximin strategies, and that a maximin strategy does not have to be part of a Nash equilibrium.

3.3. *Strict Domination*
Consider the bimatrix game

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} 6,6 & 4,4 & 1,2 & 8,5 \\ 4,5 & 6,6 & 2,8 & 4,4 \end{pmatrix} \end{array}.$$

(a) Which pure strategy of player 1 or player 2 is strictly dominated by a pure strategy?
(b) Describe all combinations of strategies W and Y of player 2 that strictly dominate X.
(c) Find all Nash equilibria of this game.

3.4. *Iterated Elimination (1)*
Consider the bimatrix game

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{pmatrix} 5,4 & 4,4 & 4,5 & 12,2 \\ 3,7 & 8,7 & 5,8 & 10,6 \\ 2,10 & 7,6 & 4,6 & 9,5 \\ 4,4 & 5,9 & 4,10 & 10,9 \end{pmatrix} \end{array}.$$

(a) Find a few different ways in which strictly dominated strategies can be iteratedly eliminated in this game.
(b) Find the Nash equilibria of this game.

3.5. *Iterated Elimination (2)*
Consider the bimatrix game

$$\begin{pmatrix} 2,0 & 1,1 & 4,2 \\ 3,4 & 1,2 & 2,3 \\ 1,3 & 0,2 & 3,0 \end{pmatrix}.$$

Find the Nash equilibria of this game.

3.6. *Weakly Dominated Strategies*
A pure strategy i of player 1 in an $m \times n$ bimatrix game (A, B) is *weakly dominated* if there a strategy $\mathbf{p} = (p_1, \ldots, p_m) \in \Delta^m$ such that $\mathbf{p}A\mathbf{e}^j \geq \mathbf{e}^i A\mathbf{e}^j$ for every $j = 1, \ldots, n$, and $\mathbf{p}A\mathbf{e}^j > \mathbf{e}^i A\mathbf{e}^j$ for at least one j. The definition of a weakly dominated strategy of player 2 is similar. In words, a pure strategy is weakly dominated if there is some pure or mixed strategy that is always at least as good, and that is better against at least one pure strategy of the opponent. Instead of iterated elimination of strictly dominated strategies one might also consider iterated elimination of weakly dominated strategies. The advantage is that in games where

no strategy is strictly dominated it might still be possible to eliminate strategies that are weakly dominated. However, some Nash equilibria of the original game may be eliminated as well, and also the order of elimination may matter. These issues are illustrated by the following examples.

(a) Consider the bimatrix game

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{pmatrix} 11,10 & 6,9 & 10,9 \\ 11,6 & 6,6 & 9,6 \\ 12,10 & 6,9 & 9,11 \end{pmatrix} \end{array}.$$

First, determine the pure Nash equilibria of this game. Next, apply iterated elimination of weakly dominated strategies to reduce the game to a 2×2 game and determine the unique Nash equilibrium of this smaller game.

(b) Consider the bimatrix game

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{pmatrix} 1,1 & 0,0 & 2,0 \\ 1,2 & 1,2 & 1,1 \\ 0,0 & 1,1 & 1,1 \end{pmatrix} \end{array}.$$

Show that different orders of eliminating weakly dominated strategies may result in different Nash equilibria.

3.7. *A Parameter Game*
Consider the bimatrix game

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 1,1 & a,0 \\ 0,0 & 2,1 \end{pmatrix} \end{array}$$

where $a \in \mathbb{R}$. Determine the Nash equilibria of this game for every possible value of a.

3.8. *Equalizing Property of Mixed Equilibrium Strategies*

(a) Consider again the game of Problem 3.3, which has a unique Nash equilibrium in mixed strategies. In this equilibrium, player 1 puts positive probability p^* on T and $1 - p^*$ on B, and player 2 puts positive probability q^* on W and $1 - q^*$ on Y. Show that, if player 2 plays this strategy, then both T and B give player 1 the same expected payoff, equal to the equilibrium payoff. Also show that, if player 1 plays his equilibrium strategy, then both W and Y give player 2 the same expected payoff, equal to the equilibrium payoff, and higher than the expected payoff from X or from Z.

(b) Generalize the observations made in (a), more precisely, give an argument for the following statement:
Let (A, B) *be an* $m \times n$ *bimatrix game and let* $(\mathbf{p}^*, \mathbf{q}^*)$ *be a Nash equilibrium. Then each row played with positive probability in this Nash equilibrium has the same expected payoff for player 1 against* $\mathbf{q}^*$ *and this payoff is at least as high as the payoff from any other row. Each column played with positive probability in this Nash equilibrium has the same expected payoff for player 2 against* $\mathbf{p}^*$ *and this payoff is at least as high as the payoff from any other column.*
You may state your argument in words, without using formulas.

3.9. *Voting*
Suppose the spectrum of political positions is described by the closed interval (line segment) $[0, 5]$. Voters are uniformly distributed over $[0, 5]$. There are two candidates, who may occupy any of the positions in the set $\{0, 1, 2, 3, 4, 5\}$. Voters will always vote for the nearest candidate. If the candidates occupy the same position they each get half of the votes. The candidates simultaneously and independently choose positions. Each candidate wants to maximize the number of votes for him/herself. Only pure strategies are considered.

(a) Model this situation as a bimatrix game between the two candidates.
(b) Determine the best replies of both players.
(c) Determine all Nash equilibria (in pure strategies), if any.

We now change the situation as follows. Candidate 1 can only occupy the positions $1, 3, 5$, and candidate 2 can only occupy the positions $0, 2, 4$.

(d) Answer questions (a), (b), and (c) for this new situation.
(e) The two games above are constant-sum games. How would you turn them into zero-sum games without changing the strategic possibilities (best replies, equilibria)? What would be the value of these games and the (pure) optimal strategies?

3.10. *Guessing Numbers*
Players 1 and 2 each choose a number from the set $\{1, \ldots, K\}$. If the players choose the same number, then player 2 pays 1 Euro to player 1; otherwise no payment is made. The players can use mixed strategies and each player's preferences are determined by his or her expected monetary payoff.

(a) Show that each player playing each pure strategy with probability $\frac{1}{K}$ is a Nash equilibrium.
(b) Show that in every Nash equilibrium, player 1 must choose every number with positive probability.
(c) Show that in every Nash equilibrium, player 2 must choose every number with positive probability.

(d) Determine all (mixed strategy) Nash equilibria of the game.
(e) This is a zero-sum game. What are the optimal strategies and the value of the game?

3.11. *Bimatrix Games*

(a) Give an example of a 2×2-bimatrix game with exactly two Nash equilibria in pure strategies and no other Nash equilibrium. For your example, determine the players' reaction functions and make a picture, showing that your example is as desired.
(b) Consider the following bimatrix game:

$$(A, B) = \begin{pmatrix} a, b & c, d \\ e, f & g, h \end{pmatrix}.$$

Assume that $a > e$. Give necessary and sufficient further conditions on the payoffs such that the game has no pure strategy Nash equilibria. Under these conditions, determine all mixed Nash equilibria of the game.

3.4 Notes

For finite two-person games, Nash (1951) proved that every game has a Nash equilibrium in mixed strategies. The term 'strategic uncertainty' can already be found in von Neumann and Morgenstern (1944/1947). For general bimatrix games, Nash equilibria can be found by Nonlinear Programming methods, see Chap. 13.

Problems 3.1(a, b) are taken from Rasmusen (1989), and Problem 3.1(c) from Gibbons (1992). Problem 3.4 is taken from Watson (2002).

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