Special Transferable Utility Games 20

In this chapter we consider a few classes of games with transferable utility which are derived from specific economic (or political) models or combinatorial problems. In particular, we study assignment and permutation games, flow games, and voting games.

20.1 Assignment and Permutation Games

An example of a permutation game is the 'dentist game' described in Sect. 1.3.4. An example of an assignment game is the following.

Example 20.1 Vladimir (player 1), Wanda (player 2), and Xavier (player 3) each own a house that they want to sell. Yolanda (player 4) and Zarik (player 5) each want to buy a house. Vladimir, Wanda, and Xavier value their houses at 1, 1.5, and 2, respectively (each unit is 100,000 Euros). The worths of their houses to Yolanda and Zarik, respectively, are 0.8 and 1.5 for Vladimir's house, 2 and 1.2 for Wanda's house, and 2.2 and 2.3 for Xavier's house.

This situation gives rise to a five-player TU-game, where the worth of each coalition is defined to be the maximal surplus that can be generated by buying and selling within the coalition. For instance, in the coalition $\{2, 3, 5\}$ the maximum surplus is generated if Zarik buys the house of Xavier, namely $2.3 - 2 = 0.3$, which is greater than the $1.2 - 1.5 = -0.3$ that results if Zarik buys Wanda's house. Each coalition can generate a payoff of at least 0 because it can refrain from trading at all. The complete game is described in Table 20.1, where coalitions with only buyers or only sellers are left out. A game like this is called an assignment game. □

We will examine such games in detail, starting with the basic definitions.

Let M and P be two finite, disjoint sets. For each pair $(i, j) \in M \times P$ the number $a_{ij} \geq 0$ is interpreted as the value of the matching between i and j. With this situation a cooperative game (N, v) can be associated, as follows. The player set N is the set

H. Peters, *Game Theory*, Springer Texts in Business and Economics,
DOI 10.1007/978-3-662-46950-7_20

Table 20.1 Worths for the assignment game in Example 20.1

S	$v(S)$	S	$v(S)$	S	$v(S)$
14	0	125	0.5	345	0.3
15	0.5	134	0.2	1,234	0.5
24	0.5	135	0.5	1,235	0.5
25	0	145	0.5	1,245	1
34	0.2	234	0.5	1,345	0.7
35	0.3	235	0.3	2,345	0.8
124	0.5	245	0.5	12,345	1

$M \cup P$. For each coalition $S \subseteq N$ the worth $v(S)$ is the maximum that S can achieve by making pairs among its own members. Formally, if $S \subseteq M$ or $S \subseteq P$ then $v(S) = 0$, because no pairs can be formed at all. Otherwise, $v(S)$ is equal to the value of the following integer programming problem.

$$\begin{aligned} \max\ & \textstyle\sum_{i \in M} \sum_{j \in P} a_{ij} x_{ij} \\ \text{subject to}\ & \textstyle\sum_{j \in P} x_{ij} \le 1_S(i) \ \text{ for all } i \in M \\ & \textstyle\sum_{i \in M} x_{ij} \le 1_S(j) \ \text{ for all } j \in P \\ & x_{ij} \in \{0,1\} \qquad \text{for all } i \in M,\ j \in P. \end{aligned} \tag{20.1}$$

Here, $1_S(i) := 1$ if $i \in S$ and equal to zero otherwise. Games defined by (20.1) are called *assignment games*. The reader may verify that in Example 20.1 the numbers a_{ij} are given by $a_{ij} = \max\{h_{ij} - c_i, 0\}$, where h_{ij} is the value of the house of player i to player j and c_i is the value of the house of player i for himself.

As will become clear below, a more general situation is the following. For each $i \in N = \{1, 2, \ldots, n\}$ let $k_{i\pi(i)}$ be the value placed by player i on the permutation $\pi \in \Pi(N)$. (The implicit assumption is that $k_{i\pi(i)} = k_{i\sigma(i)}$ whenever $\pi(i) = \sigma(i)$.) Each coalition $S \subseteq N$ may achieve a permutation π involving only the players of S, that is, $\pi(i) = i$ for all $i \notin S$. Let $\Pi(S)$ denote the set of all such permutations. Then a game v results by defining, for each nonempty coalition S, the worth by

$$v(S) := \max_{\pi \in \Pi(S)} \sum_{i \in S} k_{i\pi(i)} \ . \tag{20.2}$$

The game thus obtained is called a *permutation game*. Alternatively, the worth $v(S)$ in such a game can be defined by the following integer programming problem.

$$\begin{aligned} \max\ & \textstyle\sum_{i \in N} \sum_{j \in N} k_{ij} x_{ij} \\ \text{subject to}\ & \textstyle\sum_{j \in N} x_{ij} = 1_S(i) \ \text{ for all } i \in N \\ & \textstyle\sum_{i \in N} x_{ij} = 1_S(j) \ \text{ for all } j \in N \\ & x_{ij} \in \{0,1\} \qquad \text{for all } i, j \in N. \end{aligned} \tag{20.3}$$

The two definitions are equivalent, and both can be used to verify that the 'dentist game' of Sect. 1.3.4 is indeed a permutation game (Problem 20.1).

The relation between the class of assignment games and the class of permutation games is a simple one. The former class is contained in the latter, as the following theorem shows.

Theorem 20.2 *Every assignment game is a permutation game.*

Proof Let v be an assignment game with player set $N = M \cup P$. For all $i, j \in N$ define

$$k_{ij} := \begin{cases} a_{ij} & \text{if } i \in M,\ j \in P \\ 0 & \text{otherwise.} \end{cases}$$

Let w be the permutation game defined by (20.3) with k_{ij} as above. Note that the number of variables in the integer programming problem defining $v(S)$ is $|M| \times |P|$, while the number of variables in the integer programming problem defining $w(S)$ is $(|M| + |P|)^2$. For $S \subseteq M$ or $S \subseteq P$, $w(S) = 0 = v(S)$. Let now $S \subseteq N$ with $S \not\subseteq M$ and $S \not\subseteq P$. Let $x \in \{0,1\}^{|M| \times |P|}$ be an optimal solution for (20.1). Define $\hat{x} \in \{0,1\}^{(|M|+|P|)^2}$ by

$$\hat{x}_{ij} := x_{ij} \quad \text{if } i \in M, j \in P$$

$$\hat{x}_{ij} := x_{ji} \quad \text{if } i \in P, j \in M$$

$$\hat{x}_{ii} := 1_S(i) - \sum_{j \in P} x_{ij} \qquad \text{if } i \in M$$

$$\hat{x}_{jj} := 1_S(j) - \sum_{i \in M} x_{ij} \qquad \text{if } j \in P$$

$$\hat{x}_{ij} := 0 \ \text{ in all other cases.}$$

Then $\hat{x}$ satisfies the conditions in problem (20.3). Hence, for every S,

$$w(S) \geq \sum_{i \in N} \sum_{j \in N} k_{ij} \hat{x}_{ij} = \sum_{i \in M} \sum_{j \in P} a_{ij} x_{ij} = v(S)\ .$$

On the other hand, let $z \in \{0,1\}^{(|M|+|P|)^2}$ be an optimal solution for (20.3). Define $\hat{z} \in \{0,1\}^{|M| \times |P|}$ by

$$\hat{z}_{ij} := z_{ij} \text{ for } i \in M,\ j \in P\ .$$

Then $\hat{z}$ satisfies the conditions in problem (20.1). Hence, for every S,

$$v(S) \geq \sum_{i \in M} \sum_{j \in P} a_{ij} \hat{z}_{ij} = \sum_{i \in M} \sum_{j \in P} k_{ij} z_{ij} = w(S)\ .$$

Consequently, $v = w$. ∎

The converse of Theorem 20.2 is not true, as the following example shows. As a matter of fact, a necessary condition for a permutation game to be an assignment game is the existence of a partition of the player set N of the permutation game into two subsets N_1 and N_2, such that the value of a coalition S is 0 whenever $S \subseteq N_1$ or $S \subseteq N_2$. The example shows that this is not a sufficient condition.

Example 20.3 Let $N = \{1, 2, 3\}$ and let v be the permutation game with the numbers k_{ij} given in the following matrix:

$$\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

Then $v(i) = 0$ for every $i \in N$, $v(1, 2) = v(1, 3) = 3$, $v(2, 3) = 0$, and $v(N) = 4$. Note that this game satisfies the condition formulated above with $N_1 = \{1\}$ and $N_2 = \{2, 3\}$, but it is not an assignment game (Problem 20.2). □

The main purpose of this section is to show that permutation games and, hence, assignment games are balanced and, in fact, totally balanced. A TU-game (N, v) is *totally balanced* if the *subgame* (M, v)—where v is the restriction to M—is balanced for every $M \subseteq N$. Balanced games are exactly those games that have a non-empty core, see Chap. 16.

Theorem 20.4 *Assignment games and permutation games are totally balanced.*

Proof In view of Theorem 20.2, it is sufficient to prove that permutation games are totally balanced. Because any subgame of a permutation game is again a permutation game (see Problem 20.3), it is sufficient to prove that any permutation game is balanced.

Let (N, v) be a permutation game, defined by (20.3). By the Birkhoff–von Neumann Theorem (Theorem 22.12) the integer restriction can be dropped so that each $v(S)$ is also defined by the following program:

$$\begin{aligned} \max \ & \textstyle\sum_{i \in N} \sum_{j \in N} k_{ij} x_{ij} \\ \text{subject to } & \textstyle\sum_{j \in N} x_{ij} = 1_S(i) \quad \text{for all } i \in N \\ & \textstyle\sum_{i \in N} x_{ij} = 1_S(j) \quad \text{for all } j \in N \\ & x_{ij} \geq 0 \qquad\quad\ \ \text{for all } i, j \in N. \end{aligned} \tag{20.4}$$

Note that this is a linear programming problem of the same format as the maximization problem in Theorem 16.20. Namely, with notations as there, take

$$\mathbf{y} = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{n1}, \ldots, x_{nn})$$
$$\mathbf{b} = (k_{11}, \ldots, k_{1n}, k_{21}, \ldots, k_{2n}, \ldots, k_{n1}, \ldots, k_{nn})$$
$$\mathbf{c} = (1_S, 1_S) \ .$$

Further, let A be the $2n \times n^2$-matrix with row $\ell \in \{1, \ldots, n\}$ containing a 1 at columns $\ell, \ell+n, \ell+2n, \ldots, \ell+(n-1)n$ and zeros otherwise; and with row $\ell+n$ ($\ell \in \{1, \ldots, n\}$) containing a 1 at columns $(\ell-1)n+1, \ldots, \ell n$ and zeros otherwise. The corresponding dual problem, the minimization problem in Theorem 16.20, then has the form:

$$\begin{aligned} &\min \textstyle\sum_{i \in N} 1_S(i) y_i + \sum_{j \in N} 1_S(j) z_j \\ &\text{subject to} \qquad y_i + z_j \geq k_{ij} \qquad \text{for all } i, j \in N. \end{aligned} \tag{20.5}$$

Let $(\hat{\mathbf{y}}, \hat{\mathbf{z}})$ be an optimal solution of problem (20.5) for $S = N$. Then, by Theorem 16.20 and the fact that the maximum in problem (20.4) for $S = N$ is equal to $v(N)$ by definition, it follows that

$$\sum_{i \in N} (\hat{y}_i + \hat{z}_i) = v(N) \ .$$

Since $(\hat{\mathbf{y}}, \hat{\mathbf{z}})$ satisfies the restrictions in problem (20.5) for every $S \subseteq N$, it furthermore holds that for every $S \subseteq N$,

$$\sum_{i \in S} (\hat{y}_i + \hat{z}_i) = \sum_{i \in N} 1_S(i) \hat{y}_i + \sum_{i \in N} 1_S(i) \hat{z}_i \geq v(S) \ .$$

Therefore, $\mathbf{u} \in \mathbb{R}^N$ defined by $u_i := \hat{y}_i + \hat{z}_i$ is in the core of v. ■

20.2 Flow Games

In this section another class of balanced games is considered. These games are derived from the following kind of situation. There is a given capacitated network, the edges of which are controlled by subsets of players. These coalitions can send a flow through the network. The flow is maximal if all players cooperate, and then the question arises how to distribute the profits. One can think of an almost literal example, where the edges represent oil pipelines, and the players are in power in different countries through which these pipelines cross. Alternatively, one can think of rail networks between cities, or information channels between different users.

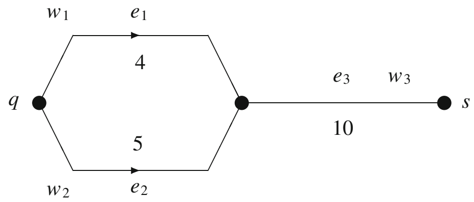

Fig. 20.1 Example 20.5

Capacitated networks are treated in Sect. 22.7, which the reader may consult before continuing.

Consider a capacitated network (V, E, k) and a set of players $N := \{1, \ldots, n\}$. Suppose that with each edge in E a simple game (cf. Sect. 16.3.) is associated. The winning coalitions in this simple game are supposed to control the corresponding edge; the capacitated network is called a *controlled* capacitated network. For any coalition $S \subseteq N$ consider the capacitated network arising from the given network by deleting the edges that are *not* controlled by S. A game can be defined by letting the worth of S be equal to the value of a maximal flow through this restricted network. The game thus arising is called a *flow game*.

Example 20.5 Consider the capacitated network in Fig. 20.1. This network has three edges denoted e_1, e_2, and e_3 with capacities 4, 5 and 10, respectively. The control games are w_1, w_2, w_3 with $N = \{1, 2, 3\}$ and

$$w_1(S) = 1 \text{ if } S \in \{\{1,2\}, N\} \text{ and } w_1(S) = 0 \text{ otherwise}$$
$$w_2(S) = 1 \text{ if } S \in \{\{1,3\}, N\} \text{ and } w_2(S) = 0 \text{ otherwise}$$
$$w_3(S) = 1 \text{ if, and only if, } 1 \in S.$$

The coalition $\{1, 2\}$ can only use the edges e_1 and e_3, so the maximal flow (per time unit) for $\{1, 2\}$ is 4. This results in $v(\{1, 2\}) = 4$ for the corresponding flow game (N, v). The complete game is given by $v(i) = 0$ for all $i \in N$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = 5$, $v(\{2, 3\}) = 0$ and $v(N) = 9$.

A minimum cut in this network corresponding to the grand coalition is $(\{q\}, V \setminus \{q\})$. By the Max Flow Min Cut Theorem of Ford and Fulkerson (Theorem 22.16), the sum of the capacities of e_1 and e_2 $(4 + 5)$ is equal to $v(N)$. Divide $v(N)$ as follows. Divide 4 equally among the veto players of w_1, and 5 equally among the veto players of w_2. The result for the players is the payoff vector $(4\frac{1}{2}, 2, 2\frac{1}{2})$. Note that this vector is in $C(v)$. □

The next theorem shows that the non-emptiness of the core of the control games is inherited by the flow game.

Theorem 20.6 *Suppose all control games in a controlled capacitated network have veto players. Then the corresponding flow game is balanced.*

Proof Take a maximal flow for the grand coalition and a minimum cut in the network for the grand coalition, consisting of the edges

$$e_1, e_2, \ldots, e_p \text{ with capacities } k_1, k_2, \ldots, k_p$$

and control games w_1, w_2, ..., w_p, respectively. Then Theorem 22.16 implies that $v(N) = \sum_{r=1}^{p} k_r$. For each r take $x^r \in C(w_r)$ and divide k_r according to the division key x^r (i.e. $k_r x_i^r$ is the amount for player i). Note that non-veto players get nothing. Then $\sum_{r=1}^{p} k_r x^r \in C(v)$. To see this, first note that

$$\sum_{i=1}^{n} \sum_{r=1}^{p} k_r x_i^r = \sum_{r=1}^{p} k_r \sum_{i=1}^{n} x_i^r = \sum_{r=1}^{p} k_r = v(N) .$$

Next, for each coalition S, the set

$$E_S := \{e_r : r \in \{1, \ldots p\}, w_r(S) = 1\}$$

is associated with a cut of the network, governed by the coalition S. Hence, $\sum_{i \in S} \left(\sum_{r=1}^{p} k_r x_i^r \right) = \sum_{r=1}^{p} k_r \sum_{i \in S} x_i^r \geq \sum_{r=1}^{p} k_r w_r (S) = \sum_{e_r \in E_S} k_r =$ capacity$(E_S) \geq v(S)$, where the last inequality follows from Theorem 22.16. ■

The next theorem is a partial converse to Theorem 20.6.

Theorem 20.7 *Each nonnegative balanced game arises from a controlled capacitated network where all control games possess veto players.*

Proof See Problem 20.5. ■

20.3 Voting Games: The Banzhaf Value

Voting games constitute another special class of TU-games. Voting games are simple games which reflect the distribution of voting power within, for instance, political systems. There is a large body of work on voting games within the political science literature. In this section we restrict ourselves to a brief discussion of a well-known example of a power index, to so-called Banzhaf–Coleman index and the associated value, the Banzhaf value.

A *power index* is a value applied to voting (simple) games. The payoff vector assigned to a game is interpreted as reflecting power distribution—e.g., the probability of having a decisive vote—rather than utility.

We start with an illustrating example.

Example 20.8 Consider a parliament with three parties 1, 2, and 3. The numbers of votes are, respectively, 50, 30, and 20. To pass any law, a two-third majority is needed. This leads to a simple game with winning coalitions $\{1,2\}$, $\{1,3\}$, and $\{1,2,3\}$. The Shapley value[1] of this game is $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$, as can easily be checked. By definition of the Shapley value this means that in four of the six orderings player 1 makes the coalition of his predecessors winning by joining them, whereas for players 2 and 3 this is only the case in one ordering each. The coalitions that are made winning by player 1 if he joins, are $\{2\}$, $\{3\}$, and $\{2,3\}$. In the Shapley value the last coalition is counted double. It might be more natural to count this coalition only once. This would lead to an outcome $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$, instead of the Shapley value. The associated value is called the *normalized Banzhaf–Coleman index*. □

For a simple game (N, v) (see Sect. 16.3), the *normalized Banzhaf–Coleman index* can be defined as follows. Define a *swing* for player i as a coalition $S \subseteq N$ with $i \in S$, S wins, and $S \setminus \{i\}$ looses. Let θ_i be the number of swings for player i, and define the numbers

$$\beta_i(N, v) := \frac{\theta_i}{\sum_{j=1}^{n} \theta_j} .$$

The vector $\beta(N, v)$ is the normalized Banzhaf–Coleman index of the simple game (N, v).

For a general game (N, v) write

$$\theta_i(v) := \sum_{S \subseteq N:\ i \notin S} [v(S \cup i) - v(S)] .$$

For a simple game v this number $\theta_i(v)$ coincides with the number θ_i above.

Next, define the value $\Psi : \mathcal{G}^N \to \mathbb{R}^N$ by

$$\Psi_i(v) := \frac{\theta_i(v)}{2^{|N|-1}} = \sum_{S \subseteq N:\ i \notin S} \frac{1}{2^{|N|-1}} [v(S \cup i) - v(S)] . \tag{20.6}$$

The value Ψ is called the *Banzhaf value*. The remainder of this section is devoted to an axiomatic characterization of this value. This characterization uses the axioms SYM (Symmetry), SMON (Strong Monotonicity), and DUM (Dummy Property), which were all introduced in Chap. 17. Besides, it uses an 'amalgamation' property, as follows.

[1] Also called the *Shapley-Shubik power index* in this context.

For a game (N, v) (with at least two players) and different players i, j put $p = \{i, j\}$ and define the game $((N \setminus p) \cup \{p\}, v_p)$ by

$$v_p(S) = v(S) \text{ and } v_p(S \cup \{p\}) = v(S \cup p), \text{ for any } S \subseteq N \setminus p\,. \tag{20.7}$$

Thus, v_p is an $(n-1)$-person game obtained by amalgamating players i and j in v into one player p in v_p.

Let ψ be an arbitrary value (on the class $\mathcal{G}$ of all games with arbitrary player set). The announced axiom is as follows.

2-Efficiency (2-EFF): $\psi_i(v) + \psi_j(v) = \psi_p(v_p)$ for all v, i, j, p, v_p as above.

The following theorem gives a characterization of the Banzhaf value.

Theorem 20.9 *The value ψ on $\mathcal{G}$ satisfies 2-EFF, SYM, DUM, and SMON, if and only if ψ is the Banzhaf value Ψ.*

Proof That the Banzhaf value satisfies the four axioms in the theorem is the subject of Problem 20.9. For the converse, let ψ be a value satisfying the four axioms. We prove that $\psi = \Psi$.

Step 1
Let u_T be a unanimity game. We first show that

$$\psi_i(u_T) = 1/2^{|T|-1} \text{ if } i \in T \text{ and } \psi_i(u_T) = 0 \text{ if } i \notin T. \tag{20.8}$$

If $|T| = 1$ then every player is a dummy, so that (20.8) follows from DUM. Suppose (20.8) holds whenever $|T| \leq k$ or $|N| \leq m$, and consider a unanimity game u_T where now the number of players is $m + 1$, and T contains $k + 1$ players. Let $i, j \in T$, put $p = \{i, j\}$ and consider the game $(u_T)_p$. Then $(u_T)_p$ is the m-person unanimity game of the coalition $T' = (T \setminus p) \cup \{p\}$, and $|T'| = k$. By the induction hypothesis

$$\psi_p((u_T)_p) = 1/2^{|T'|-1} = 1/2^{k-1}.$$

By 2-EFF this implies

$$\psi_i(u_T) + \psi_j(u_T) = 1/2^{k-1}.$$

From this and SYM it follows that

$$\psi_i(u_T) = 1/2^k = 1/2^{|T|-1}$$

for all $i \in T$, and by DUM, $\psi_j(u_T) = 0$ for all $j \notin T$. Thus, ψ is the Banzhaf value on unanimity games for any finite set of players. In the same way, one shows that this is true for any real multiple cu_T of a unanimity game.

Step 2
For an arbitrary game v write $v = \sum_{\emptyset \neq T} c_T u_T$, and let $\alpha(v)$ denote the number of nonzero coefficients in this representation. The proof will be completed by induction on the number $\alpha(v)$ and the number of players. For $\alpha(v) = 1$ Step 1 implies $\psi(v) = \Psi(v)$ independent of the number of players. Assume that $\psi(v) = \Psi(v)$ on any game v with at most n players, and also any game v with $\alpha(v) \leq k$ for some natural number k and with $n + 1$ players, and let v be a game with $n + 1$ players and with $\alpha(v) = k + 1$. There are $k + 1$ different nonempty coalitions $T_1, \ldots, T_{k+1}$ with

$$v = \sum_{r=1}^{k+1} c_{T_r} u_{T_r} ,$$

where all coefficients are nonzero. Let $T := T_1 \cap \ldots \cap T_{k+1}$. Because $k + 1 \geq 2$, it holds that $N \setminus T \neq \emptyset$. Assume $i \notin T$. Define the game w by

$$w = \sum_{r:\ i \in T_r} c_{T_r} u_{T_r} .$$

Then $\alpha(w) \leq k$ and $v(S \cup i) - v(S) = w(S \cup i) - w(S)$ for every coalition S not containing player i. By SMON and the induction hypothesis it follows that $\psi_i(v) = \psi_i(w) = \Psi_i(w) = \Psi_i(v)$. Hence,

$$\psi_i(v) = \Psi_i(v) \text{ for every } i \in N \setminus T. \tag{20.9}$$

Let $j \in T$ and $i \in N \setminus T$, put $p = \{i, j\}$, and consider the game v_p . Because the game v_p has n players the induction hypothesis implies

$$\psi_p(v_p) = \Psi_p(v_p) . \tag{20.10}$$

Applying 2-EFF to both ψ and Ψ yields

$$\psi_p(v_p) = \psi_i(v) + \psi_j(v) \text{ and } \Psi_p(v_p) = \Psi_i(v) + \Psi_j(v) . \tag{20.11}$$

Combining (20.9)–(20.11) implies $\psi_j(v) = \Psi_j(v)$ for every $j \in T$. Together with (20.9) this completes the induction argument, and therefore the proof. ■

20.4 Problems

20.1. *The Dentist Game*
Show that (20.2) and (20.3) are equivalent, and use each of these to verify that the dentist game of Sect. 1.3.4 is a permutation game.

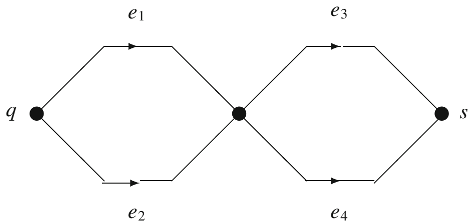

Fig. 20.2 The network of Problem 20.4

20.2. *Example 20.3*
Show that the game in Example 20.3 is not an assignment game.

20.3. *Subgames of Permutation Games*
Prove that subgames of permutation games are again permutation games. Is this also true for assignment games?

20.4. *A Flow Game*
Consider the network in Fig. 20.2. Suppose that this is a controlled capacitated network with player set $N = \{1, 2, 3, 4\}$, suppose that all edges have capacity 1 and that $w_1 = \delta_1$, $w_2 = \delta_2$, $w_3 = \delta_3$ and $w_4(S) = 1$ iff $S \in \{\{3, 4\}, N\}$. [Here, δ_i is the simple game where a coalition is winning if, and only if, it contains player i.]

(a) Calculate the corresponding flow game (N, v).
(b) Calculate $C(v)$.
(c) The proof of Theorem 20.6 describes a way to find core elements by looking at minimum cuts and dividing the capacities of edges in the minimum cut in some way among the veto players of the corresponding control game. Which elements of $C(v)$ can be obtained in this way?

20.5. *Every Nonnegative Balanced Game is a Flow Game*
Prove that every nonnegative balanced game is a flow game. [Hint: You may use the following result: every nonnegative balanced game can be written as a positive linear combination of balanced simple games.]

20.6. *On Theorem 20.6 (1)*

(a) Consider a controlled capacitated network with a minimum cut, where all control games corresponding to the edges in this minimum cut (connecting vertices between the two sets in the cut) have veto players. Prove that the corresponding flow game is balanced.
(b) Show that the flow game, corresponding to Fig. 20.3, where the winning coalitions of w_1 are $\{1, 3\}$, $\{2, 4\}$ and $N = \{1, 2, 3, 4\}$, where the winning coalitions of w_2 are $\{1, 2\}$ and N and of w_3 $\{3, 4\}$ and N and where the capacities are 1, 10, 10 respectively, has a nonempty core. Note that there is no minimum cut where all control games have veto players.

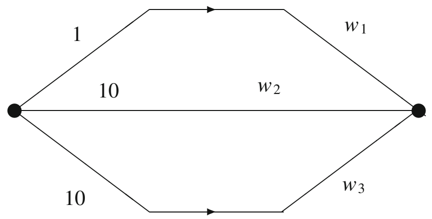

Fig. 20.3 The network of Problem 20.6

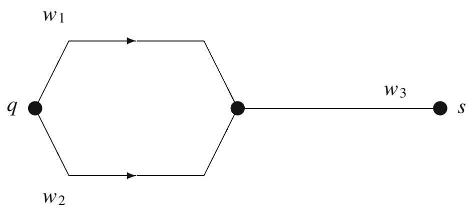

Fig. 20.4 The network of Problem 20.7

20.7. *On Theorem 20.6 (2)*
Prove that the two-person flow game corresponding to the controlled capacitated network of Fig. 20.4 has an empty core, where $w_1 = \delta_1$, $w_2 = \delta_2$, $w_3(S) = 1$ if $S \neq \emptyset$, and where the capacities of the edges are equal to 1.

20.8. *Totally Balanced Flow Games*
Let (N, v) be the flow game corresponding to a controlled capacitated network where all control games are dictatorial games (games of the form δ_i, see Problem 20.4). Prove that each subgame (S, v_S) (where v_S is the restriction of v to 2^S) has a nonempty core, i.e., that the game (N, v) is totally balanced.

20.9. *If-Part of Theorem 20.9*
Prove that the Banzhaf value satisfies 2-EFF, SYM, DUM, and SMON. Is it possible to weaken DUM to NP (the null-player property) in Theorem 20.9? Give an example showing that the Banzhaf value is not efficient.

20.5 Notes

The presentation in Sect. 20.1 is mainly based on Curiel (1997, Chap. 3). Example 20.1 is from this book. Assignment games were introduced by Shapley and Shubik (1972). Permutation games were introduced by Tijs et al. (1984).

Theorem 20.6 on flow games is due to Curiel et al. (1986).

In the literature many characterizations of power indices are available. The one presented in Sect. 20.3 is based on Nowak (1997).

For the result in the hint to Problem 20.5 see Derks (1987). Problem 20.8 refers to Kalai and Zemel (1982).

References

Curiel, I. (1997). *Cooperative game theory and applications: Cooperative games arising from combinatorial optimization problems*. Boston: Kluwer Academic.

Curiel, I., Derks, J., & Tijs, S. H. (1986). On balanced games and games with committee control. *Operations Research Spektrum, 11*, 83–88.

Derks, J. (1987). Decomposition of games with non-empty core into veto-controlled simple games. *Operations Research Spektrum, 9*, 81–85.

Kalai, E., & Zemel, E. (1982). Totally balanced games and games of flow. *Mathematics of Operations Research, 7*, 476–478.

Nowak, A. S. (1997). On an axiomatization of the Banzhaf value without the additivity axiom. *International Journal of Game Theory, 26*, 137–141.

Shapley, L. S., & Shubik, M. (1972). The assignment game I: The core. *International Journal of Game Theory 1*, 111–130.

Tijs, S. H., Parthasarathy, T., Potters, J. A. M., & Rajendra Prasad, V. (1984). Permutation games: Another class of totally balanced games. *Operations Research Spektrum, 6*, 119–123.