Finite Two-Person Zero-Sum Games

2

This chapter deals with two-player games in which each player chooses from finitely many pure strategies or randomizes among these strategies, and the sum of the players' payoffs or expected payoffs is always equal to zero. Games like the Battle of the Bismarck Sea and Matching Pennies, discussed in Sect. 1.3.1 belong to this class.

In Sect. 2.1 the basic definitions and theory are discussed. Section 2.2 shows how to solve $2\times n$ and $m\times 2$ games, and larger games by elimination of strictly dominated strategies.

2.1 Basic Definitions and Theory

Since all data of a finite two-person zero-sum game can be summarized in one matrix, such a game is usually called a 'matrix game'.

Definition 2.1 (Matrix Game) A *matrix game* is an $m\times n$ matrix A of real numbers, where m is the number of rows and n is the number of columns. A *(mixed) strategy* of player 1 is a probability distribution $\mathbf{p}$ over the rows of A, i.e., an element of the set

$$\Delta^m := \{\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} p_i = 1,\ p_i \geq 0 \text{ for all } i = 1, \ldots, m\} .$$

Similarly, a *(mixed) strategy* of player 2 is a probability distribution $\mathbf{q}$ over the columns of A, i.e., an element of the set

$$\Delta^n := \{\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} q_j = 1,\ q_j \geq 0 \text{ for all } j = 1, \ldots, n\} .$$

H. Peters, *Game Theory*, Springer Texts in Business and Economics,
DOI 10.1007/978-3-662-46950-7_2

A strategy $\mathbf{p}$ of player 1 is called *pure* if there is a row i with $p_i = 1$. This strategy is also denoted by $\mathbf{e}^i$. Similarly, a strategy $\mathbf{q}$ of player 2 is called *pure* if there is a column j with $q_j = 1$. This strategy is also denoted by $\mathbf{e}^j$. □

The interpretation of a matrix game A is as follows. If player 1 plays row i (i.e., pure strategy $\mathbf{e}^i$) and player 2 plays column j (i.e., pure strategy $\mathbf{e}^j$), then player 1 receives payoff a_{ij} and player 2 pays a_{ij} (and, thus, receives $-a_{ij}$), where a_{ij} is the number in row i and column j of matrix A. If player 1 plays strategy $\mathbf{p}$ and player 2 plays strategy $\mathbf{q}$, then player 1 receives the expected payoff

$$\mathbf{p}A\mathbf{q} = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j a_{ij} \;,$$

and player 2 receives $-\mathbf{p}A\mathbf{q}$.

Remark 2.2 (1) Note that, according to Definition 2.1, a strategy means a mixed strategy. A pure strategy is a special case of a mixed strategy. We add the adjective 'pure' if we wish to refer to a pure strategy. (2) Since no confusion is likely to arise, we do not use transpose notations like $\mathbf{p}^T A\mathbf{q}$ or $\mathbf{p}A\mathbf{q}^T$. □

For 'solving' matrix games, i.e., establishing what clever players would or should do, the concepts of maximin and minimax strategies are important, as will be explained below. First we give the definitions.

Definition 2.3 (Maximin and Minimax Strategies) A strategy $\mathbf{p}$ is a *maximin strategy* of player 1 in matrix game A if

$$\min\{\mathbf{p}A\mathbf{q} \mid \mathbf{q} \in \Delta^n\} \geq \min\{\mathbf{p}'A\mathbf{q} \mid \mathbf{q} \in \Delta^n\} \text{ for all } \mathbf{p}' \in \Delta^m \;.$$

A strategy $\mathbf{q}$ is a *minimax strategy* of player 2 in matrix game A if

$$\max\{\mathbf{p}A\mathbf{q} \mid \mathbf{p} \in \Delta^m\} \leq \max\{\mathbf{p}A\mathbf{q}' \mid \mathbf{p} \in \Delta^m\} \text{ for all } \mathbf{q}' \in \Delta^n \;.$$

□

In words: a maximin strategy of player 1 maximizes the minimal (with respect to player 2's strategies) payoff of player 1, and a minimax strategy of player 2 minimizes the maximum (with respect to player 1's strategies) that player 2 has to pay to player 1. (It can be proved by basic mathematical analysis that maximin and minimax strategies always exist.) Of course, the asymmetry in these definitions is caused by the fact that, by convention, a matrix game represents the amounts that player 2 has to pay to player 1.

In order to check if a strategy $\mathbf{p}$ of player 1 is a maximin strategy it is sufficient to check that the first inequality in Definition 2.3 holds with $\mathbf{e}^j$ for every $j = 1, \ldots, n$

instead of every $\mathbf{q} \in \Delta^n$. This is not difficult to see but the reader is referred to Chap. 12 for a formal treatment. A similar observation holds for minimax strategies. In other words, to check if a strategy is maximin (minimax) it is sufficient to consider its performance against every pure strategy, i.e., column (row).

Why would we be interested in such strategies? At first glance, these strategies seem to express a very conservative or pessimistic, worst-case scenario attitude. The reason for nevertheless considering maximin/minimax strategies is provided by the so-called *minimax theorem*, which states that for every matrix game A there is a real number $v = v(A)$ with the following properties:

(a) A strategy $\mathbf{p}$ of player 1 guarantees a payoff of at least v to player 1 (i.e., $\mathbf{p}A\mathbf{q} \geq v$ for all strategies $\mathbf{q}$ of player 2) if and only if $\mathbf{p}$ is a maximin strategy.
(b) A strategy $\mathbf{q}$ of player 2 guarantees a payment of at most v by player 2 to player 1 (i.e., $\mathbf{p}A\mathbf{q} \leq v$ for all strategies $\mathbf{p}$ of player 1) if and only if $\mathbf{q}$ is a minimax strategy.

Hence, player 1 can obtain a payoff of at least v by playing a maximin strategy, and player 2 can guarantee to pay not more than v—hence secure a payoff of at least $-v$—by playing a minimax strategy. For these reasons, the number $v = v(A)$ is also called the *value* of the game A—it represents the worth to player 1 of playing the game A—and maximin and minimax strategies are called *optimal strategies* for players 1 and 2, respectively.

Therefore, 'solving' the game A means, naturally, determining the optimal strategies and the value of the game. In the Battle of the Bismarck Sea in Sect. 1.3.1, the pure strategies N of both players guarantee the same amount 2. Therefore, this is the value of the game and N is optimal for both players. The analysis of that game is easy since it has a 'saddlepoint', namely position $(1, 1)$ with $a_{11} = 2$. The definition of a saddlepoint is as follows.

Definition 2.4 (Saddlepoint) A position (i, j) in a matrix game A is a *saddlepoint* if

$$a_{ij} \geq a_{kj} \text{ for all } k = 1, \ldots, m \text{ and } a_{ij} \leq a_{ik} \text{ for all } k = 1, \ldots, n \ ,$$

i.e., if a_{ij} is maximal in its column j and minimal in its row i. □

Clearly, if (i, j) is a saddlepoint, then player 1 can guarantee a payoff of at least a_{ij} by playing the pure strategy row i, since a_{ij} is minimal in row i. Similarly, player 2 can guarantee a payoff of at least $-a_{ij}$ by playing the pure strategy column j, since a_{ij} is maximal in column j. Hence, a_{ij} must be the value of the game A: $v(A) = a_{ij}$, $\mathbf{e}^i$ is an optimal (maximin) strategy of player 1, and $\mathbf{e}^j$ is an optimal (minimax) strategy of player 2.

2.2 Solving $2 \times n$ Games and $m \times 2$ Games

In this section we show how to solve matrix games where at least one of the players has only two pure strategies. We also show how the idea of strict domination can be of help in solving matrix games.

2.2.1 $2 \times n$ Games

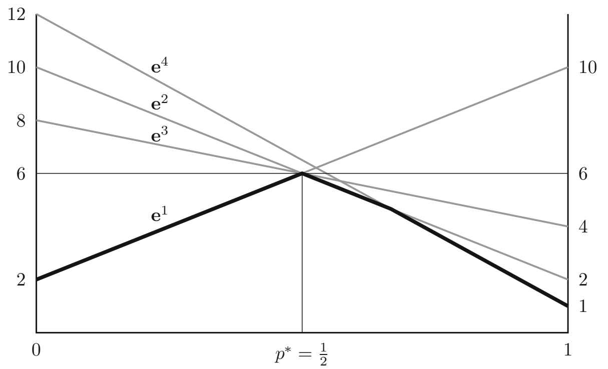
We demonstrate how to solve a matrix game with 2 rows and n columns graphically, by considering the following 2×4 example:

$$A = \begin{array}{c} \\ \end{array}\!\!\begin{array}{cccc} \mathbf{e}^1 & \mathbf{e}^2 & \mathbf{e}^3 & \mathbf{e}^4 \end{array}$$
$$A = \begin{pmatrix} 10 & 2 & 4 & 1 \\ 2 & 10 & 8 & 12 \end{pmatrix} .$$

We have labelled the columns of A, i.e., the pure strategies of player 2 for reference below. Let $\mathbf{p} = (p, 1-p)$ be an arbitrary strategy of player 1. The expected payoffs to player 1 if player 2 plays a pure strategy are equal to:

$$\begin{aligned}
\mathbf{p}A\mathbf{e}^1 &= 10p + 2(1-p) = 8p + 2 \\
\mathbf{p}A\mathbf{e}^2 &= 2p + 10(1-p) = 10 - 8p \\
\mathbf{p}A\mathbf{e}^3 &= 4p + 8(1-p) \quad = 8 - 4p \\
\mathbf{p}A\mathbf{e}^4 &= p + 12(1-p) \quad = 12 - 11p .
\end{aligned}$$

We plot these four linear functions of p in one diagram:



In this diagram the values of p are plotted on the horizontal axis, and the four straight gray lines plot the payoffs to player 1 for each of the four pure strategies of player 2. Observe that for every $0 \leq p \leq 1$ the minimum payoff that player 1 may obtain is given by the lower envelope of these curves, the thick black curve in the diagram:

for any p, any combination (q_1, q_2, q_3, q_4) of the points on $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, and $\mathbf{e}^4$ with first coordinate p would lie on or above this lower envelope. Clearly, the lower envelope is maximal for $p = p^* = \frac{1}{2}$, and the maximal value is 6. Hence, we have established that player 1 has a unique optimal (maximin) strategy, namely $\mathbf{p}^* = (\frac{1}{2}, \frac{1}{2})$, and that the value of the game, $v(A)$, is equal to 6.

What are the optimal or minimax strategies of player 2? From the theory of the previous section we know that a minimax strategy $\mathbf{q} = (q_1, q_2, q_3, q_4)$ of player 2 should guarantee to player 2 to have to pay at most the value of the game. From the diagram it is clear that q_4 should be equal to zero since otherwise the payoff to player 1 would be larger than 6 if player 1 plays $(\frac{1}{2}, \frac{1}{2})$, and thus $\mathbf{q}$ would not be a minimax strategy. So a minimax strategy has the form $\mathbf{q} = (q_1, q_2, q_3, 0)$. Any such strategy, plotted in the diagram, gives a straight line that is a combination of the lines associated with $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ and which passes through the point $(\frac{1}{2}, 6)$ since all three lines pas through this point. Moreover, for no value of p should this straight line exceed the value 6, otherwise $\mathbf{q}$ would not guarantee a payment of at most 6 by player 2. Consequently, this straight line has to be horizontal. Summarizing this argument, we look for numbers $q_1, q_2, q_3 \geq 0$ such that

$$\begin{aligned} 2q_1 + 10q_2 + 8q_3 &= 6 \quad \text{(left endpoint should be } (0, 6)) \\ 10q_1 + 2q_2 + 4q_3 &= 6 \quad \text{(right endpoint should be } (1, 6)) \\ q_1 + q_2 + q_3 &= 1 \quad (\mathbf{q} \text{ is a probability vector}) . \end{aligned}$$

By substitution, it is easy to reduce this system of equations to the two equations

$$\begin{aligned} 3q_1 - q_2 &= 1 \\ q_1 + q_2 + q_3 &= 1 . \end{aligned}$$

In fact, one of the two first equations could have been omitted from the beginning, since we already know that any combination of the three lines passes through $(\frac{1}{2}, 6)$, and two points are sufficient to determine a straight line.

From the remaining two equations, we obtain that the set of optimal strategies of player 2 is

$$\{\mathbf{q} = (q_1, q_2, q_3, q_4) \in \Delta^4 \mid q_2 = 3q_1 - 1,\ q_4 = 0\} .$$

Note that, if $q_1 = \frac{1}{3}$, then $q_2 = 0$, and if $q_1 = \frac{1}{2}$, then $q_2 = \frac{1}{2}$. Clearly, q_1 and q_2 cannot be smaller, since then their sum would be negative, and they cannot be larger since then their sum would exceed 1. Hence, the set of optimal strategies of player 2 can alternatively be described as

$$\{\mathbf{q} = (q_1, q_2, q_3, q_4) \in \mathbb{R}^4 \mid \frac{1}{3} \leq q_1 \leq \frac{1}{2},\ q_2 = 3q_1 - 1,\ q_3 = 1 - q_1 - q_2,\ q_4 = 0\} .$$

This means that the set of optimal strategies of player 2 in this game is one-dimensional, i.e., a line segment.

2.2.2 $m \times 2$ Games

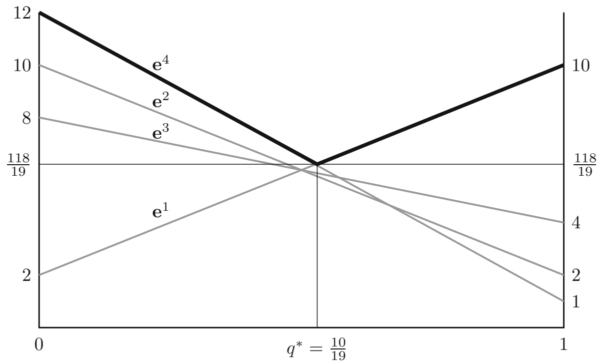
The solution method to solve $m \times 2$ games is analogous. Consider the following example:

$$A = \begin{matrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \\ \mathbf{e}^4 \end{matrix} \begin{pmatrix} 10 & 2 \\ 2 & 10 \\ 4 & 8 \\ 1 & 12 \end{pmatrix}.$$

Let $\mathbf{q} = (q, 1-q)$ be an arbitrary strategy of player 2. Again, we make a diagram in which now the values of q are put on the horizontal axis, and the straight lines indicated by $\mathbf{e}^i$ for $i = 1, 2, 3, 4$ are the payoffs to player 1 associated with his four pure strategies (rows) as functions of q. The equations of these lines are given by:

$$\begin{aligned} \mathbf{e}^1 A\mathbf{q} &= 10q + 2(1-q) = 8q + 2 \\ \mathbf{e}^2 A\mathbf{q} &= 2q + 10(1-q) = 10 - 8q \\ \mathbf{e}^3 A\mathbf{q} &= 4q + 8(1-q) \;\; = 8 - 4q \\ \mathbf{e}^4 A\mathbf{q} &= q + 12(1-q) \;\; = 12 - 11q\,. \end{aligned}$$

The resulting diagram is as follows.



Observe that the maximum payments that player 2 has to make are now located on the upper envelope, represented by the thick black curve. The minimum is reached at the point of intersection of $\mathbf{e}^1$ and $\mathbf{e}^4$ in the diagram, which has coordinates $(\frac{10}{19}, \frac{118}{19})$. Hence, the value of the game is $\frac{118}{19}$, and the unique optimal (minimax) strategy of player 2 is $\mathbf{q}^* = (\frac{10}{19}, \frac{9}{19})$.

To find the optimal strategy or strategies $\mathbf{p} = (p_1, p_2, p_3, p_4)$ of player 1, it follows from the diagram that $p_2 = p_3 = 0$, otherwise for $q = \frac{10}{19}$ the value $\frac{118}{19}$ of the game is not reached, so that $\mathbf{p}$ is not a maximin strategy. So we look for a combination of $\mathbf{e}^1$ and $\mathbf{e}^4$ that gives at least $\frac{118}{19}$ for every q, hence it has to be equal to $\frac{118}{19}$ for every q. This gives rise to the equations $2p_1 + 12p_4 = 10p_1 + p_4 = \frac{118}{19}$ and $p_1 + p_4 = 1$, with unique solution $p_1 = \frac{11}{19}$ and $p_4 = \frac{8}{19}$. So the unique optimal strategy of player 1 is $(\frac{11}{19}, 0, 0, \frac{8}{19})$.

2.2.3 Strict Domination

The idea of strict domination can be used to first eliminate pure strategies before the graphical analysis of a matrix game. Consider the game

$$\begin{array}{cc} & \begin{array}{ccccc} \mathbf{e}^1 & \mathbf{e}^2 & \mathbf{e}^3 & \mathbf{e}^4 & \mathbf{e}^5 \end{array} \\ A = & \begin{pmatrix} 10 & 2 & 5 & 1 & 6 \\ 2 & 10 & 8 & 12 & 9 \end{pmatrix} . \end{array}$$

This game is obtained from the game in Sect. 2.2.1 by adding a fifth column and changing a_{13} from 4 to 5.

In this game it cannot be optimal for player 2 to put positive probability on the fifth column, since the payoffs in the third column are always—that is, no matter what player 1's strategy is—better for player 2. So we can assume that the fifth column is played with zero probability: it is strictly dominated by the third column. This is a case where a pure strategy is strictly dominated by another pure strategy.

It can also happen that a pure strategy is strictly dominated by a mixed strategy. For instance, consider the third column. The payoff 5 in the first row is in between the payoffs 10 and 2 in the first row and first and second columns, respectively; and also the payoff 8 in the second row is in between the payoffs 2 and 10 in the second row and first and second columns. So it may be possible to find a combination of the first two columns that results in smaller payoffs than those in the third column. In order to see if such a combination is possible, suppose that probability α is put on the first column and $1-\alpha$ is put on the second column. The resulting payoffs are

$$\alpha \begin{pmatrix} 10 \\ 2 \end{pmatrix} + (1-\alpha) \begin{pmatrix} 2 \\ 10 \end{pmatrix} = \begin{pmatrix} 8\alpha + 2 \\ 10 - 8\alpha \end{pmatrix} .$$

We wish to have $8\alpha + 2 < 5$ and $10 - 8\alpha < 8$, and both inequalities hold as long as $\frac{1}{4} < \alpha < \frac{3}{8}$. This means that the pure strategy $\mathbf{e}^3$ is strictly dominated by any (mixed) strategy $(\alpha, 1-\alpha, 0, 0, 0)$, as long as α is in the computed range. This implies that, in an optimal strategy, the probability q_3 put by player 2 on the third column must be zero, otherwise player 2 could guarantee to pay less by adding αq_3 to the first column and $(1-\alpha)q_3$ to the second column, for any $\frac{1}{4} < \alpha < \frac{3}{8}$, and playing the third column with zero probability.

The preceding analysis implies that, in order to solve the above game, we can start by eliminating the third and fifth columns of the matrix. Thus, in the diagram in Sect. 2.2.1, we do not have to draw the line corresponding to $\mathbf{e}^3$. The value of the game is still 6, player 1 still has a unique optimal strategy $\mathbf{p}^* = (\frac{1}{2}, \frac{1}{2})$, and player 2 now also has a unique optimal strategy, namely the one where $q_3 = 0$, which is the strategy $(\frac{1}{2}, \frac{1}{2}, 0, 0, 0)$.

In general, strictly dominated pure strategies in a matrix game are not played with positive probability in any optimal strategy and can therefore be eliminated before solving the game. Sometimes, this idea can also be used to solve matrix games in which each player has more than two pure strategies ($m, n > 2$). Moreover, the idea can be applied iteratively, that is, after elimination of a strictly dominated pure strategy, in the smaller game perhaps another strictly dominated pure strategy can be eliminated, etc., until no more pure strategies are strictly dominated. See Example 2.7 for an illustration, and see Chap. 13 for a rigorous treatment.

We first give the formal definition of strict domination, and then discuss the announced example.

Definition 2.5 (Strict Domination) Let A be an $m \times n$ matrix game and i a row. The pure strategy $\mathbf{e}^i$ is *strictly dominated* if there is a strategy $\mathbf{p} = (p_1, \ldots, p_m) \in \Delta^m$ such that $\mathbf{p}A\mathbf{e}^j > \mathbf{e}^i A\mathbf{e}^j$ for every $j = 1, \ldots, n$. Similarly, let j be a column. The pure strategy $\mathbf{e}^j$ is *strictly dominated* if there is a strategy $\mathbf{q} = (q_1, \ldots, q_n) \in \Delta^n$ such that $\mathbf{e}^i A\mathbf{q} < \mathbf{e}^i A\mathbf{e}^j$ for every $i = 1, \ldots, m$. □

Remark 2.6 It is not difficult to see that if $\mathbf{e}^i$ is strictly dominated, then it is strictly dominated by some $\mathbf{p} \in \Delta^m$ with $p_i = 0$. Similarly, if $\mathbf{e}^j$ is strictly dominated, then it is strictly dominated by some $\mathbf{q} \in \Delta^n$ with $q_j = 0$. □

Example 2.7 Consider the following 3×3 matrix game:

$$A = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}.$$

For player 1, the third strategy $\mathbf{e}^3$ is strictly dominated by the strategy $\mathbf{p} = (\frac{7}{12}, \frac{5}{12}, 0)$, since

$$\mathbf{p}A = \begin{pmatrix} \frac{7}{2} & \frac{25}{12} & \frac{17}{6} \end{pmatrix} \text{ and } \mathbf{e}^3 A = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}.$$

Hence, in any optimal strategy player 1 puts zero probability on the third row. Elimination of this row results in the matrix

$$B = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 5 & 4 \end{pmatrix}.$$

Now, player 2's third strategy $\mathbf{e}^3$ is strictly dominated by the strategy $\mathbf{q} = (\frac{1}{4}, \frac{3}{4}, 0)$, since

$$B\mathbf{q} = \begin{pmatrix} \frac{3}{2} \\ \frac{15}{4} \end{pmatrix} \text{ and } B\mathbf{e}^3 = \begin{pmatrix} 2 \\ 4 \end{pmatrix} .$$

Hence, in any optimal strategy player 2 puts zero probability on the third column. Elimination of this column results in the matrix

$$C = \begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix} .$$

This is a 2×2 matrix game, which can be solved by the method in Sect. 2.2.1 or Sect. 2.2.2. See Problem 2.1(a). □

2.3 Problems

2.1. *Solving Matrix Games*
Solve the following matrix games, i.e., determine the optimal strategies and the value of the game. Each time, start by checking if the game has a saddlepoint.

(a)

$$\begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}$$

What are the optimal strategies in the original matrix game A in Example 2.7?

(b)

$$\begin{pmatrix} 2 & -1 & 0 & 2 \\ 2 & 0 & 0 & 3 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 16 & 12 & 2 \\ 2 & 6 & 16 \\ 8 & 8 & 6 \\ 0 & 7 & 8 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{pmatrix}$$

(f)

$$\begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 1 \\ 3 & 1 & 3 \end{pmatrix}.$$

2.2. *Saddlepoints*

(a) Let A be an arbitrary $m \times n$ matrix game. Show that any two saddlepoints must have the same value. In other words, if (i, j) and (k, l) are two saddlepoints, show that $a_{ij} = a_{kl}$.
(b) Let A be a 4×4 matrix game in which $(1, 1)$ and $(4, 4)$ are saddlepoints. Show that A has at least two other saddlepoints.
(c) Give an example of a 4×4 matrix game with exactly three saddlepoints.

2.3. *Maximin Rows and Minimax Columns*
Row i is a *maximin row* in an $m \times n$ matrix game A if $\min_{j \in \{1,\ldots,n\}} a_{ij} \geq \min_{j \in \{1,\ldots,n\}} a_{kj}$ for all $k \in \{1, \ldots, m\}$. Column j is a *minimax column* if $\max_{i \in \{1,\ldots,m\}} a_{ij} \leq \max_{i \in \{1,\ldots,m\}} a_{i\ell}$ for all $\ell \in \{1, \ldots, n\}$.
Consider the following matrix game:

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 1 \end{pmatrix}.$$

(a) Determine all maximin rows and minimax columns. What can you conclude from this about the value of this game?
(b) The value of this game is $\frac{12}{7}$. Use this to give an argument why player 2 will put zero probability on column 2 in any minimax strategy.
(c) Determine all minimax strategies of player 2 and all maximin strategies of player 1.

2.4. *Subgames of Matrix Games*
Consider the following matrix game:

$$A = \begin{pmatrix} 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{pmatrix}.$$

(a) Determine all maximin rows and minimax columns. (See Problem 2.3.) What can you conclude from this about the value of this game?

(b) Consider the six different 2×2-matrix games that can be obtained by choosing two columns from A, as follows:

$$A_1 = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \; A_2 = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix} \; A_3 = \begin{pmatrix} 3 & 0 \\ 1 & 5 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 4 \\ 2 & 0 \end{pmatrix} \; A_5 = \begin{pmatrix} 1 & 0 \\ 2 & 5 \end{pmatrix} \; A_6 = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}.$$

Determine the values of all these games. Which one must be equal to the value of A?

(c) Determine all maximin and minimax strategies of A. [Hint: Use your answer to (b).]

2.5. *Rock-Paper-Scissors*
In the famous Rock-Paper-Scissors two-player game each player has three pure strategies: Rock, Paper, and Scissors. Here, Scissors beats Paper, Paper beats Rock, Rock beats Scissors. Assign a 1 to winning, 0 to a draw, and -1 to losing. Model this game as a matrix game, try to guess its optimal strategies, and then show that these are the unique optimal strategies. What is the value of this game?

2.4 Notes

The theory of zero-sum games was developed by von Neumann (1928), who proved the minimax theorem. In general, matrix games can be solved by Linear Programming. See Chap. 12 for details.

Reference

von Neumann, J. (1928). Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen, 100*, 295–320.