6 Noncooperative Games: Extensions

In Chaps. 2–5 we have studied noncooperative games in which the players have finitely many (pure) strategies. The reason for the finiteness restriction is that in such games special results hold, such as the existence of a value and optimal strategies for two-person zero-sum games, and the existence of a Nash equilibrium in mixed strategies for finite nonzero-sum games.

The basic game-theoretical concepts discussed in these chapters can be applied to more general games. Once, in a game-theoretic situation, the players, their possible strategies, and the associated payoffs are identified, the concepts of best reply and of Nash equilibrium can be applied. Also the concepts of backward induction, subgame perfection, and perfect Bayesian equilibrium carry over to quite general extensive form games. In games of incomplete information, the concept of player types and the associated Nash equilibrium (Bayesian Nash equilibrium) can be applied also if the game has infinitely many strategies.

The bulk of this chapter consists of diverse examples verifying these claims. The main objective of the chapter is, indeed, to show how the basic game-theoretic apparatus can be applied to various different conflict situations; and, of course, to show these applications themselves.

In Sect. 6.1 we generalize some of the concepts of Chaps. 2 and 3. This section serves only as background and general framework for the examples in the following sections—most of the remainder of this chapter can also be understood without this general framework and the reader may choose to postpone reading it. Concepts specific to extensive form games and to incomplete information games are adapted later, when they are applied. In Sects. 6.2–6.7 we discuss Cournot competition with complete and incomplete information, Bertrand competition, Stackelberg equilibrium, auctions with complete and incomplete information, mixed strategies with objective probabilities, and sequential bargaining. Variations on these topics and various other topics are treated in the problem section.

H. Peters, *Game Theory*, Springer Texts in Business and Economics,
DOI 10.1007/978-3-662-46950-7_6

6.1 General Framework: Strategic Games

An *n-person strategic game* is a $2n+1$-tuple

$$G = (N, S_1, \ldots, S_n, u_1, \ldots, u_n) \ ,$$

where

- $N = \{1, \ldots, n\}$, with $n \geq 1$, is the set of *players*;
- for every $i \in N$, S_i is the *strategy set* of player i;
- for every $i \in N$, $u_i : S = S_1 \times \ldots \times S_n \to \mathbb{R}$ is the *payoff function* of player i; i.e., for every strategy combination $(s_1, \ldots, s_n) \in S$ where $s_1 \in S_1$, …, $s_n \in S_n$, $u_i(s_1, \ldots, s_n) \in \mathbb{R}$ is player i's payoff.

A *best reply* of player i to the strategy combination $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ of the other players is a strategy $s_i \in S_i$ such that

$$u_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$$

for all $s_i' \in S_i$.

A *Nash equilibrium* of G is a strategy combination $(s_1^*, \ldots, s_n^*) \in S$ such that for each player i, s_i^* is a best reply to $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$.

A strategy $s_i' \in S_i$ of player i is *strictly dominated* by $s_i \in S_i$ if

$$u_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) > u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$$

for all $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$, i.e., for all strategy combinations of players other than i. Clearly, a strictly dominated strategy is never used in a Nash equilibrium.

Finally we define weak domination. A strategy $s_i' \in S_i$ of player i is *weakly dominated* by $s_i \in S_i$ if

$$u_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \geq u_i(s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n)$$

for all $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$, such that at least once this inequality is strict.

The reader may verify that matrix games (Chap. 2) and bimatrix games (Chap. 3) are special cases of this general framework, in which the set S_i is the set of all mixed strategies of player i. The same is true for the concepts of Nash equilibrium and domination discussed in these chapters.

6.2 Cournot Quantity Competition

6.2.1 Simple Version with Complete Information

In the simplest version of the famous Cournot model, two firms producing a homogenous good compete in quantity. Each firm offers a quantity of this good on the market. The price of the good depends on the total quantity offered: the higher this quantity is, the lower the price of the good. The profit for each firm is equal to total revenue (price times quantity) minus total cost. This gives rise to a two-person game in which the players are the firms, the players' strategies are the quantities offered and the payoff functions are the profit functions. In a simple version, the price depends linearly on total quantity and marginal cost is constant and positive while there are no fixed costs. Specifically, we study the following game.

(a) The set of players is $N = \{1, 2\}$.
(b) Each player $i = 1, 2$ has set of strategies $S_i = [0, \infty)$, with typical element q_i.
(c) The payoff function of player i is $\Pi_i(q_1, q_2) = q_i P(q_1, q_2) - cq_i$, for all $q_1, q_2 \geq 0$. Here,

$$P(q_1, q_2) = \begin{cases} a - q_1 - q_2 & \text{if } q_1 + q_2 \leq a \\ 0 & \text{if } q_1 + q_2 > a \end{cases}$$

is the market price of the good, where a is a constant, and c is marginal cost, with $a > c > 0$.

A Nash equilibrium in this game is a pair (q_1^C, q_2^C), with $q_1^C, q_2^C \geq 0$, of mutually best replies, that is,

$$\Pi_1(q_1^C, q_2^C) \geq \Pi_1(q_1, q_2^C),\ \ \Pi_2(q_1^C, q_2^C) \geq \Pi_2(q_1^C, q_2) \text{ for all } q_1, q_2 \geq 0\ .$$

This equilibrium is also called *Cournot equilibrium*. To find the equilibrium, we first compute the best reply functions, also called *reaction functions*. The reaction function $\beta_1(q_2)$ of player 1 is found by solving the maximization problem

$$\max_{q_1 \geq 0} \ \Pi_1(q_1, q_2)$$

for each given value of $q_2 \geq 0$. If $q_2 > a$ then $P(q_1, q_2) = 0$ for every q_1 and the profit of firm 1 is equal to $-cq_1$ so that, clearly, the maximum is attained by setting $q_1 = 0$. For $q_2 \leq a$ we have

$$\Pi_1(q_1, q_2) = \begin{cases} q_1(a - q_1 - q_2) - cq_1 & \text{if } q_1 \leq a - q_2 \\ -cq_1 & \text{if } q_1 > a - q_2 \end{cases}$$

so that the maximum is attained for some $q_1 \le a - q_2$. In that case, function

$$q_1(a - q_1 - q_2) - cq_1 = q_1(a - c - q_1 - q_2)$$

has to be maximized with respect to $q_1 \ge 0$. If $q_2 > a - c$ then the maximum is attained for $q_1 = 0$. If $q_2 \le a - c$ then we compute the maximum by setting the derivative with respect to q_1, namely the function $a - 2q_1 - q_2 - c$, equal to zero. This yields $q_1 = (a - c - q_2)/2$. (The second derivative is equal to -2 so that, indeed, we have a maximum.) Summarizing, we have

$$\beta_1(q_2) = \begin{cases} \{\frac{a-c-q_2}{2}\} & \text{if } q_2 \le a - c \\ \{0\} & \text{if } q_2 > a - c \ . \end{cases}$$

Since this reaction function is single-valued for all q_2, we can omit the braces at the right-hand side and write

$$\beta_1(q_2) = \begin{cases} \frac{a-c-q_2}{2} & \text{if } q_2 \le a - c \\ 0 & \text{if } q_2 > a - c \ . \end{cases} \tag{6.1}$$

By symmetric arguments we obtain for the reaction function of player 2:

$$\beta_2(q_1) = \begin{cases} \frac{a-c-q_1}{2} & \text{if } q_1 \le a - c \\ 0 & \text{if } q_1 > a - c \ . \end{cases} \tag{6.2}$$

These reaction functions are drawn in Fig. 6.1. The Nash equilibrium is the point of intersection of the reaction functions. It is obtained by simultaneously solving the two equations $q_1 = (a - c - q_2)/2$ and $q_2 = (a - c - q_1)/2$, resulting in

$$(q_1^C, q_2^C) = (\frac{a-c}{3}, \frac{a-c}{3}) \ .$$

6.2.1.1 Pareto Optimality

A pair (q_1, q_2) of strategies is *Pareto optimal* if there is no other pair (q_1', q_2') such that the associated payoffs are at least as good for both players and strictly better for at least one player. Not surprisingly, the equilibrium (q_1^C, q_2^C) is not Pareto optimal. For instance, both players can strictly benefit from joint profit maximization, attained by solving the problem

$$\max_{q_1, q_2 \ge 0} \ \Pi_1(q_1, q_2) + \Pi_2(q_1, q_2) \ .$$

This amounts to solving the maximization problem

$$\max_{q_1, q_2 \ge 0} \ (q_1 + q_2)(a - c - q_1 - q_2)$$

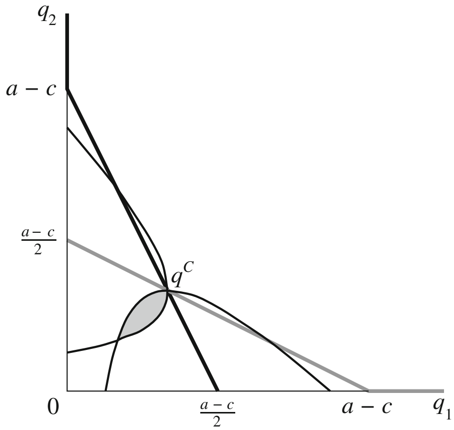

Fig. 6.1 The Cournot model: the *thick black* piecewise linear curve is the reaction function of player 1 and the *thick gray* piecewise linear curve is the reaction function of player 2. The point q^C is the Nash–Cournot equilibrium. The two isoprofit curves of the players through the Nash equilibrium are drawn. The curve intersecting the q_1-axis is the isoprofit curve of player 1: profit increases if this curve shifts downwards. The curve intersecting the q_2-axis is the isoprofit curve of player 2: profit increases if this curve shifts leftwards. The *shaded area* consists of the quantity combinations that Pareto dominate the equilibrium

or, writing $Q = q_1 + q_2$,

$$\max_{Q \geq 0} \quad Q(a - c - Q)$$

which yields $Q = (a - c)/2$. Observe that this is just the monopoly quantity. Thus, joint profit maximization is attained by any pair $(q_1, q_2) \geq 0$ with $q_1 + q_2 = (a - c)/2$. Taking, in particular, $q_1 = q_2 = (a - c)/4$ yields each player a profit of $(a - c)^2/8$, whereas in the Nash equilibrium each player obtains $(a - c)^2/9$. See also Fig. 6.1, where all points in the gray-shaded area 'Pareto dominate' the Nash equilibrium: the associated payoffs are at least as good for both agents and better for at least one agent.

6.2.2 Simple Version with Incomplete Information

Consider the Cournot model of Sect. 6.2.1 but now assume that the marginal cost of firm 2 is either high, c_H, or low, c_L, where $c_H > c_L > 0$. Firm 2 knows its marginal cost but firm 1 only knows that it is c_H with probability ϑ and c_L with probability $1 - \vartheta$. The cost of firm 1 is $c > 0$ and this is commonly known. In the terminology of Sect. 5.1, player 1 has only one type but player 2 has two types, c_H and c_L. The associated game is as follows.

(a) The player set is $\{1, 2\}$.
(b) The strategy set of player 1 is $[0, \infty)$ with typical element q_1, and the strategy set of player 2 is $[0, \infty) \times [0, \infty)$ with typical element (q_H, q_L). Here, q_H is the chosen quantity if player 2 has type c_H, and q_L is the chosen quantity if player 2 has type c_L.
(c) The payoff functions of the players are the expected payoff functions. These are

$$\Pi_i(q_1, q_H, q_L) = \vartheta \Pi_i(q_1, q_H) + (1 - \vartheta)\Pi_i(q_1, q_L) \ ,$$

for $i = 1, 2$, where $\Pi_i(\cdot, \cdot)$ is the payoff function from the Cournot model of Sect. 6.2.1.

To find the (Bayesian) Nash equilibrium, we first compute the best reply function or reaction function of player 1, by maximizing $\Pi_1(q_1, q_H, q_L)$ over $q_1 \geq 0$, with q_H and q_L regarded as given. Hence, we solve the problem

$$\max_{q_1 \geq 0} \ \vartheta \left[q_1(a - c - q_1 - q_H)\right] + (1 - \vartheta) \left[q_1(a - c - q_1 - q_L)\right] \ .$$

Assuming $q_H, q_L \leq a - c$ (this has to be checked later for the equilibrium), this problem is solved by setting the derivative with respect to q_1 equal to zero, which yields

$$q_1 = q_1(q_H, q_L) = \frac{a - c - \vartheta q_H - (1 - \vartheta)q_L}{2} \ . \tag{6.3}$$

Observe that, compared to (6.1), we now have the expected quantity $\vartheta q_H + (1 - \vartheta)q_L$ instead of q_2: this is due to the linearity of the model.

For player 2, we consider, for given q_1, the problem

$$\max_{q_H, q_L \geq 0} \ \vartheta \left[q_H(a - c_H - q_1 - q_H)\right] + (1 - \vartheta) \left[q_L(a - c_L - q_1 - q_L)\right] \ .$$

Since the first term in this function depends only on q_H and the second term only on q_L, solving this problem amounts to maximizing the two terms separately. In other words, we determine the best replies of types c_H and c_L separately.[1] Assuming $q_1 \leq a - c_H$ (and hence $q_1 \leq a - c_L$) this results in

$$q_H = q_H(q_1) = \frac{a - c_H - q_1}{2} \tag{6.4}$$

[1]This is generally so in a Bayesian, incomplete information game: maximizing the expected payoff of a player over all his types is equivalent to maximizing the payoff per type.

and

$$q_L = q_L(q_1) = \frac{a - c_L - q_1}{2} \ . \tag{6.5}$$

The Nash equilibrium is obtained by simultaneously solving (6.3)–(6.5) (using substitution or Gaussian elimination). The solution is the triple

$$\begin{aligned} q_1^C &= \frac{a - 2c + \vartheta c_H + (1-\vartheta)c_L}{3} \\ q_H^C &= \frac{a - 2c_H + c}{3} + \frac{1-\vartheta}{6}(c_H - c_L) \\ q_L^C &= \frac{a - 2c_L + c}{3} - \frac{\vartheta}{6}(c_H - c_L) \ . \end{aligned}$$

Assuming that the parameters of the game are such that these three values are nonnegative and that $q_1 \le a - c_H$ and $q_H, q_L \le a - c$, this is the Bayesian Nash–Cournot equilibrium of the game. This solution may be compared with the Nash equilibrium in the complete information model with asymmetric costs, see Problem 6.1. The high cost type of firm 2 produces more than it would in the complete information case: it benefits from the fact that firm 1 is unsure about the cost of firm 2 and therefore produces less than it would if it knew for sure that firm 2 had high costs. Similarly, the low cost firm 2 produces less.

6.3 Bertrand Price Competition

Consider two firms who compete in the price of a homogenous good. Specifically, assume that the demand q for the good is given by $q = q(p) = \max\{a - p, 0\}$ for every price $p \ge 0$, where a is a positive constant (the demand for the good if the price is zero). The firm with the lower price serves the whole market; if prices are equal the firms share the market equally. Each firm has the same marginal cost $0 \le c < a$, and no fixed cost. If firm 1 sets a price p_1 and firm 2 sets a price p_2, then the profit of firm 1 is

$$\Pi_1(p_1, p_2) = \begin{cases} (p_1 - c)(a - p_1) & \text{if } p_1 < p_2 \text{ and } p_1 \le a \\ \frac{1}{2}(p_1 - c)(a - p_1) & \text{if } p_1 = p_2 \text{ and } p_1 \le a \\ 0 & \text{in all other cases.} \end{cases}$$

Similarly, the profit of firm 2 is

$$\Pi_2(p_1, p_2) = \begin{cases} (p_2 - c)(a - p_2) & \text{if } p_2 < p_1 \text{ and } p_2 \le a \\ \frac{1}{2}(p_2 - c)(a - p_2) & \text{if } p_1 = p_2 \text{ and } p_2 \le a \\ 0 & \text{in all other cases.} \end{cases}$$

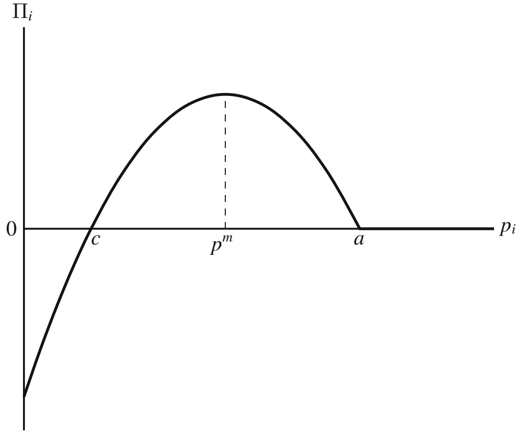

Fig. 6.2 The profit function of firm i in the monopoly situation

Thus, the two firms are the players in this game, and their profit functions are the payoff functions; the strategy sets are $[0, \infty)$ for each, with typical elements p_1 and p_2. To find a Nash equilibrium (*Bertrand equilibrium*) we first compute the best reply functions (reaction functions). An important role is played by the price that maximizes profit if there is only one firm in the market, i.e., the monopoly price $p^m = (a + c)/2$, obtained by solving the problem

$$\max_{p \geq 0} \; (p - c)(a - p) \; .$$

Note that the monopoly profit function (or the profit function of each firm in the monopoly situation) is a quadratic function, and that profit increases as the price gets closer to the monopoly price. See Fig. 6.2.

To determine player 1's best reply function $\beta_1(p_2)$ we distinguish several cases.

If $p_2 < c$, then any $p_1 \leq p_2$ yields player 1 a negative payoff, whereas any $p_1 > p_2$ yields a payoff of zero. Hence, the set of best replies in this case is the interval (p_2, ∞).

If $p_2 = c$, then any $p_1 < p_2$ yields a negative payoff for player 1, and any $p_1 \geq p_2$ yields zero payoff. So the set of best replies in this case is the interval $[c, \infty)$.

If $c < p_2 \leq p^m$, then the best reply of player 1 would be a price below p_2 (to obtain the whole market) and as close to the monopoly price as possible (to maximize payoff) but such a price does not exist: for any price $p_1 < p_2$, a price in between p_1 and p_2 would still be better. Hence, in this case the set of best replies of player 1 is empty.[2]

If $p_2 > p^m$ then the unique best reply of player 1 is the monopoly price p^m.

[2] If prices are in smallest monetary units this somewhat artificial consequence is avoided. See Problem 6.7.

Summarizing we obtain

$$\beta_1(p_2) = \begin{cases} \{p_1 \mid p_1 > p_2\} & \text{if } p_2 < c \\ \{p_1 \mid p_1 \geq c\} & \text{if } p_2 = c \\ \emptyset & \text{if } c < p_2 \leq p^m \\ \{p^m\} & \text{if } p_2 > p^m. \end{cases}$$

For player 2, similarly,

$$\beta_2(p_1) = \begin{cases} \{p_2 \mid p_2 > p_1\} & \text{if } p_1 < c \\ \{p_2 \mid p_2 \geq c\} & \text{if } p_1 = c \\ \emptyset & \text{if } c < p_1 \leq p^m \\ \{p^m\} & \text{if } p_1 > p^m. \end{cases}$$

The point(s) of intersection of these best reply functions can be found by making a diagram or by direct inspection. We follow the latter method and leave the diagram method to the reader. If $p_2 < c$ then by $\beta_1(p_2)$ a best reply p_1 satisfies $p_1 > p_2$. But then, according to $\beta_2(p_1)$, we must have $p_2 = p^m$, a contradiction since $p^m > c$. Therefore, in equilibrium, we must have $p_2 \geq c$. If $p_2 = c$, then $p_1 \geq c$; if however, $p_1 > c$ then the only possibility is $p_2 = p^m$, a contradiction. Hence, $p_1 = c$ as well and, indeed, $p_1 = p_2 = c$ is a Nash equilibrium. If $p_2 > c$, then the only possibility is $p_1 = p^m$ but then p_2 is never a best reply. We conclude that the unique Nash equilibrium (Bertrand equilibrium) is $p_1 = p_2 = c$.

It is also possible to establish this result without completely computing the best reply functions. Suppose, in equilibrium, that $p_1 \neq p_2$, say $p_1 < p_2$. If $p_1 < p^m$ then player 1 can increase his payoff by setting a higher price still below p_2. If $p_1 \geq p^m$ then player 2 can increase his payoff by setting a price below p_1, e.g., slightly below p^m if $p_1 = p^m$ and equal to p^m if $p_1 > p^m$. Hence, we must have $p_1 = p_2$ in equilibrium. If this common price is below c then each player can improve by setting a higher price. If this common price is above c then each player can improve by setting a slightly lower price. Hence, the only possibility that remains is $p_1 = p_2 = c$, and this is indeed an equilibrium, as can be verified directly.

A few remarks on this equilibrium are in order. First, it is again Pareto inferior. For example, both firms setting the monopoly price results in higher profits. Second, each firm plays a weakly dominated strategy: any price $c < p_i < a$ weakly dominates $p_i = c$, since it always results in a positive or zero profit whereas $p_i = c$ always results in zero profit. Third, the Bertrand equilibrium is beneficial from the point of view of the consumers: it maximizes consumer surplus.

See Problem 6.3(d)–(f) for an example of price competition with heterogenous goods.

6.4 Stackelberg Equilibrium

In the Cournot model of Sect. 6.2.1, the two firms move simultaneously. Consider now the situation where firm 1 moves first, and firm 2 observes this move and moves next. This situation has already been discussed in Chap. 1. The corresponding extensive form game is given in Fig. 6.3. In this game, player 1 has infinite action/strategy set $[0, \infty)$, with typical element q_1. In the diagram, we use a zigzag line to express the fact that the number of actions is infinite. Player 2 has the infinite set of actions $[0, \infty)$ with typical element q_2, again represented by a zigzag line. A strategy of player 2 assigns to each information set, hence to each decision node—the game has perfect information—an action. Since each decision node of player 2 follows an action q_1 of player 1, a strategy of player 2 is a function $s_2 : [0, \infty) \to [0, \infty)$. Hence, $q_2 = s_2(q_1)$ is the quantity that firm 2 offers if firm 1 has offered q_1. Obviously, the number of strategies of player 2 is infinite as well.[3] The appropriate solution concept is *backward induction* or *subgame perfect* equilibrium. The subgames of this game are the entire game and the infinite number of one-player games starting at each decision node of player 2, i.e., following each choice q_1 of player 1. Hence, the subgame perfect equilibrium can be found by backward induction, as follows. In each subgame for player 2, that is, after each choice q_1, player 2 should play optimally. This means that player 2 should play according to the reaction function $\beta_2(q_1)$ as derived in (6.2). Then, going back to the beginning of the game, player 1 should choose $q_1 \geq 0$ so as to maximize $\Pi_1(q_1, \beta_2(q_1))$. In other words, player 1 takes player 2's optimal reaction into account when choosing q_1. Assuming $q_1 \leq a - c$ (it is easy to verify that $q_1 > a - c$ is not optimal) player 1 maximizes the expression

$$q_1 \left(a - c - q_1 - \frac{a - c - q_1}{2} \right) .$$

The maximum is obtained for $q_1 = (a - c)/2$, and thus $q_2 = \beta_2((a - c)/2) = (a - c)/4$. Hence, the subgame perfect equilibrium of the game is:

$$q_1 = (a - c)/2, \; q_2 = \beta_2(q_1) .$$

The subgame perfect equilibrium *outcome* is by definition the resulting play of the game, that is, the actions chosen on the equilibrium path in the extensive form. In

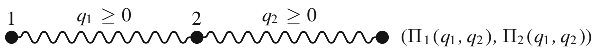

Fig. 6.3 Extensive form representation of the Stackelberg game with firm 1 as the leader

[3] In mathematical notation the strategy set of player 2 is the set $[0, \infty)^{[0,\infty)}$.

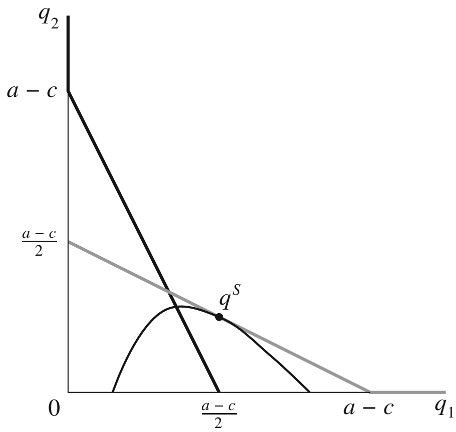

Fig. 6.4 As before, the *thick black curve* is the reaction function of player 1 and the *thick gray curve* is the reaction function of player 2. The point $q^S = (\frac{a-c}{2}, \frac{a-c}{4})$ is the Stackelberg equilibrium outcome: it is the point on the reaction curve of player 2 where player 1 maximizes profit. The associated isoprofit curve of player 1 is drawn

this case, the equilibrium outcome is:

$$q_1^S = (a - c)/2,\; q_2^S = (a - c)/4 \,.$$

The letter 'S' here is the first letter of 'Stackelberg', after whom this equilibrium is named. More precisely, this subgame perfect equilibrium (or outcome) is called the *Stackelberg equilibrium* (or *outcome*) with player 1 as the *leader* and player 2 as the *follower*. Check that player 1's profit in this equilibrium is higher and player 2's profit is lower than in the Cournot equilibrium $q_1^C = q_2^C = (a - c)/3$. See also Problem 6.9.

The Stackelberg equilibrium is depicted in Fig. 6.4. Observe that player 1, the leader, picks the point on the reaction curve of player 2 which has maximal profit for player 1. Hence, player 2 is on his reaction curve but player 1 is not.

6.5 Auctions

An auction is a procedure to sell goods among various interested parties, such that the prices are determined in the procedure. Examples range from selling a painting through an ascending bid auction (English auction) and selling flowers through a descending bid auction (Dutch auction) to tenders for public projects and selling mobile telephone frequencies.

In this section we consider a few simple, classical auction models. We start with first and second-price sealed-bid auctions under complete information, continue with a first-price sealed bid auction with incomplete information, and end with a double auction between a buyer and a seller. Some variations and extensions are discussed in Problems 6.10–6.14.

6.5.1 Complete Information

Consider n individuals who are interested in one indivisible object. Each individual i has valuation $v_i > 0$ for the object. We assume without loss of generality $v_1 \geq v_2 \geq \ldots \geq v_n$. In a *first-price sealed-bid auction* each individual submits a bid $b_i \geq 0$ for the object: the bids are simultaneous and independent ('sealed bids'). The individual with the highest bid wins the auction and obtains the object at a price equal to his own bid ('first price'). In case there are more highest bidders, the bidder among these with the lowest number wins the auction and pays his own bid—this is just a tie-breaking rule, which can be replaced by alternative tie-breaking assumptions without affecting the basic results.

This situation gives rise to a game with player set $N = \{1, 2, \ldots, n\}$, where each player i has strategy set $S_i = [0, \infty)$ with typical element b_i. The payoff function to player i is[4]

$$u_i(b_1, \ldots, b_i, \ldots, b_n) = \begin{cases} v_i - b_i & \text{if } i = \min\{k \in N \mid b_k \geq b_j \text{ for all } j \in N\} \\ 0 & \text{otherwise.} \end{cases}$$

One Nash equilibrium in this game is the strategy combination $(b_1, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$. To check this one should verify that no player has a better bid, given the bids of the other players: see Problem 6.10. In this equilibrium, player 1 obtains the object and pays v_2, the second-highest valuation. Check that this is also the outcome one would approximately expect in an auction with ascending bids (English auction) or descending bids (Dutch auction).

This game has many Nash equilibria. In each of these equilibria, however, a player with a highest valuation obtains the object. Bidding one's true valuation as well as bidding higher than one's true valuation are weakly dominated strategies. Bidding lower than one's true valuation is not weakly dominated. (See Sect. 6.1 for the definition of weak domination.) Problem 6.10 is about proving all these statements.

A *second-price sealed-bid auction* differs from a first-price sealed-bid auction only in that the winner now pays the bid of the second highest bidder. In the case that two or more players have the highest bid the player with the lowest number wins and pays his own bid. The main property of this auction is that for each player i, the strategy of bidding the true valuation v_i weakly dominates all other strategies. This property and other properties are collected in Problem 6.11.

[4] Also here the assumption is that the players know the game. This means, in particular, that the players know each other's valuations.

6.5.2 Incomplete Information

We consider the same setting as in Sect. 6.5.1 but now assume that each bidder knows his own valuation but has only a probabilistic estimate about the valuations of the other bidders. In the terminology of types (cf. Sect. 5.1), a bidder's valuation is his true type, and each bidder holds a probability distribution over the type combinations of the other bidders. To keep things simple, we assume that every bidder's type is drawn independently from the uniform distribution over the interval $[0, 1]$, that this is common knowledge, and that each bidder learns his true type. The auction is a first-price sealed-bid auction. Of course, we can no longer fix the ordering of the valuations, but we can still employ the same tie-breaking rule in case of more than one highest bid.

We discuss the case of two bidders and postpone the extension to $n > 2$ bidders until Problem 6.13. In the associated two-person game, a strategy of player $i \in \{1, 2\}$ should assign a bid to each of his possible types. Since the set of possible types is the interval $[0, 1]$ and it does not make sense to ever bid more than 1, a strategy is a function $s_i : [0, 1] \to [0, 1]$. Hence, if player i's type is v_i, then $b_i = s_i(v_i)$ is his bid according to the strategy s_i. The payoff function u_i of player i assigns to each strategy pair (s_i, s_j) (where j is the other player) player i's expected payoff if these strategies are played. In a Nash equilibrium of the game, player i maximizes this payoff function given the strategy of player j, and *vice versa*. For this, it is sufficient that *each type* of player i maximizes its expected payoff given the strategy of player j, and *vice versa*; in other words (cf. Chap. 5), it is sufficient that the strategies form a Bayesian equilibrium.[5]

We claim that $s_1^*(v_1) = v_1/2$ and $s_2^*(v_2) = v_2/2$ is a Nash equilibrium of this game. To prove this, first consider type v_1 of player 1 and suppose that player 2 plays strategy s_2^*. If player 1 bids b_1, then the probability that player 1 wins the auction is equal to the probability that the bid of player 2 is smaller than or equal to b_1. This probability is equal to the probability that $v_2/2$ is smaller than or equal to b_1, i.e., to the probability that v_2 is smaller than or equal to $2b_1$. We may assume without loss of generality that $b_1 \leq 1/2$, since according to s_2^* player 2 will never bid higher than $1/2$. Since v_2 is uniformly distributed over the interval $[0, 1]$ and $2b_1 \leq 1$, the probability that v_2 is smaller than or equal to $2b_1$ is just equal to $2b_1$. Hence, the probability that the bid b_1 of player 1 is winning is equal to $2b_1$ if player 2 plays s_2^*, and therefore the expected payoff from this bid is equal to $2b_1(v_1 - b_1)$ (if player 1 loses his payoff is zero). This is maximal for $b_1 = v_1/2$. Hence, $s_1^*(v_1) = v_1/2$ is a best reply to s_2^*. The converse is almost analogous—the only difference being that for player 2 to win player 1's bid must be strictly smaller due to the tie-breaking rule employed, but this does not change the associated probability under the uniform distribution. Hence, we have proved the claim.

[5]The difference is that, for instance, single types may not play a best reply in a Nash equilibrium since they have probability zero and therefore do not influence the expected payoffs.

Thus, in this equilibrium, each bidder bids half his true valuation, and a player with the highest valuation wins the auction.

How about the second-price sealed-bid auction with incomplete information? This is more straightforward since bidding one's true valuation ($s_i(v_i) = v_i$ for all $v_i \in [0, 1]$) is a strategy that weakly dominates every other strategy, for each player i. Hence, these strategies still form a (Bayesian) Nash equilibrium. See Problem 6.11.

6.5.3 Incomplete Information: A Double Auction

Assume there are two players, a buyer and a seller. The seller owns an object, for which he has a valuation v_s. The buyer has a valuation v_b for this object. These valuations are independently drawn from the uniform distribution over $[0, 1]$. The seller knows (learns) his own valuation, and also the buyer knows his own valuation, but each of them does not know the valuation of the other player, only that this is drawn from the uniform distribution over $[0, 1]$.

The auction works as follows. The buyer and the seller independently and simultaneously mention prices p_b and p_s. If $p_b \geq p_s$, then trade takes place at the average price $p = (p_b + p_s)/2$, and the payoffs are $v_b - p$ to the buyer and $p - v_s$ to the seller. If $p_b < p_s$ then no trade takes place and both have payoff 0.

This is a game of incomplete information. The buyer has infinitely many types $v_b \in [0, 1]$, and the seller has infinitely many types $v_s \in [0, 1]$. A strategy assigns a price to each type. A strategy for the buyer is therefore a function $p_b : [0, 1] \to [0, 1]$, where $p_b(v_b)$ is the price that the buyer offers if his type (valuation) is v_b. (Observe that we may indeed assume that the price is never higher than 1.) Similarly, a strategy for the seller is a function $p_s : [0, 1] \to [0, 1]$, where $p_s(v_s)$ is the price that the seller asks if his type (valuation) is v_s.

Suppose the seller plays a strategy $p_s(\cdot)$. Then the expected payoff to the buyer if his valuation is v_b and he offers price p_b is equal to

$$\left[v_b - \frac{p_b + E[p_s(v_s)|p_b \geq p_s(v_s)]}{2}\right] \text{Prob}[p_b \geq p_s(v_s)] \tag{6.6}$$

where $E[p_s(v_s)|p_b \geq p_s(v_s)]$ denotes the expected price asked by the seller according to his strategy $p_s(\cdot)$, conditional on this price being smaller than or equal to the price p_b of the buyer.

Similarly, suppose the buyer plays a strategy $p_b(\cdot)$. Then the expected payoff to the seller if his valuation is v_s and he asks price p_s is equal to

$$\left[\frac{p_s + E[p_b(v_b)|p_s \leq p_b(v_b)]}{2} - v_s\right] \text{Prob}[p_s \leq p_b(v_b)] \tag{6.7}$$

where $E[p_b(v_b)|p_s \leq p_b(v_b)]$ denotes the expected price offered by the buyer according to his strategy $p_b(\cdot)$, conditional on this price being larger than or equal to the price p_s of the seller.

Now the pair of strategies $(p_b(\cdot), p_s(\cdot))$ is a (Bayesian) Nash equilibrium if for each $v_b \in [0, 1]$, $p_b(v_b)$ solves

$$\max_{p_b \in [0,1]} \left[v_b - \frac{p_b + E[p_s(v_s)|p_b \geq p_s(v_s)]}{2} \right] \text{Prob}[p_b \geq p_s(v_s)] \tag{6.8}$$

and for each $v_s \in [0, 1]$, $p_s(v_s)$ solves

$$\max_{p_s \in [0,1]} \left[\frac{p_s + E[p_b(v_b)|p_s \leq p_b(v_b)]}{2} - v_s \right] \text{Prob}[p_s \leq p_b(v_b)] \ . \tag{6.9}$$

This game has many Nash equilibria. Ideally, trade should take place whenever it is efficient, i.e., whenever $v_b \geq v_s$. In Problem 6.14 we will see that not all Nash equilibria are equally efficient.

6.6 Mixed Strategies and Incomplete Information

In this section we discuss how a *mixed* strategy Nash equilibrium in a bimatrix game can be obtained as a limit of *pure* strategy Bayesian Nash equilibria in associated games of incomplete information.

Consider the bimatrix game (cf. Chap. 3)

$$G = \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 2,1 & 2,0 \\ 3,0 & 1,3 \end{pmatrix} \end{array} ,$$

which has a unique Nash equilibrium $((p^*, 1 - p^*), (q^*, 1 - q^*))$ with $p^* = 3/4$ and $q^* = 1/2$. The interpretation of mixed strategies and of a mixed strategy Nash equilibrium in particular is an old issue in the game-theoretic literature. One obvious interpretation is that a player actually plays according to the equilibrium probabilities. Although there is some empirical evidence that this may occur in reality,[6] this interpretation may not be entirely convincing, in particular since in a mixed strategy Nash equilibrium a player is indifferent between all pure strategies played with positive probability (cf. Problem 3.8). An alternative interpretation—also mentioned in Sect. 3.1—is that a mixed strategy of a player represents the belief(s) of the other player(s) about the strategic choice of that player. For instance, in the above equilibrium, player 2 believes that player 1 plays T with probability 3/4. The drawback of this interpretation is that these beliefs are *subjective*, and it is not explained how they are formed. In this section we discuss a way to obtain a mixed strategy Nash equilibrium as the limit of pure strategy (Bayesian) Nash equilibria in games obtained by adding some *objective* uncertainty about the payoffs. In this

[6] For example, Walker and Wooders (2001).

way, the strategic uncertainty of players as expressed by their beliefs is replaced by the objective uncertainty of a chance move.

In the above example, suppose that the payoff to player 1 from (T, L) is the uncertain amount $2 + \alpha$ and the payoff to player 2 from (B, R) is the uncertain amount $3 + \beta$. Assume that both α and β are (independently) drawn from a uniform distribution over the interval $[0, x]$, where $x > 0$. Moreover, player 1 learns the true value of α and player 2 learns the true value of β, and all this is common knowledge among the players. In terms of types, player 1 knows his type α and player 2 knows his type β. The new payoffs are given by

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} 2+\alpha, 1 & 2, 0 \\ 3, 0 & 1, 3+\beta \end{pmatrix} \end{array} .$$

A (pure) strategy of a player assigns an action to each of his types. Hence, for player 1 it is a map $s_1 : [0, x] \to \{T, B\}$ and for player 2 it is a map $s_2 : [0, x] \to \{L, R\}$.

To find an equilibrium of this incomplete information game, suppose that player 2 has the following rather simple strategy: play L if β is small and play R if β is large. Specifically, let $b \in [0, x]$ such that each type $\beta \leq b$ plays L and each type $\beta > b$ plays R. Call this strategy s_2^b. What is player 1's best reply against s_2^b? Suppose the type of player 1 is α. If player 1 plays T, then his expected payoff is equal to $2 + \alpha$ times the probability that player 2 plays L plus 2 times the probability that player 2 plays R. The probability that player 2 plays L, given the strategy s_2^b, is equal to the probability that β is at most equal to b, and this is equal to b/x since β is uniformly distributed over $[0, x]$. Hence, the expected payoff to player 1 from playing T is

$$(2 + \alpha) \cdot \frac{b}{x} + 2 \cdot (1 - \frac{b}{x}) = 2 + \alpha \cdot \frac{b}{x} .$$

Similarly, the expected payoff to player 1 from playing B is

$$3 \cdot \frac{b}{x} + 1 \cdot (1 - \frac{b}{x}) = 1 + 2 \cdot \frac{b}{x} .$$

From this, it easily follows that T is at least as good as B if $\alpha \geq (2b - x)/b$. Hence, the following strategy of player 1 is a best reply against strategy s_2^b of player 2: play T if $\alpha \geq a$ and play B if $\alpha < a$, where $a = (2b - x)/b$. Call this strategy s_1^a.

For the converse, assume that player 1 plays s_1^a. To find player 2's best reply against s_1^a we proceed similarly as above. If type β of player 2 plays L then the expected payoff is 1 times the probability that player 1 plays T, hence 1 times $(x - a)/x$. If type β of player 2 plays R then his expected payoff is equal to $3 + \beta$ times the probability that player 1 plays B, hence $(3 + \beta)a/x$. So L is at least as good as R if $\beta \leq (x - 4a)/a$. Hence, a best reply of player 2 against s_1^a is the strategy s_2^b with $b = (x - 4a)/a$.

Summarizing these arguments, we have that $\left(s_1^a, s_2^b\right)$ is a (Bayesian) Nash equilibrium for

$$a = (2b - x)/b,\ b = (x - 4a)/a\ .$$

Solving these two equations simultaneously for solutions $a, b \in [0, x]$ yields:

$$a = (1/4)(x + 4 - \sqrt{x^2 + 16}),\ b = (1/2)(x - 4 + \sqrt{x^2 + 16})\ .$$

In this equilibrium, the a priori probability that player 1 will play T, that is, the probability of playing T before he learns his type, is equal to $(x - a)/x$, hence to $(\sqrt{x^2 + 16} + 3x - 4)/4x$. Similarly, the a priori probability that player 2 plays L is equal to b/x, hence to $(x-4+\sqrt{x^2 + 16})/2x$. What happens with these probabilities as the amount of uncertainty decreases, i.e., for x approaching 0? For player 1,

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 16} + 3x - 4}{4x} = \lim_{x \to 0} \frac{x/\sqrt{x^2 + 16} + 3}{4} = \frac{3}{4}\ ,$$

where the first equality follows from l'Hôpital's rule. Similarly for player 2:

$$\lim_{x \to 0} \frac{x - 4 + \sqrt{x^2 + 16}}{2x} = \lim_{x \to 0} \frac{1 + x/\sqrt{x^2 + 16}}{2} = \frac{1}{2}\ .$$

In other words, these probabilities converge to the mixed strategy Nash equilibrium of the original game.

6.7 Sequential Bargaining

In its simplest version, the bargaining problem involves two parties who have to agree on one alternative within a set of feasible alternatives. If they fail to reach an agreement, a specific 'disagreement' alternative is implemented. In the game-theoretic literature on bargaining there are two main strands, namely the cooperative, axiomatic approach, also known as the Nash bargaining problem, and the noncooperative, strategic approach, with the Rubinstein alternating offers game as main representative. In this section the focus is on the strategic approach, but in Sect. 6.7.2 we also mention the connection with the Nash bargaining solution. For an introduction to the axiomatic bargaining approach see Sect. 10.1, and see Chap. 21 for a more extensive study.

6.7.1 Finite Horizon Bargaining

Consider the example in Sect. 1.3.5, where two players bargain over the division of one unit of a perfectly divisible good, e.g., 1 liter of wine. If they do not reach an agreement, we assume that no one gets anything. To keep the problem as simple as possible, assume that the preferences of the players are represented by $u_1(\alpha) = u_2(\alpha) = \alpha$ for every $\alpha \in [0, 1]$. That is, obtaining an amount α of the good has utility α for each player. In this case the picture of the feasible set in Sect. 1.3.5 would be a triangle.

To model the bargaining process we consider the following alternating offers procedure. There are $T + 1$ rounds, where $T \in \mathbb{N}$. In round $t = 0$ player 1 makes a proposal, say $(\alpha, 1 - \alpha)$, where $\alpha \in [0, 1]$, meaning that he claims an amount α for himself, so that player 2 obtains $1 - \alpha$. Player 2 can either accept this proposal, implying that the proposal is implemented and the game is over, or reject the proposal. In the latter case the next round $t = 1$ starts, and the first round is repeated with the roles of the players interchanged: player 2 makes the proposal and player 1 accepts or rejects it. If player 1 accepts the proposal then it is implemented and the game is over; if player 1 rejects the proposal then round $t = 2$ starts, and the roles of the players are interchanged again. Thus, at even rounds, player 1 proposes; at odd rounds, player 2 proposes. The last possible round is round T: if this round is reached, then the disagreement alternative $(0, 0)$ is implemented.

We assume that utilities are discounted. Specifically, there is a discount factor $0 < \delta < 1$ such that receiving an amount α at round t has utility $\delta^t \alpha$ at round 0. Or, receiving an amount α at time t has utility $\delta\alpha$ at round $t - 1$. This reflects the fact that receiving the same amount earlier is more valuable.

In Fig. 6.5 this bargaining procedure is represented as a game in extensive form. Here, we assume that T is odd, so that the last proposal at time $t = T - 1$ is made by player 1.

We look for a subgame perfect equilibrium of this game, which can be found by backward induction. Note that subgames start at each decision node of a player, and that at each such node in Fig. 6.5 where a player has to make, accept or reject

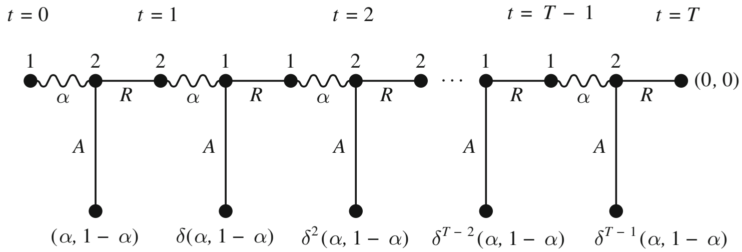

Fig. 6.5 The extensive form representation of the finite horizon bargaining procedure. The number of rounds $T + 1$ is even, α denotes the proposed amount for player 1, A is acceptance and R is rejection

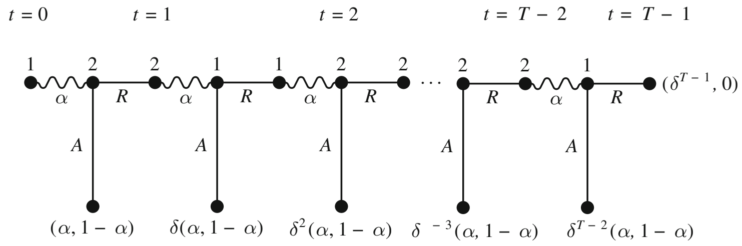

Fig. 6.6 The game of Fig. 6.5 reduced by replacing rounds $T-1$ and T by the equilibrium payoffs of the associated subgames

a proposal, actually infinitely many subgames start, since there are infinitely many possible paths leading to that node.

To start the analysis, at the final decision node, player 2 accepts if $\alpha < 1$ and is indifferent between acceptance and rejection if $\alpha = 1$: this is because rejection results in getting 0 at round T. In the subgame starting at round $T-1$ with a proposal of player 1, the only equilibrium therefore is for player 1 to propose $\alpha = 1$ and for player 2 to accept any proposal: if player 2 would reject $\alpha = 1$ then player 1 could improve by proposing $0 < \alpha < 1$; hence, we only have an equilibrium if player 2 accepts $\alpha = 1$ and, clearly, proposing $\alpha = 1$ is optimal for player 1. Hence, we can replace the part of the game from round $T-1$ on by the pair of payoffs $(\delta^{T-1}, 0)$, as in Fig. 6.6.

In this reduced game, in the subgame starting at round $T-2$, player 1 can obtain δ^{T-1} by rejecting player 2's proposal. Recall that this is the discounted utility at time $t = 0$ of receiving an amount 1 at time $T-1$; the discounted utility of the amount 1 at time $T-2$ is equal to δ. Hence, in a backward induction equilibrium player 2 proposes $\alpha = \delta$ and player 1 accepts this proposal or any higher α and rejects any lower α. Hence, we can replace this whole subgame by the pair of payoffs $(\delta^{T-2}\delta, \delta^{T-2}(1-\delta)) = (\delta^{T-1}, \delta^{T-2}(1-\delta))$. Continuing this line of reasoning, in the subgame starting at round $T-3$, player 1 proposes $\alpha = 1-\delta(1-\delta)$, which will be accepted by player 2. This results in the payoffs $(\delta^{T-3}(1-\delta(1-\delta)), \delta^{T-2}(1-\delta))$. This can be written as $\left(\delta^{T-3}(1-\delta+\delta^2), \delta^{T-3}(\delta-\delta^2)\right)$. And so on and so forth. The general principle is that each player offers the other player a share equal to δ times the share the other player can expect in the next round.

By backtracking all the way to round 0 (see Table 6.1), we find that player 1 proposes $1-\delta+\delta^2-\ldots+\delta^{T-1}$ and player 2 accepts this proposal, resulting in the payoffs $1-\delta+\delta^2-\ldots+\delta^{T-1}$ for player 1 and $\delta-\delta^2+\ldots-\delta^{T-1}$ for player 2. This is the subgame perfect equilibrium *outcome* of the game and the associated payoffs. This outcome is the path of play, induced by the following subgame perfect *equilibrium*:

- At even rounds t, player 1 proposes $\alpha = 1-\delta+\ldots+\delta^{T-1-t}$ and player 2 accepts this proposal or any smaller α, and rejects any larger α.

Table 6.1 The proposals made in the subgame perfect equilibrium

Round	Proposer	Share for player 1	Share for player 2
T		0	0
$T-1$	1	1	0
$T-2$	2	δ	$1-\delta$
$T-3$	1	$1-\delta+\delta^2$	$\delta-\delta^2$
$T-4$	2	$\delta-\delta^2+\delta^3$	$1-\delta+\delta^2-\delta^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	1	$1-\delta+\delta^2-\ldots+\delta^{T-1}$	$\delta-\delta^2+\ldots-\delta^{T-1}$

- At odd rounds t, player 2 proposes $\alpha = \delta-\delta^2+\ldots+\delta^{T-1-t}$ and player 1 accepts this proposal or any larger α, and rejects any smaller α.

In Problem 6.16 some variations on this finite horizon bargaining game are discussed.

6.7.2 Infinite Horizon Bargaining

In this subsection we consider the same bargaining problem as in the previous subsection, but now we assume $T = \infty$: the number of rounds may potentially be infinite. If no agreement is ever reached, then no player obtains anything. This game, like the finite horizon game, has many Nash equilibria: see Problem 6.16(f).

One way to analyze the game is to consider the finite horizon case and take the limit as T approaches infinity: see Problem 6.16(e). In fact, the resulting distribution is the uniquely possible outcome of a subgame perfect equilibrium, as can be seen by comparing the answer to Problem 6.16(e) with the result presented below. Of course, this claim is not proved by just taking the limit.

Note that a subgame perfect equilibrium cannot be obtained by backward induction, since the game has no final decision nodes. Here, we will just describe a pair of strategies and show that they are a subgame perfect equilibrium of the game. A proof that the associated outcome is the unique outcome resulting in any subgame perfect equilibrium can be found in the literature.

Let $\mathbf{x}^* = (x_1^*, x_2^*)$ and $\mathbf{y}^* = (y_1^*, y_2^*)$ be such that $x_1^*, x_2^*, y_1^*, y_2^* \geq 0$, $x_1^* + x_2^* = y_1^* + y_2^* = 1$, and moreover

$$x_2^* = \delta y_2^*, \quad y_1^* = \delta x_1^* \,. \tag{6.10}$$

It is not difficult to verify that $\mathbf{x}^* = (1/(1+\delta), \delta/(1+\delta))$ and $y^* = (\delta/(1+\delta), 1/(1+\delta))$. Consider the following strategies for players 1 and 2, respectively:

(σ_1^*) At $t = 0, 2, 4, \ldots$ propose $\mathbf{x}^*$; at $t = 1, 3, 5, \ldots$ accept a proposal $\mathbf{z} = (z_1, z_2)$ of player 2 if and only if $z_1 \geq \delta x_1^*$.

(σ_2^*) At $t = 1, 3, 5, \ldots$ propose $\mathbf{y}^*$; at $t = 0, 2, 4, \ldots$ accept a proposal $\mathbf{z} = (z_1, z_2)$ of player 1 if and only if $z_2 \geq \delta y_2^*$.

These strategies are stationary: the players always make the same proposals. Moreover, a player accepts any proposal that offers him at least the discounted value of his own demand. According to (6.10), player 2 accepts the proposal $\mathbf{x}^*$ and player 1 accepts the proposal $\mathbf{y}^*$. Hence, play of the strategy pair (σ_1^*, σ_2^*) results in player 1's proposal $\mathbf{x}^* = (1/(1+\delta), \delta/(1+\delta))$ being accepted at round 0, so that these are also the payoffs. We will show that (σ_1^*, σ_2^*) is a subgame perfect equilibrium of the game.

To show this, note that there are two kinds of subgames: subgames where a player has to make a proposal; and subgames where a proposal is on the table and a player has to choose between accepting and rejecting the proposal.

For the first kind of subgame we may without loss of generality consider the entire game, i.e., the game starting at $t = 0$. We have to show that (σ_1^*, σ_2^*) is a Nash equilibrium in this game. First, suppose that player 1 plays σ_1^*. By accepting player 1's proposal at $t = 0$, player 2 has a payoff of $\delta/(1+\delta)$. By rejecting this proposal, the maximum player 2 can obtain against σ_1^* is $\delta/(1+\delta)$, by proposing $\mathbf{y}^*$ in round $t = 1$. Proposals $\mathbf{z}$ with $z_2 > y_2^*$ and thus $z_1 < y_1^*$ are rejected by player 1. Hence, σ_2^* is a best reply against σ_1^*. Similarly, if player 2 plays σ_2^*, then the best player 1 can obtain is x_1^* at $t = 0$ with payoff $1/(1+\delta)$, since player 2 will reject any proposal that gives player 1 more than this, and also does not offer more.

For the second kind of subgame, again we may without loss of generality take $t = 0$ and assume that player 1 has made some proposal, say $\mathbf{z} = (z_1, z_2)$—the argument for t odd, when there is a proposal of player 2 on the table, is analogous. First, suppose that in this subgame player 1 plays σ_1^*. If $z_2 \geq \delta y_2^*$, then accepting this proposal yields player 2 a payoff of $z_2 \geq \delta y_2^* = \delta/(1+\delta)$. By rejecting, the maximum player 2 can obtain against σ_1^* is $\delta/(1+\delta)$ by proposing $\mathbf{y}^*$ at $t = 1$, which will be accepted by player 1. If, on the other hand, $z_2 < \delta y_2^*$, then player 2 can indeed better reject $\mathbf{z}$ and obtain $\delta/(1+\delta)$ by proposing $\mathbf{y}^*$ at $t = 1$. Hence, σ_2^* is a best reply against σ_1^*. Next, suppose player 2 plays σ_2^*. Then $\mathbf{z}$ is accepted if $z_2 \geq \delta y_2^*$ and rejected otherwise. In the first case it does not matter how player 1 replies, and in the second case the game starts again with player 2 as the first proposer, and by an argument analogous to the argument in the previous paragraph, player 1's best reply is σ_1^*.

We have, thus, shown that (σ_1^*, σ_2^*) is a subgame perfect equilibrium of the game. In Problem 6.17 some variations on this game are discussed.

We conclude this section with two remarks.

Remark 6.1 Nothing in the whole analysis changes if we view the number δ not as a discount factor but as the probability that the game continues to the next round. Specifically, if a proposal is rejected, then assume that with probability $1 - \delta$ the game stops and each player receives 0, and with probability δ the game continues in the usual way. Under this alternative interpretation, the game ends with probability 1 [Problem 6.17(e)], which makes the infinite horizon assumption more acceptable. □

Remark 6.2 In the subgame perfect equilibrium of the infinite horizon game the shares are $1/(1+\delta)$ for player 1 and $\delta/(1+\delta)$ for player 2. For δ converging to 1, i.e., the players becoming more patient, these shares converge to $1/2$ for each, which is the Nash bargaining outcome of the game, arising from maximizing the product $\alpha(1-\alpha)$ for $0 \le \alpha \le 1$ (cf. Sect. 1.3.5). This is true more generally, i.e., also if the utility functions are more general. See Sects. 10.1.2 and 21.4. □

6.8 Problems

6.1. *Cournot with Asymmetric Costs*
Consider the Cournot model of Sect. 6.2.1 but now assume that the firms have different marginal costs $c_1, c_2 \ge 0$. Assume $0 \le c_1, c_2 < a$ and $a \ge 2c_1 - c_2$, $a \ge 2c_2 - c_1$. Compute the Nash equilibrium.

6.2. *Cournot Oligopoly*
Consider the Cournot model of Sect. 6.2.1 but now assume that there are n firms instead of two. Each firm $i = 1, \ldots, n$ offers $q_i \ge 0$ and the market price is

$$P(q_1, \ldots, q_n) = \max\{a - q_1 - \ldots - q_n, 0\} .$$

Each firm still has marginal cost c with $a > c \ge 0$ and no fixed costs.

(a) Formulate the game corresponding to this situation. In particular, write down the payoff functions.
(b) Derive the reaction functions of the players.
(c) Derive a Nash equilibrium of the game by trying equal quantities offered. What happens if the number of firms becomes large?
(d) Show that the Nash equilibrium found in (c) is unique.

6.3. *Quantity Competition with Heterogenous Goods*
Suppose, in the Cournot model, that the firms produce heterogenous goods, which have different market prices. Specifically, suppose that these market prices are given by:

$$p_1 = \max\{5 - 3q_1 - 2q_2, 0\},\ p_2 = \max\{4.5 - 1.5\,q_1 - 3q_2, 0\} .$$

The firms still compete in quantities and have equal constant marginal costs c, with $c < 4.5$.

(a) Formulate the game corresponding to this situation. In particular, write down the payoff functions.
(b) Solve for the reaction functions and the Nash equilibrium of this game. Also compute the corresponding prices and profits.

(c) Compute the quantities at which joint profit is maximized. Also compute the corresponding prices.
In (d)–(f), we assume that the firms compete in prices.
(d) Derive the demands q_1 and q_2 as a function of the prices. Set up the associated game where the prices p_1 and p_2 are now the strategic variables.
(e) Solve for the reaction functions and the Nash equilibrium of this game. Also compute the corresponding quantities and profits.
(f) Compute the prices at which joint profit is maximized. Also compute the corresponding quantities.
(g) Compare the results found under (b) and (c) with those under (e) and (f).

6.4. *A Numerical Example of Cournot Competition with Incomplete Information*
Redo the model of Sect. 6.2.2 for the following values of the parameters: $a = 1$, $c = 0$, $\vartheta = 1/2$, $c_L = 0$, $c_H = 1/4$. Compute the Nash equilibrium and compare with what was found in the text. Also compare with the complete information case by using the answer to Problem 6.1.

6.5. *Cournot Competition with Two-Sided Incomplete Information*
Consider the Cournot game of incomplete information of Sect. 6.2.2 and assume that also firm 1 can have high costs or low costs, say c_h with probability π and c_l with probability $1 - \pi$. Set up the associated game and compute the (four) reaction functions. (Assume that the parameters of the game are such that the Nash equilibrium quantities are positive and the relevant parts of the reaction functions can be found by differentiating the payoff functions (i.e., no corner solutions).) How can the Nash equilibrium be computed? (You do not actually have to compute it explicitly.)

6.6. *Incomplete Information About Demand*
Consider the Cournot game of incomplete information of Sect. 6.2.2 but now assume that the incomplete information is not about the cost of firm 2 but about market demand. Specifically, assume that the number a can be either high, a_H, with probability ϑ, or low, a_L, with probability $1 - \vartheta$. Firm 2 knows the value for sure but firm 1 only knows these probabilities. Set up the game and compute the reaction functions and the Nash equilibrium. (Make appropriate assumptions on the parameters a_H, a_L, ϑ, and c to avoid corner solutions.)

6.7. *Variations on Two-Person Bertrand*

(a) Assume that the two firms in the Bertrand model of Sect. 6.3 have different marginal costs, say $c_1 < c_2 < a$. Derive the best reply functions and find the Nash–Bertrand equilibrium or equilibria, if any.
(b) Reconsider the questions in (a) for the case where prices and costs are restricted to integer values, i.e., $p_1, p_2, c_1, c_2 \in \{0, 1, 2, \ldots\}$. (This reflects the assumption that there is a smallest monetary unit.) Specifically, consider two cases: (i) $a = 6$, $c_1 = c_2 = 2$ and (ii) $a = 6$, $c_1 = 1$, $c_2 = 2$.

6.8. *Bertrand with More Than Two Firms*
Suppose that there are $n > 2$ firms in the Bertrand model of Sect. 6.3. Assume again that all firms have equal marginal cost c, and that the firm with the lowest price gets the whole market. In case of a tie, each firm with the lowest price gets an equal share of the market. Set up the associated game and find all its Nash equilibria.

6.9. *Variations on Stackelberg*

(a) Suppose, in the model in Sect. 6.4, that the firms have different marginal costs c_1 and c_2 (cf. Problem 6.1). Compute the Stackelberg equilibrium and outcome with firm 1 as a leader and with firm 2 as a leader.
(b) Give a logical argument why the payoff of the leader in a Stackelberg equilibrium is always at least as high as his payoff in the Cournot equilibrium. Can you generalize this to arbitrary games?
(c) Consider the situation in Sect. 6.4, but now assume that there are n firms, firm 1 moves first, firm 2 second, etc. Assume again perfect information, and compute the subgame perfect equilibrium.

6.10. *First-Price Sealed-Bid Auction*
Consider the game associated with the first-price sealed-bid auction in Sect. 6.5.1, with $v_1 \geq \ldots \geq v_n > 0$ as there.

(a) Show that $(b_1, b_2, b_3, \ldots, b_n) = (v_2, v_2, v_3, \ldots, v_n)$ is a Nash equilibrium in this game.
(b) Show that, in any Nash equilibrium of the game, a player with the highest valuation obtains the object. Exhibit at least two other Nash equilibria in this game, apart from the equilibrium in (a).
(c) Show that bidding one's true valuation as well as bidding higher than one's true valuation are weakly dominated strategies. Also show that any positive bid lower than one's true valuation is not weakly dominated. (Note: to show that a strategy is weakly dominated one needs to exhibit some other strategy that is always—that is, whatever the other players do—at least as good as the strategy under consideration and at least once—that is, for at least one strategy combination of the other players—strictly better.)
(d) Show that, in any Nash equilibrium of this game, at least one player plays a weakly dominated strategy.

6.11. *Second-Price Sealed-Bid Auction*
Consider the game associated with the second price sealed bid auction in Sect. 6.5.1.

(a) Formulate the payoff functions in this game.
(b) Show that $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$ is a Nash equilibrium in this game.
(c) Show, for each player, that bidding one's true valuation weakly dominates any other action. (Show that this holds even if each player only knows his own valuation.)

(d) Show that $(b_1, \ldots, b_n) = (v_2, v_1, 0, \ldots, 0)$ is a Nash equilibrium in this game. What about $(b_1, \ldots, b_n) = (v_1, 0, 0, \ldots, 0)$?
(e) Determine *all* Nash equilibria in the game with two players ($n = 2$). (Hint: compute the best reply functions and make a diagram.)

6.12. *Third-Price Sealed-Bid Auction*
In the auction of Sect. 6.5.1, assume that there are at least three bidders and that the highest bidder wins and pays the third highest bid.

(a) Show that for any player i bidding v_i weakly dominates any lower bid but does not weakly dominate any higher bid.
(b) Show that the strategy combination in which each player i bids his true valuation v_i is in general not a Nash equilibrium.
(c) Find some Nash equilibria of this game.

6.13. *n-Player First-Price Sealed-Bid Auction with Incomplete Information*
Consider the setting of Sect. 6.5.2 but now assume that the number of bidders/players is $n \geq 2$. Show that $(s_1^*, \ldots, s_n^*)$ with $s_i^*(v_i) = (1 - 1/n)v_i$ for every player i is a Nash equilibrium of this game. (Hence, for large n, each bidder almost bids his true valuation.)

6.14. *Double Auction*
This problem is about the auction in Sect. 6.5.3.

(a) Fix a number $x \in [0, 1]$ and consider the following strategies $p_b(\cdot)$ and $p_s(\cdot)$:

$$p_b(v_b) = \begin{cases} x & \text{if } v_b \geq x \\ 0 & \text{if } v_b < x \end{cases} \quad \text{and} \quad p_s(v_s) = \begin{cases} x & \text{if } v_s \leq x \\ 1 & \text{if } v_s > x \end{cases} .$$

Show that these strategies constitute a Nash equilibrium.
(b) For the equilibrium in (a), compute the probability that trade takes place conditional on $v_b \geq v_s$. For which value of x is this probability maximal?
(c) Consider linear strategies of the form $p_b(v_b) = a_b + c_b v_b$ and $p_s(v_s) = a_s + c_s v_s$, where a_b, c_b, a_s, c_s are positive constants. Determine the values of the four constants so that these strategies constitute a Nash equilibrium.
(d) Answer the same question as in (b) for the equilibrium in (c). Which of the equilibria in (b) and (c) is the most efficient, i.e., has the highest probability of resulting in trade conditional on $v_b \geq v_s$?

6.15. *Mixed Strategies and Objective Uncertainty*
Consider the bimatrix game

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 4,1 & 1,3 \\ 1,2 & 3,0 \end{pmatrix} \end{array} .$$

(a) Compute the Nash equilibrium of this game.
(b) Add some uncertainty to the payoffs of this game and find a pure (Bayesian) Nash equilibrium of the resulting game of incomplete information, such that the induced a priori mixed strategies converge to the Nash equilibrium of the original game as the amount of uncertainty shrinks to 0.

6.16. *Variations on Finite Horizon Bargaining*

(a) Adapt the arguments and the results of Sect. 6.7.1 for the case where T is even and the case where player 2 proposes at even rounds.
(b) Let $T = 3$ in Sect. 6.7.1 and suppose that the players have different discount factors δ_1 and δ_2. Compute the subgame perfect equilibrium and the subgame perfect equilibrium outcome.
(c) Consider again the model of Sect. 6.7.1, let $T = 3$, but now assume that the utility function of player 2 is $u_2(\alpha) = \sqrt{\alpha}$ for all $\alpha \in [0, 1]$. Hence, the utility of receiving α at time t for player 2 is equal to $\delta^t \sqrt{\alpha}$. Compute the subgame perfect equilibrium and the subgame perfect equilibrium outcome.
(d) Suppose, in the model of Sect. 6.7.1, that at time T the 'disagreement' distribution is $\mathbf{s} = (s_1, s_2)$ with $s_1, s_2 \geq 0$ and $s_1 + s_2 \leq 1$, rather than $(0, 0)$. Compute the subgame perfect equilibrium and the subgame perfect equilibrium outcome.
(e) In (d), compute the limits of the equilibrium shares for T going to infinity. Do these limits depend on $\mathbf{s}$?
(f) Show, in the game in Sect. 6.7.1, that subgame perfection really has a bite. Specifically, for every $\mathbf{s} = (s_1, s_2)$ with $s_1, s_2 \geq 0$ and $s_1 + s_2 = 1$, exhibit a Nash equilibrium of the game in Fig. 6.5 resulting in the distribution s.

6.17. *Variations on Infinite Horizon Bargaining*

(a) Determine the subgame perfect equilibrium outcome and subgame perfect equilibrium strategies in the game in Sect. 6.7.2 when the players have different discount factors δ_1 and δ_2.
(b) Determine the subgame perfect equilibrium outcome and subgame perfect equilibrium strategies in the game in Sect. 6.7.2 when player 2 proposes at even rounds and player 1 at odd rounds.
(c) Determine the subgame perfect equilibrium outcome and subgame perfect equilibrium strategies in the game in Sect. 6.7.2 when the 'disagreement' distribution is $\mathbf{s} = (s_1, s_2)$ with $s_1, s_2 \geq 0$ and $s_1 + s_2 \leq 1$, rather than $(0, 0)$, in case the game never stops.
(d) Consider the game in Sect. 6.7.2, but now assume that the utility function of player 2 is $u_2(\alpha) = \sqrt{\alpha}$ for all $\alpha \in [0, 1]$. Hence, the utility of receiving α at time t for player 2 is equal to $\delta^t \sqrt{\alpha}$. Compute the subgame perfect equilibrium and the subgame perfect equilibrium outcome. [Hint: first determine for this situation the values for $\mathbf{x}^*$ and $\mathbf{y}^*$ analogous to (6.10).]

(e) Interpret, as at the end of Sect. 6.7.2, the discount factor as the probability that the game continues to the next round. Show that the game ends with probability equal to 1.

6.18. *A Principal-Agent Game*
There are two players: a worker (the agent) and an employer (the principal). The worker has three choices: either reject the contract offered to him by the employer, or accept this contract and exert high effort, or accept the contract and exert low effort. If the worker rejects the contract then the game ends with a payoff of zero to the employer and a payoff of 2 to the worker (his reservation payoff). If the worker accepts the contract he works for the employer: if he exerts high effort the revenues for the employer will be 12 with probability 0.8 and 6 with probability 0.2; if he exerts low effort then these revenues will be 12 with probability 0.2 and 6 with probability 0.8. The employer can only observe the revenues but not the effort exerted by the worker: in the contract he specifies a high wage w_H in case the revenues equal 12 and a low wage w_L in case the revenues are equal to 6. These wages are the respective choices of the employer. The final payoff to the employer if the worker accepts the contract will be equal to revenues minus wage. The worker will receive his wage; his payoff equals this wage minus 3 if he exerts high effort and this wage minus 0 if he exerts low effort.

(a) Set up the extensive form of this game. Does this game have incomplete or imperfect information? What is the associated strategic form?
(b) Determine the subgame perfect equilibrium or equilibria of the game.

6.19. *The Market for Lemons*
A buyer wants to buy a car but does not know whether the particular car he is interested in has good or bad quality (a lemon is a car of bad quality). About half of the market consists of good quality cars. The buyer offers a price p to the seller, who is informed about the quality of the car; the seller may then either accept of reject this price. If he rejects, there is no sale and the payoff will be 0 to both. If he accepts, the payoff to the seller will be the price minus the value of the car, and to the buyer it will be the value of the car minus the price. A good quality car has a value of 15,000, a lemon has a value of 5,000.

(a) Set up the extensive as well as strategic form of this game.
(b) Compute the subgame perfect equilibrium or equilibria of this game.

6.20. *Corporate Investment and Capital Structure*
Consider an entrepreneur who has started a company but needs outside financing to undertake an attractive new project. The entrepreneur has private information about the profitability of the existing company, but the payoff of the new project cannot be disentangled from the payoff of the existing company—all that can be observed is the aggregate profit of the firm. Suppose the entrepreneur offers a potential investor an equity stake in the firm in exchange for the necessary financing. Under what

circumstances will the new project be undertaken, and what will the equity stake be? In order to model this as a game, assume that the profit of the existing company can be either high or low: $\pi = L$ or $\pi = H$, where $H > L > 0$. Suppose that the required investment for the new project is I, the payoff will be R, the potential investor's alternative rate of return is r, with $R > I(1 + r)$. The game is played as follows.

1. Nature determines the profit of the existing company. The probability that $\pi = L$ is p.
2. The entrepreneur learns π and then offers the potential investor an equity stake s, where $0 \leq s \leq 1$.
3. The investor observes s (but not π) and then decides either to accept or to reject the offer.
4. If the investor rejects then the investor's payoff is $I(1 + r) - I$ and the entrepreneur's payoff is π. If he accepts his payoff is $s(\pi + R) - I$ and the entrepreneur's is $(1 - s)(\pi + R)$.

(a) Set up the extensive form and the strategic form of this signaling game.
(b) Compute the perfect Bayesian Nash equilibrium or equilibria, if any.

6.21. *A Poker Game*
Consider the following game. There are two players, I and II. Player I deals II one of three cards—Ace, King, or Queen—at random and face down. II looks at the card. If it is an Ace, II must say "Ace", if a King he can say "King" or "Ace", and if a Queen he can say "Queen" or "Ace". If II says "Ace" player I can either believe him and give him $1 or ask him to show his card. If it is an Ace, I must pay II $2, but if it is not, II pays I $2. If II says "King" neither side looses anything, but if he says "Queen" II must pay player I $1.

(a) Set up the extensive form and the strategic form of this zerosum game.
(b) Determine its value and optimal strategies (cf. Chap. 2).

6.22. *A Hotelling Location Problem*
Consider n players each choosing a location in the interval $[0, 1]$. One may think of n shops choosing locations in a street, n firms choosing product characteristics on a continuous scale from 0 to 1, or n political parties choosing positions on the ideological scale. We assume that customers or voters are uniformly distributed over the interval, with a total of 1. The customers go to (voters vote for) the nearest shop (candidate). For example, if $n = 2$ and the chosen positions are $x_1 = 0.2$ and $x_2 = 0.6$, then 1 obtains 0.4 and 2 obtains 0.6 customers (votes). In case two or more players occupy the same position they share the customers or voters for that position equally.

In the first scenario, the players care only about winning or loosing in terms of the number of customers or votes. This scenario may be prominent for presidential elections, as an example. For each player the best alternative is to be the unique winner, the second best alternative is to be one of the winners, and the worst alternative is not to win. For this scenario, answer questions (a) and (b).

(a) Show that there is a unique Nash equilibrium for $n = 2$.
(b) Exhibit a Nash equilibrium for $n = 3$.

In the second scenario, the payoffs of the players are given by the total numbers of customers (or voters) they acquire. For this scenario, answer questions (c) and (d).

(c) Show that there is a unique Nash equilibrium for $n = 2$.
(d) Is there a Nash equilibrium for $n = 3$? How about $n = 4$?

6.23. *Median Voting*
Of the n persons in a room, each person i has a most favorite room temperature t_i, and the further away (lower or higher) the room temperature is from t_i, the worse it is for this person. Specifically, if the room temperature is x, then person i's utility is equal to $-|x - t_i|$. In order to find a compromise, the janitor asks each person to propose a room temperature, and based on the proposed temperatures a compromise is determined. The proposed temperatures are not necessarily equal to the favorite temperatures. Only temperatures (proposed and favorite) in the interval 0–30 °C are possible.

(a) Suppose the janitor announces that he will take the average of the proposed temperatures as the compromise temperature. Formulate this situation as an n-person game, that is, give the strategy sets of the players and the payoff functions. Does this game have a Nash equilibrium?
(b) Suppose n is odd, and suppose the janitor announces that he will take the median of the proposed temperatures as the compromise temperature. Formulate this situation as an n-person game, that is, give the strategy sets of the players and the payoff functions. Show that, for each player, proposing his ideal temperature weakly dominates any other strategy: thus, in particular, $(t_1, \ldots, t_n)$ is a Nash equilibrium of this game. Does the game have any other Nash equilibria?

6.24. *The Uniform Rule*
An amount $M \geq 0$ of a good (labor, green pea soup, ...) is to be distributed completely among n persons. Each person i considers an amount $t_i \geq 0$ as the ideal amount, and the further away the allocated amount is from this ideal, the worse it is. Specifically, if the amount allocated to person i is x_i, then person i's utility is equal to $-|x - t_i|$. In order to find a compromise, each person is asked to report an amount, and based on the reported amounts a compromise is determined. Let the ideal amounts be given by $t_1 \leq t_2 \leq \ldots \leq t_n$. The reported amounts are not necessarily equal to the ideal amounts.

(a) Suppose M is distributed proportionally to the reported amounts, that is, if the reported amounts are $(r_1, \ldots, r_n)$, then person i receives $x_i = \left(r_i / \sum_{j=1}^{n} r_j\right) M$.

(If all r_j are zero then take $x_i = M/n$.) Formulate this situation as a game. Does this game have a Nash equilibrium?

Consider the following division rule, called the *uniform rule*. Let $(r_1, \ldots, r_n)$ denote the reported amounts. If $M \leq \sum_{j=1}^{n} r_j$, then each person i receives

$$x_i = \min\{r_i, \lambda\} ,$$

where λ is such that $\sum_{j=1}^{n} x_j = M$. If $M \geq \sum_{j=1}^{n} r_j$, then each person i receives

$$x_i = \max\{r_i, \lambda\} ,$$

where, again, λ is such that $\sum_{j=1}^{n} x_j = M$.

(b) Suppose that $n = 3$ and $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$. Apply the uniform rule for $M = 4, M = 5, M = 5.5, M = 6, M = 6.5, M = 7, M = 8, M = 9$.
(c) Suppose, for the general case, that the uniform rule is used to distribute the amount M. Formulate this situation as a game. Show that reporting one's ideal amount weakly dominates any other strategy: thus, in particular, $(t_1, \ldots, t_n)$ is a Nash equilibrium of this game. Does the game have any other Nash equilibria?

6.25. *Reporting a Crime*
There are n individuals who witness a crime. Everybody would like the police to be called. If this happens, each individual derives satisfaction $v > 0$ from it. Calling the police has a cost of c, where $0 < c < v$. The police will come if at least one person calls. Hence, this is an n-person game in which each player chooses from $\{C, N\}$: C means 'call the police' and N means 'do not call the police'. The payoff to person i is 0 if nobody calls the police, $v - c$ if i (and perhaps others) call the police, and v if the police is called but not by person i.

(a) What are the Nash equilibria of this game in pure strategies? In particular, show that the game does not have a symmetric Nash equilibrium in pure strategies (a Nash equilibrium is symmetric if every player plays the same strategy).
(b) Compute the symmetric Nash equilibrium or equilibria in mixed strategies. (Hint: suppose, in such an equilibrium, every person plays C with probability $0 < p < 1$. Use the fact that each player must be indifferent between C and N.)
(c) For the Nash equilibrium/equilibria in (b), compute the probability of the crime being reported. What happens to this probability if n becomes large?

6.26. *Firm Concentration*
Consider a market with ten firms. Simultaneously and independently, the firms choose between locating downtown and locating in the suburbs. The profit of each firm is influenced by the number of other firms that locate in the same area. Specifically, the profit of a firm that locates downtown is given by $5n - n^2 + 50$, where n denotes the number of firms that locate downtown. Similarly, the profit of

a firm that locates in the suburbs is given by $48 - m$, where m denotes the number of firms that locate in the suburbs. In equilibrium how many firms locate in each region and what is the profit of each?

6.27. *Tragedy of the Commons*
There are n farmers, who use a common piece of land to graze their goats. Each farmer i chooses a number of goats g_i—for simplicity we assume that goats are perfectly divisible. The value to a farmer of grazing a goat when the total number of goats is G, is equal to $v(G)$ per goat. We assume that there is a number $\bar{G}$ such that $v(G) > 0$ for $G < \bar{G}$ and $v(G) = 0$ for $G \geq \bar{G}$. Moreover, v is continuous, and twice differentiable at all $G \neq \bar{G}$, with $v'(G) < 0$ and $v''(G) < 0$ for $G < \bar{G}$. The payoff to farmer i if each farmer j chooses g_j, is equal to

$$g_i v(g_1 + \ldots + g_{i-1} + g_i + g_{i+1} + \ldots + g_n) - cg_i \ ,$$

where $c \geq 0$ is the cost per goat.

(a) Interpret the conditions on the function v.
(b) Show that the total number of goats in a Nash equilibrium $(g_1^*, \ldots, g_n^*)$ of this game, $G^* = g_1^* + \ldots + g_n^*$, satisfies

$$v(G^*) + (1/n)G^* v'(G^*) - c = 0 \ .$$

(c) The socially optimal number of goats G^{**} is obtained by maximizing $Gv(G) - cG$ over $G \geq 0$. Show that G^{**} satisfies

$$v(G^{**}) + G^{**} v'(G^{**}) - c = 0 \ .$$

(d) Show that $G^* > G^{**}$. (Hence, in a Nash equilibrium too many goats are grazed.)

6.9 Notes

The Cournot model in Sect. 6.2 dates back from Cournot (1838), and the Bertrand model in Sect. 6.3 from Bertrand (1883). The occurrence of the Bertrand price equilibrium is often referred to as the *Bertrand paradox*. On Cournot versus Bertrand, see Magnan de Bornier (1992). The Stackelberg equilibrium (Sect. 6.4) is named after von Stackelberg (1934).

Our coverage of auction theory is limited, and based on Osborne (2004) and Gibbons (1992). For more extensive overviews and treatments see Milgrom (2004) and Krishna (2002). The second price auction is also called *Vickrey auction* (Vickrey, 1961). The condition that bidders bid their true valuation is an example of the *incentive compatibility* requirement.

Section 6.6 is based on Harsanyi (1973), see also Gibbons (1992).

Axiomatic bargaining theory was initiated by Nash (1950). The noncooperative, strategic approach in Sect. 6.7 is based on Rubinstein (1982). See also Nash (1953) for a noncooperative approach to the Nash bargaining solution.

The goods in the model of Problem 6.3 are *strategic substitutes*. In duopoly models such as this the distinction between strategic substitutes and *strategic complements* is important for the differences between quantity and price competition. See, e.g., Tirole (1988).

Problem 6.19 (market for lemons), exhibiting the *adverse selection* problem, is based on Akerlof (1970). Problem 6.20 is taken from Gibbons (1992). Problem 6.21 (poker game) is taken from Thomas (1986). For an axiomatization of the median voting method in Problem 6.23 see Moulin (1980), and for an axiomatization of the uniform rule in Problem 6.24 see Sprumont (1991).

Problem 6.26 is taken from Watson (2002). For the tragedy of the commons situation in Problem 6.27 see Hardin (1968) and Gibbons (1992).

References

Akerlof, G. (1970). The market for lemons: Quality uncertainty and the market mechanism. *Quarterly Journal of Economics, 84*, 488–500.

Bertrand, J. (1883). Review of Walras's 'Théorie mathématique de la richesse sociale' and Cournot's 'Recherches sur les principes mathématiques de la théorie des richesses'. *Journal des Savants*, 499–508 [translated by M. Chevaillier and reprinted in Magnan de Bornier (1992), 646–653].

Cournot, A. (1838). *Recherches sur les principes mathématiques de la théorie des richesses* [English translation (1897) *Researches into the mathematical principles of the theory of wealth*]. New York: Macmillan.

Gibbons, R. (1992). *A primer in game theory*. Hertfordshire: Harvester Wheatsheaf.

Hardin, G. (1968). The tragedy of the commons. *Science, 162*, 1243–1248.

Harsanyi, J. C. (1973). Games with randomly disturbed payoffs: A new rationale of mixed strategy equilibrium points. *International Journal of Game Theory, 2*, 1–23.

Krishna, V. (2002). *Auction theory*. San Diego: Academic.

Magnan de Bornier, J. (1992). The 'Cournot-Bertrand debate': A historical perspective. *History of Political Economy, 24*, 623–656.

Milgrom, P. (2004). *Putting auction theory to work*. Cambridge: Cambridge University Press.

Moulin, H. (1980). On strategy-proofness and single-peakedness. *Public Choice, 35*, 437–455.

Nash, J. F. (1950). The bargaining problem. *Econometrica, 18*, 155–162.

Nash, J. F. (1953). Two-person cooperative games. *Econometrica, 21*, 128–140.

Osborne, M. J. (2004). *An introduction to game theory*. New York: Oxford University Press.

Rubinstein, A. (1982). Perfect equilibrium in a bargaining model. *Econometrica, 50*, 97–109.

Sprumont, Y. (1991). The division problem with single-peaked preferences: A characterization of the uniform allocation rule. *Econometrica, 59*, 509–520.

Thomas, L. C. (1986). *Games, theory and applications*. Chichester: Ellis Horwood Limited.

Tirole, J. (1988). *The theory of industrial organization*. Cambridge: MIT Press.

Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance, 16*, 8–37.

von Stackelberg, H. F. (1934). *Marktform und Gleichgewicht*. Wien: Julius Springer.

Walker, M., & Wooders, J. (2001). Minimax play at Wimbledon. *American Economic Review, 91*, 1521–1538.

Watson, J. (2002). *Strategy, an introduction to game theory*. New York: Norton.