24 Hints, Answers and Solutions

Problems of Chapter 1

1.2 *Variant of Matching Pennies*
There are saddlepoint(s) if and only if $x \leq -1$.

1.3 *Mixed Strategies*

(b) $(3/4, 1/4)$.
(c) $(1/2, 1/2)$.
(d) By playing $(3/4, 1/4)$ player 1 obtains $10/4 = 2.5$ for sure (independent of what player 2 does). Similarly, by playing $(1/2, 1/2)$, player 2 is sure to pay 2.5. So 2.5 is the value of this game. Given a rational opponent, no player can hope to do better by playing differently.

1.4 *Sequential Cournot*

(b) $q_1 = 1/3$ and $q_2 = 1/6$.

1.6 *Glove Game*

(a) $(0, 0, 1)$ is the unique vector in the core of the glove game.

1.7 *Dentist Appointments*
The Shapley value $(9\frac{1}{2}, 6\frac{1}{2}, 8)$ is *not* in the core of this game. The nucleolus is in the core of the game.

H. Peters, *Game Theory*, Springer Texts in Business and Economics,
DOI 10.1007/978-3-662-46950-7_24

1.8 *Nash Bargaining*

(a) The problem to solve is $\max_{0\leq\alpha\leq 1} \alpha\sqrt{1-\alpha}$. Obviously, the solution must be interior: $0 < \alpha < 1$. Set the first derivative equal to 0, solve, and check that the second derivative is negative.
(b) In terms of utilities, the Nash bargaining solution is $(2/3, (1/3)\sqrt{3})$.

1.9 *Variant of Glove Game*
The worth of a coalition S depends on the minimum of the numbers of right-hand and left-hand players in the coalition.

Problems of Chapter 2

2.1 *Solving Matrix Games*

(a) The optimal strategies are $(5/11, 6/11)$ for player 1 and $(5/11, 6/11)$ for player 2. The value of the game is $30/11$. In the original game the optimal strategies are $(5/11, 6/11, 0)$ for player 1 and $(5/11, 6/11, 0)$ for player 2.
(b) The value of the game is 0. The unique maximin strategy is $(0, 1, 0)$. The minimax strategies are $(0, q, 1-q, 0)$ for any $0 \leq q \leq 1$.
(c) The value of the game is 1, the unique minimax strategy is $(1/2, 0, 1/2)$, and the maximin strategies are: $(p, (1-p)/2, (1-p)/2)$ for $0 \leq p \leq 1$.
(d) The value of the game is 9 and player 1's maximin strategy is $(1/2, 1/2, 0, 0)$. The set of all minimax strategies is $\{(\alpha, (7-14\alpha)/10, (3+4\alpha)/10) \in \mathbb{R}^3 \mid 0 \leq \alpha \leq 1/2\}$.
(e) The value is $8/5$. The unique maximin strategy is $(2/5, 3/5)$ and the unique minimax strategy is $(0, 4/5, 1/5, 0)$.
(f) The value is equal to 1, player 2 has a unique minimax strategy namely $(0, 1, 0)$, and the set of maximin strategies is $\{(0, p, 1-p) \mid 0 \leq p \leq 1\}$.

2.2 *Saddlepoints*

(b) There are saddlepoints at $(1, 4)$ and at $(4, 1)$.

2.3 *Maximin Rows and Minimax Columns*

(c) The unique maximin strategy is $(\frac{4}{7}, 0, \frac{3}{7}, 0)$ and the unique minimax strategy is $(\frac{4}{7}, 0, \frac{3}{7})$.

2.4 *Subgames of Matrix Games*

(c) The unique minimax strategy is $(0, 4/5, 1/5, 0)$ and the unique maximin strategy is $(2/5, 3/5)$.

2.5 *Rock-Paper-Scissors*
The associated matrix game is:

$$\begin{array}{c} \\ R \\ P \\ S \end{array}\begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}.$$

Problems of Chapter 3

3.1 *Some Applications*

(a) Let Smith be the row player and Brown the column player, then the bimatrix game is:

$$\begin{array}{c} \\ L \\ S \end{array}\begin{array}{c} \begin{array}{cc} L & S \end{array} \\ \begin{pmatrix} 2,2 & -1,-1 \\ -1,-1 & 1,1 \end{pmatrix} \end{array}.$$

(b) Let the government be the row player and the pauper the column player. The bimatrix game is:

$$\begin{array}{c} \\ \text{aid} \\ \text{not} \end{array}\begin{array}{c} \begin{array}{cc} \text{work} & \text{not} \end{array} \\ \begin{pmatrix} 3,2 & -1,3 \\ -1,1 & 0,0 \end{pmatrix} \end{array}.$$

(c) This game has two pure strategy Nash equilibria and one other (mixed strategy) Nash equilibrium.

(e, f) This situation can be modelled as a 3×3 bimatrix game. For (e), the expected numbers of candidates in the committee can be taken as payoffs; for (f), the payoff is the expected utility (using $\sqrt{c}$) of the lottery determining the number of candidates in the committee.

3.2 *Matrix Games*

(a) You should find the same solution, namely $(5/11, 6/11)$ for player 1 and $(5/11, 6/11)$ for player 2, as the unique Nash equilibrium.

(b) If player 2 plays a minimax strategy then 2's payoff is at least $-v$, where v is the value of the game. Hence, any strategy that gives player 1 at least v is a best reply. So a maximin strategy is a best reply. Similarly, a minimax strategy is a best reply against a maximin strategy, so any pair consisting of a maximin and a minimax strategy is a Nash equilibrium.

Conversely, in a Nash equilibrium the payoffs must be $(v, -v)$ otherwise one of the players could improve by playing an optimal (maximin or minimax) strategy. But then player 1's strategy must be a maximin strategy since otherwise player 2 would have a better reply, and player 2's strategy must be a minimax strategy since otherwise player 1 would have a better reply.

(c) The appropriate definition for player 2 would be: a maximin strategy *for player 2* in B, since now B represents the payoffs to player 2, and not what player 2 has to pay to player 1.
The Nash equilibrium of Problem 3.1(b), for instance, does not consist of maximin strategies of the players. The maximin strategy of player 1 in A is $(1/5, 4/5)$, which is not part of a (the) Nash equilibrium. The maximin strategy of player 2 (!) in B is $(1, 0)$, which is not part of a (the) Nash equilibrium.

3.3 *Strict Domination*

(c) There are three Nash equilibria: $((1,0), (1,0,0,0))$, $((0,1),(0,0,1,0))$, and $((3/7,4/7),(1/3,0,2/3,0))$.

3.4 *Iterated Elimination (1)*

(b) The unique equilibrium is (B, Y).

3.5 *Iterated Elimination (2)*
The Nash equilibria are $((1/3,2/3,0),(2/3,0,1/3))$, $((0,1,0),(1,0,0))$, and $((1,0,0),(0,0,1))$.

3.6 *Weakly Dominated Strategies*

(b) Consecutive deletion of Z, C, A results in the Nash equilibria (B, X) and (B, Y). Consecutive deletion of C, Y, B, Z results in the Nash equilibrium (A, X).

3.7 *A Parameter Game*
Distinguish three cases: $a > 2$, $a = 2$, and $a < 2$.

3.8 *Equalizing Property of Mixed Equilibrium Strategies*

(a) Check by substitution.
(b) Suppose the expected payoff (computed by using $\mathbf{q}^*$) of row i played with positive probability (p_i^*) in a Nash equilibrium $(\mathbf{p}^*, \mathbf{q}^*)$, hence the number $\mathbf{e}^i A\mathbf{q}^*$, would not be maximal. Then player 1 would improve by adding the probability p_i^* to some row j with higher expected payoff $\mathbf{e}^j A\mathbf{q}^* > \mathbf{e}^i A\mathbf{q}^*$, and in this way increase his payoff, a contradiction. A similar argument can be made for player 2 and the columns.

3.9 *Voting*

(a, b, c) Set the total number of voters equal to 10 (in order to avoid fractions). We obtain a 6×6 bimatrix game in which the sum of the payoffs per entry is always 10. This game has four Nash equilibria in pure strategies.
(d) Now we obtain a unique Nash equilibrium in pure strategies.
(e) In both games, subtract 5 from all payoffs.

3.10 *Guessing Numbers*

(d) The unique Nash equilibrium is the one where each player chooses each number with equal probability.
(e) The value of this game is $\frac{1}{K}$.

3.11 *Bimatrix Games*

(b) $e < a$, $b < d$, $c < g$, $h < f$. Then there is a unique Nash equilibrium.

Problems of Chapter 4

4.1 *Counting Strategies*
White has 20 possible opening moves, and therefore also 20 possible strategies. Black has many more strategies.

4.2 *Extensive Versus Strategic Form*
For the game with perfect information, start with a decision node of player 1. For the game with imperfect information, start with a decision node of player 2.

4.4 *Choosing Objects*

(c) In any subgame perfect equilibrium the game is played as follows: player 1 picks O_3, then player 2 picks O_2 or O_1, and finally player 1 picks O_4. These are the (two) subgame perfect equilibrium outcomes of the game. Due to ties (of player 2) there is more than one subgame perfect equilibrium, namely eight in total. All subgame perfect equilibria result in the same distribution of the objects.
(d) There is a Nash equilibrium in which player 1 obtains the objects O_4 and O_2.

4.5 *A Bidding Game*

(c) Due to ties, there are four different subgame perfect equilibria. They all result in the same outcome.

4.6 *An Extensive Form Game*
There is a unique pure strategy Nash equilibrium, which is also subgame perfect. This equilibrium is perfect Bayesian for an appropriate choice of player 2's belief.

4.7 *Another Extensive Form Game*
There is a unique Nash equilibrium (in pure strategies). This equilibrium is not perfect Bayesian.

4.8 *Still Another Extensive Form Game*

(b) There are three Nash equilibria in pure strategies.
(c) There are two subgame perfect Nash equilibria in pure strategies.
(d) Both equilibria in (c) are perfect Bayesian.

4.9 *A Centipede Game*

(b) Consider any strategy combination. The last player who has continued when playing his strategy could have improved by stopping if possible. Hence, in equilibrium the play of the game must have stopped immediately.
To exhibit a non-subgame perfect Nash equilibrium, assume that player 1 always stops, and that player 2 also always stops except at his second decision node. Check that this is a Nash equilibrium. [One can also write down the strategic form, which is an 8×8 bimatrix game.]

4.10 *Finitely Repeated Prisoners' Dilemma*

(a) There are five subgames, including the entire game.
(b) There is a unique subgame perfect equilibrium.

4.11 *A Twice Repeated* 2×2 *Bimatrix Game*

(b) Player 1: play B at the first stage; if (B, L) was played at the first stage play B at the second stage, otherwise play T at the second stage.

4.12 *Twice Repeated* 3×3 *Bimatrix Games*

(a) There are ten subgames, including the entire game.
(b) Player 1: play T at the first stage. Player 2: play L at the first stage. Second stage play is given by the following diagram:

$$\begin{array}{c} \\ T \\ C \\ B \end{array} \begin{array}{c} \begin{array}{ccc} L & M & R \end{array} \\ \begin{pmatrix} B,R & C,R & C,R \\ B,M & B,R & B,R \\ B,M & B,R & B,R \end{pmatrix} \end{array} .$$

For instance, if first stage play results in (C, L), then player 1 plays B and player 2 plays M at stage 2. Verify that this defines a subgame perfect equilibrium in which (T, L) is played at the second stage. (Other solutions are possible, as long as players 1 and 2 are punished for unilateral deviations at stage 1.)

Problems of Chapter 5

5.1 *Battle-of-the-Sexes*
The strategic form is a 4×4 bimatrix game. List the strategies of the players as in the text. We can then compute the expected payoffs. For example, if the first row corresponds to strategy FF of player 1 and strategies FF, FB, BF, and BB of player 2, then the payoffs are, respectively, $1/6$ times $(8,3)$, $(6,9)$, $(6,0)$, and $(4,6)$. The (pure strategy) Nash equilibria are (FF, FB) and (BF, BB).

5.2 *A Static Game of Incomplete Information*
There are three pure Nash equilibria: (TT, L), (TB, R), and (BB, R). (The first letter in a strategy of player 1 applies to Game 1, the second letter to Game 2.)

5.3 *Another Static Game of Incomplete Information*

(b) The unique pure strategy Nash equilibrium is: t_1 and t_1' play B, t_2 and t_2' play R.

5.4 *Job-Market Signaling*

(b) There is a separating perfect Bayesian equilibrium. There is another Nash equilibrium in which no worker takes education, but this is not perfect Bayesian.

5.5 *A Joint Venture*

(c) There is a unique Nash equilibrium (even in mixed strategies). This is also subgame perfect and perfect Bayesian.

5.6 *Entry Deterrence*
For $x \leq 100$ the strategy combination where the entrant always enters and the incumbent colludes is a perfect Bayesian equilibrium. For $x \geq 50$, the combination where the entrant always stays out and the incumbent fights is a perfect Bayesian equilibrium if the incumbent believes that, if the entrant enters, then fighting yields 0 with probability at most $1 - \frac{50}{x}$. IC applies only to the second equilibrium, which survives it.

5.7 *The Beer-Quiche Game*

(b) There are two perfect Bayesian equilibria, both of which are pooling. In the first one, player 1 always eats quiche. This equilibrium does not survive IC. In the second one, player 1 always drinks beer. This equilibrium does survive IC.

5.8 *Issuing Stock*

(b) There is a pooling equilibrium in which the manager never proposes to issue new stock, and such a proposal would not be approved of by the existing shareholders since they believe that this proposal signals a good state with high enough probability. [The background of this is that a new stock issue would dilute the value of the stock of the existing shareholders in a good state of the world, see the original article Myers and Majluf (1984) for details.] This equilibrium survives the intuitive criterion.

There is also a separating equilibrium in which a stock issue is proposed in the bad state but not in the good state. If a stock issue is proposed, then it is approved of.

Finally, there is a separating equilibrium in which a stock issue is proposed in the good state but not in the bad state. If a stock issue is proposed, then it is not approved of.

(c) In this case, a stock issue proposal would always be approved of, so the 'bad news effect' of a stock issue vanishes. The reason is that the investment opportunity is now much more attractive.

5.9 *More Signaling Games*

(a) IC does not apply, since it would rule out both types of player 1.
(b) There is a unique, pooling perfect Bayesian equilibrium. This equilibrium does not survive the intuitive criterion.
(c) There are two strategy combinations that are perfect Bayesian.

Problems of Chapter 6

6.1 *Cournot with Asymmetric Costs*
The Nash equilibrium is $q_1 = (a - 2c_1 + c_2)/3$ and $q_2 = (a - 2c_2 + c_1)/3$, given that these amounts are nonnegative.

6.2 *Cournot Oligopoly*

(b) The reaction function of player i is: $\beta_i(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) = (a - c - \sum_{j \neq i} q_j)/2$ if $\sum_{j \neq i} q_j \leq a - c$, and $\beta_i(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) = 0$ otherwise.
(c) One should compute the point of intersection of the n reaction functions. This amounts to solving a system of n linear equations in n unknowns $q_1, \ldots, q_n$. Alternatively, one may guess that there is a solution $q_1 = q_2 = \ldots = q_n$. Then $q_1 = (a - c - (n-1)q_1)/2$, resulting in $q_1 = (a-c)/(n+1)$. Hence, each firm producing $(a-c)/(n+1)$ is a Nash equilibrium. If the number of firms becomes large then this amount converges to 0, which is no surprise since demand is bounded by a.
(d) To show that this equilibrium is unique, it is sufficient to show that the determinant of the coefficient matrix associated with the system of n linear equations in n unknowns (the reaction functions) is unequal to zero.

6.3 *Quantity Competition with Heterogenous Goods*

(a) $\Pi_i(q_1, q_2) = q_i p_i(q_1, q_2) - cq_i$ for $i = 1, 2$.
(b) The equilibrium is: $q_1 = (21-4c)/33, q_2 = (13-3c)/22, p_1 = (21+7c)/11, p_2 = (39 + 13c)/22$. From this the profits are easily computed.
(c) $q_1 = (57 - 10c)/95, q_2 = (38 - 10c)/95, p_1 = (228 + 50c)/95, p_2 = (228 + 45c)/95$. From this the profits are easily computed.
(d) $q_1 = \max\{1 - \frac{1}{2}p_1 + \frac{1}{3}p_2, 0\}, q_2 = \max\{1 - \frac{1}{2}p_2 + \frac{1}{4}p_1\}$. The profit functions are now $\Pi_1(p_1, p_2) = p_1 q_1 - cq_1$ and $\Pi_2(p_1, p_2) = p_2 q_2 - cq_2$, with q_1 and q_2 as given.
(e) The equilibrium is $p_1 = (16 + 8c)/11, p_2 = (30 + 15c)/22$. Note that these prices are different from the ones in (c). Profits under price competition will be lower than those under quantity competition.
(f) These are the same prices and quantities as under (c).
(g) See the answers to (e) and (f).

6.4 *A Numerical Example of Cournot Competition with Incomplete Information*
$q_1 = 18/48$, $q_H = 9/48$, $q_L = 15/48$. In the complete information case with low cost we have $q_1 = q_2 = 16/48$, with high cost it is $q_1 = 20/48$ and $q_2 = 8/48$. Note that the low cost firm 'suffers' from incomplete information since firm 1 attaches some positive probability to firm 2 having high cost and therefore has higher supply. For the high cost firm the situation is reversed: it 'benefits' from incomplete information.

6.5 *Cournot Competition with Two-Sided Incomplete Information*
Similar to (6.3) we derive:

$$q_\ell = q_\ell(q_H, q_L) = \frac{a - c_\ell - \vartheta q_H - (1-\vartheta)q_L}{2} ,$$

$$q_h = q_h(q_H, q_L) = \frac{a - c_h - \vartheta q_H - (1-\vartheta)q_L}{2} ,$$

$$q_L = q_L(q_h, q_\ell) = \frac{a - c_L - \pi q_h - (1-\pi)q_\ell}{2} ,$$

$$q_H = q_H(q_h, q_\ell) = \frac{a - c_H - \pi q_h - (1-\pi)q_\ell}{2} .$$

Here, q_ℓ and q_h correspond to the low and high cost types of firm 1 and q_L, and q_H correspond to the low and high cost types of firm 2. The (Bayesian) Nash equilibrium follows by solving these four equations in the four unknown quantities.

6.6 *Incomplete Information About Demand*
The equilibrium is: $q_1 = (\vartheta a_H + (1-\vartheta)a_L - c)/3$, $q_H = (a_H - c)/3 + ((1-\vartheta)/6)(a_H - a_L)$, $q_L = (a_L - c)/3 - (\vartheta/6)(a_H - a_L)$. (Assume that all these quantities are positive.)

6.7 *Variations on Two-Person Bertrand*

(a) If $c_1 < c_2$ then there is no Nash equilibrium. (Write down the reaction functions or—easier—consider different cases.)
(b) In both cases (i) and (ii), there are two equilibria.

6.8 *Bertrand with More Than Two Firms*
A strategy combination is a Nash equilibrium if and only if at least two firms charge a price of c and the other firms charge prices higher than c.

6.9 *Variations on Stackelberg*

(a) With firm 1 as a leader we have $q_1 = (1/2)(a - 2c_1 + c_2)$ and $q_2 = (1/4)(a + 2c_1 - 3c_2)$. With firm 2 as a leader we have $q_2 = (1/2)(a - 2c_2 + c_1)$ and $q_1 = (1/4)(a + 2c_2 - 3c_1)$.
(b) The leader in the Stackelberg game can always play the Cournot quantity: since the follower plays the best reply, this results in the Cournot outcome. Hence, the Stackelberg equilibrium—where the leader maximizes—can only give a higher payoff. (This argument holds for an arbitrary game where one player moves first and the other player moves next, having observed the move of the first player.)
(c) $q_i = (1/2^i)(a - c)$ for $i = 1, 2, \ldots, n$.

6.10 *First-Price Sealed-Bid Auction*

(b) Suppose that in some Nash equilibrium player i wins with valuation $v_i < v_1$. Then the winning bid b_i must be at most v_i otherwise player i makes a negative profit and therefore can improve by bidding (e.g.) v_i. But then player 1 can improve by bidding higher than b_i (and win) but lower than v_1 (and make positive profit). Other Nash equilibria: $(v_1, v_1, 0, 0, \ldots, 0)$, $(b, b, b, \ldots, b)$ with $v_1 \geq b \geq v_2$, etc.
(d) If not, then there would be a Nash equilibrium in which—in view of (c)—all players bid below their valuations. By (b) a player with the highest valuation wins the auction, so this must be player 1 if each player bids below his true valuation. But then player 1 can improve if $b_1 \geq v_2$ and player 2 can improve if $b_1 < v_2$.

6.11 *Second-Price Sealed-Bid Auction*

(d) Also $(v_1, 0 \ldots, 0)$ is a Nash equilibrium.
(e) The equilibria are: $\{(b_1, b_2) \mid b_2 \geq v_1,\ 0 \leq b_1 \leq v_2\} \cup \{(b_1, b_2) \mid b_1 \geq v_2,\ b_2 \leq b_1\}$.

6.12 *Third-Price Sealed-Bid Auction*

(b) Suppose $v_1 > v_2 > v_3 > \dots$, then bidder 2 could improve by bidding higher than v_1.
(c) Everybody bidding the highest valuation v_1 is a Nash equilibrium. Also everybody bidding the second highest valuation v_2 is a Nash equilibrium. (There are many more!)

6.13 *n-Player First-Price Sealed-Bid Auction with Incomplete Information*
Suppose every player $j \neq i$ plays s_j^*. If player i's type is v_i and he bids b_i (which can be assumed to be at most $1-1/n$ since no other bidder bids higher than this) then the probability of winning the auction is equal to the probability that very bid $b_j, j \neq i$, is at most b_i (including equality since this happens with zero probability). In turn, this is equal to the probability that $v_j \leq n/(n-1)b_i$ for every $j \neq i$. Since the players's valuations are independently drawn from the uniform distribution, the probability that player i wins the auction is equal to $((n/(n-1))b_i)^{n-1}$, hence player i should maximize the expression $(v_i - b_i)((n/(n-1))b_i)^{n-1}$, resulting in $b_i = (1-1/n)v_i$.

6.14 *Double Auction*

(b) The probability of trade given that $v_s \leq v_b$ is equal to $2x(1-x)$. Note that this is maximal for $x = 1/2$, and then it is equal to $1/2$.
(c) $p_b(v_b) = (2/3)v_b + 1/12$ and $p_s(v_s) = (2/3)v_s + 1/4$.
(d) Observe that no trade occurs if $v_s > v_b$. Suppose $v_s \leq v_b$. Then the (conditional) probability that trade occurs is $9/16$. Observe that this is larger than the maximal probability in (b).

6.15 *Mixed Strategies and Objective Uncertainty*

(a) $((1/2, 1/2), (2/5, 3/5))$.

6.16 *Variations on Finite Horizon Bargaining*

(a) Adapt Table 6.1 for the various cases.
(b) The subgame perfect equilibrium *outcome* is: player 1 proposes $(1 - \delta_2 + \delta_1\delta_2, \delta_2 - \delta_1\delta_2)$ at $t = 0$ and player 2 accepts.
(c) The subgame perfect equilibrium *outcome* in shares of the good is: player 1 proposes $(1 - \delta_2^2 + \delta_1\delta_2^2, \delta_2^2 - \delta_1\delta_2^2)$ at $t = 0$ and player 2 accepts.
(d) The subgame perfect equilibrium *outcome* is: player 1 proposes $(1 - \delta + \delta^2 - \dots + \delta^{T-1} - \delta^T s_1, \delta - \delta^2 + \dots - \delta^{T-1} + \delta^T s_1)$ at $t = 0$ and player 2 accepts.
(e) The limits are $(1/(1+\delta), \delta/(1+\delta))$, independent of $\mathbf{s}$.

6.17 *Variations on Infinite Horizon Bargaining*

(a) Conditions (6.10) are replaced by $x_2^* = \delta_2 y_2^*$ and $y_1^* = \delta_1 x_1^*$. This implies $x_1^* = (1-\delta_2)/(1-\delta_1\delta_2)$ and $y_1^* = (\delta_1 - \delta_1\delta_2)/(1-\delta_1\delta_2)$. In the strategies (σ_1^*) and (σ_2^*), replace δ by δ_1 and δ_2, respectively. The equilibrium outcome is that player 1's proposal x^* at $t = 0$ is accepted.
(b) Nothing essential changes. Player 2's proposal y^* is accepted at $t = 0$.
(c) Nothing changes compared to the situation in the text, since **s** is only obtained at $t = \infty$.
(e) Let p denote the probability that the game ends. Then p is also the probability that the game ends given that it does not end at $t = 0$. Hence, $p = (1-\delta) + \delta p$, so that $p = 1$.

6.18 *A Principal-Agent Game*

(a) This is a game of complete information.
(b) The subgame perfect equilibrium can be found by backward induction. Distinguish two cases: (i) strategy h is optimal for the worker and (ii) strategy l is optimal for the worker. Show that it is optimal for the employer to induce high effort by a wage combination (w_H, w_L) with $8w_H + 2w_L = 50$ and $w_H - w_L \geq 5$.

6.19 *The Market for Lemons*

(b) There are many subgame perfect equilibria: the buyer offers $p \leq 5{,}000$ and the seller accepts any price of at least 5,000 if the car is bad and of at least 15,000 if the car is good. All these equilibria result in expected payoff of zero for both. There are no other subgame perfect equilibria.

6.20 *Corporate Investment and Capital Structure*

(b) Suppose the investor's belief that $\pi = L$ after observing s is equal to q. Then the investor accepts s if and only if

$$s[qL + (1-q)H + R] \geq I(1+r) \ . \qquad (*)$$

The entrepreneur prefers to receive the investment if and only if

$$s \leq R/(\pi + R) \ , \qquad (**)$$

for $\pi \in \{L, H\}$.

In a pooling equilibrium, $q = p$. Note that (**) is more difficult to satisfy for $\pi = H$ than for $\pi = L$. Thus, (*) and (**) imply that a pooling equilibrium exists only if

$$\frac{I(1+r)}{pL+(1-p)H+R} \leq \frac{R}{H+R} \; .$$

A separating equilibrium always exists. The low-profit type offers $s = I(1+r)/(L+R)$, which the investor accepts, and the high-profit type offers $s < I(1+r)/(H+R)$, which the investor rejects.

6.21 *A Poker Game*

(a) The strategic form of this game is as follows:

$$\begin{array}{c} \\ \text{believe} \\ \text{show} \end{array} \begin{array}{c} \begin{array}{cccc} \text{aa} & \text{aq} & \text{ka} & \text{kq} \end{array} \\ \begin{pmatrix} -1,1 & -1/3,1/3 & -2/3,2/3 & 0,0 \\ 2/3,-2/3 & 1/3,-1/3 & 0,0 & -1/3,1/3 \end{pmatrix} \end{array} .$$

Here, 'believe' and 'show' are the strategies of player I. The first letter in any strategy of player II is what player II says if the dealt card is a King, the second letter is what II says if the dealt card is a Queen—if the dealt card is an Ace player II has no choice.

(b) Player I has a unique optimal (maximin) strategy and player 2 has a unique optimal (minimax) strategy. The value of the game is $-2/9$.

6.22 *A Hotelling Location Problem*

(a) $x_1 = x_2 = \frac{1}{2}$.
(c) $x_1 = x_2 = \frac{1}{2}$.

6.23 *Median Voting*

(a) The strategy set of each player is the interval $[0, 30]$. If each player i plays x_i, then the payoff to each player i is $-|((x_1 + \ldots + x_n)/n) - t_i|$. A Nash equilibrium always exists.

(b) The payoff to player i is now $-|\operatorname{med}(x_1, \ldots, x_n) - t_i|$, where $\operatorname{med}(\cdot)$ denotes the median. For each player, proposing a temperature different from his true ideal temperature either leaves the median unchanged or moves the median farther away from the ideal temperature, whatever the proposals of the other players. Hence, proposing one's ideal temperature is a weakly dominant strategy.

6.24 *The Uniform Rule*

(b) $M = 4:\ (1, 3/2, 3/2)$, $M = 5:\ (1, 2, 2)$, $M = 5.5:\ (1, 2, 5/2)$, $M = 6:\ (1, 2, 3)$, $M = 7:\ (2, 2, 3)$, $M = 8:\ (5/2, 5/2, 3)$, $M = 9:\ (3, 3, 3)$.

(c) If player i reports t_i and receives $s_i > t_i$ then, apparently the total reported quantity is above M and thus, player i can only further increase (hence, worsen) his share by reporting a different quantity. If player i reports t_i and receives $s_i < t_i$ then, apparently the total reported quantity is below M and thus, player i can only further decrease (hence, worsen) his share by reporting a different quantity.

There exist other Nash equilibria, but they do not give different outcomes (shares). For example, if $M > \sum_{j=1}^{n} t_j$, then player 1 could just as well report 0 instead of t_1.

6.25 *Reporting a Crime*

(b) $p = 1 - (c/v)^{1/(n-1)}$.
(c) The probability of the crime being reported in this equilibrium is $1-(1-p)^n = 1-(c/v)^{n/(n-1)}$. This converges to $1-(c/v)$ for n going to infinity. Observe that both p and the the probability of the crime being reported decrease if n becomes larger.

6.26 *Firm Concentration*
Let, in equilibrium, n firms locate downtown and m firms in the suburbs, with $n = 6$ and $m = 4$.

6.27 *Tragedy of the Commons*

(d) Suppose, to the contrary, $G^* \leq G^{**}$. Then $v(G^*) \geq v(G^{**})$ since $v' < 0$, and $0 > v'(G^*) \geq v'(G^{**})$ since $v'' < 0$. Also, $G^*/n < G^{**}$. Hence

$$v(G^*) + (1/n)G^* v'(G^*) - c > v(G^{**}) + G^{**} v'(G^{**}) - c \;,$$

a contradiction since both sides should be zero.

Problems of Chapter 7

7.1 *Nash and Subgame Perfect Equilibrium in a Repeated Game (1)*

(a) $v(A) = 1$ and the minimax strategy in A is R; $v(-B) = -1$ and the maximin strategy in $-B$ is D.
(d) Player 1 plays always U but after a deviation switches to D forever. Player 2 always plays L but after a deviation switches to R forever. We need $\delta \geq 1/2$ to make this a Nash equilibrium of $G^\infty(\delta)$.

7.2 *Nash and Subgame Perfect Equilibrium in a Repeated Game (2)*

(a) The limiting average payoffs $(2,1)$, $(1,2)$, and $(2/3,2/3)$, resulting from playing, respectively, the Nash equilibria (U,L), (D,R), and $((2/3,1/3),(1/3,2/3))$ at every stage; and all payoffs (x_1,x_2) with $x_1,x_2 > 2/3$.
(b) $v(A) = 2/3$ and $-v(-B) = 2/3$. Hence, all payoffs (x_1,x_2) with $x_1,x_2 > 2/3$.
(c) The players play (U,L) at even times and (D,R) at odd times. Since at each time they play a Nash equilibrium of the stage game, no trigger strategies (describing punishment after a deviation) are needed.
(d) In this case a trigger strategy is needed. The players alternate between (U,L), (D,L), and (D,R).

7.3 *Nash and Subgame Perfect Equilibrium in a Repeated Game (3)*

(a) The stage game has a unique Nash equilibrium.
(b) $v(A) = 4$ since (D,L) is a saddlepoint in A. The minimax strategy of player 2 is L. The value of $-B$ is -1 and the maximin strategy of player 1 is $(1/2,1/2)$.
(c) These limit average payoffs are obtained, for instance, by letting the players play (U,L) at even times and (D,R) at odd times. After any deviation the players switch to playing (D,L) (or $((1/2,1/2),L)$) forever.

7.4 *Subgame Perfect Equilibrium in a Repeated Game*

(c) Alternate between (T,L) and (M,C).

7.5 *The Strategies Tr_1^* and Tr_2^**
An optimal moment for player 1 to deviate would be $t = 1$. For player 2 it would be $t = 3$.

7.6 *Repeated Cournot and Bertrand*

(a) Each player offers half of the monopoly quantity (half of $(a-c)/2$) at each time, but if a deviation from this occurs, then each player offers the Cournot quantity $(a-c)/3$ forever. This is a subgame perfect equilibrium for $\delta \geq 9/17$.
(b) In this case, each player asks the monopoly price $(a+c)/2$ at each time; if a deviation from this occurs, each player switches to the Bertrand equilibrium price $p = c$ forever. This is a subgame perfect equilibrium for $\delta \geq 1/2$.

7.7 *Repeated Duopoly*

(b) The Nash equilibrium prices are $p_1 = p_2 = 6$.
(c) Joint profit is maximized at $p_1 = p_2 = 5$.
(d) Ask prices $p_1 = p_2 = 5$, but after a deviation switch to the equilibrium prices $p_1 = p_2 = 6$. This is a subgame perfect equilibrium for $\delta \geq 25/49$.

7.8 *On Discounting*
See the solution to Problem 6.17(e).

7.9 *On Limit Average*
A sequence like $1, 3, 5, 7, \ldots$ has a limit average of infinity. More interestingly, one may construct a sequence containing only the numbers $+1$ and -1 of which the finite averages ‘oscillate’, e.g, below $-1/2$ and above $+1/2$, so that the limit does not exist.

Problems of Chapter 8

8.1 *Symmetric Games*

(a) $(0, 1)$ is the only *ESS*.
(b) Both $(1, 0)$ and $(0, 1)$ are *ESS*. The (Nash equilibrium) strategy $(1/3, 2/3)$ is not an *ESS*.

8.2 *More Symmetric Games*

(a) The replicator dynamics is $\dot{p} = p(p-1)(p-1/2)$, with rest points $p = 0, 1, 1/2$, of which only $p = 1/2$ is stable. The game (A, A^T) has a unique symmetric Nash equilibrium, namely $((1/2, 1/2), (1/2, 1/2))$. The unique *ESS* is $(1/2, 1/2)$.

8.3 *Asymmetric Games*

(b) The replicator dynamics is given by the equations $\dot{p} = pq(1-p)$ and $\dot{q} = pq(1-q)$. There is one stable rest point, namely $p = q = 1$, corresponding to the unique strict Nash equilibrium $((1, 0), (1, 0))$ of the game. The other rest points are all points in the set

$$\{(p, q) \mid p = 0 \text{ and } 0 \leq q \leq 1 \text{ or } q = 0 \text{ and } 0 \leq p \leq 1\} .$$

8.4 *More Asymmetric Games*

(a) The replicator dynamics are $dx/dt = x(1-x)(2-3y)$ and $dy/dt = 2y(1-2x)(y-1)$. There are no stable rest points.
(b) The replicator dynamics are $dx/dt = x(x-1)(2y-1)$ and $dy/dt = y(y-1)(2x-1)$.

8.5 *Frogs Call for Mates*
Note that for (a) and (b) Proposition 8.5 can be used. Similarly, for (c) we can use Proposition 8.8, by stating the conditions under which each of the four pure strategy combinations is a strict Nash equilibrium: if $z_1 < P + m - 1$ and $z_2 < P + m - 1$ then (Call, Call) is a strict Nash equilibrium, etc.

8.6 *Video Market Game*
There are four rest points, of which only one is stable.

Problems of Chapter 9

9.2 *Computing the Core*

(a) $\{(0, 0, 1)\}$; (b) polygon with vertices $(15, 5, 4)$, $(9, 5, 10)$, $(14, 6, 4)$, and $(8, 6, 10)$.

9.4 *The Core of the General Glove Game*
The Shapley value is in the core.

9.6 *Non-monotonicity of the Core*

(b) The core of (N, v') is the set $\{(0, 0, 1, 1)\}$ [use the fact that $C(N, v') \subseteq C(N, v)$]. Hence, player 1 can only obtain less in the core although the worth of coalition $\{1, 3, 4\}$ has increased.

9.7 *Efficiency of the Shapley Value*
Consider an order $i_1, i_2, \ldots, i_n$ of the players. The sum of the coordinates of the associated marginal vector is

$$\begin{aligned}
&[v(\{i_1\}) - v(\emptyset)] \\
&+[v(\{i_1, i_2\}) - v(\{i_1\})] \\
&+[v(\{i_1, i_2, i_3\}) - v(\{i_1, i_2\})] \\
&+\ldots \\
&+[v(N) - v(N \setminus \{i_n\})] \\
&= v(N) - v(\emptyset) = v(N) \ .
\end{aligned}$$

Hence, every marginal vector is efficient, so the Shapley value is efficient since it is the average of the marginal vectors.

9.8 *Computing the Shapley Value*

(a) $\Phi(N, v) = (1/6, 1/6, 2/3) \notin C(N, v)$; (b) $(9\frac{1}{2}, 6\frac{1}{2}, 8)$, not in the core.

9.9 *The Shapley Value and the Core*

(a) $a = 3$ (use Problem 9.5).
(b) $(2.5, 2, 1.5)$.
(c) The Shapley value is $(a/3 + 1/2, a/3, a/3 - 1/2)$. The minimal value of a for which this is in the core is $15/4$.

9.10 *Shapley Value in a Two-Player Game*
$\Phi(N,v) = (v(\{1\})+(v(\{1,2\})-v(\{1\})-v(\{2\}))/2, v(\{2\})+(v(\{1,2\})-v(\{1\})-v(\{2\}))/2)$.

9.11 *Computing the Nucleolus*

(a) $(0,0,1)$.
(b) $(11.5, 5.5, 7)$.
(c) $(1/5, 1/5, 1/5, 1/5, 1/5, 0, \ldots, 0) \in \mathbb{R}^{15}$.
(d) In (N,v): $(1/2, 1/2, 1/2, 1/2)$; in (N,v'): $(0,0,1,1)$.

9.12 *Nucleolus of Two-Player Games*
The nucleolus is $(v(\{1\}) + (v(\{1,2\}) - v(\{1\}) - v(\{2\}))/2, v(\{2\}) + (v(\{1,2\}) - v(\{1\}) - v(\{2\}))/2)$.

9.13 *Computing the Core, the Shapley Value, and the Nucleolus*

(a) The nucleolus and Shapley value coincide and are equal to $(1.5, 2, 2.5)$.
(c) The maximal value of $v(\{1\})$ is 2. For that value the core is the line segment with endpoints $(2,1,3)$ and $(2,3,1)$.

9.14 *Voting (1)*

(a) The Shapley value is $\Phi(N,v) = (7/12, 3/12, 1/12, 1/12)$.
(b) The nucleolus is $(1,0,0,0)$.

9.15 *Voting (2)*

(c) $\Phi(N,v) = (1/60)(9,9,14,14,14)$.
(d) The nucleolus is $(0,0,1/3,1/3,1/3)$.
(e) The nucleolus is not in the core (e.g., $v(\{1,3,4\} = 1 > 2/3)$, so the core must be empty. This can also be seen directly.

9.16 *Two Buyers and a Seller*

(c) $\Phi(N,v) = (1/6, 4/6, 7/6)$.
(d) The nucleolus is $(0, 1/2, 3/2)$.

9.17 *Properties of the Shapley Value*

(a) In $\Phi_i(N,v)$ the term $v(S \cup \{i\}) - v(S)$ occurs the same number of times as the term $v(S \cup \{j\}) - v(S)$ in $\Phi_j(N,v)$, for every coalition $S \subseteq N \setminus \{i,j\}$. Let S be a coalition with $i \in S$ and $j \notin S$. Then $v(S \setminus \{i\} \cup \{j\}) = v(S \setminus \{i\} \cup \{i\})$, so that

$$\begin{aligned} v(S \cup \{j\}) - v(S) &= v((S \setminus \{i\} \cup \{j\}) \cup \{i\}) - v((S \setminus \{i\}) \cup \{i\}) \\ &= v((S \setminus \{i\} \cup \{j\}) \cup \{i\}) - v((S \setminus \{i\}) \cup \{j\}) \,, \end{aligned}$$

and also these expressions occur the same umber of times. Similarly for coalitions S that contain j but not i.

(b) This is obvious from Definition 9.4.

(c) Observe that it is sufficient to show $\sum_{S:i\notin S} \frac{|S|!(n-|S|-1)!}{n!} = 1$. To show this, note that $\frac{|S|!(n-|S|-1)!}{n!} = \frac{1}{n}\binom{n-1}{|S|}^{-1}$, so that

$$\sum_{S:i\notin S} \frac{|S|!(n-|S|-1)!}{n!} = \frac{1}{n} \sum_{s=0,1,\ldots,n-1} \binom{n-1}{s}\binom{n-1}{s}^{-1}$$
$$= \frac{1}{n} \cdot n = 1 \, .$$

Problems of Chapter 10

10.1 *A Division Problem (1)*

(b) In terms of utilities: $(\frac{1}{3}\sqrt{3}, \frac{2}{3})$, in terms of distribution: $(\frac{1}{3}\sqrt{3}, 1 - \frac{1}{3}\sqrt{3})$.

(c) The Rubinstein outcome is x^* where $x_1^* = \sqrt{\frac{1}{1+\delta+\delta^2}}$ and $x_2^* = 1 - \frac{1}{1+\delta+\delta^2}$.

(d) $\lim_{\delta\to 1} x_1^* = \frac{1}{3}\sqrt{3}$, consistent with what was found under (a).

10.2 *A Division Problem (2)*
Use symmetry, Pareto optimality and covariance of the Nash bargaining solution.

10.3 *A Division Problem (3)*

(a) The distribution of the good is $\left(2\frac{1-\delta^3}{1-\delta^4}, 2 - 2\frac{1-\delta^3}{1-\delta^4}\right)$. In utility terms this is $\left(\frac{1-\delta^3}{1-\delta^4}, \sqrt[3]{2 - 2\frac{1-\delta^3}{1-\delta^4}}\right)$.

(b) By taking the limit for $\delta \to 1$ in (b), we obtain $(1.5, 0.5)$ as the distribution assigned by the Nash bargaining solution. In utilities: $(0.75, \sqrt[3]{0.5})$.

10.4 *An Exchange Economy*

(a) $x_1^A(p_1, p_2) = (3p_2 + 2p_1)/2p_1$, $x_2^A = (4p_1 - p_2)/2p_2$, $x_1^B = (p_1 + 6p_2)/2p_1$, $x_2^B = p_1/2p_2$.

(b) $(p_1, p_2) = (9, 5)$ (or any positive multiple thereof); the equilibrium allocation is $((33/18, 31/10), (39/18, 9/10))$.

(c) The (non-boundary part of the) contract curve is given by the equation $x_2^A = (17x_1^A + 5)/(2x_1^A + 8)$. The core is the part of this contract curve such that $\ln(x_1^A + 1) + \ln(x_2^A + 2) \geq \ln 4 + \ln 3 = \ln 12$ (individual rationality constraint for A) and $3\ln(5-x_1^A)+\ln(5-x_2^A) \geq 3\ln 2+\ln 4 = \ln 12$ (individual rationality constraint for B).

(d) The point $\mathbf{x}^A = (33/18, 31/10)$ satisfies the equation $x_2^A = (17x_1^A+5)/(2x_1^A+8)$.

(e) For the disagreement point $\mathbf{d}$ one can take the point $(\ln 12, \ln 12)$. The set S contains all points $\mathbf{u} \in \mathbb{R}^2$ that can be obtained as utilities from any distribution of the goods that does not exceed total endowments $\mathbf{e} = (4,4)$. Unlike the Walrasian equilibrium allocation, the allocation obtained by applying the Nash bargaining solution is not independent of arbitrary monotonic transformations of the utility functions. It is a 'cardinal' concept, in contrast to the Walrasian allocation, which is 'ordinal'.

10.5 *The Matching Problem of Table 10.1 Continued*

(a) The resulting matching is (w_1, m_1), (w_2, m_2), w_3 and m_3 remain single.

10.6 *Another Matching Problem*

(a) With the men proposing: (m_1, w_1), (m_2, w_2), (m_3, w_3). With the women proposing: (m_1, w_1), (m_2, w_3), (m_3, w_2).

(b) Since in any stable matching we must have (m_1, w_1), the matchings found in (a) are the only stable ones.

(c) Obvious: every man weakly or strongly prefers the men proposing matching in (a); and vice versa for the women.

10.7 *Yet Another Matching Problem: Strategic Behavior*

(b) There are no other stable matchings.

(c) The resulting matching is (m_1, w_1), (m_2, w_3), (m_3, w_2). This is clearly better for w_1.

10.8 *Core Property of Top Trading Cycle Procedure*
All players in a top trading cycle get their top houses, and thus none of these players can be a member of a blocking coalition, say S. Omitting these players and their houses from the problem, by the same argument none of the players in a top trading cycle in the second round can be a member of S: the only house that such a player may prefer is no longer available in S; etc.

10.9 *House Exchange with Identical Preferences*
Without loss of generality, assume that each player has the same preference $h_1 h_2 \ldots h_n$. Show that in a core allocation each player keeps his own house.

10.10 *A House Exchange Problem*
There are three core allocations namely: (i) $1 : h_3$, $2 : h_4$, $3 : h_1$, $4 : h_2$; (ii) $1 : h_2$, $2 : h_4$, $3 : h_1$, $4 : h_3$; (iii) $1 : h_3$, $2 : h_1$, $3 : h_4$, $4 : h_2$. Allocation (i) is in the strong core.

10.11 *Cooperative Oligopoly*
(a)–(c) Analogous to Problems 6.1, 6.2. Parts (d) and (f) follow directly from (c). For parts (e) and (g) use the methods of Chap. 9.

Problems of Chapter 11

11.1 *Preferences*

(a) If $a \neq b$ and aRb and bRa then neither aPb nor bPa, so P is not necessarily complete.
(b) I is not complete unless aRb for all $a, b \in A$. I is only antisymmetric if R is a linear order.

11.2 *Pairwise Comparison*

(a) $C(r)$ is reflexive and complete but not antisymmetric.
(c) There is no Condorcet winner in this example.

11.3 *Independence of the Conditions in Theorem 11.1*
The social welfare function based on the Borda scores is Pareto efficient but does not satisfy IIA and is not dictatorial (cf. Sect. 11.1). The social welfare function that assigns to each profile of preferences the reverse preference of agent 1 satisfies IIA and is not dicatorial but also not Pareto efficient.

11.4 *Independence of the Conditions in Theorem 11.2*
A constant social welfare function (i.e., always assigning the same fixed alternative) is strategy-proof and nondictatorial but not surjective. The social welfare function that always assigns the bottom element of agent 1 is surjective, nondictatorial, and not strategy-proof.

11.5 *Independence of the Conditions in Theorem 11.3*
A constant social welfare function (i.e., always assigning the same fixed alternative) is monotonic and nondictatorial but not unanimous. A social welfare function that assigns the common top alternative to any profile where all agents have the same top alternative, and a fixed constant alternative to any other profile, is unanimous and nondictatorial but not monotonic.

11.6 *Copeland Score and Kramer Score*

(a) The Copeland ranking is a preference. The Copeland ranking is not antisymmetric. It is easy to see that the Copeland ranking is Pareto efficient. By Arrow's Theorem therefore, it does not satisfy IIA.
(b) The Kramer ranking is a preference. The Kramer ranking is not antisymmetric and not Pareto efficient. It violates IIA.

11.7 *Two Alternatives*
Consider the social welfare function based on majority rule, i.e., it assigns aPb if $|N(a,b,r)| > |N(b,a,r)|$; bPa if $|N(a,b,r)| < |N(b,a,r)|$; and aIb if $|N(a,b,r)| = |N(b,a,r)|$.

Problems of Chapter 12

12.1 *Solving a Matrix Game*

(c) $v(A) = 12/5$ and the unique optimal strategies of players 1 and 2 are, respectively, $(0, 4/5, 1/5, 0)$ and $(0, 2/5, 0, 3/5)$.
(d) The answer is independent of y and the same as in (c).

12.3 2×2 *Games*

(a) To have no saddlepoints we need $a_{11} > a_{12}$ or $a_{11} < a_{12}$. By assume the first, the other inequalities follow.
(b) For optimal strategies $\mathbf{p} = (p, 1-p)$ and $\mathbf{q} = (q, 1-q)$ we must have $0 < p < 1$ and $0 < q < 1$. Then use that p should be such that player 2 is indifferent between the two columns and q such that player 1 is indifferent between the two rows.

12.4 *Symmetric Games*
Let $\mathbf{x}$ be optimal for player 1. Then $\mathbf{x}A\mathbf{y} \geq v(A)$ for all $\mathbf{y}$; hence $\mathbf{y}A\mathbf{x} = -\mathbf{x}A\mathbf{y} \leq -v(A)$ for all $\mathbf{y}$; hence (take $\mathbf{y} = \mathbf{x}$) $v(A) \leq -v(A)$, so $v(A) \leq 0$. Similarly, derive the converse inequality by considering an optimal strategy for player 2.

12.5 *The Duality Theorem Implies the Minimax Theorem*
Let A be an $m \times n$ matrix game. Without loss of generality assume that all entries of A are positive. Consider the associated LP as in Sect. 12.2.

Consider the vector $\bar{\mathbf{x}} = (1/m, \ldots, 1/m, \eta) \in \mathbb{R}^{m+1}$ with $\eta > 0$. Since all entries of A are positive it is straightforward to check that $\bar{\mathbf{x}} \in V$ if $\eta \leq \sum_{i=1}^{m} a_{ij}/m$ for all $j = 1, \ldots, n$. Since $\bar{\mathbf{x}} \cdot c = -\eta < 0$, it follows that the value of the LP must be negative.

Let $\mathbf{x} \in O_{\min}$ and $\mathbf{y} \in O_{\max}$ be optimal solutions of the LP. Then $-x_{m+1} = -y_{n+1} < 0$ is the value of the LP. We have $x_i \geq 0$ for every $i = 1, \ldots, m$, $\sum_{i=1}^{m} x_i \leq 1$, and $(x_1, \ldots, x_m)A\mathbf{e}^j \geq x_{m+1}$ (> 0) for every $j = 1, \ldots, n$. Optimality in particular implies $\sum_{i=1}^{m} x_i = 1$, so that $v_1(A) \geq (x_1, \ldots, x_m)A\mathbf{e}^j \geq x_{m+1}$ for all j, hence $v_1(A) \geq x_{m+1}$. Similarly, it follows that $v_2(A) \leq y_{n+1} = x_{m+1}$, so that $v_2(A) \leq v_1(A)$. The Minimax Theorem now follows.

12.6 *Infinite Matrix Games*

(a) A is an infinite matrix game with for all $i, j \in \mathbb{N}$: $a_{ij} = 1$ if $i > j$, $a_{ij} = 0$ if $i = j$, and $a_{ij} = -1$ if $i < j$.

(b) Fix a mixed strategy $\mathbf{p} = (p_1, p_2, \ldots)$ for player 1 with $p_i \geq 0$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} p_i = 1$. If player 2 plays pure strategy j, then the expected payoff for player 1 is equal to $-\sum_{i=1}^{j-1} p_i + \sum_{i=j+1}^{\infty} p_i$. Since $\sum_{i=1}^{\infty} p_i = 1$, this expected payoff converges to -1 as j approaches ∞. Hence, $\inf_{\mathbf{q}} \mathbf{p}A\mathbf{q} = -1$, so $\sup_{\mathbf{p}} \inf_{\mathbf{q}} \mathbf{p}A\mathbf{q} = -1$. Similarly, one shows $\inf_{\mathbf{q}} \sup_{\mathbf{p}} \mathbf{p}A\mathbf{q} = 1$, hence the game has no 'value'.

12.7 *Equalizer Theorem*
Assume, without loss of generality, $v = 0$. It is sufficient to show that there exists $\mathbf{q} \in \mathbb{R}^n$ with $\mathbf{q} \geq \mathbf{0}$, $A\mathbf{q} \leq \mathbf{0}$, and $q_n = 1$. The required optimal strategy is then obtained by normalization.

This is equivalent to existence of a vector $(\mathbf{q}, \mathbf{w}) \in \mathbb{R}^{n+m}$ with $\mathbf{q} \geq \mathbf{0}$, $\mathbf{w} \geq \mathbf{0}$, such that

$$\begin{pmatrix} A & I \\ \mathbf{e}^n & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} ,$$

where row vector $\mathbf{e}^n \in \mathbb{R}^n$, I is the $m \times m$ identity matrix, $\mathbf{0}$ is an $1 \times m$ vector on the left hand side and an $m \times 1$ vector on the right hand side. Thus, we have to show that the vector $\mathbf{x} := (\mathbf{0}, 1) \in \mathbb{R}^{m+1}$ is in the cone spanned by the columns of the $(m+1) \times (n+m)$ matrix on the left hand side. Call this matrix B and call this cone Z. Assume $\mathbf{x} \notin Z$ and derive a contradiction using Theorem 22.1.

Problems of Chapter 13

13.1 *Existence of Nash Equilibrium Using Brouwer*

(c) Let $\sigma^* \in \prod_{i \in N} \Delta(S_i)$. If σ^* is a Nash equilibrium of G then

$$\sigma_i^*(s_i) = \frac{\sigma_i^*(s_i) + \max\{0, u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)\}}{1 + \sum_{s_i' \in S_i} \max\{0, u_i(s_i', \sigma_{-i}^*) - u_i(\sigma^*)\}} \tag{*}$$

for all $i \in N$ and $s_i \in S_i$, so that σ^* is a fixed point of f. Conversely, let σ^* be a fixed point of f. Then (*) holds for all $i \in N$ and $s_i \in S_i$. Hence

$$\sigma_i^*(s_i) \sum_{s_i' \in S_i} \max\{0, u_i(s_i', \sigma_{-i}^*) - u_i(\sigma^*)\} = \max\{0, u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)\} .$$

Multiply both sides of this equation by $u_i(s_i, \sigma_{-i}^*) - u_i(\sigma^*)$ and next sum over all $s_i \in S_i$.

13.2 *Existence of Nash Equilibrium Using Kakutani*
For upper semi-continuity of β, take a sequence σ^k converging to σ, a sequence $\tau^k \in \beta(\sigma^k)$ converging to τ, and show $\tau \in \beta(\sigma)$.

13.3 *Lemma 13.2*
The only-if direction is straightforward from the definition of best reply.

13.4 *Lemma 13.3*
Take i such that $\mathbf{e}^i A\mathbf{q} \geq \mathbf{e}^k A\mathbf{q}$ for all $k = 1, \ldots, m$. Then, clearly, $\mathbf{e}^i A\mathbf{q} \geq \mathbf{p}' A\mathbf{q}$ for all $\mathbf{p}' \in \Delta^m$, so $\mathbf{e}^i \in \beta_1(\mathbf{q})$. The second part is analogous.

13.5 *Dominated Strategies*

(b) Denote by $NE(A, B)$ the set of Nash equilibria of (A, B). Then

$$\begin{aligned}
(\mathbf{p}^*, \mathbf{q}^*) \in NE(A, B) &\Leftrightarrow (\mathbf{p}^*, (\mathbf{q}', 0)) \in NE(A, B) \text{ where } (\mathbf{q}', 0) = \mathbf{q}^* \\
&\Leftrightarrow \forall \mathbf{p} \in \Delta^m, \mathbf{q} \in \Delta^{n-1}[\mathbf{p}^* A(\mathbf{q}', 0) \geq \mathbf{p}A(\mathbf{q}', 0), \\
&\quad\ \mathbf{p}^* B(\mathbf{q}', 0) \geq \mathbf{p}^* B(\mathbf{q}, 0)] \\
&\Leftrightarrow \forall \mathbf{p} \in \Delta^m, \mathbf{q} \in \Delta^{n-1}[\mathbf{p}^* A'\mathbf{q}' \geq \mathbf{p}A'\mathbf{q}', \\
&\quad\ \mathbf{p}^* B'\mathbf{q}' \geq \mathbf{p}^* B'\mathbf{q}] \\
&\Leftrightarrow (\mathbf{p}^*, \mathbf{q}') \in NE(A', B')\ .
\end{aligned}$$

Note that the first equivalence follows by part (a).

13.6 *A* 3×3 *Bimatrix Game*

(c) The unique Nash equilibrium is $((0, 0, 1), (0, 0, 1))$.

13.7 *A* 3×2 *Bimatrix Game*
The set of Nash equilibria is $\{(\mathbf{p}, \mathbf{q}) \in \Delta^3 \times \Delta^2 \mid p_1 = 0,\ q_1 \geq \frac{1}{2}\} \cup \{((1, 0, 0), (0, 1))\}$.

13.8 *The Nash Equilibria in Example 13.18*

(a) Let $\mathbf{p} = (p_1, p_2, p_3)$ be the strategy of player 1. We distinguish two cases: (i) $p_2 = 0$ (ii) $p_2 > 0$.

In case (i), reduce the game to

$$\begin{array}{c} \\ p_1 \\ p_3 \end{array}
\begin{array}{c} \begin{array}{ccc} q_1 & q_2 & q_3 \end{array} \\ \begin{pmatrix} 1,1 & 0,0 & 2,0 \\ 0,0 & 1,1 & 1,1 \end{pmatrix} \end{array}$$

where $\mathbf{q} = (q_1, q_2, q_3)$ is player 2's strategy. Solve this game graphically. As long as player 1 gets at least 1 (the payoff from playing M) theobtained equilibria are also equilibria of the original game G.

In case (ii), R gives a lower expected payoff to player 2 than C, so the game can be reduced to

$$\begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} 1,1 & 0,0 \\ 1,2 & 1,2 \\ 0,0 & 1,1 \end{pmatrix} \end{array} .$$

Solve this game graphically and extend to G.

(b) Consider again the perturbed games $G(\varepsilon)$ as in Example 13.18. For $q = 0$ consider the strategy combination $((\varepsilon, 1-2\varepsilon, \varepsilon), (\varepsilon, 1-2\varepsilon, \varepsilon))$ in $G(\varepsilon)$. For $q = 1$ consider, similarly, $((\varepsilon, 1-2\varepsilon, \varepsilon), (1-2\varepsilon, \varepsilon, \varepsilon))$ in $G(\varepsilon)$; for $0 < q < 1$ consider $((\varepsilon, 1-2\varepsilon, \varepsilon), (q-\varepsilon/2, 1-q-\varepsilon/2, \varepsilon))$.

13.9 *Proof of Theorem 13.8*

'If': conditions (13.1) are satisfied and $f = 0$, which is optimal since $f \leq 0$ always.

'Only-if': clearly we must have $a = \mathbf{p}A\mathbf{q}$ and $b = \mathbf{p}B\mathbf{q}$ (otherwise $f < 0$ which cannot be optimal). From the conditions (13.1) we have $\mathbf{p}'A\mathbf{q} \leq a = \mathbf{p}A\mathbf{q}$ and $\mathbf{p}B\mathbf{q}' \leq b = \mathbf{p}B\mathbf{q}$ for all $\mathbf{p}' \in \Delta^m$ and $\mathbf{q}' \in \Delta^n$, which implies that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium.

13.10 *Matrix Games*

This is a repetition of the proof of Theorem 12.5. Note that the solutions of program (13.3) give exactly the value of the game a and the optimal (minimax) strategies of player 2. The solutions of program (13.4) give exactly the value of the game $-b$ and the optimal (maximin) strategies of player 1.

13.11 *Tic-Tac-Toe*

(a) Start by putting a cross in the center square. Then player 2 has essentially two possibilities for the second move, and it is easy to see that in each of the two cases player 1 has a forcing third move. After this, it is equally easy to see that player 1 can always enforce a draw.

(b) If player 1 does not start at the center, then player 2 can put his first circle at the center and then can place his second circle in such a way that it becomes forcing. If player 1 starts at the center then either a pattern as in (a) is followed, leading to a draw, or player 2's second circle becomes forcing, also resulting in a draw.

13.12 *Iterated Elimination in a Three-Player Game*

The resulting strategy combination is (D, l, L).

13.13 *Never a Best Reply and Domination*
First argue that strategy Y is not strictly dominated. Next assume that Y is a best reply to strategies $(p, 1-p)$ of player 1 and $(q, 1-q)$ of player 2, and derive a contradiction.

13.15 *A 3-Player Game with an Undominated But Not Perfect Equilibrium*

(a) First observe that the set of Nash equilibria is $\{((p, 1-p), l, L) \mid 0 \leq p \leq 1\}$, where p is the probability with which player 1 plays U.

13.16 *Existence of Proper Equilibrium*
Tedious but straightforward.

13.17 *Strictly Dominated Strategies and Proper Equilibrium*

(a) The only Nash equilibria are $(U, l(, L))$ and $(D, r(, L))$. Obviously, only the first one is perfect and proper.
(b) (D, r, L), is a proper Nash equilibrium.

13.18 *Strictly Perfect Equilibrium*

(a) Identical to the proof of Lemma 13.16, see Problem 13.14: note that any sequence of perturbed games converging to the given game must eventually contain any given completely mixed Nash equilibrium σ.
(c) The set of Nash equilibria is $\{((p, 1-p), L) \mid 0 \leq p \leq 1\}$, where p is the probability on U. Every Nash equilibrium of the game (A, B) is perfect and proper. No Nash equilibrium is strictly perfect.

13.19 *Correlated Equilibria in the Two-Driver Example (1)*
Use inequalities (13.5) and (13.6) to derive the conditions: $p_{11}+p_{12}+p_{21}+p_{22} = 1$, $p_{ij} \geq 0$ for all $i, j \in \{1, 2\}$, $p_{11} \leq \frac{3}{5}\min\{p_{12}, p_{21}\}$, $p_{22} \leq \frac{5}{3}\min\{p_{12}, p_{21}\}$.

13.20 *Nash Equilibria are Correlated*
Check that (13.5) and (13.6) are satisfied for P.

13.21 *The Set of Correlated Equilibria is Convex*
Let P and Q be correlated equilibria and $0 \leq t \leq 1$. Check that (13.5) and (13.6) are satisfied for $tP + (1-t)Q$.

13.22 *Correlated vs. Nash Equilibrium*

(a) The Nash equilibria are: $((1, 0), (0, 1))$, $((0, 1), (1, 0))$, and $((2/3, 1/3), (2/3, 1/3))$.

13.23 *Correlated Equilibria in the Two-Driver Example (2)*
The matrix C is:

$$\begin{array}{c} \\ (1,1') \\ (1,2') \\ (2,1') \\ (2,2') \end{array} \begin{array}{c} \begin{array}{cccc} (1,2) & (2,1) & (1',2') & (2',1') \end{array} \\ \begin{pmatrix} -10 & 0 & -10 & 0 \\ 6 & 0 & 0 & 10 \\ 0 & 10 & 6 & 0 \\ 0 & -6 & 0 & -6 \end{pmatrix} \end{array} .$$

13.24 *Finding Correlated Equilibria*
There is unique correlated equilibrium

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} .$$

13.25 *Nash, Perfect, Proper, Strictly Perfect, and Correlated Equilibria*

(d) $\{((0,1,0,(q_1,q_2,q_3)) \mid q_3 \le 2q_1 \le 4q_3\} \cup \{((p_1,p_2,p_3),(0,1,0)) \mid 0 < p_1 \le 2p_3 \le 4p_1\}$.
(e) The only perfect equilibrium is $((0,1,0),(0,1,0))$.
(f) $((0,1,0),(0,1,0))$ is also the only proper and strictly perfect equilibrium.
(h) For instance $\beta = \gamma = 3/8, \alpha = \delta = 1/8$.

13.26 *Independence of the Axioms in Corollary 13.40*
Not OPR: take the set of all strategy combinations in every game. Not CONS: in games with maximal player set take all strategy combinations, in other games take the set of Nash equilibria. Not COCONS: drop a Nash equilibrium in some game with maximal player set, but otherwise always take the set of all Nash equilibria.

13.27 *Inconsistency of Perfect Equilibria*
First show that the perfect equilibria in G_0 are all strategy combinations where player 2 plays L, player 3 plays D, and player 1 plays any mixture between T and B. Next consider the reduced game by fixing player 3's strategy at D.

Problems of Chapter 14

14.2 *An Extensive Form Structure without Perfect Recall*

(a) The paths $\{(x_0,x_1)\}$ and $\{(x_0,x_2)\}$ contain different player 1 actions.

14.3 *Consistency Implies Bayesian Consistency*
With notations as in Definition 14.13, for $h \in H$ with $\mathbb{P}_b(h) > 0$ and $x \in h$ we have: $\beta_h(x) = \lim_{m\to\infty} \beta_h^m(x) = \lim_{m\to\infty} \mathbb{P}_{b^m}(x)/\mathbb{P}_{b^m}(h) = \mathbb{P}_b(x)/\mathbb{P}_b(h)$. Here, the second equality follows from Bayesian consistency of the (b^m, β^m).

14.4 *(Bayesian) Consistency in Signaling Games*
The idea of the proof is as follows. Let (b, β) be a Bayesian consistent assessment. This means that β is determined on every information set of player 2 that is reached with positive probability, given b_1. Take $m \in \mathbb{N}$. Assign the number $1/m^2$ to action a of a type i of player 1 if that type does not play a but some other type of player 1 plays a with positive probability. Assign the number $1/m^2$ also to action a of type i if no type of player 1 plays a and player 2 attaches zero belief probability to type i conditional on player 1 having played a. To every other action of a of player 1, assign the number $\beta(i,a)/m$, where $\beta(i,a)$ is the (positive) belief that player 2 attaches to player 1 being of type i conditional on having played a. Next, normalize all these numbers to behavioral strategies b_1^m of player 1. For player 2, just take completely mixed behavioral strategies b_2^m converging to b_2. Then $(b^m, \beta^m) \to (b, \beta)$, where the β^m are determined by Bayesian consistency.

14.5 *Sequential Equilibria in a Signaling Game*
There is one pure and one completely mixed sequential equilibrium.

14.6 *Computation of Sequential Equilibrium (1)*
The unique sequential equilibrium consists of the behavioral strategies where player 1 plays B with probability 1 and C with probability $1/2$, and player 2 plays L with probability $1/2$; and player 1 believes that x_3 and x_4 are equally likely.

14.7 *Computation of Sequential Equilibrium (2)*

(b) The Nash equilibria are (L, l), and $(R, (\alpha, 1-\alpha))$ for all $\alpha \leq 1/2$, where α is the probability with which player 2 plays l.
(c) Let π be the belief player 2 attaches to node y_1. Then the sequential equilibria are: (L, l) with belief $\pi = 1$; (R, r) with belief $\pi \leq 1/2$; and $(R, (\alpha, 1-\alpha))$ for any $\alpha \leq 1/2$ with belief $\pi = 1/2$.

14.8 *Computation of Sequential Equilibrium (3)*

(b) The Nash equilibria are $(R, (q, 1-q))$ with $1/3 \leq q \leq 2/3$. (The conditions on q keep player 1 from deviating to L or M.)

14.9 *Computation of Sequential Equilibrium (4)*
The Nash equilibria in this game are: $(R, (q_1, q_2, q_3))$ with $q_3 \leq 1/3$ and $q_1 \leq 1/2 - (3/4)q_3$, where q_1, q_2, q_3 are the probabilities put on l, m, r, respectively; and $((1/4, 3/4, 0), (1/4, 0, 3/4))$ (probabilities on L, M, R and l, m, r, respectively).

Let π be the belief attached by player 2 to y_1. Then with $\pi = 1/4$ the equilibrium $((1/4, 3/4, 0), (1/4, 0, 3/4))$ becomes sequential. The first set of equilibria contains no equilibrium that can be extended to a sequential equilibrium.

14.10 *Computation of Sequential Equilibrium (5)*
The Nash equilibria are: (DB, r); $((R, (s, 1-s)), (q, 1-q))$ with $0 \leq s \leq 1$ and $q \geq 1/3$, where s is the probability on A and q is the probability on l. The subgame perfect equilibria are: (DB, r); (RA, l); $((R, (3/4, 1/4)), (3/5, 2/5))$. The first one

becomes sequential with $\beta = 0$; the second one with $\beta = 1$; and the third one with $\beta = 3/5$.

Problems of Chapter 15

15.1 *Computing ESS in* 2×2 *Games (1)*
$ESS(A)$ can be computed using Proposition 15.4.
(a) $ESS(A) = \{\mathbf{e}^2\}$. (b) $ESS(A) = \{\mathbf{e}^1, \mathbf{e}^2\}$. (c) $ESS(A) = \{(2/3, 1/3)\}$.

15.2 *Computing ESS in* 2×2 *Games (2)*
Case (1): $ESS(A') = \{\mathbf{e}^2\}$; case (2): $ESS(A') = \{\mathbf{e}^1, \mathbf{e}^2\}$; case (3): $ESS(A') = \{\hat{\mathbf{x}}\} = \{(a_2/(a_1 + a_2), a_1/(a_1 + a_2))\}$.

15.3 *Rock-Paper-Scissors (1)*
The unique Nash equilibrium is $((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))$, which is symmetric. But $(1/3, 1/3, 1/3)$ is not an ESS.

15.4 *Uniform Invasion Barriers*
Case (1), $\mathbf{e}^2$: maximal uniform invasion barrier is 1.
Case (2), $\mathbf{e}^1$: maximal uniform invasion barrier is $a_1/(a_1 + a_2)$.
Case (2), $\mathbf{e}^2$: maximal uniform invasion barrier is $a_2/(a_1 + a_2)$.
Case (3), $\hat{\mathbf{x}}$: maximal uniform invasion barrier is 1.

15.5 *Replicator Dynamics in Normalized Game (1)*
Straightforward computation.

15.6 *Replicator Dynamics in Normalized Game (2)*
The replicator dynamics can be written as $\dot{x} = [x(a_1 + a_2) - a_2]x(1 - x)$, where $\dot{x} = \dot{x}_1$. So $x = 0$ and $x = 1$ are always stationary points. In case (1) the graph of $\dot{x}$ on $(0, 1)$ is below the horizontal axis. In case (2) there is another stationary point, namely at $x = a_2/(a_1 + a_2)$; on $(0, a_2/(a_1 + a_2))$ the function $\dot{x}$ is negative, on $(a_2/(a_1 + a_2), 1)$ it is positive. In case (3) the situation of case (2) is reversed: the function $\dot{x}$ is positive on $(0, a_2/(a_1 + a_2))$ and negative on $((a_2/(a_1 + a_2), 1)$.

15.7 *Weakly Dominated Strategies and Replicator Dynamics*
(b) The stationary points are $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, and all points with $x_3 = 0$. Except $\mathbf{e}^3$, all stationary points are Lyapunov stable. None of these points is asymptotically stable. Also, $\mathbf{e}^3$ is strictly dominated (by $\mathbf{e}^1$)). [One can also derive $d(x_1/x_2)/dt = x_1x_3/x_2 > 0$ at completely mixed strategies, i.e., at the interior of Δ^3. Hence, the share of subpopulation 1 grows faster than that of 2 but this difference goes to zero if x_3 goes to zero ($\mathbf{e}^2$ is weakly dominated by $\mathbf{e}^1$).]

15.8 *Stationary Points and Nash Equilibria (1)*

(a) $NE(A) = \{(\alpha, \alpha, 1 - 2\alpha) \mid 0 \leq \alpha \leq 1/2\}$.
(b) By Proposition 15.18 and (a) it follows that $\{(\alpha, \alpha, 1 - 2\alpha) \mid 0 \leq \alpha \leq 1/2\} \cup \{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\} \subseteq ST(A)$, and that possibly other stationary points must be boundary

points of Δ^3. By considering the replicator dynamics it follows that there are no additional stationary points. All stationary points except $\mathbf{e}^1$ and $\mathbf{e}^2$ are Lyapunov stable, but no point is asymptotically stable.

15.9 *Stationary Points and Nash Equilibria (2)*

(a) The Nash equilibrium strategies are: $(0, 1, 0)$, $(1/2, 0, 1/2)$, $(0, 2/3, 1/3)$, and $(0, 0, 1)$.
(b) Use Proposition 15.18. This implies that $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1/2, 0, 1/2)$, and $(0, 2/3, 1/3)$ all are stationary states. Any other stationary state must be on the boundary of Δ^3 and have exactly one zero coordinate. Using this it can be shown that there are no other stationary states.
(c) The state $(0, 0, 1)$ is asymptotically stable. All other stationary states are not Lyapunov stable.

15.10 *Lyapunov Stable States in* 2×2 *Games*
Case (1): $\mathbf{e}^2$; case (2): $\mathbf{e}^1$ and $\mathbf{e}^2$; case (3): $\hat{\mathbf{x}}$. (Cf. Problem 15.6.)

15.11 *Nash Equilibrium and Lyapunov Stability*
$NE(A) = \{\mathbf{e}^1\}$. If we start at a completely mixed strategy close to $\mathbf{e}^1$, then first x_3 increases, and we can make the solution trajectory pass $\mathbf{e}^3$ as closely as desired. This shows that $\mathbf{e}^1$ is not Lyapunov stable.

15.12 *Rock-Paper-Scissors (2)*

(e) Follows from (d). If $a > 0$ then any trajectory converges to the maximum point of $x_1x_2x_3$, i.e. to $(1/3, 1/3, 1/3)$. If $a = 0$ then the trajectories are orbits ($x_1x_2x_3$ constant) around $(1/3, 1/3, 1/3)$. If $a < 0$ then the trajectories move outward, away from $(1/3, 1/3, 1/3)$.

Problems of Chapter 16

16.1 *Imputation Set of an Essential Game*
Note that $I(v)$ is a convex set and $\mathbf{f}^i \in I(v)$ for every $i = 1, \ldots, n$. Thus, $I(v)$ contains the convex hull of $\{\mathbf{f}^i \mid i \in N\}$. Now let $\mathbf{x} \in I(v)$, and write $\mathbf{x} = (v(1), \ldots, v(n)) + (\alpha_1, \ldots, \alpha_n)$, where $\sum_{i \in N} \alpha_i = v(N) - \sum_{i \in N} v(i) =: \alpha$.

16.2 *Convexity of the Domination Core*
First prove the following claim: For each $\mathbf{x} \in I(v)$ and $\emptyset \neq S \subseteq N$ we have

$$\exists \mathbf{z} \in I(v) : \mathbf{z}\, \mathrm{dom}_S \mathbf{x} \Leftrightarrow x(S) < v(S) \text{ and } x(S) < v(N) - \sum_{i \notin S} v(i) \ .$$

Use this claim to show that $I(v) \setminus D(S)$ is a convex set. Finally, conclude that $DC(v)$ must be convex.

16.3 *Dominated Sets of Imputations*

(b) In both games, $D(ij) = \{\mathbf{x} \in I(v) \mid x_i + x_j < v(ij)\}$, $i, j \in \{1, 2, 3\}$, $i \neq j$.

16.7 *A Glove Game*

(b) The core and the domination core are both equal to $\{(0, 1, 0)\}$, cf. Theorem 16.12.

16.11 *Core and D-Core*
Condition (16.1) is not a necessary condition for equality of the core and the D-core. To find a counterexample, first note that if $C(v) \neq \emptyset$ then (16.1) must hold. Therefore, a counterexample has to be some game with empty core and D-core.

16.12 *Strategic Equivalence*
Straightforward using the definitions.

16.13 *Proof of Theorem 16.20*
Write $B = \begin{pmatrix} A \\ -A \end{pmatrix}$. Then

$$\begin{aligned}
\max\{\mathbf{b} \cdot \mathbf{y} \mid A\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0}\} &= \max\{\mathbf{b} \cdot \mathbf{y} \mid B\mathbf{y} \leq (\mathbf{c}, -\mathbf{c}), \mathbf{y} \geq \mathbf{0}\} \\
&= \min\{(\mathbf{c}, -\mathbf{c}) \cdot (\mathbf{x}, \mathbf{z}) \mid (\mathbf{x}, \mathbf{z})B \geq \mathbf{b}, (\mathbf{x}, \mathbf{z}) \geq \mathbf{0}\} \\
&= \min\{\mathbf{c} \cdot (\mathbf{x} - \mathbf{z}) \mid (\mathbf{x} - \mathbf{z})A \geq \mathbf{b}, (\mathbf{x}, \mathbf{z}) \geq \mathbf{0}\} \\
&= \min\{\mathbf{c} \cdot \mathbf{x}' \mid \mathbf{x}'A \geq \mathbf{b}\} .
\end{aligned}$$

The second equality follows from Theorem 22.6.

16.14 *Infeasible Programs in Theorem 16.20*
Follow the hint.

16.15 *Proof of Theorem 16.22 Using Lemma 22.5*
Follow the hint and investigate (b) of Lemma 22.5.

16.17 *Minimum of Balanced Games*
Follows by using the definition of balancedness or by Theorem 16.22.

16.18 *Balanced Simple Games*
Let (N, v) be a simple game.

Suppose i is a veto player. Let B be a balanced collection with balanced map λ. Then

$$\sum_{S \in B} \lambda(S) v(S) = \sum_{S \in B : i \in S} \lambda(S) v(S) \leq 1 = v(N) ,$$

since i is a veto player. Hence, v is balanced.

For the converse, suppose v is balanced, and distinguish two cases:

Case 1: There is an i with $v(\{i\}) = 1$. Show that i is a veto player.
Case 2: $v(\{i\}) = 0$ for every $i \in N$. Show that also in this case v has veto players.

Problems of Chapter 17

17.1 *The Games* 1_T

(c) For $i \in T$: $\Phi_i(1_T) = \frac{(|T|-1)!(n-|T|)!}{n!}$.

17.2 *Unanimity Games*

(a) Suppose $\sum_{T \neq \emptyset} \alpha_T u_T = 0$, where 0 means the zero-game, for some $\alpha_T \in \mathbb{R}$. Show that all α_T are zero by induction, starting with one-person coalitions.
(b) Let $W \in 2^N$, then show

$$\sum_{T \neq \emptyset} c_T u_T(W) = v(W) + \sum_{S:\ S \subsetneq W} v(S) \sum_{T: S \subseteq T \subseteq W} (-1)^{|T|-|S|} \ .$$

It is sufficient to show that the second term of the last expression is equal to 0, hence that $\sum_{T:S \subseteq T \subseteq W} (-1)^{|T|-|S|} = 0$.

17.3 *If-Part of Theorem 17.4*
EFF, NP and ADD are straightforward. SYM needs more attention. Let i, j be symmetric in v. Note that for $S \subseteq N$ with $i \notin S$ and $j \in S$ we have $v((S \cup i) \setminus j) = v(S)$ by symmetry of i and j, since $v((S \cup i) \setminus j) = v((S \setminus j) \cup i)$ and $v(S) = v((S \setminus j) \cup j)$. Use this to show $\Phi_i(v) = \Phi_j(v)$ by collecting terms in a clever way.

17.4 *Dummy Player Property and Anonymity*
That DUM implies NP and the Shapley value satisfies DUM is straightforward. AN implies SYM: Let i and j be symmetric players, and let the value ψ satisfy AN. Then consider the permutation σ with $\sigma(i) = j$, $\sigma(j) = i$, and $\sigma(k) = k$ otherwise.

17.5 *Shapley Value, Core, and Imputation Set*
In the case of two players the core and the imputation set coincide. If the core is not empty then the Shapley value is in the core, cf. Example 17.2. In general, consider any game with $v(1) = 2$, $v(N) = 3$, and $v(S) = 0$ otherwise. Then the Shapley value is not even in the imputation set as soon as $n \geq 3$.

17.6 *Shapley Value as a Projection*
If a is an additive game then $\Phi(a) = (a(1), a(2), \ldots, a(n))$. For a general game v let a^v be the additive game generated by $\Phi(v)$. Then $\Phi(a^v) = (a^v(1), \ldots, a^v(n)) = \Phi(v)$.

17.7 *Shapley Value of Dual Game*
Follow the hint, or give a direct proof by using (17.4).

17.8 *Multilinear Extension*

(b) Let g be another multilinear extension of $\tilde{v}$ to $[0,1]^n$, say $g(\mathbf{x}) = \sum_{T\subseteq N} b_T \left(\prod_{i\in T} x_i\right)$. Show $b_T = c_T$ for all T by induction, starting with one-player coalitions.

17.9 *The Beta-Integral Formula*
Apply partial integration.

17.10 *Path Independence of* Φ
Use Theorem 17.12(c).

17.11 *An Alternative Characterization of the Shapley Value*
The Shapley value satisfies all these conditions. Conversely, (b)–(d) imply standardness for two-person games, so the result follows from Theorem 17.18.

Problems of Chapter 18

18.1 *Marginal Vectors and Dividends*

(b) For each $i \in N$, $m_i^\pi = \sum_{T\subseteq P_\pi(i)\cup i,\ i\in T} \Delta_v(T)$.

18.2 *Convexity and Marginal Vectors*
Use Theorems 18.3 and 18.6.

18.3 *Strictly Convex Games*
Let π and σ be two different permutations and suppose that $k \geq 1$ is the minimal number such that $\pi(k) \neq \sigma(k)$. Then show that $m^\pi_{\pi(k)}(v) < m^\sigma_{\pi(k)}(v)$. Hence, $m^\pi \neq m^\sigma$.

18.4 *Sharing Profits*

(a) For the landlord: $\Phi_0(v) = \frac{1}{n+1}\left[\sum_{s=0}^n f(s)\right]$.
(c) Extend f to a piecewise linear function on $[0,n]$. Then v is convex if and only if f is convex.

18.5 *Sharing Costs*

(a) For every nonempty coalition S, $v(S) = \sum_{i\in S} c_i - \max\{c_i \mid i \in S\}$. If we regard $c = (c_1,\ldots,c_2)$ as an additive game we can write $v = c - c_{\max}$, where $c_{\max}(S) = \max\{c_i \mid i \in S\}$.

18.6 *Independence of the Axioms in Theorem 18.8*

(a) Consider the value which, for every game v, gives each dummy player his individual worth and distributes the rest, $v(N) - \sum_{i\in D} v(i)$ where D is the subset of dummy players, evenly among the players. This value satisfies all axioms except LIN.

(b) Consider the value which, for every game v, distributes $v(N)$ evenly among all players. This value satisfies all axioms except DUM.

(c) The value which gives each player his individual worth satisfies all axioms except EFF.

(d) Consider any set of weights $\{\alpha_\pi \mid \pi \in \Pi(N)\}$ with $\alpha_\pi \in \mathbb{R}$ for all π and $\sum_{\pi \in \Pi(N)} \alpha_\pi = 1$. The value $\sum_{\pi \in \Pi(N)} \alpha_\pi m^\pi$ satisfies LIN, DUM and EFF, but not MON unless all weights are nonnegative.

18.7 *Null-Player in Theorem 18.8*
Check that the dummy axiom in the proof of this theorem is only used for unanimity games. In those games, dummy players are also null-players, so it is sufficient to require NP. Alternatively, one can show that DUM is implied by ADD (and, thus, LIN), EFF and NP.

18.8 *Characterization of Weighted Shapley Values*
Check that every weighted Shapley value satisfies the Partnership axiom. Conversely, let ψ be a value satisfying the Partnership axiom and the four other axioms. Let $S^1 := \{i \in N \mid \psi_i(u_N) > 0\}$ and for every $i \in S^1$ let $\omega_i := \psi_i(u_N)$. Define, recursively, $S^k := \{i \in N \setminus (S^1 \cup \ldots \cup S^{k-1}) \mid \psi_i(u_{N\setminus(S^1\cup\ldots\cup S^{k-1})}) > 0\}$ and for every $i \in S^k$ let $\omega_i := \psi_i(u_{N\setminus(S^1\cup\ldots\cup S^{k-1})})$. This results in a partition $(S^1, \ldots, S^m)$ of N. Now define the weight system w by the partition $(S_1, \ldots, S_m)$ with $S_1 := S^m$, $S_2 := S^{m-1}$, ..., $S_m := S^1$, and the weights $\boldsymbol{\omega}$. Then it is sufficient to prove that for each coalition S and each player $i \in S$ we have $\psi_i(u_S) = \Phi_i^w(u_S)$. Let $h := \max\{j \mid S \cap S_j \neq \emptyset\}$, then with $T = N \setminus (S_{h+1} \cup \ldots \cup S_m)$ we have by the Partnership axiom: $\psi_i(u_S) = \frac{1}{\psi(u_T)(S)}\psi_i(u_T)$. If $i \notin S_h$ then $\psi_i(u_T) = 0$, hence $\psi_i(u_S) = 0 = \Phi_i^w(u_S)$. If $i \in S_h$ then $\psi_i(u_S) = \frac{\omega_i}{\sum_{j \in S \cap S_h} \omega_j} = \Phi_i^w(u_S)$.

18.9 *Core and Weighted Shapley Values in Example 18.2*
First write the game as a sum of unanimity games:

$$v = u_{\{1,2\}} + u_{\{1,3\}} - u_{\{2,3\}} + 2u_N \ .$$

Then consider all possible ordered partitions of N, there are 13 different ones, and associated weight vectors. This results in a description of all payoff vectors associated with weighted Shapley values, including those in the core of the game.

Problems of Chapter 19

19.1 *Binary Relations*

Not (4): $\succeq$ on $\mathbb{R}$ defined by $x \succeq y \Leftrightarrow x^2 \geq y^2$.
Not (3): $\geq$ on $\mathbb{R}^2$.

Not (2): $\succeq$ on $\mathbb{R}$ defined by: for all $x, y, x \geq y$, let $x \succeq y$ if $x - y \geq 1$, and let $y \succeq x$ if $x - y < 1$.
Not (1): $>$ on $\mathbb{R}$.

The ordering on $\mathbb{R}$, defined by $[x \succeq y] \Leftrightarrow [x = y \text{ or } 0 \leq x, y \leq 1]$ is reflexive and transitive but not complete and not anti-symmetric.

19.2 *Linear Orders*
If $\mathbf{x} \succ \mathbf{y}$ then by definition $\mathbf{x} \succeq \mathbf{y}$ and not $\mathbf{y} \succeq \mathbf{x}$: hence $\mathbf{x} \neq \mathbf{y}$ since otherwise $\mathbf{y} \succeq \mathbf{x}$ by reflexivity.
If $\mathbf{x} \succeq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ then not $\mathbf{y} \succeq \mathbf{x}$ since otherwise $\mathbf{x} = \mathbf{y}$ by anti-symmetry. Hence $\mathbf{x} \succ \mathbf{y}$.

19.3 *The Lexicographic Order (1)*
Check (1)–(4) in Sect. 19.2 for $\succeq_{\text{lex}}$. Straightforward.

19.4 *The Lexicographic Order (2)*
This is the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid [x_1 = 3,\ x_2 \geq 1] \text{ or } [x_1 > 3]\}$. This set is not closed.

19.5 *Representability of Lexicographic Order (1)*
Consider Problem 19.4. Since $(\alpha, 0) \succeq_{\text{lex}} (3, 1)$ for all $\alpha > 3$, we have $u(\alpha, 0) \geq u(3, 1)$ for all $\alpha > 3$ and hence, by continuity of u, $\lim_{\alpha \downarrow 3} u(\alpha, 0) \geq u(3, 1)$. Therefore $(3, 0) \succeq_{\text{lex}} (3, 1)$, a contradiction.

19.6 *Representability of Lexicographic Order (2)*
Suppose that u represents $\succeq_{\text{lex}}$ on $\mathbb{R}^2$, that is, $\mathbf{x} \succeq_{\text{lex}} \mathbf{y}$ if and only if $u(\mathbf{x}) \geq u(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Then for any $t \in \mathbb{R}$ let $q(t)$ be a rational number in the interval $[u(t, 0), u(t, 1)]$. Since $(t, \alpha) \succ_{\text{lex}} (s, \beta)$ and hence $u(t, \alpha) > u(s, \beta)$ for all $t > s$ and all $\alpha, \beta \in [0, 1]$, we have $[u(t, 0), u(t, 1)] \cap [u(s, 0), u(s, 1)] = \emptyset$ for all $t \neq s$. Hence, $q(t) \neq q(s)$ for all $t \neq s$. Therefore, we have found uncountably many different rational numbers, a contradiction.

19.7 *Single-Valuedness of the Pre-nucleolus*
Consider the pre-nucleolus on a suitable compact convex subset and apply Theorem 19.3.

19.8 *(Pre-)Nucleolus and Core*
Use the fact that core elements have all excesses non-positive.

19.9 *Kohlberg Criterion for the Nucleolus*
First observe that the following modification of Theorem 19.4 holds:
Theorem 19.4′ Let (N, v) be a game and $\mathbf{x} \in I(N, v)$. Then the following two statements are equivalent.

(1) $\mathbf{x} = \nu(N, v)$.
(2) *For every α such that $\mathcal{D}(\alpha, \mathbf{x}, v) \neq \emptyset$ and for every side-payment $\mathbf{y}$ with $y(S) \geq 0$ for every $S \in \mathcal{D}(\alpha, \mathbf{x}, v)$ and with $y_i \geq 0$ for all $i \in N$ with $x_i = v(i)$ we have $y(S) = 0$ for every $S \in \mathcal{D}(\alpha, \mathbf{x}, v)$.*

The proof of this theorem is almost identical to the proof of Theorem 19.4. In the second sentence of the proof, note that $\mathbf{z}_\varepsilon \in I(N, v)$ for ε small enough. In the second part of the proof, (2)⇒(1), note that $y_i = z_i - x_i \geq 0$ whenever $x_i = v(i)$.

For the 'if'-part of the statement in this problem, let $\mathbf{x} \in I(N, v)$, $\mathcal{D}(\alpha, \mathbf{x}, v) \neq \emptyset$, and $\mathcal{E}(\alpha, \mathbf{x}, v)$ such that $\mathcal{D}(\alpha, \mathbf{x}, v) \cup \mathcal{E}(\alpha, \mathbf{x}, v)$ is balanced. Consider a side-payment $\mathbf{y}$ with $y(S) \geq 0$ for every $S \in \mathcal{D}(\alpha, \mathbf{x}, v)$ and $y_i \geq 0$ for every i with $x_i = v(i)$ [hence in particular for every i with $\{i\} \in \mathcal{E}(\alpha, \mathbf{x}, v)$]. The argument in the first part of the proof of Theorem 19.5 now applies to $\mathcal{D}(\alpha, \mathbf{x}, v) \cup \mathcal{E}(\alpha, \mathbf{x}, v)$, and Theorem 19.4′ implies $\mathbf{x} = \nu(N, v)$.

For the 'only-if' part, consider the program (19.4) in the second part of the proof of Theorem 19.5 but add the constraints $-y_i \leq 0$ for every $i \in N$ with $x_i = v(i)$. Theorem 19.4′ implies that the dual of this program is feasible, that is, there are $\lambda(S) \geq 0$, $S \in \mathcal{D}(\alpha, \mathbf{x}, v)$, $\lambda(\{i\}) \geq 0$, i such that $x_i = v(i)$, and $\lambda(N) \in \mathbb{R}$ such that

$$-\sum_{i \in N:\ x_i = v(i)} \lambda(\{i\})\mathbf{e}^{\{i\}} - \sum_{S \in \mathcal{D}(\alpha, \mathbf{x}, v)} \lambda(S)\mathbf{e}^S + \lambda(N)\mathbf{e}^N = \sum_{S \in \mathcal{D}(\alpha, \mathbf{x}, v)} \mathbf{e}^S .$$

Hence $\lambda(N)\mathbf{e}^N = \sum_{S \in \mathcal{D}(\alpha, \mathbf{x}, v)} (1 + \lambda(S))\mathbf{e}^S + \sum_{i \in N:\ x_i = v(i)} \lambda(\{i\})\mathbf{e}^{\{i\}}$. Let $\mathcal{E}(\alpha, \mathbf{x}, v)$ consist of those one-person coalitions $\{i\}$ with $x_i = v(i)$ and $\lambda(\{i\}) > 0$, then $\mathcal{D}(\alpha, \mathbf{x}, v) \cup \mathcal{E}(\alpha, \mathbf{x}, v)$ is balanced.

19.10 *Proof of Theorem 19.5*
To formulate the dual program, use for instance the formulation in Theorem 16.20. For instance, the primal (19.4) can be converted to the minimization problem in Theorem 16.20; then the dual corresponds to the maximization problem in Theorem 16.20. Feasibility of the dual follows from Problem 16.14.

19.11 *Nucleolus of a Three-Person Game (1)*
The nucleolus is $(5, 4, 3)$.

19.12 *Nucleolus of a Three-Person Game (2)*
The (pre-)nucleolus is $(5, 3, 2)$.

19.14 *Individual Rationality Restrictions for the Nucleolus*
The nucleolus is $(1, 0, 0)$. The pre-nucleolus is $(5/3, -1/3, -1/3)$.

19.15 *Example 19.7*
The set $\mathcal{B}_1 = \{123, 124, 34\}$ is balanced with weights all equal to $1/2$. The set $\mathcal{B}_1 \cup \mathcal{B}_2 = \{123, 124, 34, 134, 234\}$ is balanced with weights, respectively, equal to $5/12, 5/12, 3/12, 2/12, 2/12$.

19.16 *(Pre-)Nucleolus of a Symmetric Game*

(a) $\nu(v) = \nu^*(v) = (v(N)/n)\mathbf{e}^N$.

19.17 *COV and AN of the Pre-nucleolus*

Covariance of the pre-nucleolus follows since applying a transformation as in the definition of this property changes all excesses (only) by the same positive (multiplicative) factor.
Anonymity of the pre-nucleolus follows since a permutation of the players does not change the ordered vectors $\theta(\mathbf{x})$, but only permutes the coalitions to which the excesses correspond.

19.18 *Apex Game*
The (pre-)nucleolus is $(3/7, 1/7, 1/7, 1/7, 1/7)$. This can easily be verified using the Kohlberg criterion.

19.19 *Landlord Game*

(a) By anonymity, each worker is assigned $\frac{1}{2}[f(n) - f(n-1)]$. By computing the excesses, it follows that among all coalitions containing the landlord, with this payoff vector the maximal excesses are reached by the coalitions containing $n-1$ workers, and further also by the coalitions consisting of a single worker and not the landlord. By the Kohlberg criterion this immediately implies that the given vector is the (pre-)nucleolus. For the Shapley value, see Problem 18.4.
(b) Compute the excesses for the payoff vector $\frac{f(n)}{n+1}\mathbf{e}^{\{0,1,\ldots,n\}}$, and apply the Kohlberg criterion.

19.20 *Game in Sect. 19.1*
The first linear program is: minimize α subject to the constraints $x_i + \alpha \geq 4$ for $i = 1, 2, 3$, $x_1 + x_2 + \alpha \geq 8$, $x_1 + x_3 + \alpha \geq 12$, $x_2 + x_3 + \alpha \geq 16$, $x_1 + x_2 + x_3 = 24$. The program has optimal value $\alpha = -2$, reached for $x_1 = 6$ and $x_2, x_3 \geq 6$.

In the second program x_1 has been eliminated. This program reduces to: minimize α subject to $x_2 + \alpha \geq 4$, $x_2 \leq 12 + \alpha$, $x_2 + x_3 = 18$. This has optimal value $\alpha = -4$, reached for $x_2 = 8$, $x_3 = 10$.

19.21 *The Prekernel*
For $i, j \in N$ denote by $\mathcal{T}_{ij}$ those coalitions that contain player i and not player j. For a payoff vector $\mathbf{x}$ denote by $s_{ij}(\mathbf{x}, v)$ the maximum of $e(S, \mathbf{x}, v)$ over all $S \in \mathcal{T}_{ij}$.

Let now $\mathbf{x}$ be the pre-nucleolus and suppose, contrary to what has to be proved, that there are two distinct players k, ℓ such that $s_{k\ell}(\mathbf{x}, v) > s_{\ell k}(\mathbf{x}, v)$. Let $\delta = (s_{k\ell}(\mathbf{x}, v) - s_{\ell k}(\mathbf{x}, v))/2$ and define $\mathbf{y}$ by $y_k = x_k + \delta$, $y_\ell = x_\ell - \delta$, and $y_i = x_i$ for all $i \neq k, \ell$. Denote $\mathcal{S} = \{S \in 2^N \setminus \mathcal{T}_{k\ell} \mid e(S, \mathbf{x}, v) \geq s_{kl}(\mathbf{x}, v)\}$ and $s = |S|$. Then $\theta_{s+1}(\mathbf{x}) = s_{k\ell}(\mathbf{x}, v)$. For $S \in 2^N \setminus (\mathcal{T}_{k\ell} \cup \mathcal{T}_{\ell k})$, we have $e(S, \mathbf{x}, v) = e(S, \mathbf{y}, v)$. For $S \in \mathcal{T}_{k\ell}$ we have $e(S, \mathbf{y}, v) = e(S, \mathbf{x}, v) - \delta$. Finally, for $S \in \mathcal{T}_{\ell k}$ we have

$$e(S, \mathbf{y}, v) = e(S, \mathbf{x}, v) + \delta \leq s_{\ell k}(\mathbf{x}, v) + \delta = s_{k\ell}(\mathbf{x}, v) - \delta \ .$$

Thus, $\theta_t(\mathbf{y}) = \theta_t(\mathbf{x})$ for all $t \leq s$ and $\theta_{s+1}(\mathbf{y}) < s_{k\ell}(\mathbf{x}, v) = \theta_{s+1}(\mathbf{x})$. Hence $\theta(\mathbf{x}) \succ_{\text{lex}} \theta(\mathbf{y})$, a contradiction.

Problems of Chapter 20

20.2 *Example 20.3*
Argue that $a_{12} = a_{13} = 3$ if v were an assignment game. Use this to derive a contradiction.

20.3 *Subgames of Permutation Games*
That a subgame of a permutation game is again a permutation game follows immediately from the definition: in (20.3) the worth $v(S)$ depends only on the numbers k_{ij} for $i, j \in S$. By a similar argument [consider (20.1)] this also holds for assignment games.

20.4 *A Flow Game*

(c) $(1, 1, 0, 0)$, corresponding to the minimum cut through e_1 and e_2; $\{(0, 0, 1 + \alpha, 1 - \alpha) \mid 0 \leq \alpha \leq 1\}$, corresponding to the minimum cut through e_3 and e_4.

20.5 *Every Nonnegative Balanced Game is a Flow Game*
Let v be a nonnegative balanced game, and write (following the hint to the problem) $v = \sum_{r=1}^{k} \alpha_r w_r$, where $\alpha_r > 0$ and w_r a balanced simple game for each $r = 1, \ldots, k$. Consider the controlled capacitated network with two vertices, the source and the sink, and k edges connecting them, where each edge e_r has capacity α_r and is controlled by w_r. Then show that the associated flow game is v.

20.6 *On Theorem 20.6 (1)*

(a) This follows straightforwardly from the proof of Theorem 20.6.
(b) For example, each player receiving $5\frac{1}{4}$ is a core element.

20.7 *On Theorem 20.6 (2)*
In any core element, player should 1 receive at least 1 and player 2 also, but $v(N) = 1$. Hence the game has an empty core.

20.8 *Totally Balanced Flow Games*
This follows immediately from Theorem 20.6, since every dictatorial game is balanced, i.e., has veto players.

20.9 *If-Part of Theorem 20.9*
We show that the Banzhaf value satisfies 2-EFF (the other properties are obvious). With notations as in the formulation of 2-EFF, we have

$$\begin{aligned}\psi_p(v_p) &= \sum_{S \subseteq (N \setminus p) \cup \{p\}:\, p \notin S} \frac{1}{2^{|N|-2}} \left[v_p(S \cup \{p\}) - v_p(S) \right] \\ &= \sum_{S \subseteq N \setminus \{i,j\}} \frac{1}{2^{|N|-2}} \left[v(S \cup \{ij\}) - v(S) \right]\end{aligned}$$

$$= \sum_{S \subseteq N \setminus \{i,j\}} \frac{1}{2^{|N|-1}} \left[2v(S \cup \{ij\}) - 2v(S)\right] .$$

The term in brackets can be written as

$$[v(S \cup \{i,j\}) - v(S \cup \{i\}) + v(S \cup \{j\}) - v(S)]$$
$$+ [v(S \cup \{i,j\}) - v(S \cup \{j\}) + v(S \cup \{i\}) - v(S)] ,$$

hence $\psi_p(v_p) = \psi_j(v) + \psi_i(v)$.
Show that DUM cannot be weakened to NP by finding a different value satisfying 2-EFF, SYM, NP, and SMON.

Problems of Chapter 21

21.1 *Anonymity and Symmetry*
An example of a symmetric but not anonymous solution is as follows. To symmetric problems, assign the point in $W(S)$ with equal coordinates; otherwise, assign the point of S that is lexicographically (first player 1, then player 2) maximal.

21.3 *The Nash Solution is Well-Defined*
The function $\mathbf{x} \mapsto (x_1 - d_1)(x_2 - d_2)$ is continuous on the compact set $\{\mathbf{x} \in S \mid \mathbf{x} \geq \mathbf{d}\}$ and hence attains a maximum on this set. We have to show that this maximum is attained at a unique point. In general, consider two points $\mathbf{z}, \mathbf{z}' \in \{\mathbf{x} \in S \mid \mathbf{x} \geq \mathbf{d}\}$ with $(z_1 - d_1)(z_2 - d_2) = (z_1' - d_1)(z_2' - d_2) = \alpha$. Then one can show that at the point $\mathbf{w} = \frac{1}{2}(\mathbf{z} + \mathbf{z}') \in S$ one has $(w_1 - d_1)(w_2 - d_2) > \alpha$. This implies that the maximum is attained at a unique point.

21.4 $(a) \Rightarrow (b)$ *in Theorem 21.1*
WPO and IIA are straightforward by definition, and SC follows from an easy computation. For SYM, note that if $N(S, \mathbf{d}) = \mathbf{z}$ for a symmetric problem $(S, \mathbf{d})$, then also $(z_2, z_1) = N(S, \mathbf{d})$ by definition of the Nash bargaining solution. Hence, $z_1 = z_2$ by uniqueness.

21.5 *Geometric Characterization of the Nash Bargaining Solution*
Let $(S, \mathbf{d}) \in B$ and $N(S, \mathbf{d}) = \mathbf{z}$. The slope of the tangent line ℓ to the graph of the function $x_1 \mapsto (z_1 - d_1)(z_2 - d_2)/(x_1 - d_1) + d_2$ (which describes the level set of $\mathbf{x} \mapsto (x_1 - d_1)(x_2 - d_2)$ through $\mathbf{z}$) at $\mathbf{z}$ is equal to $-(z_2 - d_2)/(z_1 - d_1)$, i.e., the negative of the slope of the straight line through d and z. Clearly, ℓ supports S at $\mathbf{z}$: this can be seen by invoking a separating hyperplane theorem, but also as follows. Suppose there were some point $\mathbf{z}'$ of S on the other side of ℓ than $\mathbf{d}$. Then there is a point $\mathbf{w}$ on the line segment connecting $\mathbf{z}'$ and $\mathbf{z}$ (hence, $\mathbf{w} \in S$) with $(w_1 - d_1)(w_2 - d_2) > (z_1 - d_1)(z_2 - d_2)$, contradicting $\mathbf{z} = N(S, \mathbf{d})$. The existence of such a point $\mathbf{w}$ follows since otherwise the straight line through $\mathbf{z}'$ and $\mathbf{z}$ would also be a tangent line to the level curve of the Nash product at $\mathbf{z}$.

For the converse, suppose that at a point $\mathbf{z}$ there is a supporting line of S with slope $-(z_2 - d_2)/(z_1 - d_1)$. Clearly, this line is tangent to the graph of the function $x_1 \mapsto (z_1 - d_1)(z_2 - d_2)/(x_1 - d_1) + d_2$ at $\mathbf{z}$. It follows that $\mathbf{z} = N(S, \mathbf{d})$.

21.6 *Strong Individual Rationality*
The implication $(a) \Rightarrow (b)$ is straightforward. For $(b) \Rightarrow (a)$, if F is also weakly Pareto optimal, then $F = N$ by Theorem 21.1. So it is sufficient to show that, if F is not weakly Pareto optimal then $F = D$. Suppose that F is not weakly Pareto optimal. Then there is an $(S, \mathbf{d}) \in B$ with $F(S, \mathbf{d}) \notin W(S)$. By IR, $F(S, \mathbf{d}) \geq \mathbf{d}$. Suppose $F(S, \mathbf{d}) \neq \mathbf{d}$. By SC, we may assume w.l.o.g. $\mathbf{d} = (0, 0)$. Let $\alpha > 0$ be such that $F(S, (0, 0)) \in W((\alpha, \alpha)S)$. Since $F(S, (0, 0)) \notin W(S)$, $\alpha < 1$. So $(\alpha, \alpha)S \subseteq S$. By IIA, $F((\alpha, \alpha)S, (0, 0)) = F(S, (0, 0))$, so by SC, $F((\alpha, \alpha)S, (0, 0)) = (\alpha, \alpha)F(S, (0, 0)) = F(S, (0, 0))$, contradicting $\alpha < 1$. So $F(S, (0, 0)) = (0, 0)$. Suppose $F(T, (0, 0)) \neq (0, 0)$ for some $(T, (0, 0)) \in B$. By SC we may assume $(0, 0) \neq F(T, (0, 0)) \in S$. By IIA applied twice, $(0, 0) = F(S \cap T, (0, 0)) = F(T, (0, 0)) \neq (0, 0)$, a contradiction. Hence, $F = D$.

21.7 $(a) \Rightarrow (b)$ *in Theorem 21.2*
Straightforward. Note in particular that in a symmetric game the utopia point is also symmetric, and that the utopia point is 'scale covariant'.

21.8 *Restricted Monotonicity*

(a) Follows from applying IM twice.
(b) For $(S, \mathbf{d})$ with $\mathbf{d} = (0, 0)$ and $u(S, \mathbf{d}) = (1, 1)$, let $F(S, \mathbf{d})$ be the lexicographically (first player 1, then player 2) maximal point of $S \cap \mathbb{R}^2_+$. Otherwise, let F be equal to R. This F satisfies RM but not IM.

21.9 *Global Individual Monotonicity*
It is straightforward to verify that G satisfies WPO, SYM, SC, and GIM. For the converse, suppose that F satisfies these four axioms, let $(S, \mathbf{d}) \in B$ and $\mathbf{z} := G(S, \mathbf{d})$. By SC, w.l.o.g. $\mathbf{d} = (0, 0)$ and $g(S) = (1, 1)$. Let $\alpha \leq 0$ such that $S \subseteq \tilde{S}$ where $\tilde{S} := \{\mathbf{x} \in \mathbb{R}^2 \mid (\alpha, \alpha) \leq \mathbf{x} \leq \mathbf{y}$ for some $\mathbf{y} \in S\}$. In order to prove $F(S, (0, 0)) = G(S, (0, 0))$ it is sufficient to prove that $F(\tilde{S}, (0, 0)) = G(\tilde{S}, (0, 0))$ (in view of GIM and WPO). Let $T = \text{conv}\{\mathbf{z}, (\alpha, g_2(\tilde{S})), (g_1(\tilde{S}), \alpha)\} = \text{conv}\{\mathbf{z}, (\alpha, 1), (1, \alpha)\}$. By SYM and WPO, $F(T, (0, 0)) = \mathbf{z}$. By GIM, $F(\tilde{S}, (0, 0)) \geq F(T, (0, 0)) = \mathbf{z} = G(S, (0, 0)) = G(\tilde{S}, (0, 0))$, so by WPO: $F(\tilde{S}, (0, 0)) = G(\tilde{S}, (0, 0))$. (Make pictures. Note that this proof is analogous to the proof of Theorem 21.2.)

21.10 *Monotonicity and (Weak) Pareto Optimality*

(a) Consider problems of the kind $(\text{conv}\{\mathbf{d}, \mathbf{a}\}, \mathbf{d})$ for some $\mathbf{a} > \mathbf{d}$.
(b) The egalitarian solution E satisfies MON and WPO on B_0.

21.11 *The Egalitarian Solution (1)*
Straightforward.

21.12 *The Egalitarian Solution (2)*
Let $\mathbf{z} := E(S, \mathbf{d}) + E(T, \mathbf{e})$. Then it is straightforward to derive that $z_1 - (d_1 + e_1) = z_2 - (d_2 + e_2)$. Since $E(S + T, d + e)$ is the maximal point $\mathbf{x}$ such that $x_1 - (d_1 + e_1) = x_2 - (d_2 + e_2)$, it follows that $E(S + T, d + E) \geq \mathbf{z}$.

21.13 *Independence of Axioms*
Theorem 21.1:
WPO, SYM, SC: $F = R$; WPO, SYM, IIA: $F = L$, where $L(S, \mathbf{d})$ is the point of $P(S)$ nearest to the point $\mathbf{z} \geq \mathbf{d}$ with $z_1 - d_1 = z_2 - d_2$ measured along the boundary of S; WPO, SC, IIA: $F = D^1$, where $D^1(S, \mathbf{d})$ is the point of $\{\mathbf{x} \in P(S) \mid \mathbf{x} \geq \mathbf{d}\}$ with maximal first coordinate; SYM, SC, IIA: $F = D$ (disagreement solution).
Theorem 21.2:
WPO, SYM, SC: $F = N$; WPO, SYM, IM: $F = L$; WPO, SC, IM: if $\mathbf{d} = (0, 0)$ and $u(S, \mathbf{d}) = (1, 1)$, let F assign the point of intersection of $W(S)$ and the line segment connecting $(1/4, 3/4)$ and $(1, 1)$ and, otherwise, let F be determined by SC; SYM, SC, IM: $F = D$.
Theorem 21.3:
WPO, MON, SYM: $F(S, \mathbf{d})$ is the maximal point of S on the straight line through $\mathbf{d}$ with slope $1/3$ if $\mathbf{d} = (1, 0)$, $F(S, \mathbf{d}) = E(S, \mathbf{d})$ otherwise; WPO, MON, TC: $F(S, \mathbf{d})$ is the maximal point of S on the straight line through $\mathbf{d}$ with slope $1/3$; WPO, SYM, TC: $F = N$; MON, SYM, TC: $F = D$.

21.14 *Nash and Rubinstein*

(b) The Nash bargaining solution outcome is $(\frac{1}{3}\sqrt{3}, \frac{2}{3})$, hence $(\frac{1}{3}\sqrt{3}, 1 - \frac{1}{3}\sqrt{3})$ is the resulting distribution of the good.
(c) The Rubinstein bargaining outcome is $\left(\sqrt{\frac{1-\delta}{1-\delta^3}}, \frac{\delta-\delta^3}{1-\delta^3}\right)$.
(d) The outcome in (c) converges to the outcome in (b) if δ converges to 1.

Problems of Chapter 22

22.1 *Convex Sets*
The only-if part is obvious. For the if-part, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in Z the condition implies that $\frac{k}{2^m}\mathbf{x} + \frac{2^m-k}{2^m}\mathbf{y} \in Z$ for every $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^m\}$. By closedness of Z, this implies that $\text{conv}\{\mathbf{x}, \mathbf{y}\} \subseteq Z$, hence Z is convex.

22.2 *Proof of Lemma 22.3*
Suppose that both systems have a solution, say $(\mathbf{y}, \mathbf{z}) \geq \mathbf{0}$, $(\mathbf{y}, \mathbf{z}) \neq \mathbf{0}$, $A\mathbf{y} + \mathbf{z} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$, $\mathbf{x}A > \mathbf{0}$. Then $\mathbf{x}A\mathbf{y} + \mathbf{x} \cdot \mathbf{z} = \mathbf{x}(A\mathbf{y} + \mathbf{z}) = 0$, hence $\mathbf{y} = \mathbf{0}$ and $\mathbf{z} = \mathbf{0}$ since $\mathbf{x} > \mathbf{0}$, $\mathbf{x}A > \mathbf{0}$. This contradicts $(\mathbf{y}, \mathbf{z}) \neq \mathbf{0}$.

22.3 *Rank of AA^T*
This follows from basic linear algebra. Prove that the null spaces of A and A^TA are equal and use the Rank Theorem to conclude that the rank of A^TA, and thus that of AA^T, is equal to the rank of A.

22.4 *Proof of Lemma 22.5*
Suppose that both systems have a solution, say $\mathbf{x} > \mathbf{0}$, $\mathbf{x}A = \mathbf{b}$, $A\mathbf{y} \geq \mathbf{0}$, $\mathbf{b} \cdot \mathbf{y} < 0$. Then $\mathbf{x}A\mathbf{y} < 0$, contradicting $\mathbf{x} > \mathbf{0}$ and $A\mathbf{y} \geq \mathbf{0}$.

22.5 *Proof of Lemma 22.7*

(a) If $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x}A \leq \mathbf{b}$, $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b} \cdot \mathbf{y} < 0$ then $\mathbf{x}A\mathbf{y} \leq \mathbf{b} \cdot \mathbf{y} < 0$, so $A\mathbf{y} \ngeq \mathbf{0}$. This shows that at most one of the two systems has a solution.
(b) Suppose the system in (a) has no solution. Then also the system $\mathbf{x}A + \mathbf{z}I = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, $\mathbf{z} \geq \mathbf{0}$ has no solution. Hence, by Farkas' Lemma the system $\begin{pmatrix} A \\ I \end{pmatrix} \mathbf{y} \geq \mathbf{0}$, $\mathbf{b} \cdot \mathbf{y} < 0$ has a solution. Therefore, the system in (b) has a solution.

22.6 *Extreme Points*
The implication (b) $\Rightarrow$ (a) follows by definition of an extreme point.
For the implication (a) $\Rightarrow$ (c), let $x, y \in C \backslash \{e\}$ and $0 < \lambda < 1$. Let $z = \lambda x + (1-\lambda)y$. If $z \neq e$ then $z \in C \setminus \{e\}$ by convexity of C. Suppose now that $z = e$. W.l.o.g. let $\lambda \geq 1/2$. Then $e = \lambda x + (1-\lambda)y = (1/2)x + (1/2)[\mu x + (1-\mu)y]$ for $\mu = 2\lambda - 1$. Since $\mu x + (1-\mu)y \in C$, this implies that e is not an extreme point of C. This proves the implication (a) $\Rightarrow$ (c).
For the implication (c) $\Rightarrow$ (b), let $x, y \in C$, $x \neq y$, and $0 < \alpha < 1$. If $x = e$ or $y = e$ then clearly $\alpha x + (1-\alpha)y \neq e$. If $x \neq e$ and $y \neq e$ then $\alpha x + (1-\alpha)y \in C \setminus \{e\}$ by convexity of $C \setminus \{e\}$, hence $\alpha x + (1-\alpha)y \neq e$ as well.

22.7 *Affine Subspaces*
Let $A = a + L$ be an affine subspace, $x, y \in A$, and $\lambda \in \mathbb{R}$. Write $x = a + \bar{x}$ and $y = a + \bar{y}$ for $\bar{x}, \bar{y} \in L$, then $\lambda x + (1-\lambda)y = a + \lambda\bar{x} + (1-\lambda)\bar{y} \in A$ since $\lambda\bar{x} + (1-\lambda)\bar{y} \in L$ (L is a linear subspace).
Conversely, suppose that A contains the straight line through any two of its elements. Let a be an arbitrary element of A and let $L := \{x - a \mid x \in A\}$. Then it follows straightforwardly that L is a linear subspace of V, and thus $A = a + L$ is an affine subspace.

22.8 *The Set of Sup-points of a Linear Function on a Convex Set*
In general, linearity of f implies that, if $f(\mathbf{x}) = f(\mathbf{y})$, then $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = f(\mathbf{x}) = f(\mathbf{y})$ for any two points of C and $0 < \lambda < 1$. It follows, in particular, that the set D is convex.
Let $\mathbf{e} \in \text{ext}(D)$ and suppose $\mathbf{e} = (1/2)\mathbf{x} + (1/2)\mathbf{y}$ for some $\mathbf{x}, \mathbf{y} \in C$. Then by linearity of f, $f(\mathbf{e}) = (1/2)f(\mathbf{x}) + (1/2)f(\mathbf{y})$, hence $\mathbf{x}, \mathbf{y} \in D$ since $\mathbf{e} \in D$. So $\mathbf{e} = \mathbf{x} = \mathbf{y}$ since $\mathbf{e}$ is an extreme point of D. Thus, $\mathbf{e}$ is also an extreme point of C.

Problems of Chapter 23

RP 1 *Matrix Games (1)*

(a) All rows are (pure) maximin strategies (with minimum 0) and all columns are pure minimax strategies (with maximum 2). The value of the game is between 0 and 2 (which is obvious anyway in this case).
(b) The third column is strictly dominated by the second column and the third row is strictly dominated by the second row. Entry $(1,2)$ is a saddlepoint, hence the value of the game is 2. The unique maximin strategy is $(1,0,0)$, and the minimax strategies are the strategies in the set $\{(q, 1-q, 0) \mid 0 \leq q \leq 1/2\}$.
(c) The second and third rows are the maximin rows. The second column is the unique minimax column. From this we can conclude that the value of the game is between 1 and 2. The first and fourth columns are strictly dominated by the second. Next, the first row is strictly dominated by the last row. The unique maximin strategy is $(0, 2/3, 1/3)$ and the unique minimax strategy is $(0, 2/3, 1/3, 0)$. The value of the game is $5/3$.

RP 2 *Matrix Games (2)*

(a) The first row is the unique maximin row (with minimum 2) and both columns are minimax columns (with maximum 5). So the value is between 2 and 5. The game has no saddlepoint.
(b) $v(A_1) = 5/2$, $v(A_2) = 20/7$, $v(A_3) = 2$ (saddlepoint), $v(A_4) = 1$ (saddlepoint), $v(A_5) = 7/3$, $v(A_6) = 25/9$. Since player 1 can pick rows, the value must be the maximum of these amounts, hence $20/7$, the value of A_2.
(c) The unique maximin strategy is $(5/7, 0, 2/7, 0)$ and the unique minimax strategy is $(3/7, 4/7)$.

RP 3 *Matrix Games (3)*

(a) The unique maximin row is the first row, with minimum 8. The unique minimax column is the first column, with maximum 12. So the value of the game is between 8 and 12. The game has no saddlepoint.
(b) The second row is strictly dominated by for instance putting probability $1/2$ on the first row and $1/2$ on the third row. After eliminating the second row, the third column is strictly dominated by the first column.
(c) The unique maximin strategy is $(1/2, 0, 1/2)$ and the unique minimax strategy is $(3/4, 1/4, 0)$. The value of the game is 10.

RP 4 *Bimatrix Games (1)*

(a) D is strictly dominated by $3/5 \cdot U + 2/5 \cdot M$. Next, C is strictly dominated by R.
(b) In the reduced (two by two) game, the best reply function of player 1 is: play U if player 2 puts less than probability $2/5$ on L, play M if player 2 puts more

than probability $2/5$ on L, and play any combination of U and M if player 2 puts probability $2/5$ on L. The best reply function of player 2 is: play R if player 1 puts positive probability on U, and play any combination of L and R if player 1 plays M. The set of Nash equilibria is: $\{((1,0),(0,1))\} \cup \{((0,1),(q,1-q)) \mid 1 \geq q \geq 2/5\}$.

(c) The set of Nash equilibria in the original game is: $\{((1,0,0),(0,0,1))\} \cup \{((0,1,0),(q,0,1-q)) \mid 1 \geq q \geq 2/5\}$.

RP 5 *Bimatrix Games (2)*

(a) For $x > 2$: $\{((1,0),(1,0))\}$. For $x = 2$: $\{((1,0),(1,0))\} \cup \{((p,1-p),(0,1)) \mid 0 \leq p \leq 1/2\}$. For $0 < x < 2$: $\{((1/2,1/2),((2-x)/2,x/2))\}$. For $x = 0$: $\{((0,1),(0,1))\} \cup \{((p,1-p),(1,0)) \mid 1 \geq p \geq 1/2\}$. For $x < 0$: $\{((0,1),(0,1))\}$.

(b) f is strictly dominated by $1/3 \cdot e + 2/3 \cdot g$. Next: b is strictly dominated by c, e by g, a by d. The remaining two by two game has a unique Nash equilibrium. In the original game the unique Nash equilibrium is $((0,0,4/9,5/9),(0,0,1/2,1/2))$.

RP 6 *Voting*

(a)

$$\begin{array}{c} \\ (4,0) \\ (3,1) \\ (2,2) \end{array} \begin{array}{c} \begin{array}{ccc} (4,0) & (3,1) & (2,2) \end{array} \\ \begin{pmatrix} 3/2,3/2 & 1,2 & 1,2 \\ 2,1 & 3/2,3/2 & 1,2 \\ 2,1 & 2,1 & 3/2,3/2 \end{pmatrix} \end{array}.$$

(b) By iterated elimination of strictly dominated strategies it follows that the unique Nash equilibrium in this game is $((2,2),(2,2))$. (This is a constant sum game: $(2,2)$ is the optimal strategy for each party.)

RP 7 *A Bimatrix Game*

(a) For $a \neq 0$ the unique Nash equilibrium is $((1/2,1/2),(1/2,1/2))$. For $a = 0$ the set of Nash equilibria is $\{((p,1-p),(0,1)) \mid 1 \geq p > 1/2\} \cup \{((p,1-p),(1,0)) \mid 0 \leq p < 1/2\} \cup \{((1/2,1/2),(q,1-q)) \mid 0 \leq q \leq 1\}$.

(b) The strategic form of this game is

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cccc} LL & LR & RL & RR \end{array} \\ \begin{pmatrix} a,0 & a,0 & 0,1 & 0,1 \\ 0,1 & a,0 & 0,1 & a,0 \end{pmatrix} \end{array}.$$

There are two subgame perfect equilibria in pure strategies: player 1 plays T and player 2 plays RL (i.e., R after T and L after B); and player 1 plays B and player 2 plays RL.

RP 8 *An Ice-Cream Vendor Game*

(a) There are four different situations: (i) all vendors in the same location: each gets 400; (ii) two in the same location and the third vendor in a neighboring location: the first two get 300 and the third gets 600; (iii) two in the same location and the third vendor in the opposite location: the first two get 300 and the third gets 600; and (iv) all vendors in different locations: the middle one gets 300 and the others get 450 each. From this it is easily seen that (iii) and (iv) are Nash equilibria but (i) and (ii) are not Nash equilibria.

(b) There are many subgame perfect Nash equilibria, but they can be reduced to three types: (i) player 1 chooses arbitrarily, player 2 chooses the opposite location of player 1, and player 3 chooses a remaining optimal open location; (ii) player 1 chooses arbitrarily, player 2 chooses one of the neighboring locations of player 1, and player 3 chooses the opposite location of player 2 if that is unoccupied, and otherwise the same location as player 2; (iii) player 1 chooses arbitrarily, player 2 chooses the same location as player 1, and player 3 chooses the opposite location of player 1.

RP 9 *A Repeated Game*

(a) (U,L,B) and (D,R,B).

(b) In the second period, after each action combination of the first period, one of the two equilibria in (a) has to be played.

(c) In the first period player 1 plays U, player 2 plays R, and player 3 plays A. In the second period, if the first period resulted in (U,R,A) then player 1 plays D, player 2 plays R, and player 3 plays B; in all other cases, player 1 plays U, player 2 plays L, and player 3 plays B.

(d) In the first period player 1 plays U, player 2 plays R, and player 3 plays B. In the second period, if the first period resulted in (U,R,B) then player 1 plays U, player 2 plays L, and player 3 plays B; in all other cases, player 1 plays D, player 2 plays R, and player 3 plays B.

RP 10 *Locating a Pub*

(a) Player 1 has three pure strategies and player 2 has eight pure strategies.

(b) Player 1 chooses B. Player 2 chooses B, C, B, if player 1 chooses A, B, C respectively.

(c) Player 1 has 24 pure strategies and player 2 has 8 pure strategies.

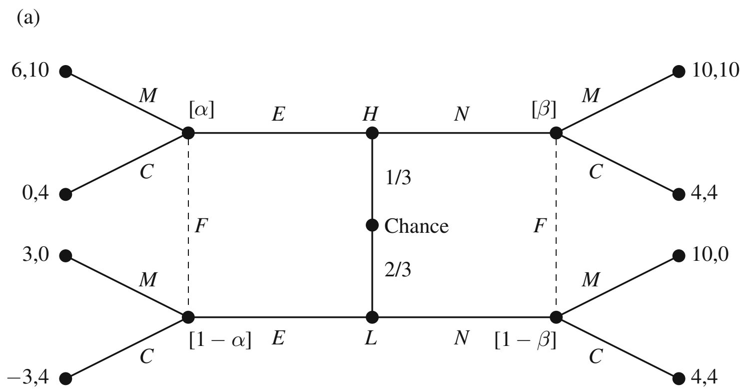
(d) (i) Player 1 plays A; after A the subgame equilibrium (B,C) is played, after B the subgame equilibrium (A,C), and after C the subgame equilibrium (A,B). (ii) Player 1 plays B; after A the subgame equilibrium (B,C) is played, after B the subgame equilibrium (C,A), and after C the subgame equilibrium (A,B). (iii) Player 1 plays C; after A the subgame equilibrium (B,C) is played, after B the subgame equilibrium (C,A), and after C the subgame equilibrium (B,A).

RP 11 *A Two-Stage Game*

(a) In G_1: (D,R); in G_2: (T,X), (M,Y), and (B,Z).
(b) Each player has $2 \cdot 3^4 = 162$ pure strategies.
(c) In G_1 player 1 plays U and player 2 plays L. In G_2 the players play as follows. If (U,L) was played, then player 1 plays M and player 2 plays Y. If (D,L) was played, then player 1 plays B and player 2 plays Z. If (U,R) was played, then player 1 plays T and player 2 plays X. If (D,R) was played, then player 1 plays M and player 2 plays Y.
(d) In the second stage (in G_1) always (U,L) has to be played. Hence, there are three subgame perfect equilibria, corresponding to the three Nash equilibria of G_2.

RP 12 *Job Market Signaling*

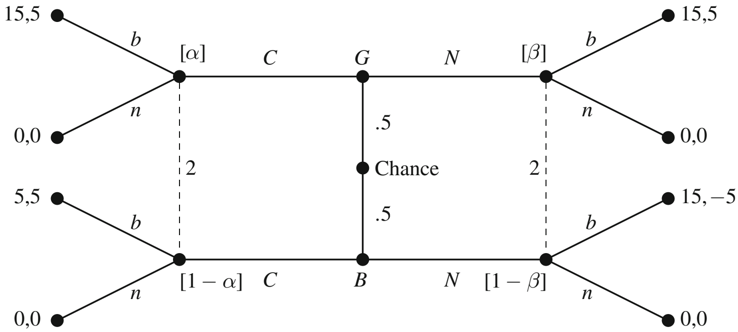
(a)



(b) The Nash equilibria are: (i) type H plays E, type L plays N, F plays M after E and C after N; (ii) both types play N, F always plays C.
(c) The equilibrium in (i) is separating with (forced) beliefs $\alpha = 1$ and $\beta = 0$. The equilibrium in (ii) is pooling with $\beta = 1/3$ (forced) and $\alpha \leq 2/5$. According to the intuitive criterion we must have $\alpha = 1$, so that the intuitive criterion is not satisfied by the latter equilibrium. (It does not apply to the first equilibrium.)

RP 13 *Second-Hand Cars (1)*

(a, b) The extensive form of this signaling game is as follows:



The strategic form is:

$$\begin{array}{c} \\ CC \\ CN \\ NC \\ NN \end{array} \begin{array}{c} \begin{array}{cccc} bb & bn & nb & nn \end{array} \\ \begin{pmatrix} 10, \underline{5} & \underline{10}, \underline{5} & 0, 0 & \underline{0}, 0 \\ \underline{15}, 0 & 7.5, \underline{2.5} & 7.5, -2.5 & \underline{0}, 0 \\ 10, \underline{5} & 2.5, 2.5 & 7.5, 2.5 & \underline{0}, 0 \\ \underline{15}, \underline{0} & 0, \underline{0} & \underline{15}, \underline{0} & \underline{0}, \underline{0} \end{pmatrix} \end{array}.$$

The Nash equilibria are: (CC, bn), (NN, bb), (NN, nb), (NN, nn).

(c) (CC, bn) is pooling with $\beta \leq 1/2$, (NN, bb) is pooling for all α. The other two equilibria are not perfect Bayesian, since player 2 will play b after C.

RP 14 *Second-Hand Cars (2)*

(a) This is a static game of incomplete information, represented by the pair G_1, G_2:

$$G_1 = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{pmatrix} 1, -1 & 0, 0 & 0, 0 \\ 0, 0 & -1, 1 & 0, 0 \\ -1, 1 & -2, 2 & -3, 3 \end{pmatrix} \end{array} \quad G_2 = \begin{array}{c} \\ 1 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{pmatrix} 3, -3 & 0, 0 & 0, 0 \\ 2, -2 & 1, -1 & 0, 0 \\ 1, -1 & 0, 0 & -1, 1 \end{pmatrix} \end{array}$$

where G_1 is played with probability 25 % and G_2 with probability 75 %. (The numbers should be multiplied by 1,000, the buyer is the row and the seller the column player.)

(b) The buyer has one type and three pure strategies, the seller has two types and nine pure strategies.

(c) Strategy "5" is strictly dominated by strategy "3".

(d) Against strategy "3" of the buyer the best reply of the seller is the combination $(3, 5)$, but against this combination the best reply of the buyer is "1".

(e) Against strategy "1" of the buyer the seller has four best replies: $(3,3)$, $(3,5)$, $(5,3)$, and $(5,5)$. In turn, (only) against $(3,5)$ and $(5,5)$ is "1" a best reply. Hence there are two Nash equilibra in pure stategies: (i) $(1,(3,5))$ and (ii) $(1,(5,5))$. No trade is going to take place.

RP 15 *Signaling Games*

(a) The strategic form with best replies underlined is:

$$\begin{array}{c} \\ LL \\ LR \\ RL \\ RR \end{array} \begin{array}{c} \begin{array}{cccc} uu & ud & du & dd \end{array} \\ \begin{pmatrix} 2,\underline{1} & \underline{2},\underline{1} & 1.5,0.5 & \underline{1.5},0.5 \\ \underline{2.5},\underline{1.5} & 1.5,1 & \underline{2},0.5 & 1,0 \\ 1,0 & 0.5,0.5 & 1,0.5 & 0.5,\underline{1} \\ 1.5,\underline{0.5} & 0,\underline{0.5} & 1.5,\underline{0.5} & 0,\underline{0.5} \end{pmatrix} \end{array}.$$

(LR, uu) is a separating perfect Bayesian equilibrium with beliefs $\alpha = 1$ and $\beta = 0$. (LL, ud) is a pooling Bayesian equilibrium with beliefs $\alpha = 1/2$ and $\beta \geq 1/2$. For the latter, the intuitive criterion requires $\beta = 0$, so that this equilibrium does not satisfy it.

(b) The strategic form with best replies underlined is:

$$\begin{array}{c} \\ LL \\ LR \\ RL \\ RR \end{array} \begin{array}{c} \begin{array}{cccc} uu & ud & du & dd \end{array} \\ \begin{pmatrix} \underline{3},\underline{1.5} & 3,\underline{1.5} & 0.5,1 & 0.5,1 \\ 2,1 & 2.5,0 & \underline{1},\underline{1.5} & 1.5,0.5 \\ 1.5,1.5 & \underline{3.5},\underline{2} & 0,0.5 & 2,1 \\ 0.5,\underline{1} & 3,0.5 & 0.5,\underline{1} & \underline{3},0.5 \end{pmatrix} \end{array}.$$

(LL, uu) is a pooling perfect Bayesian equilibrium with beliefs $\alpha = 1/2$ and $\beta \leq 2/3$. The intuitive criterion requires $\beta = 1$, so this pooling equilibrium does not satisfy it. (LR, du) is a separating perfect Bayesian equilibrium with beliefs $\alpha = 1$ and $\beta = 0$, and (RL, ud) is a separating perfect Bayesian equilibrium with beliefs $\alpha = 0$ and $\beta = 1$.

RP 16 *A Game of Incomplete Information*

(a) Start with the decision node of player 1. Player 1 has four actions/strategies: *AA*, *AB*, *BA*, *BB*. All these actions lead to one and the same information set of player 2, who has three actions/strategies: *C*, *D*, *E*.

(b) The strategic form is:

$$\begin{array}{c} \\ AA \\ AB \\ BA \\ BB \end{array} \begin{array}{c} \begin{array}{ccc} C & D & E \end{array} \\ \begin{pmatrix} 3, \underline{2} & 1.5, 1.5 & \underline{2.5}, 1.5 \\ \underline{4}, 2.5 & \underline{2.5}, \underline{3} & 1, 1.5 \\ 3, \underline{2} & 1.5, 1.5 & \underline{2.5}, 1.5 \\ \underline{4}, 2.5 & \underline{2.5}, \underline{3} & 1, 1.5 \end{pmatrix} \end{array}.$$

The Nash equilibria in pure strategies are (AB, D) and (BB, D).

(c) Player 1 has now two pure strategies, namely A and B. If player 1 plays A then the best reply of player 2 is EC. Against EC, the payoff of A is 1.5 and the payoff of B is 2.5, so that A is not a best reply against EC. Against B, the best reply of player 2 is ED. In turn, B is player 1's best reply against ED (yields 2 whereas A only yields 1). So the unique Nash equilibrium in pure strategies is (B, ED).

RP 17 *A Bayesian Game*

(a) This is the game

$$\begin{array}{c} \\ F \\ Y \end{array} \begin{array}{c} \begin{array}{cc} F & Y \end{array} \\ \begin{pmatrix} -1, 1 & 1, 0 \\ 0, 1 & 0, 0 \end{pmatrix} \end{array}$$

with (Y, F) as unique Nash equilibrium (also in mixed strategies).

(b) Start with the decision node for player 1, who has two actions/strategies: F and Y. Player 2 has a singe information set and four actions/strategies: FF, FY, YF, YY.

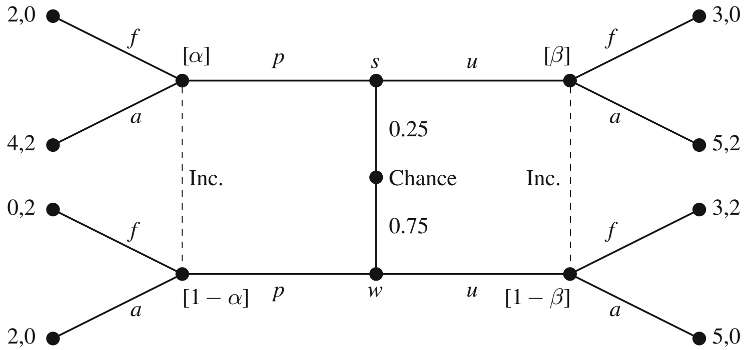
(c) The strategic form is:

$$\begin{array}{c} \\ F \\ Y \end{array} \begin{array}{c} \begin{array}{cccc} FF & FY & YF & YY \end{array} \\ \begin{pmatrix} 1-2\alpha, 2\alpha-1 & 1-2\alpha, \alpha & 1, \alpha-1 & 1, 0 \\ 0, 1 & 0, \alpha & 0, 1-\alpha & 0, 0 \end{pmatrix} \end{array}.$$

For $\alpha = 0$ the Nash equilibria in pure strategies are (F, FY) and (F, YY). For $0 < \alpha < 1/2$: (F, FY). For $\alpha = 1/2$: (F, FY) and (Y, FF). For $1/2 < \alpha < 1$: (Y, FF). For $\alpha = 1$: (Y, FF) and (Y, FY).

RP 18 *Entry as a Signaling Game*

(a) The extensive form of this signaling game is:



(b, c) The strategy combination (pu, af) (strong type p, incumbent a after p) is a Nash equilibrium. It is a separating perfect Bayesian equilibrium for $\alpha = 1$ and $\beta = 0$. Also (uu, ff) is a Nash equilibrium. It is pooling perfect Bayesian for $\beta = 1/2$ and $\alpha \leq 1/2$. It does not satisfy the intuitive criterion since that requires $\alpha = 1$.

RP 19 *Bargaining (1)*

(a) Player 1 has only one type. Player 2 has infinitely many types, namely each $v \in [0, 1]$ is a possible type of player 2. A typical strategy of player 1 consists of a price $p_1 \in [0, 1]$ and a yes/no decision depending on the price p_2 of player 2 if that player rejects p_1—in principle, the yes/no decision may also depend on p_1.
(b) A typical strategy of player 2 consists, for every type $v \in [0, 1]$, of a yes/no decision depending on the price p_1 asked by player 1 and a price p_2 in case the decision was 'no'. In principle, p_2 may also depend on p_1 (not only via the yes/no decision).
(c) Player 2 accepts if $v - p_1 \geq \delta v$ (noting that he can offer $p_2 = 0$ if he does not accept the price p_1 of player 1); rejects and offers $p_2 = 0$ if $v - p_1 < \delta v$.
(d) Using (c) player 1 asks the price p_1 that maximizes $p_1 \cdot Pr[p_1 \leq (1 - \delta)v]$, i.e., his expected payoff—note that his payoff is 0 if player 2 rejects. Hence, player 1 solves $\max_{p_1 \in [0,1]} p_1 \cdot [1 - p_1/(1 - \delta)]$, which has solution $p_1 = (1 - \delta)/2$. So the equilibrium is, that player 1 asks this price and accepts any price of player 2; and player 2 accepts any price at most $(1 - \delta)/2$, and rejects any higher price and then offers $p_2 = 0$.

RP 20 *Bargaining (2)*

(a) The (Pareto) boundary of the feasible set consists of all pairs $(x, 1-x^2)$ for $x \in [0,1]$.

(b) The Nash bargaining solution outcome is found by maximizing the expression $x(1-x^2)$ over all $x \in [0,1]$. The solution is $((1/3)\sqrt{3}, 2/3)$. In distribution of the good: $((1/3)\sqrt{3}, 1-(1/3)\sqrt{3})$.

(c, d) Let $(x, 1-x^2)$ be the proposal of player 1 and $(y, 1-y^2)$ that of player 2. Then the equations $1-x^2 = \delta(1-y^2)$ and $y = \delta x$ hold for the Rubinstein outcome. This results in $x = 1/\sqrt{1+\delta+\delta^2}$; taking the limit for $\delta \to 1$ gives $(1/3)\sqrt{3}$, which is indeed the Nash bargaining solution outcome for player 1.

RP 21 *Bargaining (3)*

(a) Player 1 proposes $(1-\delta+(1/2)\delta^2, \delta-(1/2)\delta^2)$ at $t=0$ and player 2 accepts. Note that $1-\delta+(1/2)\delta^2 > \delta-(1/2)\delta^2$, so the beginning player has an advantage.

(b) If the utility function of player 2 were the same as that of player 1, then the Nash bargaining solution would result in equal split. This is still the case if player 2's utility function is multiplied by 2, as is the case here: the maximum of $u(x)\cdot 2u(1-x)$ is attained at the same point as the maximum of $u(x)\cdot u(1-x)$. So the division of the good is $(1/2, 1/2)$. In terms of utilities, this gives $(u(1/2), 2u(1/2))$. (The Nash bargaining solution is symmetric, Pareto optimal, and scale covariant: see Chap. 10.)

RP 22 *Ultimatum Bargaining*

(a) Player 1 chooses an action/strategy $(1-m, m)$. Player 2 decides for each strategy of player 1 whether to accept or reject the offer. If he accepts, the payoffs are $(1-m, m+a(2m-1))$, otherwise the payoffs are $(0,0)$.

(b) Player 1 proposes $(1-a/(1+2a), a/(1+2a))$, and player 2 accepts $(1-m, m)$ if and only if $m \geq a/(1+2a)$. Hence, the outcome is $(1-a/(1+2a), a/(1+2a))$.

(c) If a becomes large, then this outcome converges to equal split: this is because then player 2 cares mainly about the division and not so much about what he gets.

RP 23 *An Auction (1)*

(a) The game has imperfect but complete information.

(b) The unique Nash equilibrium is each bidder bidding $v_1 = v_2$.

(c) There is no Nash equilibrium.
(d) The associated bimatrix game is:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/2, 3/2 & 0, 2 & 0, 1 & 0, 0 \\ 0, 0 & 0, 1 & 0, 1 & 0, 0 \\ -1, 0 & -1, 0 & -1/2, 1/2 & 0, 0 \\ -2, 0 & -2, 0 & -2, 0 & -1, 0 \end{pmatrix} \end{array}.$$

The Nash equilibria are $(0, 1)$, $(1, 1)$, and $(1, 2)$.

RP 24 *An Auction (2)*

(a) Let $b_i < v_i$. If b_i wins then v_i is equally good. If b_i loses and the winning bid is below v_i then v_i is a strict improvement. If b_i loses and the winning bid is at least v_i then v_i is at least as good. If, on the other hand, $b_i > v_i$, then, if b_i wins, the fourth-highest bid is below v_i and the second highest bid is above v_i, then bidding v_i results in zero instead of positive payoff.
(b) For instance, player 2 can improve by any bid above v_1.
(c) All bidders bid $\tilde{v}$ where $\tilde{v} \in [v_2, v_1]$.

RP 25 *An Auction (3)*

(a) The best reply function b_2 of player 2 is given by: $b_2(0) = \{1\}$, $b_2(1) = \{2\}$, $b_2(2) = \{3\}$, $b_2(3) = b_2(4) = \{0, \ldots, 4\}$, $b_2(5) = \{0, \ldots, 5\}$, $b_2(6) = \{0, \ldots, 6\}$. The best reply function b_1 of player 1 is given by: $b_1(0) = \{0\}$, $b_1(1) = \{1\}$, $b_1(2) = \{2\}$, $b_1(3) = \{3\}$, $b_1(4) = \{4\}$, $b_1(5) = \{5\}$, $b_1(6) = \{0, \ldots, 6\}$.
(b) The Nash equilibria are: $(3, 3)$, $(4, 4)$, $(5, 5)$, and $(6, 6)$.

RP 26 *Quantity Versus Price Competition*

(a) The profit functions are $q_1(4 - 2q_1 - q_2)$ and $q_2(4 - q_1 - 2q_2)$ respectively (or zero in case an expression is negative). The first-order conditions (best reply functions) are $q_1 = (4 - q_2)/4$ and $q_2 = (4 - q_1)/4$ (or zero) and the equilibrium is $q_1 = q_2 = 4/5$ with associated prices equal to $8/5$ and profits equal to $32/25$.
(b) Follows by substitution.
(c, d) The profit functions are $(1/3)p_1(p_2 - 2p_1 + 4)$ and $(1/3)p_2(p_1 - 2p_2 + 4)$ (or zero) respectively. The first-order conditions (best reply functions) are $p_1 = (p_2 + 4)/4$ and $p_2 = (p_1 + 4)/4$. The equilibrium is $p_1 = p_2 = 4/3$ with associated quantities $q_1 = q_2 = 8/9$ and profits equal to $32/27$. Price competition is tougher.

RP 27 *An Oligopoly Game (1)*

(a, b) Player 1 chooses $q_1 \geq 0$. Players 2 and 3 then choose q_2 and q_3 simultaneously, depending on q_1. The best reply functions of players 2 and 3 in the subgame following q_1 are $q_2 = (a-q_1-q_3-c)/2$ and $(a-q_1-q_2-c)/2$ (or zero), and the equilibrium in the subgame is $q_2 = q_3 = (a-q_1-c)/3$. Player 1 takes this into account and maximizes $q_1(a-c-q_1-2(a-q_1-c)/3)$, which gives $q_1 = (a-c)/2$. Hence, the subgame perfect equilibrium is: player 1 plays $q_1 = (a-c)/2$; players 2 and 3 play $q_2 = q_3 = (a-q_1-c)/3$. The outcome is player 1 playing $(a-c)/6$ and players 2 and 3 playing $(a-c)/6$.

RP 28 *An Oligopoly Game (2)*

(a) The best-reply functions are $q_1 = (10-q_2-q_3)/2$, $q_2 = (10-q_1-q_3)/2$, $q_3 = (9-q_1-q_2)/2$.
(b) The equilibrium is $q_1 = q_2 = 11/4$, $q_3 = 7/4$.
(c) To maximize joint profit, $q_3 = 0$ and $q_1 + q_2 = 5$. (This follows by using intuition: firm 3 has higher cost, or by solving the problem as a maximization problem under nonnegativity constraints.)

RP 29 *A Duopoly Game with Price Competition*

(a) The monopoly price of firm 1 is $p_1 = 65$ and the monopoly price of player 2 is $p_2 = 75$.
(b)

$$p_1(p_2) = \begin{cases} \{x \mid x > p_2\} & \text{if } p_2 < 30 \\ \{x \mid x \geq 30\} & \text{if } p_2 = 30 \\ \{31\} & \text{if } p_2 = 31 \\ \{p_2 - 1\} & \text{if } p_2 \in [32, 65] \\ \{65\} & \text{if } p_2 \geq 66 \end{cases} \qquad p_2(p_1) = \begin{cases} \{x \mid x > p_1\} & \text{if } p_1 < 50 \\ \{x \mid x \geq 50\} & \text{if } p_1 = 50 \\ \{51\} & \text{if } p_1 = 51 \\ \{p_1 - 1\} & \text{if } p_1 \in [52, 75] \\ \{75\} & \text{if } p_1 \geq 76 \end{cases}$$

(c) $(p_1, p_2) = (31, 32)$.
(d) $(p_1, p_2) = (50, 51)$.

RP 30 *Contributing to a Public Good*

(a) The Nash equilibria in pure strategies are all strategy combinations where exactly two persons contribute.
(b) The expected payoff of contributing is equal to $-3 + 8(1-(1-p)^2)$, which in turn is equal to $16p - 8p^2 - 3$.
(c) A player should be indifferent between contributing or not if the other two players contribute, hence $16p - 8p^2 - 3 = 8p^2$. This holds for $p = 1/4$ and for $p = 3/4$.

RP 31 *A Demand Game*

(a) Not possible: each player can gain by raising his demand by 0.1. (b) Not possible: at least one player has $x_i > 0.2$ and can gain by decreasing his demand by 0.2. (c) The unique Nash equilibrium is $(0.5, 0.5, 0.5)$. (d) A Nash equilibrium is for instance $(0.6, 0.6, 0.6)$.

(e) All triples with sum equal to one, and all triples such that the sum of each pair is at least one.

RP 32 *A Repeated Game (1)*

(a) The unique Nash equilibrium in the stage game is $((2/3, 1/3), (1/2, 1/2))$, with payoffs $(8, 22)$. Therefore, all payoffs pairs in the quadrangle with vertices $(16, 24)$, $(0, 25)$, $(0, 18)$, and $(16, 16)$ which are strictly larger than $(8, 22)$, as well as $(8, 22)$, can be reached as long run average payoffs in a subgame perfect equilibrium in the repeated game, for suitable choices of δ.

(b) Write $G = (A, B)$, then $v(A) = 8$ and $-v(-B) = 18$. Therefore, all payoffs pairs in the quadrangle with vertices $(16, 24)$, $(0, 25)$, $(0, 18)$, and $(16, 16)$ which are strictly larger than $(8, 20)$, can be reached as long run average payoffs in a Nash equilibrium in the repeated game, for suitable choices of δ.

(c) The players alternate between (T, L) and (B, R). Player 1 has no incentive to deviate, but uses the eternal punishment strategy B to keep player 2 from deviating. Player 2 will not deviate provided

$$25 + 18\delta/(1-\delta) \leq 24/(1-\delta^2) + 16\delta/(1-\delta^2)$$

and

$$18 + 18\delta/(1-\delta) \leq 16/(1-\delta^2) + 24\delta/(1-\delta^2)\ .$$

The first inequality is satisfied if δ is at least (approximately) 0.55, and the second inequality if $\delta \geq 1/3$. Hence, this is a Nash equilibrium for $\delta \geq 0.55$. It is not subgame perfect since player 2 can obtain 22 by playing the stage game equilibrium strategy.

RP 33 *A Repeated Game (2)*

(a) (D, C), (D, R), and (M, R).

(b) Let $((p_1, p_2, p_3), (q_1, q_2, q_3))$ be a Nash equilibrium. First consider the case $q_3 < 1$. Then $p_1 = 0$ and therefore $q_1 = 0$. If $p_2 > 0$ then $q_2 = 0$ and $q_3 = 1$, a contradiction. Hence, $p_2 = 0$, and then $p_3 = 1$. We obtain the set of Nash equilibria $\{((0, 0, 1), (0, q_2, q_3)) \mid q_2, q_3 \geq 0, q_2 + q_3 = 1, q_3 < 1\}$.

Next, consider the case $q_3 = 1$. Then $9p_1 + p_2 + 4p_3 \leq p_1 + 2p_2 + 4p_3$, hence $8p_1 \leq p_2$. We obtain another set of Nash equilibria $\{((p_1, p_2, p_3), (0, 0, 1)) \mid p_1 \geq 0, 8p_1 \leq p_2, p_1 + p_2 = 1\}$.

(c) Each player has $3 \times 3^9 = 3^{10}$ pure strategies. In the first stage the players play (U, L) and in the second stage they play (for instance) according to the table

$$\begin{array}{c} \\ U \\ M \\ D \end{array} \begin{array}{c} \begin{array}{ccc} L & C & R \end{array} \\ \begin{pmatrix} D,R & M,R & D,R \\ D,C & D,R & D,R \\ D,C & D,R & D,R \end{pmatrix} \end{array} .$$

(d) Always play (U, L) but after a deviation by player 1, player 2 reverts to C forever, to which player 1 replies by D, and after a deviation by player 2, player 1 reverts to M forever, to which player 2 replies by R. This is a subgame perfect equilibrium provided that

$$10 + 2\delta/(1-\delta) \leq 8/(1-\delta) \Leftrightarrow \delta \geq 1/4$$

and

$$9 + 2\delta/(1-\delta) \leq 8/(1-\delta) \Leftrightarrow \delta \geq 1/7$$

hence if $\delta \geq 1/4$.

RP 34 *A Repeated Game (3)*

(a) (D, L), (U, R), and (D, R).
(b) The second row and next the second column can be deleted by iterated elimination of strictly dominated strategies. This results in the sets of Nash equilibria $\{((0, 0, 1), (q_1, 0, q_3)) \mid q_1, q_3 \geq 0, q_1 + q_3 = 1\}$ and $\{((p_1, 0, p_3), (0, 0, 1)) \mid p_1, p_3 \geq 0, p_1 + p_3 = 1\}$.
(c) In the first stage the players play (M, C) and in the second stage they play (for instance) according to the table

$$\begin{array}{c} \\ U \\ M \\ D \end{array} \begin{array}{c} \begin{array}{ccc} L & C & R \end{array} \\ \begin{pmatrix} D,R & D,L & D,R \\ U,R & D,R & D,R \\ D,R & D,L & D,R \end{pmatrix} \end{array} .$$

(d) Always play (M, C) but after a deviation by player 1 player 2 reverts to L forever, to which player 1 replies by D, and after a deviation by player 2 player 1 reverts to U, to which player 2 replies by R. This is a subgame perfect equilibrium provided that

$$12 + \delta/(1-\delta) \leq 10/(1-\delta)$$

which holds for $\delta \geq 2/11$.

RP 35 *A Repeated Game (4)*

(a) Player 1 plays B and player 2 plays L in both stages.
(b) They play (T, L) in the first stage. If player 1 would deviate to B, then player 2 plays R in the second stage, otherwise L. Player 1 plays B in the second stage.
(c) Since (B, L) is the unique Nash equilibrium in the stage game and there are no payoff pairs better for both players, the only possibility is that player 1 plays B and player 2 plays L forever. This is a subgame perfect equilibrium for any value of δ, with long run average payoffs $(5, 5)$.

RP 36 *A Repeated Game (5)*

(a) Only (T, L).
(b) The payoff pair $(2, 1)$, and all payoff pairs larger for both players in the triangle with vertices $(5, 0)$, $(0, 6)$, and $(1, 1)$.
(c) At even times play (B, L) and at odd times play (T, R). After a deviation revert to T (player 1) and L (player 2) forever. This is a subgame perfect Nash equilibrium provided that

$$2 + 2\delta/(1-\delta) \leq 5\delta/(1-\delta^2)$$

and

$$1 + \delta/(1-\delta) \leq 6\delta/(1-\delta^2)$$

which is equivalent to $\delta \geq \max\{2/3, 1/5\} = 2/3$.

RP 37 *An Evolutionary Game*

(a) The species consists of $100p\%$ animals of type C and $100(1-p)\%$ animals of type D.
(b) $\dot{p} = p(0p + 2(1-p) - 2p(1-p) - 3(1-p)p - (1-p)^2)$ which after simplification yields $\dot{p} = 4p(p-1)(p-1/4)$. Hence the rest points are $p = 0, 1/4, 1$ and $p = 1/4$ is stable.
(c) The unique symmetric Nash equilibrium strategy is $(1/4, 3/4)$. One has to check that $(1/4, 3/4)A(q, 1-q) > (q, 1-q)A(q, 1-q)$ for all $q \neq 1/4$, which follows readily by computation.

RP 38 *An Apex Game*

(a) Suppose $(x_1, \ldots, x_5)$ is in the core. Since $x_1 + x_2 \geq 1$, and all x_i are nonnegative and sum to one, we must have $x_3 = x_4 = x_5 = 0$. Similarly, $x_2 = 0$, but this contradicts $x_2 + \ldots + x_5 \geq 1$. So the core is empty.
(b) $\Phi_2(v) = 1!3!/5! + 3!1!/5! = 1/10$, hence $\Phi(v) = (6/10, 1/10, 1/10, 1/10, 1/10)$.

(c) Let $(1-4a, a, a, a, a)$ be the nucleolus of this game. The relevant (maximal) excesses to consider are $1-(1-4a)-a = 3a$ (e.g., $\{1,2\}$) and $1-4a$ ($\{2,\ldots,5\}$). Equating these yields $a = 1/7$.

RP 39 *A Three-person Cooperative Game (1)*

(a) For $a > 10$ the core is empty. For $a = 10$, a core element is for instance $(0, 5, 5)$. Hence, $a \leq 10$.
(b) The Shapley value is $((25-2a)/6, (19+a)/6, (16+a)/6)$. By writing down the core constraints, it follows that this is in the core for $-13 \leq a \leq 8.75$.
(c) At this vector, the excesses of the three two-player coalitions are equal, namely to $(a-14)/3$. For this tobe the nucleolus we need that the excesses of the one-person coalitions are not larger than this, i.e.,

$$(2a-16)/3 \leq (a-14)/3, (-a-4)/3 \leq (a-14)/3, (-a-7)/3 \leq (a-14)/3$$

and it is straightforward to check that this is true for no value of a.

RP 40 *A Three-person Cooperative Game (2)*

(a) The core is nonempty for $a \leq 1$. In that case, the core is the quadrangle (or line segment if $a = 1$) with vertices $(1, 2, 2)$, $(a, 2, 3-a)$, $(1, 1, 3)$, and $(a, 2-a, 3)$.
(b) The Shapley value is $((2a+7)/6, (10-a)/6, (13-a)/6)$, which is in the core for $-2 \leq a \leq -1/2$.
(c) By equating the excesses of the two-person coalitions we obtain the vector $(2/3, 5/3, 8/3)$ with excess $-1/3$. This is the nucleolus if $a - 2/3 \leq -1/3$, hence if $a \leq 1/3$.

RP 41 *Voting*

(a) The winning coalitions (omitting set braces) are *AB*, *AC*, *ABC*, *ABD*, *ACD*, *ABCD*, and *BCD*. Then $\Phi_A(v) = 1!2!/4! + 1!2!/4! + 2!1!/4! + 2!1!/4! + 2!1!/4! = 5/12$. Similarly, one computes the other values to obtain $\Phi(v) = (1/12)(5, 3, 3, 1)$. (In fact, it is sufficient to compute $\Phi_B(v)$ and $\Phi_C(v)$.)
(b) $p_A = 5, p_B = 3, p_C = 3, p_D = 1$; $\beta(A) = 5/12$, $\beta(B) = 3/12$, $\beta(C) = 3/12$, $\beta(D) = 1/12$.
(c) The winning coalitions are *AB*, *AC*, *ABC*. The Shapley value is $(2/3, 1/6, 1/6)$. Further, $p_A = 3, p_B = p_C = 1$; $\beta(A) = 3/5$, $\beta(B) = \beta(C) = 1/5$.

RP 42 *An Airport Game*

(a) $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = c_1$, $v(23) = c_2$, and $v(N) = c_1 + c_2$.
(b) The core is the quadrangle with vertices $(c_1, c_2, 0)$, $(0, c_2, c_1)$, $(0, c_1, c_2)$, and $(c_1, 0, c_2)$.

(c) $\Phi(v) = (1/6)(4c_1, 3c_2 + c_1, 3c_2 + c_1)$. This is a core element (check the constraints).
(d) The nucleolus is of the form $(a, (c_1 + c_2 - a)/2, (c_1 + c_2 - a)/2)$. By equating the excesses of the two-person coalitions it follows that $a = (3c_1 - c_2)/3$, hence the nucleolus would be $((3c_1 - c_2)/3, 2c_2/3, 2c_2/3)$ and the excess of the two-person coalitions is then $-c_2/3$. We need that the excesses of the one-person coalitions are not larger, that is, $-(3c_1 - c_2)/3 \leq -c_2/3$ and $-(2/3)c_2 \leq -c_2/3$. This results in the condition $c_1 \geq 2c_2/3$.

RP 43 *A Glove Game*

(a) By straightforward computation, $\Phi(v) = (1/60)(39, 39, 14, 14, 14)$: note that it is sufficient to compute one of these values.
(b) $C(v) = \{(1, 1, 0, 0, 0)\}$.
(c) By (b) and the fact that the nucleolus is in the core whenever the core is nonempty, the nucleolus is $(1, 1, 0, 0, 0,)$.

RP 44 *A Four-Person Cooperative Game*

(a) $C(v) = \{\mathbf{x} \in \mathbb{R}^4 \mid x_i \geq 0\, \forall i, x_1 + x_2 = x_3 + x_4 = 2, x_1 + x_3 \geq 3\}$. In the intended diagram, the core is a triangle with vertices (2, 1), (2, 2), and (1, 2).
(b) $\Phi(v) = (1/4)(5, 3, 5, 3)$ (it is sufficient to compute one of these values).

RP 45 *A Matching Problem*

(a) The resulting matching is (x_1, y_4), (x_2, y_3), (x_3, y_2), (x_4, y_1).
(b) The resulting matching is (x_1, y_4), (x_2, y_3), (x_3, y_1), (x_4, y_2).
(c) x_1 prefers y_4 over y_1 and y_4 prefers x_1 over y_4.
(d) In any core matching, x_2 and y_3 have to be paired, since they are each other's top choices. Given this, x_1 and y_4 have to be paired. This leaves only the two matchings in (a) and (b).

RP 46 *House Exchange*

(a) There are two core allocations: $1 : h_1, 2 : h_3, 3 : h_2$ and $1 : h_2, 2 : h_3, 3 : h_1$.
(b) The unique top trading cycle is 1, 2, 3, with allocation $1 : h_2, 2 : h_3, 3 : h_1$.
(c) Take preference h_1, h_2, h_3 with unique core allocation $1 : h_1, 2 : h_3, 3 : h_2$.

RP 47 *A Marriage Market*

(a) m_1 must be paired to his favorite woman in the core. Next, m_2 must be paired to his favorite of the remaining women, etc.
(b) (m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4).
(c) (m_1, w_4), (m_2, w_3), (m_3, w_2), (m_4, w_1).

(d) (m_1, w_2), (m_2, w_1), (m_3, w_3), (m_4, w_4) (one can reason about this but also just try the six possibilities).

Reference

Myers, S. C., & Majluf, N. S. (1984). Corporate financing and investment decisions when firms have information that investors do not have. *Journal of Financial Economics, 13*, 187–221.