7 Repeated Games

In the famous prisoners' dilemma game the bad (Pareto inferior) outcome, resulting from each player playing his dominant action, cannot be avoided in a Nash equilibrium or subgame perfect Nash equilibrium even if the game is repeated a finite number of times, cf. Problem 4.10. As we will see in this chapter, this bad outcome can be avoided if the game is repeated an infinite number of times. This, however, is coming at a price, namely the existence of a multitude of outcomes attainable in equilibrium. Such an *embarrassment of riches* is expressed by a so-called *folk theorem*.

As was illustrated in Problem 4.11, also *finite* repetitions of a game may sometimes lead to outcomes that are better than (repeated) Nash equilibria of the original game.

In this chapter we consider two-person *infinitely* repeated games and formulate folk theorems both for subgame perfect and for Nash equilibrium. The approach is somewhat informal, and mainly based on examples. In Sect. 7.1 we consider subgame perfect equilibrium and in Sect. 7.2 we consider Nash equilibrium.

7.1 Subgame Perfect Equilibrium

7.1.1 The Prisoners' Dilemma

Consider the prisoners' dilemma game (in the form of the 'marketing game' of Problem 3.1(d))

$$G = \begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} 50,50 & 30,60 \\ 60,30 & 40,40 \end{pmatrix} \end{array}.$$

H. Peters, *Game Theory*, Springer Texts in Business and Economics,
DOI 10.1007/978-3-662-46950-7_7

In G each player has a strictly dominated action, namely C, and (D, D) is the unique Nash equilibrium of the game, also if mixed strategies are allowed.

We assume now that G is played infinitely many times, at times $t = 0, 1, 2, \ldots$, and that after each play of G the players learn what has been played, i.e., they learn which element of the set $\{(C, C), (C, D), (D, C), (D, D)\}$ has occurred. For instance, in the marketing game, one can think of the game being played once per period—a week, month—each player observing in each period whether his opponent has advertised or not. Note that a player does not learn the exact, possibly mixed, action of his opponent, but only its realization. These realizations induce an infinite stream of associated payoffs, and we assume that there is a common discount factor $0 < \delta < 1$ such that the final payoff to each player is the δ-discounted value of the infinite stream. That is, player i ($i = 1, 2$) obtains

$$\sum_{t=0}^{\infty} (\text{payoff from play of the stage game } G \text{ at time } t) \cdot \delta^t \ .$$

Here, the expression *stage game* is used for the one-shot game G, in order to distinguish the one-shot game from the repeated game.

As always, a strategy of a player is a complete plan to play the game. This means that, at each moment t, this plan should prescribe an action of a player—a mixed or pure strategy in the stage game G—for each possible history of the game up to time t, that is, an action for each sequence of length t (namely, at $0, \ldots, t-1$) of elements from the set $\{(C, C), (C, D), (D, C), (D, D)\}$. Clearly, such a strategy can be quite complicated and the number of possible strategies is enormous. We will be able, however, to restrict attention to relatively simple strategies.

The infinite extensive form game just defined is denoted by $G^{\infty}(\delta)$. A natural solution concept for this game is subgame perfect (Nash) equilibrium. Each subgame in $G^{\infty}(\delta)$ is, basically, equal to the game $G^{\infty}(\delta)$ itself: the difference between two subgames is the difference between the two histories leading to those subgames. For instance, at $t = 6$, there are 4^6 possible histories of play and therefore there are 4^6 different subgames; each of these subgames, however, looks exactly like $G^{\infty}(\delta)$.

We will now exhibit a few subgame perfect equilibria of $G^{\infty}(\delta)$. First consider the simple strategy

> D^{∞}: play D at each moment $t = 0, 1, 2, \ldots$, independent of the history of the game, i.e., independent of what was played before t.

Then D^{∞} is a well-defined strategy. If both players play D^{∞} then the resulting payoff is

$$\sum_{t=0}^{\infty} 40\,\delta^t = 40/(1-\delta)$$

for each player. We claim that (D^∞, D^∞) is a subgame perfect equilibrium in $G^\infty(\delta)$. Consider any $t = 0, 1, \ldots$ and any subgame starting at time t. Then (D^∞, D^∞) induces a Nash equilibrium in this subgame: given that player 2 always plays D, player 1 cannot do better than always play D as well, and *vice versa*. Hence, (D^∞, D^∞) is a subgame perfect equilibrium. In this subgame perfect equilibrium, the players just play the Nash equilibrium of the stage game at every time t.

We next exhibit another subgame perfect equilibrium. Consider the following strategy:

> $Tr(C)$: at $t = 0$ and at every time t such that in the past only (C, C) has occurred in the stage game: play C. Otherwise, play D.

Strategy $Tr(C)$ is an example of a so-called *trigger strategy*. In general, if the players play trigger strategies, they follow some fixed pattern of play until a deviation occurs: then a Nash equilibrium action of the stage game is played forever. In the present example, $Tr(C)$, a player starts by playing C and keeps on playing C as long as both players have only played C in the past, i.e., as long as the history of play is $(C, C), \ldots, (C, C)$; after any deviation from this, i.e., if the history of play is *not* $(C, C), \ldots, (C, C)$, the player plays D and keeps on playing D forever. Again, $Tr(C)$ is a well-defined strategy, and if both players play $Tr(C)$, then each player obtains the payoff

$$\sum_{t=0}^{\infty} 50\, \delta^t = 50/(1-\delta)\ .$$

Is $(Tr(C), Tr(C))$ also a subgame perfect equilibrium? The answer is a qualified yes: if δ is large enough, then it is. The crux of the argument is as follows. At each stage of the game, a player has an incentive to deviate from C and play his dominant action D, thereby obtaining a momentary gain of 10. Deviating, however, triggers eternal 'punishment' by the other player, who is going to play D forever. The best reply to this punishment is to play D as well, entailing a loss of 10 at each moment from the next moment on. The discounted value, at the moment of deviation, of this loss is equal to $10\delta/(1-\delta)$, and to keep a player from deviating this loss should be at least as large as the momentary gain of 10. This is the case if and only if $\delta \geq 1/2$.

More formally, we can distinguish two kinds of subgames that are relevant for the strategy combination $(Tr(C), Tr(C))$. One kind are those subgames where *not* always (C, C) has been played in the past. In such a subgame, $Tr(C)$ tells a player to play D forever, and therefore the best reply of the other player is to do so as well, which means indeed to play according to $Tr(C)$. Thus, in this kind of subgame, $(Tr(C), Tr(C))$ is a Nash equilibrium.

In the other kind of subgame, no deviation has occurred so far: in the past always (C, C) has been played. Consider this subgame at some time T and suppose that player 2 plays $Tr(C)$. If player 1 plays $Tr(C)$ as well, his payoff is equal to

$$\sum_{t=0}^{T-1} 50\,\delta^t + \sum_{t=T}^{\infty} 50\,\delta^t\ .$$

If, instead, he deviates at time T to D, he obtains maximally

$$\sum_{t=0}^{T-1} 50\,\delta^t + 60\,\delta^T + \sum_{t=T+1}^{\infty} 40\,\delta^t\ .$$

The first term in this expression is the discounted payoff from (C, C) at $t = 0, \ldots, T-1$. The second term is the discounted payoff from (D, C) at time T. The third term is the discounted payoff from (D, D) from time $t = T+1$ on. Note that from $t = T+1$ on player 2 plays D, according to his strategy $Tr(C)$, and the best that the (deviating) player 1 can do is to play D as well.

Hence, to avoid deviation (and make $Tr(C)$ player 1's best reply in the subgame) we need that the first payoff is at least as high as the second one, resulting in the inequality

$$50\,\delta^T/(1-\delta) \geq 60\,\delta^T + 40\,\delta^{T+1}/(1-\delta)$$

or, equivalently, $\delta \geq 1/2$—as found before. The arguments for the roles of the players reversed are exactly equal. We conclude that for every $\delta \geq 1/2$, $(Tr(C), Tr(C))$ is a subgame perfect equilibrium of the game $G^\infty(\delta)$. The existence of this equilibrium is a major reason to study infinitely repeated games. In popular terms, it shows that cooperation is sustainable if deviations can be credibly punished, which is the case if the future is sufficiently important (i.e., δ large enough).

To exhibit yet another subgame perfect equilibrium, different from (D^∞, D^∞) and $(Tr(C), Tr(C))$, consider the following strategies for players 1 and 2, respectively.

Tr_1: As long as the sequence (C, D), (D, C), (C, D), (D, C), (C, D), (D, C), ... has occurred in the past from time 0 on, play C at $t \in \{0, 2, 4, 6, \ldots\}$; play D at $t \in \{1, 3, 5, 7, \ldots\}$. Otherwise, play D.

Tr_2: As long as the sequence (C, D), (D, C), (C, D), (D, C), (C, D), (D, C), ... has occurred in the past from time 0 on, play D at $t \in \{0, 2, 4, 6, \ldots\}$; play C at $t \in \{1, 3, 5, 7, \ldots\}$. Otherwise, play D.

Note that these are again 'trigger strategies': the players 'tacitly' agree on a certain sequence (pattern) of play, but revert to playing D forever after a deviation. If player

1 plays Tr_1 and player 2 plays Tr_2, then the sequence (C, D), (D, C), (C, D), (D, C), ..., results. To see why (Tr_1, Tr_2) might be a subgame perfect equilibrium, note that on average a player obtains 45 per stage, which is more than the 40 that would be obtained from deviating from this sequence and playing D forever. More precisely, suppose player 2 plays Tr_2 and suppose player 1 considers a deviation from Tr_1. It is optimal to deviate at an even moment, say at $t = 0$, since then player 1 receives a payoff of 30, and can obtain 40 by deviating to D. Since, after this deviation, player 2 plays D forever, the maximal total discounted payoff to player 1 from deviating is obtained by also playing D forever after his deviation to D at time $t = 0$, and this payoff is equal to

$$40 + 40(\delta + \delta^2 + \ldots) = 40/(1 - \delta) .$$

If player 1 does not deviate and sticks to the strategy Tr_1 he obtains

$$30(1 + \delta^2 + \delta^4 + \ldots) + 60(\delta + \delta^3 + \delta^5 + \ldots) = (30 + 60\delta)/(1 - \delta^2) .$$

To keep player 1 from deviating we need $40/(1-\delta) \leq (30+60\delta)/(1-\delta^2)$, which yields $\delta \geq 1/2$. The argument if the roles of the players are reversed is similar, and we conclude that for each $\delta \geq 1/2$, (Tr_1, Tr_2) is a subgame perfect equilibrium in $G^\infty(\delta)$.

More generally, by playing appropriate sequences of elements from the set of possible outcomes $\{(C, C), (C, D), (D, C), (D, D)\}$ of the stage game G, the players can on average reach any convex combination of the associated payoffs in the long run. That is, take any such combination

$$\alpha_1(50, 50) + \alpha_2(30, 60) + \alpha_3(60, 30) + \alpha_4(40, 40) , \tag{*}$$

where $\alpha_i \in \mathbb{R}$, $\alpha_i \geq 0$ for every $i = 1, \ldots, 4$, and $\sum_{i=1}^{4} \alpha_i = 1$. By choosing a sequence of possible outcomes such that (C, C) occurs on average in a fraction α_1 of the stages, (C, D) in a fraction α_2, (D, C) in a fraction α_3, and (D, D) in a fraction α_4, then the payoffs (*) are reached as averages in the limit, i.e., as $t \to \infty$. These are indeed average payoffs, independent of the discount factor δ. If these limit average payoffs exceed 40 (the Nash equilibrium payoff of the stage game) for each player, associated trigger strategies can be formulated that result in these (limit average) payoffs and that trigger eternal play of (D, D) after a deviation, similar to the strategies $Tr(C)$, Tr_1 and Tr_2 above.

Note that for $\alpha_1 = 1$ we can take the strategy pair $(Tr(C), Tr(C))$. For $\alpha_2 = \alpha_3 = 1/2$ we can take the strategy pair (Tr_1, Tr_2).

To exhibit yet another example, consider the average payoff pair $(42, 48)$, which is equal to

$$\frac{1}{5}(50, 50) + \frac{2}{5}(30, 60) + \frac{1}{5}(60, 30) + \frac{1}{5}(40, 40) .$$

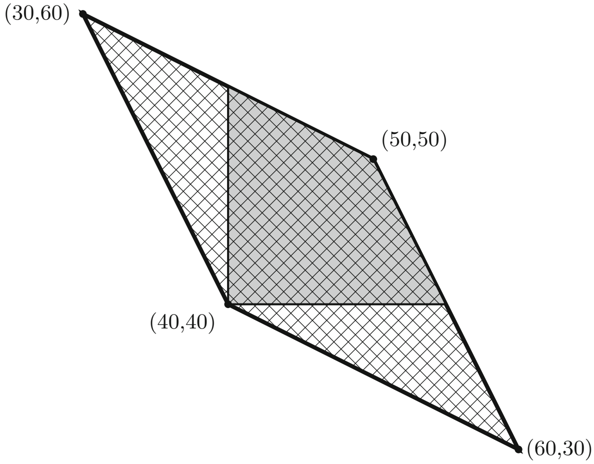

Fig. 7.1 For every payoff pair in the *shaded area* there is a δ large enough such that this payoff pair can be obtained as the limit average in a subgame perfect equilibrium of $G^{\infty}(\delta)$, where G is the prisoners' dilemma game of Sect. 7.1.1

This means that the payoffs $(42, 48)$ can be obtained on average by playing (C, C) at $t = 0, 5, 10, \ldots$; (C, D) at $t = 1, 2, 6, 7, 11, 12, \ldots$; (D, C) at $t = 3, 8, 13, \ldots$; and (D, D) at $t = 4, 9, 14, \ldots$ Translated to trigger strategies we have

Tr_1^*: As long as the sequence (C, C), (C, D), (C, D), (D, C), (D, D); (C, C), (C, D), (C, D), (D, C), (D, D); ... has occurred in the past from time 0 on, play C at $t \in \{0, 1, 2, 5, 6, 7, 10, 11, 12, \ldots\}$; play D at $t \in \{3, 4, 8, 9, 13, 14, \ldots\}$. Otherwise, play D.

Tr_2^*: As long as the sequence (C, C), (C, D), (C, D), (D, C), (D, D); (C, C), (C, D), (C, D), (D, C), (D, D); ... has occurred in the past from time 0 on, play C at $t \in \{0, 3, 5, 8, 10, 13, \ldots\}$; play D at $t \in \{1, 2, 4, 6, 7, 9, 11, 12, 14, \ldots\}$. Otherwise, play D.

For δ sufficiently high, these strategies form a subgame perfect equilibrium of $G^{\infty}(\delta)$. (See Problem 7.5.)

Figure 7.1 shows all limit average payoffs that can be reached in this way.

7.1.2 Some General Observations

For the prisoners' dilemma game we have established that each player playing always D is a subgame perfect equilibrium of $G^{\infty}(\delta)$ for every $0 < \delta < 1$. The logic is simple. If player 2 *always* plays the Nash equilibrium action D in the stage game, then player 1 can never do better than playing a best reply action in the stage game, i.e., playing D—'never' means: independent of the history, i.e., independent of the play in the stage game thus far, i.e., in any subgame. The same logic holds for *any* stage game, that is, with finitely or infinitely many actions, with any arbitrary number of players, and for *any* Nash equilibrium of the stage game. The following proposition merely states this more formally.[1]

Proposition 7.1 *Let G be any arbitrary (not necessarily finite) n-person game, and let the strategy combination* $\mathbf{s} = (s_1, \ldots, s_i, \ldots, s_n)$ *be a Nash equilibrium in G. Let* $0 < \delta < 1$. *Then each player i playing* s_i *at every moment t is a subgame perfect equilibrium in* $G^\infty(\delta)$.

In particular, this proposition holds for any bimatrix game (see Definition 3.1) and any (not necessarily pure) Nash equilibrium in this bimatrix game. But it also holds, for instance, for the Cournot or Bertrand games (cf. Chap. 6) with two or more players (see Problems 7.6 and 7.7).

Let $G = (A, B)$ be an $m \times n$-bimatrix game with $A = (a_{ij})$ and $B = (b_{ij})$. Let $P(G)$ be the convex hull of the set $\{(a_{ij}, b_{ij}) \in \mathbb{R}^2 \mid i = 1, \ldots, m,\ j = 1, \ldots, n\}$. That is,

$$P(G) = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}(a_{ij}, b_{ij}) \;\middle|\; \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij} = 1,\ \alpha_{ij} \geq 0 \text{ for all } i, j \right\} .$$

Equivalently, to obtain the set $P(G)$, just plot all $m \times n$ payoff pairs in $\mathbb{R}^2$, and take the smallest convex polygon containing all these points.

For the prisoners' dilemma game G, $P(G)$ is the quadrangle with vertices $(40, 40)$, $(30, 60)$, $(60, 30)$, and $(50, 50)$, see Fig. 7.1. The elements (payoff pairs) of $P(G)$ can be obtained as limit average payoffs in the infinitely repeated game G by an appropriate sequence of play, as demonstrated before in Sect. 7.1.1. The following proposition says that every payoff pair in $P(G)$ that strictly dominates the payoffs associated with a Nash equilibrium of G can be obtained as limit average payoffs in a subgame perfect equilibrium of $G^\infty(\delta)$ for δ large enough. Such a proposition is known as a *folk theorem*. Its proof (omitted here) is somewhat technical but basically consists of formulating trigger strategies in a similar way as for the prisoners' dilemma game above. In these strategies, after a deviation from the pattern leading to the desired limit average payoffs, players revert to the Nash equilibrium under consideration of the stage game forever.

Proposition 7.2 (Folk Theorem for Subgame Perfect Equilibrium) *Let* $(\mathbf{p}^*, \mathbf{q}^*)$ *be a Nash equilibrium of* $G = (A, B)$, *and let* $\mathbf{x} = (x_1, x_2) \in P(G)$ *such that* $x_1 > \mathbf{p}^* A \mathbf{q}^*$ *and* $x_2 > \mathbf{p}^* B \mathbf{q}^*$. *Then there is a* $0 < \delta^* < 1$ *and a subgame perfect equilibrium in* $G^\infty(\delta^*)$ *with limit average payoffs* $\mathbf{x}$.

[1] In this proposition it is assumed that $G^\infty(\delta)$ is well-defined, in particular that the discounted payoff sums are finite.

Remark 7.3 For the purpose of this chapter, a limit average payoff pair is just a payoff pair in the set $P(G)$, i.e., in the polygon with the payoff pairs in (A, B) as vertices. More formally, if $\xi_0, \xi_1, \xi_2, \ldots$ is a sequence of real numbers, then the limit average of this sequence is the number $\lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} \xi_t$, assuming that this limit exists. In other words, we take the average of the first $T + 1$ numbers of the sequence and let T go to infinity. □

7.1.3 Another Example

In order to illustrate Propositions 7.1 and 7.2 we consider another example. Let the bimatrix game (stage game) $G = (A, B)$ be given by

$$G = \begin{matrix} & L & R \\ U & 4,4 & 0,2 \\ D & 3,6 & 1,8 \end{matrix} .$$

This game has two pure strategy Nash equilibria (U, L) and (D, R) and a mixed equilibrium $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$, with associated payoffs respectively $(4, 4)$, $(1, 8)$, and $(2, 5)$. The set of limit average payoff pairs $P(G)$ is depicted in Fig. 7.2.

Proposition 7.1 applies to the three Nash equilibria of the stage game:

- Player 1 always playing U and player 2 always L is a subgame perfect Nash equilibrium of $G^\infty(\delta)$ for any value of δ. The payoffs are $4/(1-\delta)$ for each. The limit average payoffs are 4 for each.

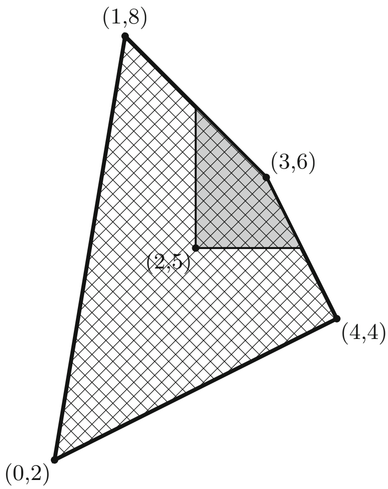

Fig. 7.2 The *crosshatched area* is the set $P(G)$ of the game G in Sect. 7.1.3. The *shaded area*—all points strictly above $(2, 5)$—as well as $(2, 5)$, $(4, 4)$, and $(1, 8)$, are the limit average payoffs pairs attainable in a subgame perfect equilibrium of $G^\infty(\delta)$ according to Propositions 7.1 and 7.1, for a sufficiently high value of δ

- Player 1 always playing D and player 2 always L is a subgame perfect Nash equilibrium of $G^{\infty}(\delta)$ for any value of δ. The payoff is $1/(1-\delta)$ for player 1 and $8/(1-\delta)$ for player 2. The limit average payoffs are 1 and 8, respectively.
- Player 1 always playing $(\frac{1}{2}, \frac{1}{2})$ and player 2 always $(\frac{1}{2}, \frac{1}{2})$ is a subgame perfect Nash equilibrium of $G^{\infty}(\delta)$ for any value of δ. The payoff is $2/(1-\delta)$ for player 1 and $5/(1-\delta)$ for player 2. The limit average payoffs are 2 and 5, respectively.

Proposition 7.2 says that all payoffs in the shaded area strictly above $(2, 5)$ in Fig. 7.2 can be obtained as limit average payoffs in $G^{\infty}(\delta)$, provided δ is sufficiently high.

As a first example we consider the payoffs $(3, 6)$. Clearly, these payoffs can be obtained as limit average payoffs if the players play (D, L) always. In order to obtain them in a subgame perfect equilibrium we can use the mixed Nash equilibrium of the stage game as a punishment after a deviation. Specifically, consider the following trigger strategy pair (S_1^*, S_2^*).

S_1^*: Start with D and keep playing D as long as (D, L) has been played so far. After any deviation, play $(\frac{1}{2}, \frac{1}{2})$ forever.

S_2^*: Start with L and keep playing L as long as (D, L) has been played so far. After any deviation, play $(\frac{1}{2}, \frac{1}{2})$ forever.

In this case, the maximal payoff to player 1 from deviating to U, given that player 2 plays S_2^*, is equal to $4 + 2 \cdot \delta/(1-\delta)$, since player 2 switches to $(\frac{1}{2}, \frac{1}{2})$ forever: to this, player 1's best reply is to also play $(\frac{1}{2}, \frac{1}{2})$ forever, resulting in the payoff 2 at each stage. This maximal payoff is smaller than or equal to $3/(1-\delta)$ if and only if $\delta \geq 1/2$. Similarly, the maximal payoff to player 2 from deviating to R, given that player 1 plays S_1^*, is equal to $8 + 5 \cdot \delta/(1-\delta)$, and this is smaller than or equal to $6/(1-\delta)$ if and only if $\delta \geq 2/3$. We conclude that (S_1^*, S_2^*) is a subgame perfect equilibrium of $G^{\infty}(\delta)$ for $\delta \geq 2/3$.

As a second example, consider the limit average payoff pair $(3, 5\frac{1}{3})$. From Proposition 7.2 we can conclude that these payoffs can be obtained as limit average payoffs in a subgame perfect equilibrium of $G^{\infty}(\delta)$. Note that we can write $(3, 5\frac{1}{3}) = \frac{2}{3}(4, 4) + \frac{1}{3}(1, 8)$, so that these limit averages can be achieved by the sequence of play

$$(U, L), (U, L), (D, R), (U, L), (U, L), (D, R), \ldots$$

Consider the following strategies.

$\bar{S}_1$: Play U at times $t = 0, 1, 3, 4, 6, 7, \ldots$ and play D at times $t = 2, 5, 8, \ldots$.

$\bar{S}_2$: Play L at times $t = 0, 1, 3, 4, 6, 7, \ldots$ and play R at times $t = 2, 5, 8, \ldots$.

Observe that, with these strategies, the players play a Nash equilibrium of the stage game at every time t, which implies that they can only loose by deviating. Thus, we do not need trigger strategies to punish deviations. The pair $(\bar{S}_1, \bar{S}_2)$ is a subgame perfect equilibrium of $G^\infty(\delta)$ for every value of $\delta \in (0,1)$.

As a final example, consider following the strategies in $G^\infty(\delta)$ for players 1 and 2, respectively.

$\hat{S}_1$: Play D at times $t = 4, 9, 14, 19, \ldots$ and play U at all other times.

$\hat{S}_2$: Play R at times $t = 4, 9, 14, 19, \ldots$ and play L at all other times.

These strategies result in the sequence of play

$$(U,L), (U,L), (U,L), (U,L), (D,R), (U,L), (U,L), (U,L), (U,L), (D,R), \ldots$$

resulting in limit average payoffs $\frac{4}{5}(4,4) + \frac{1}{5}(1,8) = (3\frac{2}{5}, 4\frac{4}{5})$. This pair is outside the shaded region in Fig. 7.2, i.e., does not dominate the pair $(2,5)$. This means that the mixed equilibrium $((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$ of the stage game cannot serve as a punishment for deviations by player 2, since the payoff 5 to player 2 from this equilibrium is larger than $4\frac{4}{5}$. Nevertheless, by the same logic as for the pair $(\bar{S}_1, \bar{S}_2)$, the strategy pair $(\hat{S}_1, \hat{S}_2)$ is a subgame perfect equilibrium of $G^\infty(\delta)$ for every value of $\delta \in (0,1)$: at each time t the players play a Nash equilibrium of the stage game. This example shows that Proposition 7.2 is not exhaustive: it does not necessarily give *all* limit average payoff pairs attainable in a subgame perfect equilibrium.

7.2 Nash Equilibrium

In this section we consider the consequences of relaxing the subgame perfection requirement for a Nash equilibrium in an infinitely repeated game. When thinking of trigger strategies as in Sect. 7.1, this means that deviations can be punished more severely, since the equilibrium does not have to induce a Nash equilibrium in the 'punishment subgame'.

For the infinitely repeated prisoners' dilemma game of Sect. 7.1 this has no consequences. In this game each player can guarantee to obtain at least 40 at each stage, so that more severe punishments are not possible. In the following subsection we consider a different example.

7.2.1 An Example

Consider the bimatrix game

$$G = (A,B) = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & -1,4 \end{pmatrix} \end{array}.$$

The set $P(G)$ (see Sect. 7.1.2) is the triangle with vertices $(1,1)$, $(0,0)$, and $(-1,4)$. In the game G the strategy D is a strictly dominated strategy for player 1. The unique Nash equilibrium of the stage game is (U,L). Player 1 always playing U and player 2 always playing L is a subgame perfect equilibrium in $G^\infty(\delta)$ for every $0 < \delta < 1$, cf. Proposition 7.1. Note that Proposition 7.2 does not add anything to this observation, since $P(G)$ does not contain any payoff pair larger than $(1,1)$ for each player.

Now consider the following strategy pair (N_1, N_2) in the infinitely repeated game $G^\infty(\delta)$.

N_1: At $t = 0$ play D. After a history where (D,R) was played at $t = 0, 4, 8, 12, \ldots$ and (U,L) at all other times: play D at $t = 0, 4, 8, 12, \ldots$ and play U at all other times. After any other history play the mixed action $(\frac{4}{5}, \frac{1}{5})$, that is, play U with probability $\frac{4}{5}$ and D with probability $\frac{1}{5}$.

N_2: At $t = 0$ play R. After a history where (D,R) was played at $t = 0, 4, 8, 12, \ldots$ and (U,L) at all other times: play R at $t = 0, 4, 8, 12, \ldots$ and play L at all other times. After any other history play R.

These strategies are again trigger strategies. They induce a sequence of play in which within each four times, (D,R) is played once and (U,L) is played thrice. After a deviation player 1 plays the mixed action $(\frac{4}{5}, \frac{1}{5})$ and player 2 the pure action R forever. Thus, in a subgame following a deviation the players do not play a Nash equilibrium: if player 2 plays R always, then player 1's best reply is to play U always. Hence, (N_1, N_2) is not a subgame perfect equilibrium.

We claim, however, that (N_1, N_2) is a Nash equilibrium if δ is sufficiently large.

First observe that player 2 can never achieve a momentary gain from deviating since, if player 1 plays N_1, then N_2 requires player 2 to play a best reply in the stage game at every moment t. Moreover, after any deviation player 1 plays $(\frac{4}{5}, \frac{1}{5})$ at any moment t, so that both L and R have an expected payoff of $\frac{4}{5}$ for player 2, which is less than 1 and less than 4.

Suppose player 2 plays N_2. If player 1 wants to deviate from N_1, the best moment to do so is one where he is supposed to play D, so at $t = 0, 4, \ldots$. Without loss of generality suppose player 1 deviates at $t = 0$. Then (U,R) is played at $t = 0$, yielding payoff 0 to player 1. After that, player 2 plays R forever, and the best reply of player 1 to this is to play U forever, again yielding 0 each time. So his total payoff from deviating is 0. Without deviation player 1's total discounted payoff is equal to

$$-1(\delta^0 + \delta^4 + \delta^8 + \ldots) + 1(\delta^1 + \delta^2 + \delta^3 + \delta^5 + \delta^6 + \delta^7 + \ldots) \,.$$

In order to keep player 1 from deviating this expression should be at least 0, i.e.

$$\frac{-1}{1-\delta^4} + \left[\frac{\delta}{1-\delta} - \frac{\delta^4}{1-\delta^4}\right] \geq 0$$

which holds if and only if $\delta \geq \delta^*$ with $\delta^* \approx 0.54$. Hence, for these values of δ, (N_1, N_2) is a Nash equilibrium in $G^\infty(\delta)$. The limit average payoffs in this equilibrium are equal to $\frac{3}{4}(1,1) + \frac{1}{4}(-1,4)$, hence to $(\frac{1}{2}, \frac{7}{4})$.

The actions played in this equilibrium after a deviation are, in fact, the actions that keep the opponent to his maximin payoff. To see this, first consider the action of player 2, R. The payoff matrix of player 1 is the matrix A with

$$A = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}.$$

The value of the *matrix game* A (cf. Chap. 2) is equal to 0—in fact, (U, R) is a saddlepoint of A—and, thus, player 1 can always obtain at least 0. By playing R, which is player 2's optimal strategy in A, player 2 can hold player 1 down to 0. Hence, this is the most severe punishment that player 2 can inflict upon player 1 after a deviation.

Similarly, if we view the payoff matrix B for player 2 as a zerosum game with payoffs to player 2 and, following convention, convert this to a matrix game giving the payoffs to player 1, we obtain

$$-B = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \end{array}.$$

In this game, $(\frac{4}{5}, \frac{1}{5})$ is an (the) optimal strategy for player 1, yielding the value of the game, which is equal to $-\frac{4}{5}$. Hence, player 2 can guarantee to obtain a payoff of $\frac{4}{5}$, but player 1 can make sure that player 2 does not obtain more than this by playing $(\frac{4}{5}, \frac{1}{5})$. Again, this is the most severe punishment that player 1 can inflict upon player 2 after a deviation.

By using these punishments in a trigger strategy, the same logic as in Sect. 7.1 tells us that any pair of payoffs in $P(G)$ that strictly dominates the pair $\mathbf{v} = (v(A), -v(-B)) = (0, \frac{4}{5})$ can be obtained as limit average payoffs in a Nash equilibrium of the game $G^\infty(\delta)$ for δ sufficiently large. This is illustrated in Fig. 7.3.

7.2.2 A Folk Theorem for Nash Equilibrium

Let $G = (A, B)$ be an arbitrary $m \times n$ bimatrix game. Let $v(A)$ be the value of the matrix game A and let $v(-B)$ be the value of the matrix game $-B$. Let the set $P(G)$ be defined as in Sect. 7.1.2. The following proposition generalizes what we found above.

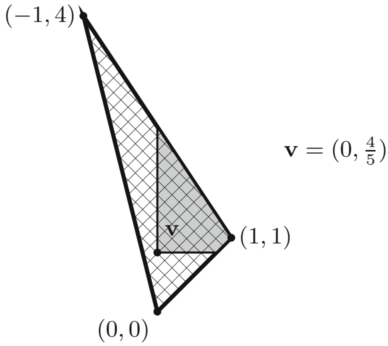

Fig. 7.3 For every payoff pair in the *shaded area* there is a δ large enough such that this payoff pair can be obtained as the pair of limit averages in a Nash equilibrium of $G^{\infty}(\delta)$

Proposition 7.4 (Folk Theorem for Nash Equilibrium) *Let* $\mathbf{x} = (x_1, x_2) \in P(G)$ *such that* $x_1 > v(A)$ *and* $x_2 > -v(-B)$. *Then there is a* $0 < \delta^* < 1$ *and a Nash equilibrium in* $G^{\infty}(\delta^*)$ *with limit average payoffs* $\mathbf{x}$.

The set of payoff pairs that can be reached as limit average payoff pairs in a Nash equilibrium (Proposition 7.4) contains the set of payoff pairs that can be obtained this way in a subgame perfect equilibrium (Proposition 7.2). This follows because the payoffs in a Nash equilibrium of the stage game (A, B) are at least as large as the payoffs $(v(A), -v(-B))$: if not then a player could improve by playing an optimal strategy in the associated matrix game, guaranteeing $v(A)$ (player 1) or $-v(-B)$ (player 2).

7.2.3 Another Example

Consider the bimatrix game

$$G = (A, B) = \begin{matrix} \\ U \\ D \end{matrix} \begin{matrix} L \quad\quad R \\ \begin{pmatrix} 5,1 & 1,2 \\ 4,2 & 2,4 \end{pmatrix} \end{matrix} .$$

The matrix game A has a saddlepoint at (D, R). Thus, $v(A) = 2$ and R is the unique optimal strategy of player 2 in A. The matrix game $-B$, given by

$$-B = \begin{matrix} \\ U \\ D \end{matrix} \begin{matrix} L \quad\quad R \\ \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \end{matrix} ,$$

has a saddlepoint at (U, R). Its value is $v(-B) = -2$, and U is the unique optimal strategy of player 1 in $-B$. The set of limit average payoff pairs and the point $\mathbf{v} = (v(A), -v(-B)) = (2, 2)$ are depicted in Fig. 7.4.

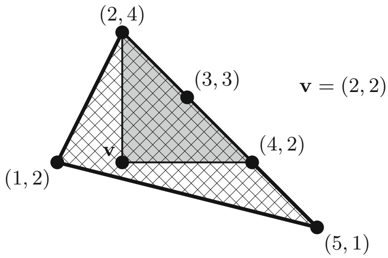

Fig. 7.4 For every payoff pair in the *shaded area* there is a δ large enough such that this payoff pair can be obtained as the pair of limit averages in a Nash equilibrium of $G^{\infty}(\delta)$

Consider, for instance, the limit average payoffs $(3,3)$, and the following strategy pair in $G^{\infty}(\delta)$.

N_1^*: Start by playing D and keep playing D as long as (D,L) was played at even moments and (D,R) at odd moments. After any deviation from this, play U forever.

N_2^*: Play L at even moments and R at odd moments as long as (D,L) was played at even moments and (D,R) at odd moments. After any deviation from this, play R forever.

These strategies result in the limit average payoffs $\frac{1}{2}(4,2) + \frac{1}{2}(2,4) = (3,3)$.

Suppose player 2 plays N_2^*. If player 1 deviates from N_1^*, it is optimal to do so at even moments, say at $t = 0$. His payoff from deviating is $5 + 2 \cdot \delta/(1-\delta)$, since after this deviation player 2 plays R forever and player 1's best reply is to play D forever. If player 1 plays according to N_1^* he obtains $4/(1-\delta^2) + 2 \cdot \delta/(1-\delta^2)$. To obtain an equilibrium, this should be at least as large as the maximal payoff from deviating, hence

$$4/(1-\delta^2) + 2 \cdot \delta/(1-\delta^2) \geq 5 + 2 \cdot \delta/(1-\delta) \ ,$$

which holds if and only if $\delta \geq \frac{1}{3}\sqrt{3} \approx 0.58$.

Suppose player 1 plays N_1^*. If player 2 deviates from N_2^*, it is optimal to do so at even moments, say at $t = 0$. His payoff from deviating is $4 + 2 \cdot \delta/(1-\delta)$, since after this deviation player 1 plays U forever and player 2's best reply is to play R forever. If player 2 plays according to N_2^* he obtains $2/(1-\delta^2) + 4 \cdot \delta/(1-\delta^2)$. To obtain an equilibrium, this should be at least as large as the maximal payoff from deviating, hence

$$2/(1-\delta^2) + 4 \cdot \delta/(1-\delta^2) \geq 4 + 2 \cdot \delta/(1-\delta) \ ,$$

which holds if and only if $\delta \geq \frac{1}{2}\sqrt{5} - \frac{1}{2} \approx 0.62$.

Thus, (N_1^*, N_2^*) is a Nash equilibrium of $G^{\infty}(\delta)$ for $\delta \geq 0.62$. Note that this equilibrium is not subgame perfect: in a subgame following a deviation the players end up playing (U,R) forever, which is not a Nash equilibrium in such a subgame. Also note that the only Nash equilibrium in the stage game is (D,R). Hence, Propositions 7.1 and 7.2 only imply that the limit average payoffs $(2, 4)$ can be obtained in a subgame perfect Nash equilibrium of $G^{\infty}(\delta)$.

7.3 Problems

7.1. *Nash and Subgame Perfect Equilibrium in a Repeated Game (1)*
Consider the following bimatrix game:

$$G = (A, B) = \begin{matrix} & \begin{matrix} L & R \end{matrix} \\ \begin{matrix} U \\ D \end{matrix} & \begin{pmatrix} 2,3 & 1,5 \\ 0,1 & 0,1 \end{pmatrix} \end{matrix} .$$

(a) Determine all Nash equilibria of this game. Also determine the value $v(A)$ of the matrix game A and the value $v(-B)$ of the matrix game $-B$. Determine the optimal strategies of player 2 in A and of player 1 in $-B$.
(b) Consider the repeated game $G^{\infty}(\delta)$. Which limit average payoffs can be obtained in a subgame perfect equilibrium of this repeated game according to Proposition 7.1 or Proposition 7.2? Does this depend on δ?
(c) Which limit average payoffs can be obtained in a Nash equilibrium in $G^{\infty}(\delta)$ according to Proposition 7.4?
(d) Describe a pair of Nash equilibrium strategies in $G^{\infty}(\delta)$ that result in the limit average payoffs $(2, 3)$. What is the associated minimum value of δ?

7.2. *Nash and Subgame Perfect Equilibrium in a Repeated Game (2)*
Consider the following bimatrix game:

$$G = (A, B) = \begin{matrix} & \begin{matrix} L & R \end{matrix} \\ \begin{matrix} U \\ D \end{matrix} & \begin{pmatrix} 2,1 & 0,0 \\ 0,0 & 1,2 \end{pmatrix} \end{matrix} .$$

(a) Which payoffs can be reached as limit average payoffs in a subgame perfect equilibrium of the infinitely repeated game $G^{\infty}(\delta)$ for suitable choices of δ according to Propositions 7.1 and 7.2?
(b) Which payoffs can be reached as limit average payoffs in a Nash equilibrium of the infinitely repeated game $G^{\infty}(\delta)$ for suitable choices of δ according to Proposition 7.4?
(c) Describe a subgame perfect Nash equilibrium of $G^{\infty}(\delta)$ resulting in the limit average payoffs $(\frac{3}{2}, \frac{3}{2})$. Also give the corresponding restriction on δ.
(d) Describe a subgame perfect Nash equilibrium of $G^{\infty}(\delta)$ resulting in the limit average payoffs $(1, 1)$. Also give the corresponding restriction on δ.

7.3. *Nash and Subgame Perfect Equilibrium in a Repeated Game (3)*
Consider the following bimatrix game:

$$G = (A, B) = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 3,2 & 8,0 \\ 4,0 & 6,2 \end{pmatrix} \end{array}$$

(a) Which payoffs can be reached as limit average payoffs in a subgame perfect Nash equilibrium of the infinitely repeated discounted game $G^{\infty}(\delta)$ for suitable choices of δ according to Propositions 7.1 and 7.2?
(b) Which payoffs can be reached as limit average payoffs in a Nash equilibrium of the infinitely repeated game $G^{\infty}(\delta)$ for suitable choices of δ according to Proposition 7.4?
(c) Describe a Nash equilibrium of $G^{\infty}(\delta)$ resulting in the limit average payoffs $(4\frac{1}{2}, 2)$. Is there any value of δ for which this equilibrium is subgame perfect? Why or why not?

7.4. *Subgame Perfect Equilibrium in a Repeated Game*
Consider the following bimatrix game:

$$G = \begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{ccc} L & C & R \end{array} \\ \begin{pmatrix} 6,4 & 0,7 & 0,0 \\ 8,0 & 4,6 & 0,0 \\ 0,0 & 0,0 & 1,1 \end{pmatrix} \end{array} .$$

(a) What are the pure strategy Nash equilibria of this game?
(b) Which limit average payoffs can be obtained in a subgame perfect equilibrium of $G^{\infty}(\delta)$ according to your answer to (a) and Propositions 7.1 and 7.2?
(c) Describe a subgame perfect equilibrium in the infinitely repeated game which results in the limit average payoffs of 5 for each player. Also give the minimum value of δ for which your strategy combination is a subgame perfect equilibrium.

7.5. *The Strategies Tr_1^* and Tr_2^**
Give the inequalities that determine the lower bound on δ for the strategy combination (Tr_1^*, Tr_2^*) to be a subgame perfect equilibrium in the infinitely repeated prisoners' dilemma game in Sect. 7.1.1.

7.6. *Repeated Cournot and Bertrand*

(a) Reconsider the duopoly (Cournot) game of Sect. 6.2.1. Suppose that this game is repeated infinitely many times, and that the two firms discount the streams of payoffs by a common discount factor δ. Describe a subgame perfect Nash equilibrium of the repeated game that results in each firm receiving half of the monopoly profits at each time. Also give the corresponding restriction on δ. What could be meant by the expression 'tacit collusion'?

(b) Answer the same questions as in (a) for the Bertrand game of Sect. 6.3.

7.7. *Repeated Duopoly*
Two firms (1 and 2) offer heterogenous goods at prices

$$p_1 = \max\{10 - 2q_1 + q_2, 0\}$$
$$p_2 = \max\{10 - 2q_2 + q_1, 0\}$$

where q_1, q_2 are the quantities offered. All costs are assumed to be zero, and the firms are engaged in price competition.

(a) Show that the market clearing quantity of firm 1 at prices p_1 and p_2 is given by

$$q_1 = \max\{10 - \frac{2}{3}p_1 - \frac{1}{3}p_2, 0\} .$$

Also derive the quantity q_2 of firm 2 as a function of the prices, and set up the profit functions of the two firms.

(b) Derive the reaction functions of the firms and use these to compute the Nash equilibrium prices. Compute the associated profits of both firms.

(c) Compute the prices at which joint profit is maximized. Compute the associated profits of both firms.

(d) Now suppose that this price competition game is played infinitely many times, and that the firms' payoffs are discounted by a common factor δ. Describe a subgame perfect equilibrium in which joint profit is maximized in each period. Also give the associated lower bound for the discount factor δ.

7.8. *On Discounting*
In a repeated game, interpret the discount factor $0 < \delta < 1$ as the probability that the game will continue, i.e., that the stage game will be played again. Show that, with this interpretation, the repeated game will end with probability 1. (Cf. Problem 6.17(e).)

7.9. *On Limit Average*
Can you give an example in which the limit that defines the long run average payoffs, does not exist? (Cf. Remark 7.3.)

7.4 Notes

For finitely repeated games as in Problems 4.10–4.12 see also Benoit and Krishna (1985) and Friedman (1985).

Fudenberg and Maskin (1986) show that Proposition 7.4—the folk theorem for Nash equilibrium in infinitely repeated games—also holds for subgame perfect Nash equilibrium if the dimension of the 'cooperative payoff space' (the set $P(G)$ in the text) is equal to the number of players. This, however, requires more sophisticated strategies. See Fudenberg and Tirole (1991a) for further references.

The expression 'folk theorem' refers to the fact that results like this had been known among game theorists even before they were formulated and written down explicitly. They belonged to the *folklore* of game theory.

For more advanced and elaborate treatments of repeated games see, e.g., Fudenberg and Tirole (1991a), Mailath and Samuelson (2006), or Myerson (1991).

References

Benoit, J.-P., & Krishna, V. (1985). Finitely repeatd games. *Econometrica, 53*, 905–922.

Friedman, J. W. (1985). Cooperative equilibria in finite horizon noncooperative supergames. *Journal of Economic Theory, 35*, 390–398.

Fudenberg, D., & Maskin, E. (1986). The folk theorem in repeated games with discounting or with incomplete information. *Econometrica, 54*, 533–556.

Fudenberg, D., & Tirole, J. (1991a). *Game theory*. Cambridge: MIT Press.

Mailath, G. J., & Samuelson, L. (2006). *Repeated games and reputations*. New York: Oxford University Press.

Myerson, R. B. (1991). *Game theory, analysis of conflict*. Cambridge: Harvard University Press.