1 Introduction

The best introduction to game theory is by way of examples. This chapter starts with a global definition in Sect. 1.1, collects some historical facts in Sect. 1.2, and presents examples in Sect. 1.3. Section 1.4 briefly comments on the distinction between cooperative and noncooperative game theory.

1.1 A Definition

Game theory studies situations of competition and cooperation between several involved parties by using mathematical methods. This is a broad definition but it is consistent with the large number of applications. These applications range from strategic questions in warfare to understanding economic competition, from economic or social problems of fair distribution to behavior of animals in competitive situations, from parlor games to political voting systems—and this list is certainly not exhaustive.

Game theory is an official mathematical discipline (American Mathematical Society Classification code 91A) but it is developed and applied mostly by economists. In economics, articles and books on game theory and applications are found in particular under the Journal of Economic Literature codes C7x. The list of references at the end of this book contains many textbooks and other books on game theory.

1.2 Some History

In terms of applications, game theory is a broad discipline, and it is therefore not surprising that game-theoretic situations can be recognized in the Bible (Brams, 1980) or the Talmud (Aumann and Maschler, 1985). Also the literature on strategic

© Springer-Verlag Berlin Heidelberg 2015

H. Peters, *Game Theory*, Springer Texts in Business and Economics,
DOI 10.1007/978-3-662-46950-7_1

warfare contains many situations that could have been modelled using game theory: a very early reference, over 2,000 years old, is the work of the Chinese warrior-philosopher Sun Tzu (1988). Early works dealing with economic problems are Cournot (1838) on quantity competition and Bertrand (1883) on price competition. Some of the work of Dodgson—better known as Lewis Carroll, the writer of *Alice's Adventures in Wonderland*—is an early application of zero-sum games to the political problem of parliamentary representation, see Dodgson (1884) and Black (1969).

One of the first formal works on game theory is Zermelo (1913). The logician Zermelo proved that in the game of chess either White has a winning strategy (i.e., can always win), or Black has a winning strategy, or each player can always enforce a draw—see Sect. 13.2.5. Up to the present, however, it is still not known which of these three cases is the true one. A milestone in the history of game theory is Von Neumann (1928). In this article von Neumann proved the famous minimax theorem for zero-sum games. This work was the basis for the book *Theory of Games and Economic Behavior* by von Neumann and Morgenstern (1944/1947), by many regarded as the starting point of game theory. In this book the authors extended von Neumann's work on zero-sum games and laid the groundwork for the study of cooperative (coalitional) games. See Dimand and Dimand (1996) for a comprehensive study of the history of game theory up to 1945.

The title of the book of von Neumann and Morgenstern reveals the intention of the authors that game theory was to be applied to economics. Nevertheless, in the 1950s and 1960s the further development of game theory was mainly the domain of mathematicians. Seminal articles in this period were the papers by Nash[1] on Nash equilibrium (Nash, 1951) and on bargaining (Nash, 1950), and Shapley on the Shapley value and the core for games with transferable utility (Shapley, 1953, 1967). See also Bondareva (1962) on the core. Apart from these articles, the foundations of much that was to follow later were laid in the contributed volumes edited by Kuhn and Tucker (1950, 1953), Dresher et al. (1957), Luce and Tucker (1958), and Dresher et al. (1964). A classical work in game theory is Luce and Raiffa (1957): many examples still used in game theory can be traced back to this source, like the Prisoners' Dilemma and the Battle of the Sexes.

In the late 1960s and 1970s of the previous century game theory became accepted as a new formal language for economics in particular. This development was stimulated by the work of Harsanyi (1967/1968) on modelling games with incomplete information and Selten (1965, 1975) on (sub)game perfect Nash equilibrium.

In 1994, Nash, Harsanyi and Selten jointly received the Nobel prize in economics for their work in game theory. Since then, many Nobel prizes in economics have been awarded for achievements in game theory or closely related to game theory: Mirrlees and Vickrey (in 1996), Sen (in 1998), Akerlof, Spence and Stiglitz (in 2001), Aumann and Schelling (in 2005), Hurwicz, Maskin and Myerson (in 2007), and Roth and Shapley (in 2012).

[1] See Nasar (1998) for a biography, and the later movie with the same title *A Beautiful Mind*.

From the 1980s on, large parts of economics have been rewritten and further developed using the ideas, concepts and formal language of game theory. Articles on game theory and applications can be found in many economic journals. Journals explicitly focusing on game theory include the *International Journal of Game Theory*, *Games and Economic Behavior*, and *International Game Theory Review*. Game theorists are organized within the *Game Theory Society*, see http://www.gametheorysociety.org/.

1.3 Examples

Every example in this section is based on a *story*. Each time this story is presented first and, next, it is translated into a formal mathematical *model*. Such a mathematical model is an alternative description, capturing the essential ingredients of the story with the omission of details that are considered unimportant: the mathematical model is an abstraction of the story. After having established the model, we spend some lines on how to *solve* it: we say something about how the players should or would act. In more philosophical terms, these solutions can be normative or positive in nature, or somewhere in between, but often such considerations are left as food for thought for the reader. As a general remark, a basic distinction between optimization theory and game theory is that in optimization it is usually clear when some action or choice is optimal, whereas in game theory we deal with human (or, more generally, animal) behavior and then it may be less clear when an action is optimal or even what optimality means.[2]

Each example is concluded by further *comments*, possibly including a short preview on the treatment of the exemplified game in the book. The examples are grouped in subsections on zero-sum games, nonzero-sum games, extensive form games, cooperative games, and bargaining games.

1.3.1 Zero-Sum Games

The first example is based on a military situation staged in World War II.

1.3.1.1 The Battle of the Bismarck Sea

Story The game is set in the South-Pacific in 1943. The Japanese admiral Imamura has to transport troops across the Bismarck Sea to New Guinea, and the American admiral Kenney wants to bomb the transport. Imamura has two possible choices: a short Northern route (2 days) or a long Southern route (3 days), and Kenney must choose one of these routes to send his planes to. If he chooses the wrong route he

[2]Feyerabend's (1974) 'anything goes' adage reflects a workable attitude in a young science like game theory.

can call back the planes and send them to the other route, but the number of bombing days is reduced by 1. We assume that the number of bombing days represents the payoff to Kenney in a positive sense and to Imamura in a negative sense.

Model The Battle of the Bismarck Sea problem can be modelled using the following table:

$$\begin{array}{c} \\ \text{North} \\ \text{South} \end{array} \begin{array}{c} \begin{array}{cc} \text{North} & \text{South} \end{array} \\ \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \end{array} .$$

This table represents a game with two players, namely Kenney and Imamura. Each player has two possible choices; Kenney (player 1) chooses a row, Imamura (player 2) chooses a column, and these choices are to be made independently and simultaneously. The numbers represent the payoffs to Kenney. For instance, the number 2 up left means that if Kenney and Imamura both choose North, the payoff to Kenney is 2 and the payoff to Imamura is -2. Thus, the convention is to let the numbers denote the payments *from* player 2 (the column player) *to* player 1 (the row player). This game is an example of a *zero-sum game* because the sum of the payoffs is always equal to zero.

Solution In this particular example, it does not seem difficult to predict what will happen. By choosing North, Imamura is always at least as well off as by choosing South, as is easily inferred from the above table of payoffs. So it is safe to assume that Imamura chooses North, and Kenney, being able to perform this same kind of reasoning, will then also choose North, since that is the best reply to the choice of North by Imamura. Observe that this game is easy to analyze because one of the players (Imamura) has a dominated choice, namely South: no matter what the opponent (Kenney) decides to do, North is at least as good as South, and sometimes better.

Another way to look at this game is to observe that the payoff 2 resulting from the combination (North, North) is maximal in its column ($2 \geq 1$) and minimal in its row ($2 \leq 2$). Such a position in the matrix is called a *saddlepoint*. In such a saddlepoint, neither player has an incentive to deviate unilaterally. (As will become clear later, this implies that the combination (North, North) is a *Nash equilibrium*.) Also observe that, in such a saddlepoint, the row player maximizes his minimal payoff (because $2 = \min\{2, 2\} \geq 1 = \min\{1, 3\}$), and the column player (who has to pay according to our convention) minimizes the maximal amount that he has to pay (because $2 = \max\{2, 1\} \leq 3 = \max\{2, 3\}$). The resulting payoff of 2 from player 2 to player 1 is called the *value* of the game.

Comments Two-person zero-sum games with finitely many choices, like the one above, are also called *matrix games* since they can be represented by a single matrix.

Matrix games are studied in Chaps. 2 and 12. The combination (North, North) in the example above corresponds to what happened in reality back in 1943.

1.3.1.2 Matching Pennies

Story In the two-player game of *matching pennies*, both players have a coin and simultaneously show heads or tails. If the coins match, player 2 gives his coin to player 1; otherwise, player 1 gives his coin to player 2.

Model This is a zero-sum game with payoff matrix

$$\begin{array}{c} \\ \text{Heads} \\ \text{Tails} \end{array} \begin{array}{c} \begin{array}{cc} \text{Heads} & \text{Tails} \end{array} \\ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}.$$

Solution Observe that in this game no player has a dominated choice: Heads can be better or worse than Tails, depending on the choice of the opponent. Also, there is no *saddlepoint*: there is no position in the matrix at which there is simultaneously a minimum in the row and a maximum in the column. Thus, there does not seem to be a natural way to solve the game. One way to overcome this difficulty is by allowing players to randomize between their choices: player 1 chooses Heads with a certain probability p and Tails with probability $1 - p$, and player 2 chooses Heads and Tails with probabilities q and $1 - q$ respectively. From considerations of symmetry, a good guess would be to suppose that player 1 chooses Heads or Tails both with probability $\frac{1}{2}$. As above, suppose that player 2 plays Heads with probability q and Tails with probability $1 - q$. In that case the expected payoff for player 1 is equal to

$$\frac{1}{2}[q \cdot 1 + (1-q) \cdot -1] + \frac{1}{2}[q \cdot -1 + (1-q) \cdot 1]$$

which is independent of q, namely, equal to 0. So by randomizing in this way between his two choices, player 1 can guarantee to obtain 0 in expectation (of course, the actually realized outcome is always $+1$ or -1). Analogously, player 2, by playing Heads or Tails each with probability $\frac{1}{2}$, can guarantee to pay 0 in expectation. Thus, the amount of 0 plays a role similar to that of a saddlepoint. Again, we will say that 0 is the *value* of this game.

Comments The randomized choices of the players are usually called *mixed strategies*. Randomized choices are often interpreted as *beliefs* of the other player(s) about the choice of the player under consideration. See also Sect. 3.1.

1.3.2 Nonzero-Sum Games

1.3.2.1 Prisoners' Dilemma

Story Two prisoners (players 1 and 2) have committed a crime together and are interrogated separately. Each prisoner has two possible choices: he may 'cooperate' (C) which means 'not betray his partner' or he may 'defect' (D), which means 'betray his partner'. The punishment for the crime is 10 years of prison. Betrayal yields a reduction of 1 year for the defector (traitor). If a prisoner is not betrayed, he is convicted to 1 year for a minor offense.

Model This situation can be summarized as follows:

$$\begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} -1,-1 & -10,0 \\ 0,-10 & -9,-9 \end{pmatrix} \end{array} .$$

This table is read in the same way as before, but now there are two payoffs at each position: by convention the first number is the payoff for player 1 (the row player) and the second number is the payoff for player 2 (the column player). Observe that the game is no longer zero-sum, and we have to write down both numbers at each matrix position.

Solution For both players C is a strictly dominated choice: D is better than C, whatever the other player does. So it is natural to argue that the outcome of this game will be the pair of choices (D, D), leading to the payoffs $-9, -9$. Thus, due to the existence of strictly dominated choices, the Prisoners' Dilemma game is easy to analyze.

Comments The payoffs $(-9, -9)$ are inferior: they are not *Pareto optimal*, the players could obtain the higher payoff of -1 for each by cooperating, i.e., both playing C. There is a large literature on how to establish cooperation, e.g. by reputation effects in a repeated play of the game. If the game is played repeatedly, other (higher) payoffs are possible, see Chap. 7.

The Prisoners' Dilemma is a metaphor for many economic situations. An outstanding example is the so-called 'tragedy of the commons', see Problem 6.27 in this book.

1.3.2.2 Battle of the Sexes

Story A man and a woman want to go out together, either to a football match or to a ballet performance. They forgot to agree where they would go to that night, are in different places and have to decide on their own where to go; they have no means to communicate. Their main concern is to be together. The man has a preference for football and the woman for ballet.

Model A table reflecting the situation is as follows.

$$\begin{array}{c} \\ \text{Football} \\ \text{Ballet} \end{array} \begin{array}{c} \begin{array}{cc} \text{Football} & \text{Ballet} \end{array} \\ \begin{pmatrix} 2,1 & 0,0 \\ 0,0 & 1,2 \end{pmatrix} \end{array}.$$

Here, the man chooses a row and the woman a column.

Solution Observe that no player has a dominated choice. The players have to coordinate without being able to communicate. Now it may be possible that the night before they discussed football at length; each player remembers this, may think that the other remembers this, and so this may serve as a focal point for both. In the absence of such considerations it is hard to give a unique prediction for this game. We can, however, say that the combinations (Football, Football) and (Ballet, Ballet) are special in the sense that the players' choices are *best replies* to each other; if the man chooses Football (Ballet), then it is optimal for the woman to choose Football (Ballet) as well, and vice versa. In the literature, such choice combinations are called *Nash equilibria*. The concept of Nash equilibrium is without doubt the main solution concept developed in game theory.

Comments The Battle of the Sexes game is metaphoric for problems of coordination.

1.3.2.3 Matching Pennies

Every zero-sum game is, trivially, a special case of a nonzero-sum game. For instance, the Matching Pennies game discussed in Sect. 1.3.1 can be represented as a nonzero-sum game as follows:

$$\begin{array}{c} \\ \text{Heads} \\ \text{Tails} \end{array} \begin{array}{c} \begin{array}{cc} \text{Heads} & \text{Tails} \end{array} \\ \begin{pmatrix} 1,-1 & -1,1 \\ -1,1 & 1,-1 \end{pmatrix} \end{array}.$$

Clearly, no player has a dominated choice and there is no combination of a row and a column such that each player's choice is optimal given the choice of the other player—there is no Nash equilibrium. If mixed strategies are allowed, then it can be checked that if player 2 plays Heads and Tails each with probability $\frac{1}{2}$, then for player 1 it is optimal to do so too, and vice versa. Such a combination of mixed strategies is again called a Nash equilibrium. See Chaps. 3 and 13.

1.3.2.4 A Cournot Game

Story Two firms produce a similar ('homogenous') product. The market price of this product is equal to $p = 1 - Q$ or zero (whichever is larger), where Q is the total quantity produced. There are no production costs.

Model The two firms are the players, 1 and 2. Each player $i = 1, 2$ chooses a quantity $q_i \geq 0$, and makes a profit of $K_i(q_1, q_2) = q_i(1 - q_1 - q_2)$ (or zero if $q_1 + q_2 \geq 1$).

Solution Suppose player 2 produces $q_2 = \frac{1}{3}$. Then player 1 maximizes his own profit $q_1(1 - q_1 - \frac{1}{3})$ by choosing $q_1 = \frac{1}{3}$. Also the converse holds: if player 1 chooses $q_1 = \frac{1}{3}$ then $q_2 = \frac{1}{3}$ maximizes profit for player 2. This combination of strategies consists of mutual best replies and is therefore again called a Nash equilibrium.

Comments This particular Nash equilibrium is often called Cournot equilibrium. It is easy to check that the Cournot equilibrium in this example is again not Pareto optimal: if the firms each would produce $\frac{1}{4}$, then they would both be better off. The main difference between this example and the preceding ones is, that each player here has infinitely many choices, also if no mixed strategies are included. See further Chap. 6.

1.3.3 Extensive Form Games

All examples in Sects. 1.3.1 and 1.3.2 are examples of one-shot games: the players choose only once, independently and simultaneously. In parlor games as well as in games derived from real-life economic or political situations, this is often not what happens. Players may move sequentially, and observe or partially observe each others' moves. Such situations are better modelled by so-called extensive form games.

1.3.3.1 Sequential Battle of the Sexes

Story The story is similar to the one in Sect. 1.3.2, but we now assume that the man chooses first and the woman can observe the choice of the man.

Model This situation can be represented by the decision tree in Fig. 1.1.

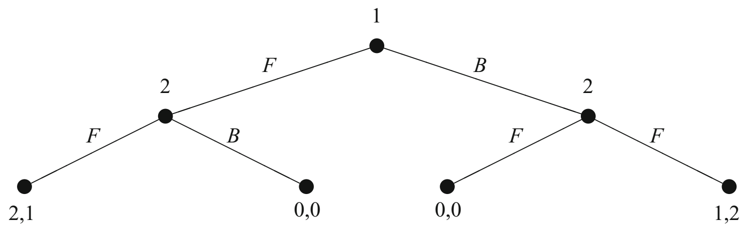


Fig. 1.1 The decision tree of sequential battle of the sexes

Player 1 (the man) chooses first, player 2 (the woman) observes player 1's choice and then makes her own choice. The first number in each pair of numbers is the payoff to player 1, and the second number is the payoff to player 2. Filled circles denote decision nodes (of a player) or end nodes (followed by payoffs).

Solution An obvious way to analyze this game is by working backwards. If player 1 chooses F, then it is optimal for player 2 to choose F as well, and if player 1 chooses B, then it is optimal for player 2 to choose B as well. Given this choice behavior of player 2 and assuming that player 1 performs this line of reasoning about the choices of player 2, player 1 should choose F.

Comments What this simple example shows is that in such a so-called extensive form game, there is a distinction between a play plan of a player and an actual move or choice of that player. Player 2 has the plan to choose F (B) if player 1 has chosen F (B). Player 2's actual choice is F—assuming as above that player 1 has chosen F. We use the word *strategy* to denote a play plan, and the word *action* to denote a particular move. In a one-shot game there is no difference between the two, and then the word 'strategy' is used.

Games in extensive form are studied in Chaps. 4, 5, and 14. The solution described above is an example of a so-called backward induction (or subgame perfect) (Nash) equilibrium. There are other Nash equilibria as well. Suppose player 1 chooses B and player 2's plan (strategy) is to choose B always, independent of player 1's choice. Observe that, given the strategy of the opponent, no player can do better, and so this combination is a Nash equilibrium, although player 2's plan is only partly 'credible': if player 1 would choose F instead of B, then player 2 would be better off by changing her choice to F.

1.3.3.2 Sequential Cournot

Story The story is similar to the one in Sect. 1.3.2, but we now assume that firm 1 chooses first and firm 2 can observe the choice of firm 1.

Model Since each player $i = 1, 2$ has infinitely many actions $q_i \geq 0$, we cannot draw a picture like Fig. 1.1 for the sequential Battle of the Sexes. Instead of straight lines we use zigzag lines to denote a continuum of possible actions. For this example we obtain Fig. 1.2.
Player 1 moves first and chooses $q_1 \geq 0$. Player 2 observes player 1's choice of q_1 and then chooses $q_2 \geq 0$.

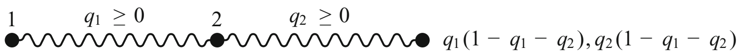


Fig. 1.2 The extensive form of sequential Cournot

Solution Like in the sequential Battle of the Sexes game, an obvious way to solve this game is by working backwards. Given the observed choice q_1, player 2's optimal (profit maximizing) choice is $q_2 = \frac{1}{2}(1 - q_1)$ or $q_2 = 0$, whichever is larger. Given this *reaction function* of player 2, the optimal choice of player 1 is obtained by maximizing the profit function $q_1 \mapsto q_1\left(1 - q_1 - \frac{1}{2}(1 - q_1)\right)$. The maximum is obtained for $q_1 = \frac{1}{2}$. Consequently, player 2 chooses $q_2 = \frac{1}{4}$.

Comments The solution described here is another example of a backward induction or subgame perfect equilibrium. It is also called *Stackelberg equilibrium*. See Chap. 6.

1.3.3.3 Entry Deterrence

Story An old question in industrial organization is whether an incumbent monopolist can maintain his position by threatening to start a price war against any new firm that enters the market. In order to analyze this question, consider the following situation. There are two players, the entrant and the incumbent. The entrant decides whether to Enter (E) or to Stay Out (O). If the entrant enters, the incumbent can Collude (C) with him, or Fight (F) by cutting the price drastically. The payoffs are as follows. Market profits are 100 at the monopoly price and 0 at the fighting price. Entry costs 10. Collusion shares the monopoly profits evenly.

Model This situation can be represented by the decision tree in Fig. 1.3.

Solution By working backward, we find that the entrant enters and the incumbent colludes.

Comments As in the sequential battle of the sexes there exists another Nash equilibrium. If the entrant stays out and the incumbent's plan is to fight if the entrant would enter, then also this is a combination where no player can do better given the strategy of the other player. Again, one might argue that the 'threat' of the incumbent firm to start a price war in case the potential entrant would enter, is not credible since the incumbent hurts himself by carrying out the threat.

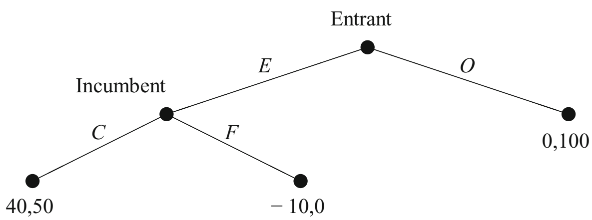


Fig. 1.3 The game of entry deterrence. Payoffs: entrant, incumbent

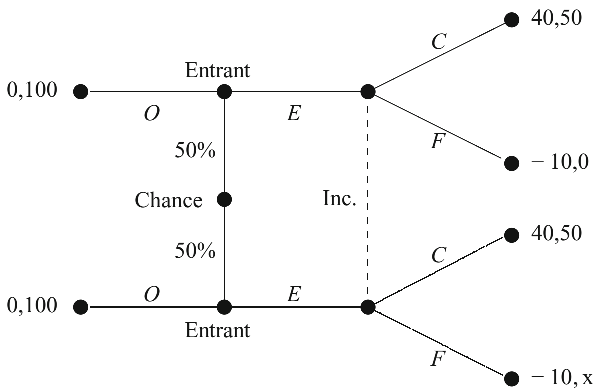
1.3.3.4 Entry Deterrence with Incomplete Information

Story Consider the following variation on the foregoing entry deterrence model. Suppose that with 50 % probability the incumbent's payoff from Fight (F) is equal to some amount x rather than the 0 above. (Here, x is a given number. It is not a variable but a parameter.) The entrant still moves first and knows whether this payoff is x or 0. The incumbent moves last and does not know whether this payoff is x or 0 when he moves. Both firms know the probabilities of the payoff being x or 0. This situation might arise, for instance, if the capacity of the entrant firm is private information. A positive value of x might be associated with the entrant having a capacity constraint which leaves a larger share of the market to the incumbent if he fights. The incumbent estimates the probability of the entrant having this capacity constraint at 50 %, and the entrant knows this.

Model This situation can be modelled by including a chance move in the game tree. Moreover, the tree should express the asymmetric information between the players. Consider the game tree in Fig. 1.4. First there is a chance move. The entrant learns the outcome of the chance move and decides to enter or not. If he enters, then the incumbent decides to collude or fight, without however knowing the outcome of the chance move: this is indicated by the dashed line. Put otherwise, the incumbent has two decision nodes where he should choose, but he does not know at which node he actually is. Thus, he can only choose between 'collude' and 'fight', without making this choice contingent on the outcome of the chance move.

Solution If $x \leq 50$ then an obvious solution is that the incumbent colludes and the entrant enters. Also the combination of strategies where the entrant stays out no matter what the outcome of the chance move is, and the incumbent fights, is a Nash equilibrium. A complete analysis is more subtle and may include a consideration of the probabilistic information that the incumbent might derive from the action of the entrant in a so-called *perfect Bayesian equilibrium*, see Chaps. 5 and 14.

Fig. 1.4 Entry deterrence with incomplete information



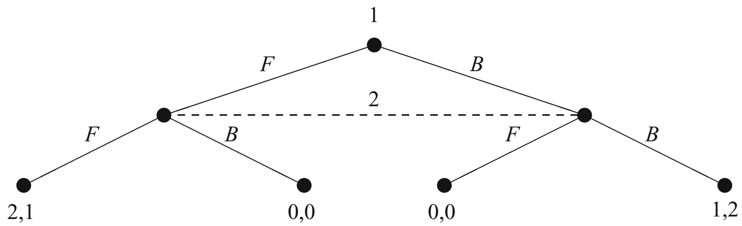


Fig. 1.5 Simultaneous battle of the sexes in extensive form

Comments The collection of the two nodes of the incumbent, connected by the dashed line, is called an *information set*. In general, information sets are used to model imperfect information. In the present example imperfect information arises since the incumbent does not know the outcome of the chance move. Imperfect information can also arise if some player does not observe some move of some other player. As a simple example, consider again the simultaneous move Battle of the Sexes game of Sect. 1.3.2. This can be modelled as a game in extensive form as in Fig. 1.5. Hence, player 2, when he moves, does not know what player 1 has chosen. This is equivalent to players 1 and 2 moving independently and simultaneously.

1.3.4 Cooperative Games

In a cooperative game the focus is on payoffs and coalitions, rather than on strategies. The prevailing analyses have an axiomatic flavor, in contrast to the equilibrium analysis of noncooperative theory. The implicit assumption is that players can make binding agreements.

1.3.4.1 Three Cooperating Cities

Story Cities 1, 2 and 3 want to be connected with a nearby power source. The possible transmission links and their costs are shown in the following figure. Each city can hire any of the transmission links. If the cities cooperate in hiring the links they save on the hiring costs (the links have unlimited capacity). The situation is represented in Fig. 1.6.

Model The players in this situation are the three cities. Denote the player set by $N = \{1, 2, 3\}$. These players can form coalitions: any subset S of N is called a *coalition*. Table 1.1 presents the costs as well as the savings of each coalition.

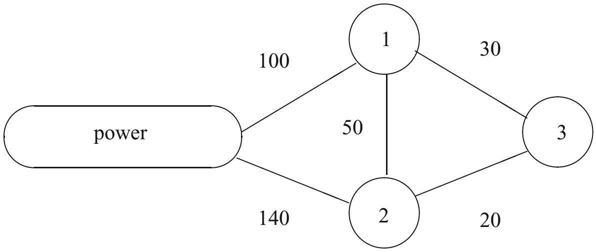


Fig. 1.6 Situation leading to the three cities game

Table 1.1 The three cities game

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c(S)$	100	140	130	150	130	150	150
$v(S)$	0	0	0	90	100	120	220

The costs $c(S)$ are obtained by calculating the cheapest routes connecting the cities in the coalition S with the power source. The cost savings $v(S)$ are determined by

$$v(S) := \sum_{i \in S} c(\{i\}) - c(S) \text{ for each nonempty } S \subseteq N.$$

The cost savings $v(S)$ for coalition S are equal to the difference in costs corresponding to the situation where all members of S work alone and the situation where all members of S work together. The pair (N, v) is called a *cooperative game*.

Solution Basic questions in a cooperative game (N, v) are: which coalitions will actually be formed, and how should the worth (savings) of such a coalition be distributed among its members? To form a coalition the consent of every member is needed, but it is likely that the willingness of a player to participate in a coalition depends on what that player obtains in that coalition. Therefore, the second question seems to be the more basic one, and in this book attention is focussed on that question. Specifically, it is usually assumed that the *grand coalition* N of all players is formed, and the question is then reduced to the problem of distributing the amount $v(N)$ among the players. In the present example, how should the amount 220 $(= v(N))$ be distributed among the three cities? In other words, we look for vectors $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1 + x_2 + x_3 = 220$, where player $i \in \{1, 2, 3\}$ obtains x_i. One obvious candidate is to choose $x_1 = x_2 = x_3 = 220/3$, but this does not really reflect the asymmetry of the situation: some coalitions save more than others. The literature offers many quite different solutions to this distribution problem, among which are the *core*, the *Shapley value*, and the *nucleolus*.

The core consists of those payoff distributions that cannot be improved upon by any smaller coalition. For the three cities example, this means that the core consists of those vectors (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 = 220, x_1, x_2, x_3 \geq 0, x_1 + x_2 \geq 90,$

$x_1 + x_3 \geq 100$, and $x_2 + x_3 \geq 120$. Hence, this is a large set and therefore it is rather indeterminate as a solution to the game.

In contrast, the Shapley value consists by definition of one point (vector). Roughly, according to the Shapley value, each player receives his average contribution to the worth (savings) of coalitions. More precisely, imagine the players entering the 'bargaining room' one at a time, say first player 1, then player 2, and finally player 3. When player 1 enters, he forms a coalition on his own, which has worth 0. When player 2 enters, they form the coalition $\{1, 2\}$, so that the contribution of player 2 is equal to $v(\{1, 2\}) - v(\{1\}) = 90 - 0 = 90$. Finally, player 3 enters and they form the grand coalition. Player 3's contribution is equal to $v(N) - v(\{1, 2\}) = 220 - 90 = 130$. Hence, this results in the payoff vector $(0, 90, 130)$. Now the Shapley value is obtained by repeating this argument for the five other possible orderings in which the players can enter the bargaining room, and then taking the average of the six payoff vectors. In this example this results in the distribution $(65, 75, 80)$.

Also the nucleolus consists of one point, in this case the vector $(56\frac{2}{3}, 76\frac{2}{3}, 86\frac{2}{3})$. The nucleolus is more complicated to define and harder to compute, and at this stage the reader should take these numbers for granted.

Formal definitions of all these concepts are provided in Chap. 9. See also Chaps. 16–20.

Comments The implicit assumptions for a game like this are, first, that a coalition which is actually formed, can make binding agreements on the distribution of its payoff and, second, that any payoff distribution which distributes (or, at least, does not exceed) the savings (or, more generally, *worth*) of the coalition is possible. For these reasons, such games are called *cooperative games with transferable utility*.

1.3.4.2 The Glove Game

Story Assume that there are three players, 1, 2, and 3. Players 1 and 2 each possess a right-hand glove, while player 3 has a left-hand glove. A pair of gloves has worth 1. The players cooperate in order to generate a profit.

Model The associated cooperative game is described by Table 1.2.

Solution The core of this game consists of exactly one vector. The Shapley value assigns $2/3$ to player 3 and $1/6$ to both player 1 and player 2 (see Problem 1.6). The nucleolus is the unique element of the core.

Table 1.2 The glove game

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	1	1	1

Table 1.3 Preferences for dentist appointments

	Mon	Tue	Wed
Adams	2	4	8
Benson	10	5	2
Cooper	10	6	4

Table 1.4 The dentist game: a permutation game

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	2	5	4	14	18	9	24

1.3.4.3 A Permutation Game

Story Mr. Adams, Mrs. Benson, and Mr. Cooper have appointments with the dentist on Monday, Tuesday, and Wednesday, respectively. This schedule not necessarily matches their preferences, due to different urgencies and other factors. These preferences (expressed in numbers) are given in Table 1.3.

Model This situation gives rise to a game in which the coalitions can gain by reshuffling their appointments. For instance, Adams (player 1) and Benson (player 2) can change their appointments and obtain a total of 14 instead of 7. A complete description of the resulting game is given in Table 1.4.[3]

Solution The core of this game is the convex hull of the vectors $(15, 5, 4)$, $(14, 6, 4)$, $(8, 6, 10)$, and $(9, 5, 10)$, i.e., it is the quadrangle with these points as vertices, together with its inside. The Shapley value is the vector $(9\frac{1}{2}, 6\frac{1}{2}, 8)$, and the nucleolus is the vector $(11\frac{1}{2}, 5\frac{1}{2}, 7)$.

Comments See Chap. 20 for an analysis of permutation games.

1.3.4.4 A Voting Game

The United nations Security Council consists of five permanent members (United States, Russia, Britain, France, and China) and ten other members. Motions must be approved by at least nine members, including all the permanent members. This situation gives rise to a 15-player so-called voting game (N, v) with $v(S) = 1$ if the coalition S contains the five permanent members and at least four nonpermanent members, and $v(S) = 0$ otherwise. Games with worths 0 or 1 are also called *simple games*. Coalitions with worth equal to 1 are called winning, the other coalitions are called losing. Simple games are studied in Chap. 16.

A solution to such a voting game is interpreted as representing the power of a player, rather than payoff (money) or utility.

[3]The numbers in this table are the total payoffs to coalitions and not the net payoffs compared to the coalition members staying alone instead of cooperating. These would be, respectively, 0, 0, 0, 7, 12, 0, and 13.

1.3.5 Bargaining Games

Bargaining theory focusses on agreements between individual players.

1.3.5.1 A Division Problem

Story Consider the following situation. Two players have to agree on the division of one unit of a perfectly divisible good, say a liter of wine. If they reach an agreement, say (α, β) where $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$, then they split up the one unit according to this agreement; otherwise, they both receive nothing. The players have preferences for the good, described by utility functions.

Model To fix ideas, assume that player 1 has a utility function $u_1(\alpha) = \alpha$ and player 2 has a utility function $u_2(\alpha) = \sqrt{\alpha}$. Thus, a distribution $(\alpha, 1-\alpha)$ of the good leads to a corresponding pair of utilities $(u_1(\alpha), u_2(1-\alpha)) = (\alpha, \sqrt{1-\alpha})$. By letting α range from 0 to 1 we obtain all utility pairs corresponding to all distributions of the whole unit of the good: this is the bold curve in Fig. 1.7. It is assumed that also distributions summing to less than the whole unit are theoretically possible. This yields the whole shaded region.

Solution According to the *Nash bargaining solution* this bargaining problem should be solved as follows: maximize the product of the players' utilities on the shaded area. Since this maximum will be reached on the boundary, the problem is equivalent to

$$\max_{0 \leq \alpha \leq 1} \alpha\sqrt{1-\alpha}.$$

The maximum is obtained for $\alpha = \frac{2}{3}$. So the solution of the bargaining problem in utilities equals $(\frac{2}{3}, \frac{1}{3}\sqrt{3})$, which is the point z in Fig. 1.7. This implies that player 1 obtains $\frac{2}{3}$ of the 1 unit of the good, whereas player 2 obtains $\frac{1}{3}$. As described

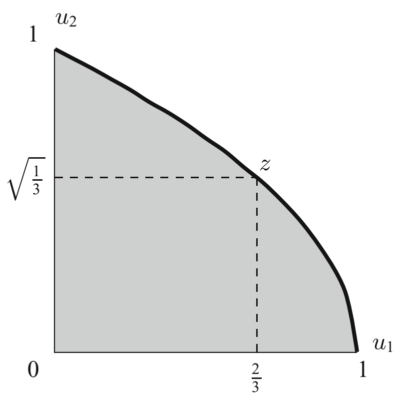


Fig. 1.7 A bargaining game

here, this solution may seem to come out of the blue, but it can be backed up axiomatically, see below.

Comments For an axiomatic characterization of the Nash bargaining solution, see Chaps. 10 and 21. The bargaining literature also includes many noncooperative, strategic approaches to the bargaining problem, notably the Rubinstein alternating offers model: see Chaps. 6 and 21. The bargaining game can be seen as a special case of a cooperative game without transferable utility. Also games with transferable utility form a subset of the more general class of games without transferable utility. See Chap. 21.

1.4 Cooperative Versus Noncooperative Game Theory

The usual distinction between cooperative and noncooperative game theory is that in a cooperative game binding agreements between players are possible, whereas this is not the case in noncooperative games. This distinction is informal and also not very clear-cut: for instance, the core of a cooperative game has a clear noncooperative flavor; a concept such as correlated equilibrium for noncooperative games (see Sect. 13.7) has a clear cooperative flavor. Moreover, quite some game-theoretic literature is concerned with viewing problems both from a cooperative and a noncooperative perspective. The latter approach is sometimes called the *Nash program*; the bargaining problem discussed above is a typical example. In a much more precise sense, the theory of *implementation* is concerned with representing outcomes from cooperative solutions as equilibrium outcomes of specific noncooperative solutions.

A workable distinction between cooperative and noncooperative games can be based on the 'modelling technique' that is used: in a noncooperative game players have explicit strategies, whereas in a cooperative game players and coalitions are characterized, more abstractly, by the outcomes and payoffs that they can reach. The examples in Sects. 1.3.1–1.3.3 are examples of noncooperative games, whereas those in Sects. 1.3.4 and 1.3.5 are examples of cooperative games.

1.5 Problems

1.1. *Battle of the Bismarck Sea*

(a) Represent the 'Battle of the Bismarck Sea' as a game in extensive form.
(b) Now assume that Imamura moves first, and Kenney observes Imamura's move and moves next. Represent this situation in extensive form and solve by working backwards.
(c) Answer the same questions as under (b) with now Kenney moving first.

1.2. *Variant of Matching Pennies*
Consider the following variant of the 'Matching Pennies' game:

$$\begin{array}{c} \\ \text{Heads} \\ \text{Tails} \end{array} \begin{array}{c} \begin{array}{cc} \text{Heads} & \text{Tails} \end{array} \\ \begin{pmatrix} x & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

where x is a real number. For each value of x, determine all saddlepoints of the game, if any.

1.3. *Mixed Strategies*
Consider the following zero-sum game:

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \end{array} .$$

(a) Show that this game has no saddlepoint.
(b) Find a mixed strategy (randomized choice) of (the row) player 1 that makes his expected payoff independent of player 2's strategy.
(c) Find a mixed strategy of player 2 that makes his expected payoff independent of player 1's strategy.
(d) Consider the expected payoffs found under (b) and (c). What do you conclude about how the game could be played if randomized choices are allowed?

1.4. *Sequential Cournot*
Consider the sequential Cournot model in Sect. 1.3.3, but now based on the market price $p = 2 - 3Q$ (or zero, whichever is larger), where Q is the total quantity produced.

(a) Represent this game in extensive form, similar as in Fig. 1.2.
(b) Solve this game by working backwards.

1.5. *Three Cooperating Cities*

(a) Complete the computation of the Shapley value of the 'Three Cooperating Cities Game'. Is it an element of the core? Why or why not?
(b) Show that the nucleolus of this game is an element of the core.

1.6. *Glove Game*

(a) Compute the core of the glove game.
(b) Compute the Shapley value of this game. Is it an element of the core?

1.7. *Dentist Appointments*

(a) For the permutation (dentist appointments) game, compute the Shapley value and check if it is an element of the core.
(b) Show that the nucleolus of this game is in the core.

1.8. *Nash Bargaining*

(a) Verify the computation of the Nash bargaining solution for the division problem in Sect. 1.3.5.
(b) Compute the Nash bargaining outcome, both in utilities and in division of the good, when the utility functions are $u_1(\alpha) = 2\alpha - \alpha^2$ and $u_2(\alpha) = \alpha$.

1.9. *Variant of Glove Game*
Suppose there are $n = \ell + r$ players, where ℓ players own a left-hand glove and r players own a right-hand glove. Let N be the set of all players, let L be the subset of N consisting of the players who own a left-hand glove, and let R be the subset of N consisting of the players who own a right-hand glove. Let $S \subseteq N$ denote an arbitrary coalition and let $|S|$ denote the number of players in S. For instance, $|L| = \ell$, and $|R \cap S|$ is the number of players in S who own a right-hand glove. As before, each pair of gloves has worth 1. Find an expression for $v(S)$, i.e., the maximal profit that S can generate by cooperation of its members.

1.6 Notes

The 'Battle of the Bismarck Sea' example is taken from Rasmusen (1989). Also see the memoirs of Churchill (1983): in 1953, Churchill received the Nobel prize in literature for this work.

Von Neumann (1928) proved that every two-person finite zero-sum game in which the players can use mixed strategies, has a value. This result is known as the *minimax theorem*.

The Prisoners' Dilemma game has been widely studied in the literature. Axelrod (1984) described the results of a tournament for which players could submit strategies for repeated play of the Prisoners' Dilemma: the so-called tit-for-tat strategy emerged as a winning strategy. For the 'tragedy of the commons', see Hardin (1968), or Gibbons (1992, p. 27).

The concept of focal points was developed by Schelling (1960). Nash equilibria were first explicitly proposed by Nash (1951), who proved that every game in which each player has finitely many choices—zero-sum or nonzero-sum—has a Nash equilibrium in mixed strategies. The basic idea of a Nash equilibrium is much older. For instance, the Cournot equilibrium was developed in Cournot (1838).

Subgame perfect equilibria in extensive form games were first explicitly studied by Selten (1965). Again, the basic idea occurs earlier, for instance in von Stackelberg (1934).

The Three Cooperating Cities is an example of a minimum cost spanning tree game, see Bird (1976). The core was first introduced in Gillies (1953), the Shapley value in Shapley (1953), and the nucleolus in Schmeidler (1969). The dentist example is taken from Curiel (1997, p. 54). The United Nations Security Council game is taken from Owen (1995).

The Nash bargaining solution was proposed and axiomatically characterized in Nash (1950). Nash (1953) proposed a noncooperative game to back up the Nash bargaining solution. Rubinstein (1982) modelled the bargaining problem as an alternating offers extensive form game. Binmore et al. (1986) observed the close relationship between the Nash bargaining solution and the strategic approach of Rubinstein. See Chap. 10.

The reference list contains a number of textbooks on game theory, notably: Fudenberg and Tirole (1991a), Gardner (1995), Gibbons (1992), Maschler et al. (2013), Morris (1994), Moulin (1988), Moulin (1995), Myerson (1991), Osborne (2004), Owen (1995), Peleg and Sudhölter (2003), Perea (2012), Rasmusen (1989), Thomas (1986), and Watson (2002).

References

Aumann, R. J., & Maschler, M. (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory, 36*, 195–213.

Axelrod, R. (1984). *The evolution of cooperation*. New York: Basic Books.

Bertrand, J. (1883). Review of Walras's 'Théorie mathématique de la richesse sociale' and Cournot's 'Recherches sur les principes mathématiques de la théorie des richesses'. *Journal des Savants*, 499–508 [translated by M. Chevaillier and reprinted in Magnan de Bornier (1992), 646–653].

Binmore, K., Rubinstein, A., & Wolinsky, A. (1986). The Nash bargaining solution in economic modelling. *Rand Journal of Economics, 17*, 176–188.

Bird, C. G. (1976). On cost allocation for a spanning tree: a game theoretic approach. *Networks, 6*, 335–350.

Black, D. (1969). Lewis Carroll and the theory of games. In *American Economic Review, Proceedings*.

Bondareva, O. N. (1962). Theory of the core in the *n*-person game. *Vestnik Leningradskii Universitet, 13*, 141–142 (in Russian).

Brams, S. J. (1980). *Biblical games: A strategic analysis of stories in the Old Testament*. Cambridge: MIT Press.

Churchill, W. (1983). *Second world war*. Boston: Houghton Mifflin.

Cournot, A. (1838). *Recherches sur les principes mathématiques de la théorie des richesses* [English translation (1897) *Researches into the mathematical principles of the theory of wealth*]. New York: Macmillan.

Curiel, I. (1997). *Cooperative game theory and applications: Cooperative games arising from combinatorial optimization problems*. Boston: Kluwer Academic.

Dimand, M. A., & Dimand, R. W. (1996). *The history of game theory, volume 1: from the beginnings to 1945*. London: Routledge.

Dodgson, C. L. (1884). *The principles of parliamentary representation*. London: Harrison & Sons.

Dresher, M., Tucker. A. W., & Wolfe, P. (Eds.), (1957). *Contributions to the theory of games III. Annals of mathematics studies* (Vol. 39). Princeton: Princeton University Press.

Dresher, M., Shapley, L. S., & Tucker, A. W., (Eds.), (1964). *Advances in game theory. Annals of mathematics studies* (Vol. 52). Princeton: Princeton University Press.

Feyerabend, P. K. (1974). *Against method*. London: New Left Books.
Fudenberg, D., & Tirole, J. (1991a). *Game theory*. Cambridge: MIT Press.
Gardner, R. (1995). *Games for business and economics*. New York: Wiley.
Gibbons, R. (1992). *A primer in game theory*. Hertfordshire: Harvester Wheatsheaf.
Gillies, D. B. (1953). *Some theorems on n-person games*, Ph.D. Thesis. Princeton: Princeton University Press.
Hardin, G. (1968). The tragedy of the commons. *Science, 162*, 1243–1248.
Harsanyi, J. C. (1967/1968) Games with incomplete information played by “Bayesian” players I, II, and III. *Management Science, 14*, 159–182, 320–334, 486–502.
Kuhn, H. W., & Tucker, A. W. (Eds.), (1950). *Contributions to the theory of games I*. *Annals of mathematics studies* (Vol. 24). Princeton: Princeton University Press.
Kuhn, H. W., & Tucker, A. W. (Eds.), (1953). *Contributions to the theory of games II*. *Annals of mathematics studies* (Vol. 28). Princeton: Princeton University Press.
Luce, R. D., & Raiffa, H. (1957). *Games and decisions: Introduction and critical survey*. New York: Wiley.
Luce, R. D., & Tucker, A. W. (Eds.), (1958). *Contributions to the theory of games IV*. *Annals of mathematics studies* (Vol. 40). Princeton: Princeton University Press.
Maschler, M., Solan, E., & Zamir, S. (2013). *Game theory*. Cambridge: Cambridge University Press.
Morris, P. (1994). *Introduction to game theory*. New York: Springer.
Moulin, H. (1988). *Axioms of cooperative decision making*. Cambridge: Cambridge University Press.
Moulin, H. (1995). *Cooperative microeconomics; a game-theoretic introduction*. Hemel Hempstead: Prentice Hall/Harvester Wheatsheaf.
Myerson, R. B. (1991). *Game theory, analysis of conflict*. Cambridge: Harvard University Press.
Nasar, S. (1998). *A beautiful mind*. London: Faber and Faber Ltd.
Nash, J. F. (1950). The bargaining problem. *Econometrica, 18*, 155–162.
Nash, J. F. (1951). Non-cooperative games. *Annals of Mathematics, 54*, 286–295.
Nash, J. F. (1953). Two-person cooperative games. *Econometrica, 21*, 128–140.
Osborne, M. J. (2004). *An introduction to game theory*. New York: Oxford University Press.
Owen, G. (1995). *Game theory* (3rd ed.). San Diego: Academic.
Peleg, B., & Sudhölter, P. (2003). *Introduction to the theory of cooperative games*. Boston: Kluwer Academic.
Perea, A. (2012). *Epistemic game theory: Reasoning and choice*. Cambridge: Cambridge University Press.
Rasmusen, E. (1989). *Games and information: An introduction to game theory* (2nd ed., 1994/1995). Oxford: Basil Blackwell.
Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica, 50*, 97–109.
Schelling, T. C. (1960). *The strategy of conflict*. Cambridge: Harvard University Press.
Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics, 17*, 1163–1170.
Selten, R. (1965). Spieltheoretische Behandlung eines Oligopolmodels mit Nachfragezeit. *Zeitschrift für Gesammte Staatswissenschaft, 121*, 301–324.
Selten, R. (1975). Reexamination of the perfectness concept for equilibrium points in extensive games. *International Journal of Game Theory, 4*, 25–55.
Shapley, L. S. (1953). A value for *n*-person games. In: A. W. Tucker & H. W. Kuhn (Eds.), *Contributions to the theory of games II* (pp. 307–317). Princeton: Princeton University Press.
Shapley, L. S. (1967). On balanced sets and cores. *Naval Research Logistics Quarterly, 14*, 453–460.
Sun Tzu (1988) *The art of war* [Translated by Thomas Cleary]. Boston: Shambala.
Thomas, L. C. (1986). *Games, theory and applications*. Chichester: Ellis Horwood Limited.
von Neumann, J. (1928). Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen, 100*, 295–320.

von Neumann, J., & Morgenstern, O. (1944/1947) *Theory of games and economic behavior*. Princeton: Princeton University Press.

von Stackelberg, H. F. (1934). *Marktform und Gleichgewicht*. Wien: Julius Springer.

Watson, J. (2002). *Strategy, an introduction to game theory*. New York: Norton.

Zermelo, E. (1913). Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels. In *Proceedings Fifth International Congress of Mathematicians* (Vol. 2, pp. 501–504).