12 Matrix Games

In this chapter we study finite two-person zero-sum games—matrix games—more rigorously. In particular, von Neumann's Minimax Theorem is proved. The chapter extends Chap. 2 in Part I. Although it is self-contained, it may be useful to (re)read Chap. 2 first.

Section 12.1 presents a proof of the Minimax Theorem, and Sect. 12.2 shows how a matrix game can be solved—i.e., optimal strategies and the value of the game can be found—by solving an associated linear programming problem.

12.1 The Minimax Theorem

A two-person zero-sum game is completely determined by a single matrix. We repeat Definition 2.1.

Definition 12.1 (Matrix Game) A *matrix game* is an $m \times n$ matrix A of real numbers, where m is the number of rows and n is the number of columns. A (*mixed*) *strategy* of player 1 is a probability distribution $\mathbf{p}$ over the rows of A, i.e., an element of the set

$$\Delta^m := \{\mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} p_i = 1,\ p_i \geq 0 \text{ for all } i = 1, \ldots, m\}\ .$$

Similarly, a (*mixed*) *strategy* of player 2 is a probability distribution $\mathbf{q}$ over the columns of A, i.e., an element of the set

$$\Delta^n := \{\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} q_j = 1,\ q_j \geq 0 \text{ for all } j = 1, \ldots, n\}\ .$$

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A strategy $\mathbf{p}$ of player 1 is called *pure* if there is a row i with $p_i = 1$. This strategy is also denoted by $\mathbf{e}^i$. Similarly, a strategy $\mathbf{q}$ of player 2 is called *pure* if there is a column j with $q_j = 1$. This strategy is also denoted by $\mathbf{e}^j$. $\square$

Let A be an $m \times n$ matrix game. For any strategy $\mathbf{p} \in \Delta^m$ of player 1, let $v_1(\mathbf{p}) = \min_{\mathbf{q}\in\Delta^n} \mathbf{p}A\mathbf{q}$. It is easy to see that $v_1(\mathbf{p}) = \min_{j\in\{1,\ldots,n\}} \mathbf{p}A\mathbf{e}^j$, since $\mathbf{p}A\mathbf{q}$ is a convex combination of the numbers $\mathbf{p}A\mathbf{e}^j$. In the matrix game A player 1 can guarantee a payoff of at least

$$v_1(A) := \max_{\mathbf{p}\in\Delta^m} v_1(\mathbf{p}) .$$

Similarly, for any strategy $\mathbf{q} \in \Delta^n$ of player 2 let $v_2(\mathbf{q}) = \max_{\mathbf{p}\in\Delta^m} \mathbf{p}A\mathbf{q} = \max_{i\in\{1,\ldots,m\}} \mathbf{e}^i A\mathbf{q}$, then player 2 can guarantee to have to pay at most

$$v_2(A) := \min_{\mathbf{q}\in\Delta^n} v_2(\mathbf{q}) .$$

Intuitively, player 1 should not be able to guarantee to obtain more than what player 2 can guarantee to pay maximally. Indeed, we have the following lemma.

Lemma 12.2 *For any $m \times n$ matrix game, $v_1(A) \leq v_2(A)$.*

Proof Problem 12.2. $\blacksquare$

The following theorem is due to von Neumann. The proof is based on Lemma 22.3, which is equivalent to Farkas' Lemma.

Theorem 12.3 (Minimax Theorem for Matrix Games) *For any $m \times n$ matrix game A, $v_1(A) = v_2(A)$.*

Proof Let A be an $m \times n$ matrix game. In view of Lemma 12.2 it is sufficient to prove that $v_1(a) \geq v_2(A)$. Suppose, to the contrary, that $v_1(A) < v_2(A)$. We derive a contradiction, which completes the proof of the theorem.

Let B be any arbitrary $m \times n$ matrix game. Then either (i) or (ii) in Lemma 22.3 has to hold for B, i.e., exactly one of the following holds:

(i) There are $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^m$ with $(\mathbf{y}, \mathbf{z}) \geq \mathbf{0}$, $(\mathbf{y}, \mathbf{z}) \neq \mathbf{0}$ and $B\mathbf{y} + \mathbf{z} = 0$.
(ii) There is an $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{x} > \mathbf{0}$ and $\mathbf{x}B > \mathbf{0}$.

First suppose that (i) holds and let $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^m$ with $(\mathbf{y}, \mathbf{z}) \geq \mathbf{0}$, $(\mathbf{y}, \mathbf{z}) \neq \mathbf{0}$ and $B\mathbf{y} + \mathbf{z} = 0$. It cannot be the case that $\mathbf{y} = \mathbf{0}$, since that would imply that also $\mathbf{z} = 0$, a contradiction. Hence $\sum_{k=1}^n y_k > 0$. Define $\mathbf{q} \in \Delta^n$ by $q_j = y_j / \sum_{k=1}^n y_k$ for every $j = 1, \ldots, n$. Then $B\mathbf{q} = -\mathbf{z}/\sum_{k=1}^n y_k \leq \mathbf{0}$. Hence $v_2(\mathbf{q}) \leq 0$, and therefore $v_2(B) \leq 0$.

Suppose instead that (ii) holds. Then there is an $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{x} > \mathbf{0}$ and $\mathbf{x}B > \mathbf{0}$. Define $\mathbf{p} \in \Delta^m$ by $\mathbf{p} = \mathbf{x}/\sum_{i=1}^m x_i$, then $v_1(\mathbf{p}) > 0$ and therefore $v_1(B) > 0$.

We conclude that, for any matrix game B, it is not possible to have $v_1(B) \le 0 < v_2(B)$.

Let now B be the matrix game arising by subtracting the number $v_1(A)$ from all entries of A. Then, clearly, $v_1(B) = v_1(A) - v_1(A) = 0$ and $v_2(B) = v_2(A) - v_1(A) > 0$. Hence, $v_1(B) \le 0 < v_2(B)$, which is the desired contradiction. ■

In view of Theorem 12.3 we can define the *value* of the game A by $v(A) = v_1(A) = v_2(A)$. An *optimal strategy* of player 1 is a strategy $\mathbf{p}$ such that $v_1(\mathbf{p}) \ge v(A)$. Similarly, an *optimal strategy* of player 2 is a strategy $\mathbf{q}$ such that $v_2(\mathbf{q}) \le v(A)$. Theorem 12.3 implies that $v_1(\mathbf{p}) = v_2(\mathbf{q}) = v(A)$ for such optimal strategies. Thus, if $\mathbf{p}$ is an optimal strategy for player 1, then $v_1(\mathbf{p}) = \max_{\mathbf{p}' \in \Delta^m} v_1(\mathbf{p}')$, so that $\mathbf{p}$ is a maximin strategy (cf. Definition 2.3). Conversely, every maximin strategy is an optimal strategy for player 1. Similarly, the optimal strategies for player 2 are exactly the minimax strategies.

For computation of optimal strategies and the value of matrix games in some special cases, see Chap. 2 and Problems 12.3 and 12.4. In general, matrix games can be solved by linear programming. This is demonstrated in the next section.

12.2 A Linear Programming Formulation

Let A be an $m \times n$ matrix game:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Adding the same number to all entries of A changes the value by that same number but not the optimal strategies of the players. So we may assume without loss of generality that all entries of A are positive. We define the $(m+1) \times (n+1)$ matrix B as follows:

$$B = \begin{pmatrix} a_{11} & \cdots & a_{1n} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & -1 \\ -1 & \cdots & -1 & 0 \end{pmatrix}.$$

Let $\mathbf{b} = (0, \ldots, 0, -1) \in \mathbb{R}^{n+1}$ and $\mathbf{c} = (0, \ldots, 0, -1) \in \mathbb{R}^{m+1}$. Define $V := \{\mathbf{x} \in \mathbb{R}^{m+1} \mid \mathbf{x}B \ge \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}$ and $W := \{\mathbf{y} \in \mathbb{R}^{n+1} \mid B\mathbf{y} \le \mathbf{c}, \mathbf{y} \ge \mathbf{0}\}$. It is easy to check that $V, W \ne \emptyset$. The Duality Theorem of Linear Programming (Theorem 22.6) therefore implies:

Corollary 12.4 $\min\{\mathbf{x}\cdot\mathbf{c} \mid \mathbf{x}\in V\} = \max\{\mathbf{b}\cdot\mathbf{y} \mid \mathbf{y}\in W\}$.

The minimum and maximum problems in this corollary are so-called linear programming (LP) problems. If we call the minimization problem the *primal* problem, then the maximization problem is the *dual* problem—or *vice versa*. The common minimum/maximum is called the *value* of the LP, and $\mathbf{x}$ and $\mathbf{y}$ achieving the value are called *optimal solutions*. Denote the sets of optimal solutions by $O_{\min}$ and $O_{\max}$, respectively.

We have the following result. The proof uses Lemma 22.9, which states that if $\hat{\mathbf{x}}\in V$ and $\hat{\mathbf{y}}\in W$ satisfy $\hat{\mathbf{x}}\cdot\mathbf{c} = \mathbf{b}\cdot\hat{\mathbf{y}}$, then $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are optimal solutions.

Theorem 12.5 *Let A be an $m\times n$ matrix game with all entries positive.*

(*i*) *If $\mathbf{p}\in\Delta^m$ is an optimal strategy for player 1 and $\mathbf{q}\in\Delta^n$ is an optimal strategy for player 2 in A, then $(\mathbf{p}, v(A))\in O_{min}$ and $(\mathbf{q}, v(A))\in O_{max}$. The value of the LP is $-v(A)$.*

(*ii*) *If $\mathbf{x} = (x_1,\ldots,x_m,x_{m+1})\in O_{min}$ and $\mathbf{y} = (y_1,\ldots,y_n,y_{n+1})\in O_{max}$, then $(x_1,\ldots,x_m)$ is an optimal strategy for player 1 in A, $(y_1,\ldots,y_n)$ is an optimal strategy for player 2 in A, and $v(A) = x_{m+1} = y_{n+1}$.*

Proof

(i) Let $\mathbf{p}\in\Delta^m$ and $\mathbf{q}\in\Delta^n$ be optimal strategies in the matrix game A. Then $\mathbf{p}A\mathbf{e}^j\geq v(A)$ and $\mathbf{e}^iA\mathbf{q}\leq v(A)$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$. Since all entries of A are positive and therefore $v(A) > 0$, this implies $(\mathbf{p}, v(A))\in V$ and $(\mathbf{q}, v(A))\in W$. Since $(\mathbf{p}, v(A))\cdot\mathbf{c} = -v(A)$ and $(\mathbf{q}, v(A))\cdot\mathbf{b} = -v(A)$, Lemma 22.9 implies that the value of the LP is $-v(A)$, $(\mathbf{p}, v(A))\in O_{\min}$ and $(\mathbf{q}, v(A))\in O_{\max}$.

(ii) Let $\mathbf{x} = (x_1,\ldots,x_{m+1})\in O_{\min}$. Since $\mathbf{x}\cdot\mathbf{c} = -v(A)$ by (i), we have $x_{m+1} = v(A)$. Since $\mathbf{x}B\geq\mathbf{b}$, we have $(x_1,\ldots,x_m)A\mathbf{e}^j\geq v(A)$ for all $j = 1,\ldots,n$, $x_i\geq 0$ for all $i = 1,\ldots,m$, and $\sum_{i=1}^m x_i\leq 1$. Suppose that $\sum_{i=1}^m x_i < 1$. Obviously, $\sum_{i=1}^m x_i > 0$, otherwise $\mathbf{x} = (0,\ldots,0,v(A))\notin V$ since $v(A) > 0$. Then, letting $t = (\sum_{i=1}^m x_i)^{-1} > 1$, we have $t\mathbf{x}\in V$ and $t\mathbf{x}\cdot\mathbf{c} = -t\,v(A) < -v(A)$, contradicting $\mathbf{x}\in O_{\min}$. Hence, $\sum_{i=1}^m x_i = 1$, and $(x_1,\ldots,x_m)$ is an optimal strategy of player 1 in A.

The proof of the second part of (ii) is analogous. ■

The interest of this theorem derives from the fact that solving linear programming problems is a well established area. Thus, one can apply any (computer) method for solving LPs to find the value and optimal strategies of a matrix game.

By slightly modifying part (ii) of the proof of Theorem 12.5, we can in fact derive the Minimax Theorem from the Duality Theorem (Problem 12.5). Conversely, with each LP we can associate a matrix game and thereby derive the Duality

Theorem from the Minimax Theorem. This confirms the close relationship between linear programming (Duality Theorem) and the theory of matrix games (Minimax Theorem).

12.3 Problems

12.1. *Solving a Matrix Game*
Consider the matrix game

$$A = \begin{pmatrix} 6 & 4 & 2 & 1 \\ 5 & 3 & 3 & 2 \\ 1 & 0 & 3 & 4 \\ 2 & -3 & 2 & 3 \end{pmatrix} .$$

(a) Reduce the game by iterated elimination of strictly dominated strategies. Describe exactly which pure strategy you eliminate each time, and by which pure or mixed strategy the strategy to be eliminated is strictly dominated. [See Chap. 2, also for the following questions.]

Denote the reduced game derived in (a) by B.

(b) Solve B graphically. Explicitly compute $v_1(\mathbf{p})$ and $v_2(\mathbf{q})$ for strategies $\mathbf{p}$ of player 1 and $\mathbf{q}$ of player 2. Determine the value of B and the optimal strategy or strategies of players 1 and 2 in B.
(c) Determine the value of A and the optimal strategy or strategies of players 1 and 2 in A.

Now change the entry a_{11} from 6 to $y \in \mathbb{R}$, so that we obtain the matrix game

$$A_y = \begin{pmatrix} y & 4 & 2 & 1 \\ 5 & 3 & 3 & 2 \\ 1 & 0 & 3 & 4 \\ 2 & -3 & 2 & 3 \end{pmatrix} .$$

(d) Compute $v(A_y)$ and the optimal strategies in A_y for every $y \in \mathbb{R}$.

12.2. *Proof of Lemma 12.2*
Prove Lemma 12.2.

12.3. 2×2 *Games*
Consider the 2×2 matrix game

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} .$$

Assume that A has no saddlepoints. [A saddlepoint is an entry (i,j) such that a_{ij} is maximal in column j and minimal in row i, cf. Definition 2.4.]

(a) Assume that $a_{11} > a_{12}$. Show that

$$a_{12} < a_{22},\ a_{21} < a_{22},\ a_{11} > a_{21}\ .$$

(b) Show that the unique optimal strategies $\mathbf{p}$ and $\mathbf{q}$ and the value of the game are given by:

$$\mathbf{p} = \frac{\mathbf{J}A^*}{\mathbf{J}A^*\mathbf{J}^T},\ \ \mathbf{q} = \frac{A^*\mathbf{J}^T}{\mathbf{J}A^*\mathbf{J}^T},\ \ v(A) = \frac{|A|}{\mathbf{J}A^*\mathbf{J}^T}\ ,$$

where A^* is the adjoint matrix of A, i.e.,

$$A^* = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} ,$$

$|A|$ is the determinant of A, and $\mathbf{J} := (1,1)$.[1]

12.4. *Symmetric Games*
An $m \times n$ matrix game $A = (a_{ij})$ is called *symmetric* if $m = n$ and $a_{ij} = -a_{ji}$ for all $i,j = 1,\ldots,m$. Prove that the value of a symmetric game is zero and that the sets of optimal strategies of players 1 and 2 coincide.

12.5. *The Duality Theorem Implies the Minimax Theorem*
Modify the proof of part (ii) of Theorem 12.5 in order to derive the Minimax Theorem from the Duality Theorem. [Hint: first show that the value of the LP must be negative.]

12.6. *Infinite Matrix Games*
Consider the following two-player game. Each player mentions a natural number. The player with the higher number receives one Euro from the player with the lower number. If the numbers are equal then no player receives anything.

[1] $\mathbf{J}$ denotes the row vector and $\mathbf{J}^T$ the transpose, i.e., the column vector. In general, we omit the transpose notation if confusion is unlikely.

(a) Write this game in the form of an infinite matrix game A.
(b) Compute $\sup_{\mathbf{p}} \inf_{\mathbf{q}} \mathbf{p}A\mathbf{q}$ and $\inf_{\mathbf{q}} \sup_{\mathbf{p}} \mathbf{p}A\mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are probability distributions over the rows and the columns of A, respectively. (Conclude that this game has no 'value'.)

12.7. *Equalizer Theorem*
Let v be the value of the $m \times n$-matrix game A, and suppose that $\mathbf{p}A\mathbf{e}^n = v$ for every optimal strategy $\mathbf{p}$ of player 1. Show that player 2 has an optimal strategy $\mathbf{q}$ with $q_n > 0$.

12.4 Notes

The Minimax Theorem was first proved in von Neumann (1928). The simplex algorithm was developed by George Dantzig in 1947; see Dantzig (1963) or any textbook on linear programming or operations research.

With each linear programming problem we can associate a matrix game and thereby derive the Duality Theorem from the Minimax Theorem: see, e.g., Owen (1995).

For Problem 12.7 see also Raghavan (1994).

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