23 Review Problems for Part I

This chapter contains Review Problems to Chaps. 2–10, organized per chapter.

Chapter 2

RP 1 *Matrix Games (1)*

(a) Give the maximin rows and minimax columns in the following matrix game:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

What can you infer from this about the value of the game?

(b) Calculate all (mixed) maximin and minimax strategies and the value of the following matrix game:

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 3 & 1 & 4 \\ 2 & 0 & 2 \end{pmatrix} .$$

(c) Answer the same questions as in (a) and (b) for the following game:

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 4 & 2 & 1 & 3 \\ 3 & 1 & 3 & 3 \end{pmatrix} .$$

H. Peters, *Game Theory*, Springer Texts in Business and Economics,
DOI 10.1007/978-3-662-46950-7_23

RP 2 *Matrix Games (2)*

(a) Consider the following matrix game:

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \\ 0 & 5 \\ 5 & 1 \end{pmatrix}.$$

Determine all maximin rows and minimax columns, and also all saddlepoint(s), if any. What can you conclude from this about the value of this game?

(b) Consider the six different 2×2-matrix games that can be obtained by choosing two rows from A, as follows:

$$A_1 = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \quad A_2 = \begin{pmatrix} 4 & 2 \\ 0 & 5 \end{pmatrix} \quad A_3 = \begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 3 \\ 0 & 5 \end{pmatrix} \quad A_5 = \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix} \quad A_6 = \begin{pmatrix} 0 & 5 \\ 5 & 1 \end{pmatrix}.$$

Determine the values of all these games. Which one must be equal to the value of A? (Give an argument for your answer!)

(c) Determine all (possibly mixed) maximin and minimax strategies of A. [Hint: Use your answer to (b).]

RP 3 *Matrix Games (3)*

(a) Consider the following matrix game:

$$A = \begin{pmatrix} 8 & 16 & 12 \\ 9 & 8 & 4 \\ 12 & 4 & 16 \end{pmatrix}.$$

Determine all maximin rows and minimax columns, and also all saddlepoint(s), if any. What can you conclude from this about the value of this game?

(b) Give an argument why player 1 (the row player) will never put probability on the second row in a maximin strategy. Give also an argument why, consequently, player 2 will not put probability on the third column in a minimax strategy.

(c) Determine all maximin and minimax strategies and the value of A. [Hint: Use part (b).]

Chapter 3

RP 4 *Bimatrix Games (1)*
Consider the following bimatrix game:

$$\begin{array}{c} \\ U \\ M \\ D \end{array}\begin{array}{c} \begin{array}{ccc} L & C & R \end{array} \\ \begin{pmatrix} 0,0 & 1,1 & 2,2 \\ 3,1 & 0,0 & 0,1 \\ 1,1 & 0,0 & 1,0 \end{pmatrix} \end{array}.$$

(a) Simplify this game by iterated elimination of strictly dominated strategies. Each time you eliminate a pure strategy, say by which pure or mixed strategy the eliminated strategy is strictly dominated and why.
(b) Find all Nash equilibria in the remaining game by computing the best reply functions and making a diagram.
(c) Determine all Nash equilibria in the original game.

RP 5 *Bimatrix Games (2)*

(a) Consider the following bimatrix game:

$$(A,B) = \begin{pmatrix} x,1 & x,0 \\ 0,0 & 2,1 \end{pmatrix}.$$

For every $x \in \mathbb{R}$, determine all Nash equilibria (in mixed and pure strategies) of this game.
(b) Consider the following bimatrix game:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} e & f & g & h \end{array} \\ \begin{pmatrix} 8,0 & 0,1 & 0,2 & 2,10 \\ 1,7 & 3,2 & 2,0 & 3,1 \\ 3,0 & 0,3 & 4,5 & 4,0 \\ 0,2 & 1,1 & 1,3 & 7,7 \end{pmatrix} \end{array}.$$

Show that strategy f for player 2 is strictly dominated by a mixed strategy, but not by a pure strategy. Which strategies survive iterated elimination of strictly dominated strategies in this game? Find all Nash equilibria in pure and mixed strategies of this game.

RP 6 *Voting*
Two political parties, I and II, each have four votes that they can distribute over two party-candidates each. A committee is to be elected, consisting of three members. Each political party would like to see as many as possible of their own candidates elected in the committee. Of the total of four candidates those three that have most

of the votes will be elected; in case of ties, tied candidates are drawn with equal probabilities.

(a) Model this situation as a bimatrix game between the two parties.
(b) Determine all (possibly mixed) Nash equilibria in this game.

Chapter 4

RP 7 *A Bimatrix Game*
Consider the following bimatrix game, where a can be any real number:

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} a, 0 & 0, 1 \\ 0, 1 & a, 0 \end{pmatrix} \end{array}$$

(a) Determine the set of (mixed and pure) Nash equilibria of this game for every value of a.
Suppose now that player 1 (the row player) chooses first, and that player 2 (the column player) observes this move and chooses next.
(b) Write down the extensive form of this game, determine the associated strategic form, and for every $a \in \mathbb{R}$ determine all subgame perfect equilibria (in pure strategies) in this game.

RP 8 *An Ice-cream Vendor Game*
Three ice-cream vendors choose a location on the beach. This beach has the following form:

A	*B*
C	*D*

Each of the four regions has 300 customers. Each customer goes to the nearest ice-cream vendor, but can only move vertically or horizontally (hence not diagonally). In case of ties customers are distributed equally over vendors.

(a) Suppose the three vendors simultaneously and independently choose one of the four regions. Determine the Nash equilibrium or equilibria of this game, if any.
(b) Suppose vendor 1 chooses first, vendor 2 observes this and chooses next, and vendor 3 observes both choices and chooses last. Determine the subgame perfect Nash equilibrium or equilibria of this game, if any.

RP 9 *A Repeated Game*
Consider the three player game where player 1 chooses between U and D, player 2 between L and R, and player 3 between A and B. Choices are made

simultaneously and independently, and only pure strategies are considered. The payoffs are given by:

$$A: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 2,2,0 & 5,5,5 \\ 8,6,8 & 0,7,4 \end{pmatrix} \end{array} \quad B: \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 4,4,1 & 4,2,8 \\ 0,2,9 & 4,2,5 \end{pmatrix} \end{array}.$$

Hence, if player 3 plays A the left matrix applies and if player 3 plays B the right matrix applies. The triples are the payoffs for players 1, 2, and 3, respectively.

(a) Find the (pure strategy) Nash equilibria of this game, if any.
(b) Consider the twice repeated game with as payoffs the sums of the payoffs in the two periods. Suppose the players play a subgame perfect Nash equilibrium. What can you say about the actions chosen in the second period?
(c) Consider again the twice repeated game of (b). Is there a subgame perfect Nash equilibrium of the game in which (U, R, A) is played in the first period? If not, argue why not; otherwise, give a complete description of such an equilibrium.
(d) Answer the same question as in (b) for the combination (U, R, B).

RP 10 *Locating a Pub*
Two pub owners, player 1 and player 2, must choose a location for their pubs in a long street. In fact, player 1 wants to open two new pubs in the street, but player 2 only one. The possible locations are A, B and C, where A is right at the beginning of the street, B is somewhere in between A and C, and C is at the very end. Between A and B there are 200 potential customers (uniformly distributed), and between B and C there are 300 potential customers (uniformly distributed as well). Every customer will always go to the pub that is closest to his house.
The two pub owners have agreed on the following procedure. First, player 1 chooses a location for his first pub. Next, player 2 observes player 1's choice and chooses a location for his pub. Player 2 must choose a different location than player 1. Finally, player 1 obtains the remaining location for his second pub.
The objective for both pub owners is to maximize the number of customers.

(a) Formulate this situation as an extensive form game. How many strategies does player 1 have? And player 2?
(b) Find the unique subgame perfect equilibrium for this game.

Suppose now that they change the procedure as follows. After player 1 has chosen the location for his first pub, players 1 and 2 simultaneously choose one of the two remaining locations. If they happen to choose the same location, then they will enter into a fierce fight and the overall utility for both pub owners is 0.

(c) Formulate this situation as an extensive form game. How many strategies does player 1 have? And player 2?

(d) Find a subgame perfect equilibrium in which player 1 starts by choosing A. Find another subgame perfect equilibrium in which player 1 starts by choosing B. Find yet another subgame perfect equilibrium in which player 1 starts by choosing C.

RP 11 *A Two-stage Game*
Consider the two bimatrix games

$$G_1 = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 3,3 & 0,4 \\ 4,0 & 1,1 \end{pmatrix} \end{array}, \quad G_2 = \begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{pmatrix} 3,1 & 0,0 & 0,0 \\ 0,0 & 3,3 & 0,0 \\ 0,0 & 0,0 & 1,3 \end{pmatrix} \end{array}.$$

(a) Compute all the pure strategy Nash equilibria in G_1 and all the pure strategy Nash equilibria in G_2.

Now suppose that G_1 is played first, then the players learn the outcome of G_1 and next play the game G_2. For each player the payoff is the sum of the payoffs of each game separately.

(b) In this two-stage game, how many (pure) strategies does each player have?
(c) Is there a subgame perfect Nash equilibrium in which (U, L) is played in G_1? If so, then describe such an equilibrium. If not, explain why.
(d) Now suppose that the order of play is reversed: first G_2 is played, then G_1. Determine all subgame perfect Nash equilibria of this two-stage game.

Chapter 5

RP 12 *Job Market Signaling*[1]
A worker (W) has private information about his level of ability. With probability $\frac{1}{3}$ he is a high type (H) and with probability $\frac{2}{3}$ he is a low type (L). After observing his own type, the worker decides whether to obtain a costly education (E) or not (N); think of E as getting a degree. The firm (F) observes the worker's education decision but not the worker's ability. The firm then decides whether to employ the worker in an important managerial job (M) or in a much less important clerical job (C). The payoffs are as follows. If the worker gets job M then he receives a payoff of 10 but has to subtract the cost of education if he has chosen E: this is 4 for a worker of type H but 7 for a worker of type L. If the worker gets job C then he receives a payoff of 4 minus the cost of education if he has chosen E. The firm has a payoff of 10 if it offers the job M to a H type worker but 0 if it offers M to an L type worker; and it has payoff 4 if it offers job C to any type worker.

[1]From Watson (2002).

(a) Represent this game in extensive form.
(b) Find all the pure strategy Nash equilibria of this game, if any. (You may but do not have to use the strategic form.)
(c) Which one(s) is (are) perfect Bayesian? Pooling or separating? Provide the associated beliefs and discuss whether or not the intuitive criterion is satisfied.

RP 13 *Second-hand Cars (1)*
We consider the following variation on the 'lemons' (second hand car) market. A seller wants to sell his car. He knows the quality, which can be good or bad with equal probabilities. There is also a buyer, who does not know the quality. The seller has the choice between offering a guarantee certificate or not, next the buyer has the choice between buying the car or not, having observed the action of the seller. The car has a fixed price of 15. The value of a bad or a good car for the buyer equals 10 or 20, respectively (some time after the sale, the buyer will discover the quality of the car). The car does not have any value for the seller. Offering a guarantee certificate represents an expected transfer from the seller to the buyer of 0 if the car is good, and of 10 if the car is bad (if the buyer buys the car). If the buyer does not buy the car then payoffs are 0 to both.

(a) Model this situation as a game in extensive form. Is this a signaling game? Why or why not?
(b) Determine the strategic form of this game, and all Nash equilibria in pure strategies, if any.
(c) Determine all perfect Bayesian Nash equilibria. Which ones are pooling or separating, if any?

RP 14 *Second-hand Cars (2)*
You have seen a nice-looking second-hand car, and you are considering buying it. The problem is that you do not know the precise value of the car. Suppose that there is a 25 % chance that the value of the car is 2,000, and there is a 75 % chance that the value is 4,000. You are aware of these probabilities. The car seller, on the other hand, knows the precise value of the car. You and the car seller simultaneously name a price, and a transaction only takes place if your price is higher or equal than the price named by the seller. In that case, the eventual price to be paid will be the average of your price and the price named by the seller.

For convenience, assume that both you and the seller can only name prices 1,000, 3,000 and 5,000. If you buy the car, your utility would be the value of the car minus the price paid. For the seller the utility would be the price received minus the value of the car. If no transaction takes place, the utility for both would be zero.

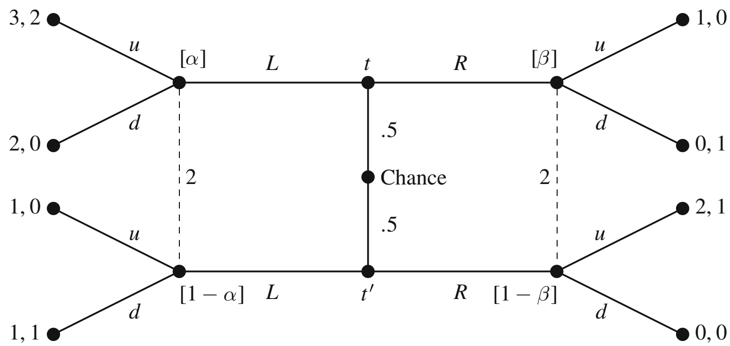
(a) Model this situation as a game with incomplete information.
(b) How many types and how many strategies do you have? What about the seller?
(c) One of your strategies is strictly dominated. Which one? Find a pure strategy, or mixed strategy, for you that strictly dominates it.

(d) Show that there is no Nash equilibrium in which you choose a price of 3,000.
(e) Find the Nash equilibrium, or equilibria, in pure strategies of this game. Will you eventually buy the car?

RP 15 *Signaling Games*
Compute the strategic form and all the pure strategy equilibria in the games in (a) and (b) below. Also determine all perfect Bayesian equilibria in pure strategies. Which ones are pooling or separating, if any? Give the corresponding beliefs. Apply the intuitive criterion.

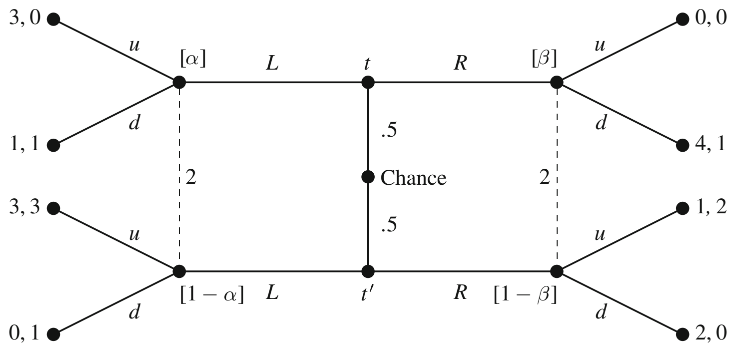
(a)

(a)



(b)

(b)



RP 16 *A Game of Incomplete Information*
Consider the following two games G_1 and G_2.

$$G_1 = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} C & D & E \end{array} \\ \begin{pmatrix} 5,2 & 3,2 & 2,3 \\ 5,2 & 3,2 & 2,3 \end{pmatrix} \end{array} \quad G_2 = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} C & D & E \end{array} \\ \begin{pmatrix} 1,2 & 0,1 & 3,0 \\ 3,3 & 2,4 & 0,0 \end{pmatrix} \end{array} .$$

At the beginning, a chance move determines whether the payoffs are as in G_1 or as in G_2. Both events happen with probability 0.5. Player 1 knows whether G_1 is played or G_2, but player 2 does not know which of the two games is played. Finally, both players simultaneously choose an action.

(a) Draw the extensive form representation of this game. (That is, draw the game tree.)
(b) Compute the strategic form of this game with incomplete information, and find all Nash equilibria in pure strategies.

Suppose now that player 2 knows whether G_1 or G_2 is played, but that player 1 does not know.

(c) Find all Nash equilibria in pure strategies of this new game with incomplete information. Use reasoning to find these equilibria, without making the strategic form.

RP 17 *A Bayesian Game*[2]
Two persons are involved in a dispute, and each person can either fight (F) or yield (Y). Each person has a payoff of 0 from yielding, regardless of the other person's action, and a payoff of 1 from fighting if the other person yields. If both persons fight, then the payoffs are -1 to person 1 and 1 to person 2. (This reflects the fact that person 2 is *strong*.)

(a) Represent this game in bimatrix from and compute all the Nash equilibria (so, in pure and mixed strategies).

Now assume that person 2 could also be *weak*, in which case the payoffs from both persons fighting are equal to 1 for person 1 and -1 for person 2; all other payoffs stay the same. Suppose that person 2 knows whether he is weak or strong, but person 1 only knows that person 2 is strong with probability α.

(b) Represent this game in extensive form.
(c) Determine the strategic form of the game in (b) and compute all Nash equilibria in pure strategies, if any. (Your answer may depend on the value of α, so you may have to distinguish cases.)

[2]From Osborne (2004).

RP 18 *Entry as a Signaling Game*[3]
A challenger contests an incumbent's market. The challenger is *strong* with probability $1/4$ and *weak* with probability $3/4$; it knows its type, but the incumbent does not. The challenger may either *prepare* itself for battle, or remain *unprepared* (it does not have the option of staying out). The incumbent observes whether the challenger is prepared or not, but not its type, and chooses whether to *fight* or *acquiesce*. An unprepared challenger's payoff is 5 if the incumbent acquiesces to its entry. Preparations cost a strong challenger 1 unit of payoff and a weak one 3 units, and fighting entails a loss of 2 units for each type. The incumbent prefers to fight (payoff 2) rather than to acquiesce to (payoff 0) a weak challenger (who is quickly dispensed with), and prefers to acquiesce to (payoff 2) rather than to fight (payoff 0) a strong one.

(a) Represent this game in extensive form.
(b) Find all the pure strategy Nash equilibria of this game, if any. (You may but do not have to use the strategic form.)
(c) Which one(s) is (are) perfect Bayesian? Pooling or separating? Provide the associated beliefs and discuss whether or not the intuitive criterion is satisfied.

Chapter 6

RP 19 *Bargaining (1)*
Consider the following two-player bargaining game. Player 1 owns an object which has worth 0 to him. The object has worth $v \in [0, 1]$ to player 2: player 2 knows v but player 1 only knows that v has been drawn according to the uniform distribution over $[0, 1]$. There are two periods. In period 1, player 1 makes a price offer p_1 and player 2 responds with "yes" or "no". If player 2 rejects player 1's offer, then player 2 makes the price offer p_2 in the second period, to which player 1 can say "yes" or "no". Agreement in the first period yields the payoff p_1 to player 1 and $v - p_1$ to player 2. Agreement in the second period yields δp_2 to player 1 and $\delta(v - p_2)$ to player 2 ($0 < \delta < 1$ is a discount factor). No agreement yields 0 to both.

(a) Explain that this is a game of incomplete information. What are the type sets of the players? Describe the strategy set of player 1.
(b) Describe the strategy set of player 2.
(c) Suppose player 1 offers a price p_1 in period 1. Give the best response of player 2 (assume that players 1 and 2 accept in case of indifference).
(d) Compute the perfect Bayesian Nash equilibrium of this game.

RP 20 *Bargaining (2)*
Suppose two players (bargainers) bargain over the division of one unit of a perfectly divisible good. Player 1 has utility function $u_1(\alpha) = \alpha$ and player 2

[3]From Osborne (2004).

has utility function $u_2(\beta) = 1 - (1 - \beta)^2$ for amounts $\alpha, \beta \in [0,1]$ of the good.

(a) Determine the set of feasible utility pairs. Make a picture.
(b) Determine the Nash bargaining solution outcome, in terms of utilities as well as of the physical distribution of the good.
(c) Suppose the players' utilities are discounted by a factor $\delta \in [0,1)$. Calculate the Rubinstein bargaining outcome.
(d) Determine the limit of the Rubinstein bargaining outcome, for δ approaching 1, in two ways: by using the result of (b) and by using the result of (c).

RP 21 *Bargaining (3)*
Suppose two players (bargainers) bargain over the division of one unit of a perfectly divisible good. Assume that utilities are just equal to the amounts of the good obtained, discounted by a common discount factor $0 < \delta < 1$. The players play a finite Rubinstein bargaining game over three periods $t = 0, 1, 2$, player 1 starting to make an offer at time $t = 0$, player 2 making an offer at time $t = 1$ (if reached), and the game ending at time $t = 2$ (if reached) with equal split $(\frac{1}{2}, \frac{1}{2})$.

(a) Calculate the backwards induction outcome of this game. Argue that player 1 has a beginner's advantage.

Suppose again that two players (bargainers) bargain over the division of one unit of a perfectly divisible good. Assume that player 1 has utility function $u(x)$ $(0 \leq x \leq 1)$ and player 2 has utility function $v(x) = 2u(x)$ $(0 \leq x \leq 1)$.

(b) Determine the physical distribution of the good according to the Nash bargaining solution. Can you say something about the resulting utilities?

RP 22 *Ultimatum Bargaining*
Consider the ultimatum bargaining game where player 1 offers a division $(1-m, m)$ of one Euro, and player 2 can accept (receiving m whereas player receives $1-m$) or reject (in which both players receive zero). Suppose that player 1 only cares about how much money he receives, but player 2 also cares about the division: specifically, if the game results in monetary payoffs (x_1, x_2), then the (utility) payoff to player 2 is $x_2 + a(x_2 - x_1)$, where a is some positive constant.

(a) Represent this game in extensive form, writing the payoffs in terms of m and a.
(b) Determine the subgame perfect Nash equilibrium.
(c) What happens to the equilibrium monetary split as a becomes large? What is the explanation for this?

RP 23 *An Auction (1)*
Two players (bidders) have valuations $v_1 > 0$ and $v_2 > 0$ for an object. They simultaneously and independently announce bids $b_1, b_2 \in [0, \infty)$. The player with the highest bid obtains the object and has a payoff $v_i - b_i$, the other player has

payoff 0. If the bids are equal, then each obtains the object and associated payoff with probability $\frac{1}{2}$.

(a) Does this game have perfect or imperfect information? Complete or incomplete information?
(b) Suppose that $v_1 = v_2$. Determine all Nash equilibria (in pure strategies), if any.
(c) Suppose that $v_1 > v_2$. Determine all Nash equilibria (in pure strategies), if any.
(d) Suppose that $v_1 = 1$ and $v_2 = 3$. Also suppose that only integer numbers up to 3 are allowed as bids: $b_1, b_2 \in \{0, 1, 2, 3\}$. Represent this game as a bimatrix game and solve for all pure Nash equilibria.

RP 24 *An Auction (2)*
$n \geq 4$ players participate in an auction for an object for which their evaluations are $v_1 > v_2 > \ldots > v_n$. It is a sealed-bid auction, and the highest bidder obtains the object and pays the fourth-highest bid. In case of a tie among highest bidders, the player with the lowest number among the highest bidders obtains the object.

(a) Show that for any player i the bid of v_i weakly dominates any lower bid but does not weakly dominate any higher bid.
(b) Show that a strategy profile in which each player bids his true valuation is not a Nash equilibrium.
(c) Find all Nash equilibria in which all players submit the same bid.

RP 25 *An Auction (3)*
Two individuals (players) participate in the auction of a painting. The painting has worth 6 for player 1 and 4 for player 2. The individuals simultaneously and independently submit their bids b_1 and b_2, where these bids can only be whole numbers: $b_1, b_2 \in \{0, \ldots, 6\}$ (higher bids do not make sense). The highest bidder wins the auction, receives the painting and pays his bid, whereas the other player receives and pays nothing. In case of a draw player 1 wins. Hence, the payoff to player 1 is $6 - b_1$ if $b_1 \geq b_2$ and 0 if $b_1 < b_2$, and the payoff to player 2 is $4 - b_2$ if $b_2 > b_1$ and 0 if $b_2 \leq b_1$.

(a) Compute the best reply functions of both players, and draw a diagram.
(b) Compute the Nash equilibrium or equilibria of this game, if any.

RP 26 *Quantity Versus Price Competition*
Suppose, in the Cournot model, that the two firms produce heterogenous goods, which have different market prices. Specifically, suppose that these market prices are given by

$$p_1 = \max\{4 - 2q_1 - q_2, 0\}, \quad p_2 = \max\{4 - q_1 - 2q_2, 0\} . \qquad (*)$$

These are the prices of the goods of firms 1 and 2, respectively. The firms compete in quantities. Both fixed and marginal costs are assumed to be zero.

(a) Write down the payoff (= profit) functions of the two firms and compute the Nash equilibrium quantities.
(b) Use (*) to show that

$$q_1 = \max\{\frac{1}{3}(p_2 - 2p_1 + 4), 0\}, \quad q_2 = \max\{\frac{1}{3}(p_1 - 2p_2 + 4), 0\}\ . \qquad (**)$$

Assume now that the firms compete in prices, with demands given by (**).

(c) Write down the payoff (= profit) functions of the two firms in terms of prices and compute the Nash equilibrium prices.
(d) Compare the equilibria found under (a) and (c). Specifically, compare the quantities and (associated) prices under (a) to the (associated) quantities and prices under (c). What about the associated profits? Which is tougher: quantity or price competition?

RP 27 *An Oligopoly Game (1)*
Three oligopolists operate in a market with inverse demand function given by $P(Q) = a - Q$, where $Q = q_1 + q_2 + q_3$ and q_i is the quantity produced by firm i. Each firm has a constant marginal cost of production, $0 < c < a$, and no fixed cost. The firms choose their quantities as follows: (1) firm 1 chooses $q_1 \geq 0$; (2) firms 2 and 3 observe q_1 and then simultaneously choose q_2 and q_3, respectively.

(a) Draw a picture of the extensive form of this game. Also give the strategic form: describe the strategy spaces of the players and the associated payoff functions.
(b) Determine the subgame perfect Nash equilibrium of this game.

RP 28 *An Oligopoly Game (2)*
Consider the Cournot model with three firms. Each firm $i = 1, 2, 3$ offers $q_i \geq 0$, and the market price of the good is $P(q_1, q_2, q_3) = 10 - q_1 - q_2 - q_3$ (or zero if this amount should be negative). Firms 1 and 2 have marginal costs equal to 0 while firm 3 has marginal cost equal to 1. We assume that the firms are involved in Cournot quantity competition.

(a) Derive the reaction functions of the three firms.
(b) Compute the Nash equilibrium of this game.
(c) Determine the maximal joint profit the three firms can achieve by making an agreement on the quantities q_1, q_2, and q_3. Is such an agreement unique?

RP 29 *A Duopoly Game with Price Competition*
Two firms (1 and 2) sell one and the same good. They engage in price competition à la Bertrand. The demand for the good at price p is $100 - p$. The firm that sets the lower price gains the whole market; in case of equal prices the market is split evenly.

The unit marginal cost for firm 1 is $c_1 = 30$, whereas for firm 2 it is $c_2 = 50$. Prices are in whole units, i.e., $p \in \{0, 1, \ldots\}$. (So, e.g., a price of 20.5 is not allowed.)

(a) Write down the profit functions of both firms, and compute the monopoly prices.
(b) Derive the reaction functions of both firms.
(c) Compute the Nash equilibrium in which the price of firm 1 is minimal among all Nash equilibria.
(d) Compute the Nash equilibrium in which the price of firm 1 is maximal among all Nash equilibria.

RP 30 *Contributing to a Public Good*
Three people simultaneously decide whether or not to contribute to a public good. At least two contributions are needed in order to provide the public good. Suppose that the public good has a value of 8 units to every person if it is provided, and that the contribution is fixed at 3 units. If a person decides to contribute, then the contribution must be paid also if the good is not provided.

(a) Find all Nash equilibria in pure strategies of this game.
(b) Suppose now that players 2 and 3 use the same mixed strategy, in which they contribute with probability p. Show that player 1's expected payoff of contributing is equal to

$$16p - 8p^2 - 3 \ .$$

(c) Compute the two symmetric mixed strategy Nash equilibria of this game.

RP 31 *A Demand Game*
Three players divide one perfectly divisible Euro. The players simultaneously submit their demands, x_1, x_2, and x_3, respectively, where $0 \le x_i \le 1$ for each player $i = 1, 2, 3$. If the sum of these demands is smaller than or equal to 1, i.e., $x_1 + x_2 + x_3 \le 1$, then each player i obtains his demand x_i; otherwise, i.e., if $x_1 + x_2 + x_3 > 1$, then each player obtains 0.
First, for each of the following amounts, either exhibit a Nash equilibrium of this game with sum of demands equal to that amount, or show that such a Nash equilibrium does not exist.

(a) 0.9; (b) 1.2; (c) 1.5; (d) 1.8

(e) Determine all Nash equilibria of this game, i.e., all triples (x_1, x_2, x_3) with $0 \le x_1, x_2, x_3 \le 1$ so that no player can do better given the demands of the other players.

Chapter 7

RP 32 *A Repeated Game (1)*
Consider the following bimatrix game

$$G = \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 16, 24 & 0, 25 \\ 0, 18 & 16, 16 \end{pmatrix} \end{array}.$$

(a) Which set of payoffs can be reached as the long run average payoffs in subgame perfect Nash equilibria of the infinitely repeated discounted game $G^{\infty}(\delta)$ for suitable choices of δ?
(b) Same question as under (a), but now for Nash equilibrium (not necessarily subgame perfect).
(c) Describe a Nash equilibrium of $G^{\infty}(\delta)$ resulting in the long run average payoffs $(16, 20)$. Is there any value of δ for which this equilibrium is subgame perfect? Why or why not?

RP 33 *A Repeated Game (2)*
Consider the following stage game G:

$$\begin{array}{c} \\ U \\ M \\ D \end{array} \begin{array}{c} \begin{array}{ccc} L & C & R \end{array} \\ \begin{pmatrix} 8, 8 & 0, 9 & 4, 1 \\ 9, 1 & 2, 1 & 4, 2 \\ 10, 3 & 2, 4 & 4, 4 \end{pmatrix} \end{array}.$$

Player 1 is the row player and player 2 the column player.

(a) Compute all the pure strategy Nash equilibria of G.
(b) Compute all mixed-strategy Nash equilibria of the game G.
(c) Suppose that the game G is played twice, with as payoffs the sums of the payoffs of the two stages. Determine the number of (pure) strategies of each player. Is there a subgame perfect (pure strategy) Nash equilibrium of the twice repeated game in which (U, L) is played in the first stage? (Describe such an equilibrium or argue that it cannot exist.)
(d) Consider the infinite repetition of the game G in which the players use the discounted sums of payoffs to evaluate the outcome. The players have a common discount factor $0 < \delta < 1$. Describe a subgame perfect Nash equilibrium of the repeated game in which always (U, L) is played, using trigger strategies. Also compute the minimal value of δ for which this equilibrium exists.

RP 34 *A Repeated Game (3)*
Consider the following stage game G:

$$\begin{array}{c} \\ U \\ M \\ D \end{array} \begin{array}{c} \begin{matrix} L & C & R \end{matrix} \\ \begin{pmatrix} 0,0 & 11,0 & 6,1 \\ 0,12 & 10,10 & 5,5 \\ 1,6 & 12,4 & 6,6 \end{pmatrix} \end{array}.$$

Player 1 is the row player and player 2 the column player.

(a) Compute all the pure strategy Nash equilibria of G.
(b) Compute all mixed-strategy Nash equilibria of the game G.
(c) Suppose that the game G is played twice, with as payoffs the sums of the payoffs of the two stages. Determine the number of (pure) strategies of each player. Is there a subgame perfect (pure strategy) Nash equilibrium of the twice repeated game in which (M, C) is played in the first stage? (Describe such an equilibrium or argue that it cannot exist.)
(d) Consider the infinite repetition of the game G in which the players use the discounted sums of payoffs to evaluate the outcome. The players have a common discount factor $0 < \delta < 1$. Describe a subgame perfect Nash equilibrium of the repeated game in which always (M, C) is played, using trigger strategies. Also compute the minimal value of δ for which this equilibrium exists.

RP 35 *A Repeated Game (4)*
Consider the bimatrix game

$$G = \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{matrix} L & R \end{matrix} \\ \begin{pmatrix} 4,10 & 3,9 \\ 5,5 & 3,4 \end{pmatrix} \end{array}.$$

Suppose this game is played twice. After each play of the game the players learn the outcome. The total payoff is the sum of the payoffs of each of the two plays of the game.

(a) Describe the subgame perfect equilibrium or equilibria in pure strategies of this twice repeated game, if any. Also give the associated outcome(s) and payoffs.
(b) Is there a Nash equilibrium of the twice repeated game in which the combination (T, L) is played the first time? If so, describe such an equilibrium and the associated outcome and payoffs. If not, explain why.

Now assume that the game is repeated infinitely many times. Payoffs are the discounted sums of stage payoffs, with common discount factor $0 < \delta < 1$.

(c) Which long run average payoffs can be obtained in a subgame perfect equilibrium in (pure) trigger strategies of this infinitely repeated game for a suitably chosen value of δ? Give an example of such a subgame perfect equilibrium, and also give the associated outcome, payoffs, and long-run average payoffs, as well as the values of δ for which this is an equilibrium.

RP 36 *A Repeated Game (5)*
Consider the following bimatrix game:

$$\begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 2,1 & 5,0 \\ 0,6 & 1,1 \end{pmatrix} \end{array}.$$

(a) Determine all Nash equilibria (in pure or mixed strategies) in this game.

Now suppose that the game is infinitely repeated, at times $t = 0, 1, 2, \ldots$, and that the players learn the outcome after each play of the game. There is a common discount factor $0 < \delta < 1$, and the payoffs are the discounted streams of payoffs.

(b) Which pairs of payoffs can be reached as long-run average payoffs in a subgame perfect equilibrium (in trigger strategies) in this game, assuming that we can take δ as close to 1 as desired?
(c) Give a subgame perfect equilibrium of the infinitely repeated game resulting in the average payoffs $(2\frac{1}{2}, 3)$. Give the values of δ for which your strategies indeed form a subgame perfect equilibrium.

Chapter 8

RP 37 *An Evolutionary Game*
Consider the following matrix

$$A = \begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \end{array}.$$

(a) Suppose this matrix represents an evolutionary game between animals of the same species. Give an interpretation of a mixed strategy $(p, 1-p)$.
(b) Determine the replicator dynamics, rest points and stable rest points, and evolutionary stable strategies. Include a phase diagram for the replicator dynamics.
(c) For the evolutionary stable strategy or strategies, show directly (that is, without using the replicator dynamics) that the strategy (or strategies) is (or are) evolutionary stable.

Chapter 9

RP 38 *An Apex Game*

A voting committee consists of five members. Player 1 is a major player, called the *apex player*. The other players are called *minor players*. In order to pass a decision one needs the consent of either the apex player and at least one minor player, or of the four minor players. Therefore, a coalition that contains either the apex player and at least one minor player, or all minor players, is called *winning*. We model this situation as a so-called *simple game*: winning coalitions obtain worth 1, all other coalitions worth 0.

(a) Is the core of this game empty or not? If not, then compute it.
(b) Compute the Shapley value of this game. (Use the symmetry.)
(c) Compute the nucleolus of this game. (Use the symmetry.)

RP 39 *A Three-person Cooperative Game (1)*
A three-person cooperative game with player set $\{1,2,3\}$ is described in the following table:

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	1	0	3	3	a	10

(a) Determine all values of a for which the core of this game is not empty.
(b) Determine the Shapley value of this game. For which values of a is it in the core?
(c) Determine all values of a, if any, for which the vector $(\frac{16-2a}{3}, \frac{7+a}{3}, \frac{7+a}{3})$ is the nucleolus of this game.

RP 40 *A Three-person Cooperative game (2)*
For each $a \in \mathbb{R}$ the three-person cooperative game $(\{1,2,3\}, v_a)$ is given by $v_a(1) = a$, $v_a(2) = v_a(3) = 0$, $v_a(12) = 2$, $v_a(13) = 3$, $v_a(23) = 4$, and $v_a(N) = v_a(123) = 5$.

(a) For which values of a is the core of v_a non-empty? For these values, compute the core.
(b) Compute the Shapley value of v_a. For which values of a is it in the core of v_a?
(c) For which values of a are the maximal excesses at the nucleolus of v_a reached by the two-person coalitions? For those values, compute the nucleolus.

RP 41 *Voting*
Suppose in Parliament there are four parties A, B, C, D with numbers of votes equal to $40, 30, 20, 10$, respectively. To pass any law an absolute majority ($>50\,\%$) is needed.

(a) Formulate this situation as a four-person cooperative game where winning coalitions (coalitions that have an absolute majority) have worth 1 and losing coalitions worth 0. Determine the Shapley value of this game.

For every party X in $\{A, B, C, D\}$ let p_X denote the number of coalitions, containing X, that are winning but would be losing without X. Define $\beta(X) = \frac{p_X}{p_A+p_B+p_C+p_D}$ for every party X.

(b) Compute $\beta(X)$ for every $X \in \{A, B, C, D\}$.

Consider now a Parliament with three parties A, B, C and numbers of votes equal to 20, 10, 10, respectively. To pass any law a two-third majority is needed.

(c) Answer the same questions as in (a) and (b) for this Parliament. (Of course, now $p_X = \frac{p_X}{p_A+p_B+p_C}$ for $X \in \{A, B, C\}$.)

RP 42 *An Airport Game*
Three airline companies share the cost of a runway. To serve the planes of company $i \in \{1, 2, 3\}$ the length of the runway must be c_i, which is also the cost of a runway of that length. The airline companies can form coalitions, and the cost of a coalition is the cost of the smallest runway long enough to serve the planes of all companies in the coalition. The costs c_i are given by $c_i = i$ for each company $i \in \{1, 2, 3\}$, and we assume $c_1 \le c_2 \le c_3$.

(a) Model this situation as a three-player cost savings (TU) game.
(b) Compute the core of this game.
(c) Compute the Shapley value of this game. Is it in the core?
(d) Suppose that at the nucleolus of this game the excesses of the two-person coalitions are equal and maximal. What does this imply for c_1, c_2, and c_3?

RP 43 *A Glove Game*
There are five players. Players 1 and 2 each possess a right-hand glove, while players 3, 4, and 5 each possess a left-hand glove. The players can form coalitions, and the worth of each coalition is equal to the number of pairs of gloves that the coalition can make.

(a) Compute the Shapley value of this game.
(b) Compute the core of this game.
(c) Compute the nucleolus of this game.

RP 44 *A Four-person Cooperative Game*
Consider the four-person game (N, v) with $N = \{1, 2, 3, 4\}$, $v(\{1, 2\}) = v(\{3, 4\}) = 2$, $v(\{1, 3\}) = 3$, $v(N) = 4$, and $v(S) = 0$ for all other coalitions S.

(a) Compute the core of this game. Plot the possible core payoffs of players 1 and 3 in a two-dimensional diagram with the payoffs of player 1 on the horizontal axis and the payoffs of player 3 on the vertical axis.
(b) Compute the Shapley value of this game.

Chapter 10

RP 45 *A Matching Problem*

Let $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ be two groups of people, and let their preferences concerning possible partners be given by the following table.

x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
y_3	y_3	y_3	y_4	x_2	x_3	x_2	x_1
y_4	y_2	y_1	y_1	x_1	x_4	x_1	x_4
y_1	y_4	y_2	y_2	x_3	x_2	x_3	x_2
y_2	y_1	y_4	y_3	x_4	x_1	x_4	x_3

(a) Apply the deferred acceptance procedure with proposals by members of X. Which matching do you obtain?
(b) Apply the deferred acceptance procedure with proposals by members of Y. Which matching do you obtain?
(c) Explain why the matching $(x_1, y_1), (x_2, y_3), (x_3, y_2), (x_4, y_4)$ is not in the core.
(d) How many matchings are in the core? Explain your answer.

RP 46 *House Exchange*

Player i owns house h_i, $i = 1, 2, 3$. The preferences of the players over the houses are given by the following table:

Player 1	Player 2	Player 3
h_2	h_3	h_1
h_1	h_2	h_2
h_3	h_1	h_3

(a) Compute all core allocations of this game.
(b) Which of these allocations are in the strong core?
(c) Give a new preference of player 1 such that the new game has a unique core allocation.

RP 47 *A Marriage Market*

Consider a marriage market with four men (m_1, m_2, m_3, and m_4) and four women (w_1, w_2, w_3, and w_4).

(a) Suppose that every woman has the same preference $m_1 > m_2 > m_3 > m_4$ over the men. Argue that there is a unique matching in the core, and describe this matching.

For (b)–(d) assume that the preferences are given by the following table.

m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
w_1	w_2	w_3	w_4	m_4	m_3	m_2	m_1
w_2	w_1	w_4	w_3	m_3	m_4	m_1	m_2
w_3	w_4	w_1	w_2	m_2	m_1	m_4	m_3
w_4	w_3	w_2	w_1	m_1	m_2	m_3	m_4

(b) Compute the core matching that is optimal from the point of view of the men.
(c) Compute the core matching that is optimal from the point of view of the women.
(d) Find a core matching in which m_1 and w_2 are coupled.

References

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