TU-Games: Domination, Stable Sets, and the Core

16

In a game with transferable utility (TU-game) each coalition (subset of players) is characterized by its worth, i.e., a real number representing the payoff or utility that the coalition can achieve if it forms. It is assumed that this payoff can be freely distributed among the members of the coalition in any way desired.

For some examples the reader is referred to Chap. 1. Chapter 9 presents a first acquaintance with transferable utility games. Although the present chapter and the following ones are self-contained, it may be helpful to study the relevant parts of Chaps. 1 and 9 first.

In this chapter the focus is on the core of a transferable utility game. Section 16.1 starts with a weaker concept, the imputation set, and introduces the concept of domination. Section 16.2 introduces the domination core and the core. Section 16.3 studies these solution concepts for a special class of TU-games called simple games. In Sect. 16.4 we briefly review von Neumann and Morgenstern's stable sets, which are also based on the concept of domination. Section 16.5, finally, presents a characterization of games with non-empty cores in terms of balancedness.

16.1 Imputations and Domination

We start with repeating the definition of a game with transferable utility (cf. Definition 9.1).

Definition 16.1 A *cooperative game with transferable utility* or *TU-game* is a pair (N, v), where $N = \{1, \ldots, n\}$ with $n \in \mathbb{N}$ is the set of *players*, and v is a function assigning to each *coalition* S, i.e., to each subset $S \subseteq N$ a real number $v(S)$, such that $v(\emptyset) = 0$. The function v is called the *characteristic function* and $v(S)$ is called the *worth* of S. The coalition N is called the *grand coalition*. A *payoff distribution for* coalition S is a vector of real numbers $(x_i)_{i \in S}$. ☐

The set of coalitions is also denoted by 2^N, so that a TU-game is a pair (N, v) with $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. The game (N, v) is often denoted by v if no confusion about the set of players is likely to arise. Also, for a coalition $\{i, j, \ldots, k\}$ we sometimes write $i, j, \ldots, k$ or $ij \ldots k$ instead of $\{i, j, \ldots, k\}$. By $|S|$ we denote the cardinality of a coalition S. By $\mathcal{G}^N$ the set of all TU-games with player set N is denoted.

We frequently use the notation $x(S) := \sum_{i \in S} x_i$ for a payoff distribution $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^N$ and a coalition $S \subseteq N$.

Let (N, v) be a TU-game. A vector $\mathbf{x} \in \mathbb{R}^N$ is called an *imputation* if

(a) $\mathbf{x}$ is *individually rational* i.e.

$$x_i \geq v(i) \text{ for all } i \in N ,$$

(b) $\mathbf{x}$ is *efficient* i.e.

$$x(N) = v(N) .$$

The set of imputations of (N, v) is denoted by $I(v)$. An element $\mathbf{x} \in I(v)$ is a payoff distribution of the worth $v(N)$ of the grand coalition N which gives each player i a payoff x_i which is at least as much as he can obtain when he operates alone.

Example 16.2 A game v is called *additive* if $v(S \cup T) = v(S) + v(T)$ for all disjoint coalitions S and T. Such a game is completely determined by the worths of the one-person coalitions $v(i)$ ($i \in N$), since $v(S) = \sum_{i \in S} v(i)$ for every coalition S. For an additive game v, $I(v)$ consists of one point: $I(v) = \{(v(1), v(2), \ldots, v(n))\}$.   □
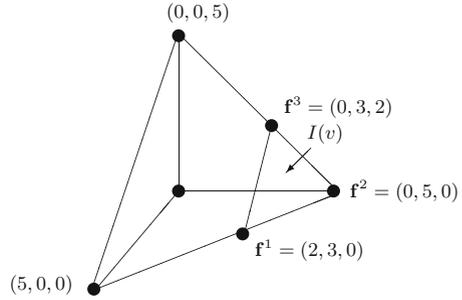
Note that for a game v

$$I(v) \neq \emptyset \text{ if and only if } v(N) \geq \sum_{i=1}^{n} v(i) .$$

For an *essential* game v, that is, a game with $v(N) \geq \sum_{i=1}^{n} v(i)$, $I(v)$ is the convex hull of the points: $\mathbf{f}^1, \mathbf{f}^2, \ldots, \mathbf{f}^n$ where $f_k^i := v(k)$ if $k \neq i$ and $f_i^i := v(N) - \sum_{k \in N \setminus \{i\}} v(k)$. (See Problem 16.1.)

Example 16.3 Let (N, v) be a three-person game with $v(1) = v(3) = 0$, $v(2) = 3$, $v(1, 2, 3) = 5$. Then $I(v)$ is the triangle with vertices $\mathbf{f}^1 = (2, 3, 0)$, $\mathbf{f}^2 = (0, 5, 0)$ and $\mathbf{f}^3 = (0, 3, 2)$. (See Fig. 16.1.)   □

Fig. 16.1 Example 16.3

Definition 16.4 Let (N, v) be a game. Let $\mathbf{y}, \mathbf{z} \in I(v)$, $S \in 2^N \setminus \{\emptyset\}$. Then $\mathbf{y}$ *dominates* $\mathbf{z}$ *via coalition* S, denoted by $\mathbf{y} \operatorname{dom}_S \mathbf{z}$, if

(1) $y_i > z_i$ for all $i \in S$,
(2) $y(S) \le v(S)$.

For $\mathbf{y}, \mathbf{z} \in I(v)$, $\mathbf{y}$ is said to *dominate* $\mathbf{z}$ (notation: $\mathbf{y} \operatorname{dom} \mathbf{z}$) if there is an $S \in 2^N \setminus \{\emptyset\}$ such that $\mathbf{y} \operatorname{dom}_S \mathbf{z}$.   □

Thus, imputation $\mathbf{y}$ dominates imputation $\mathbf{z}$ via coalition S if $\mathbf{y}$ is better than $\mathbf{z}$ for all members $i \in S$—this is condition (1)—and the payoffs $(y_i)_{i \in S}$ are attainable for the members of S by cooperation—this is condition (2). Against each $\mathbf{z}$ in

$$D(S) := \{\mathbf{z} \in I(v) \mid \text{ there exists } \mathbf{y} \in I(v) \text{ with } \mathbf{y} \operatorname{dom}_S \mathbf{z}\}$$

the players of S can protest successfully. The set $D(S)$ consists of the imputations which are dominated via S. Note that always $D(N) = \emptyset$ (see Problem 16.3). We call $\mathbf{x} \in I(v)$ *undominated* if $\mathbf{x} \in I(v) \setminus \bigcup_{S \in 2^N \setminus \{\emptyset\}} D(S)$.

Example 16.5 Let (N, v) be the three-person game with $v(1, 2) = 2$, $v(N) = 1$ and $v(S) = 0$ if $S \neq \{1, 2\}, N$. Then $D(S) = \emptyset$ if $S \neq \{1, 2\}$ and $D(\{1, 2\}) = \{\mathbf{x} \in I(v) \mid x_3 > 0\}$. The elements $\mathbf{x}$ in $I(v)$ which are undominated are those that satisfy $x_3 = 0$.   □

16.2   The Core and the Domination-Core

The concept of domination defined in the preceding section gives rise to the following definition.

Definition 16.6 The *domination core* (*D-core*) of a game (N, v) is the set

$$DC(v) := I(v) \setminus \bigcup_{S \in 2^N \setminus \{\emptyset\}} D(S),$$

i.e., the set of all undominated elements in $I(v)$. The *core* of a game (N, v) is the set

$$C(v) := \{\mathbf{x} \in I(v) \mid x(S) \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\}\}. \qquad \square$$

If $\mathbf{x} \in C(v)$, then no coalition $S \neq N$ has an incentive to split off if $\mathbf{x}$ is the proposed payoff distribution in N, because the total amount $x(S)$ allocated to S is not smaller than the amount $v(S)$ which the players in S can obtain by forming the coalition S.

For the game in Example 16.5 the D-core is nonempty and the core is empty. In general the following holds.

Theorem 16.7 *The core is a subset of the D-core for each TU-game.*

Proof Let (N, v) be a game and $\mathbf{x} \in I(v)$, $\mathbf{x} \notin DC(v)$. Then there is a $\mathbf{y} \in I(v)$ and a coalition $S \neq \emptyset$ such that $\mathbf{y} \text{ dom}_S \mathbf{x}$. Thus, $v(S) \geq y(S) > x(S)$, which implies that $\mathbf{x} \notin C(v)$. $\blacksquare$

Elements of $C(v)$ can easily be obtained because the core is defined with the aid of linear inequalities. The core is a polytope. Also the D-core is a convex set: see Problem 16.2.

A natural question that arises is: for which games is the core equal to the D-core? Consider the following condition on a game (N, v):

$$v(N) \geq v(S) + \sum_{i \in N \setminus S} v(i) \text{ for all } S \in 2^N \setminus \{\emptyset\}. \qquad (16.1)$$

It turns out that this condition is sufficient for the equality of core and D-core.

Theorem 16.8 *Let (N, v) be a game satisfying (16.1). Then $DC(v) = C(v)$.*

Proof In view of Theorem 16.7 it is sufficient to show that $DC(v) \subseteq C(v)$.

Claim Let $\mathbf{x} \in I(v)$ with $x(S) < v(S)$ for some S, then there is a $\mathbf{y} \in I(v)$ such that $\mathbf{y} \text{ dom}_S \mathbf{x}$.

To prove this claim, define $\mathbf{y}$ as follows. If $i \in S$, then $y_i := x_i + |S|^{-1}(v(S) - x(S))$. If $i \notin S$, then $y_i := v(i) + (v(N) - v(S) - \sum_{i \in N \setminus S} v(i))|N \setminus S|^{-1}$. Then $\mathbf{y} \in I(v)$, where $y_i \geq v(i)$ for $i \in N \setminus S$ follows from (16.1). Furthermore, $\mathbf{y} \text{ dom}_S \mathbf{x}$. This proves the claim.

To prove $DC(v) \subseteq C(v)$, suppose $\mathbf{x} \in DC(v)$. Then there is no $\mathbf{y} \in I(v)$ with $\mathbf{y} \text{ dom } \mathbf{x}$. In view of the Claim it follows that $x(S) \geq v(S)$ for all $S \in 2^N \setminus \{\emptyset\}$. Hence, $\mathbf{x} \in C(v)$. $\blacksquare$

Remark 16.9 Condition (16.1) is satisfied if the game v has a non-empty core. So in that case, $C(v) = DC(v)$. See also Problem 16.11. $\square$

Many games v derived from practical situations have the following property:

$$v(S \cup T) \geq v(S) + v(T) \text{ for all disjoint } S, T \subseteq N. \tag{16.2}$$

A game satisfying (16.2) is called *super-additive*. Observe that (16.2) implies (16.1), so that Theorem 16.8 holds for super-additive games in particular.

16.3   Simple Games

In this section we study the core and D-core of simple games. Simple games arise in particular in political situations, see for instance the United Nations Security Council example in Chap. 1.

Definition 16.10 A *simple game* (N, v) is a game where every coalition has either worth 0 or worth 1, and the grand coalition N has worth 1. Coalitions with worth 1 are called *winning*, the other coalitions are called *losing*. A *minimal* winning coalition is a winning coalition for which every proper subset is losing. A player i is called a *dictator* in a simple game (N, v) if $v(S) = 1$ if and only if $i \in S$. A player i is called a *veto player* in a simple game (N, v) if i belongs to all winning coalitions. The *set of veto players* of v is denoted by veto(v). Hence,

$$\text{veto}(v) = \bigcap \{S \in 2^N \mid v(S) = 1\}. \qquad \square$$

The next example suggests that non-emptiness of the core has something to do with the existence of veto players.

For each $i \in N$ let $\mathbf{e}^i \in \mathbb{R}^n$ denote the vector with i-th coordinate equal to 1 and all other coordinates equal to 0.

Example 16.11 (1) Let $i \in N$. For the *dictator game* δ_i, which is the simple game with $\delta_i(S) = 1$ if and only if $i \in S$ one has $I(\delta_i) = \{\mathbf{e}^i\}$, veto$(\delta_i) = \{i\}$ and $C(\delta_i) = DC(\delta_i) = \{\mathbf{e}^i\}$.

(2) For the three-person *majority game* with $v(S) = 1$ if $|S| \in \{2, 3\}$ and $v(S) = 0$ if $|S| \in \{0, 1\}$ one has:

$$\{1, 2\} \cap \{1, 3\} \cap \{2, 3\} \cap \{1, 2, 3\} = \emptyset = \text{veto}(v)$$

and

$$C(v) = DC(v) = \emptyset .$$

(3) Let T be a nonempty coalition. For the *T-unanimity game* u_T, which is the simple game with $u_T(S) = 1$ if and only if $T \subseteq S$, veto$(u_T) = T$ and

$$C(u_T) = DC(u_T) = \text{conv}\{\mathbf{e}^i \mid i \in T\} . \qquad \square$$

The following theorem shows that the core of a simple game is nonempty if and only if the game has veto players. Furthermore, core elements divide the total amount $v(N) = 1$ of the grand coalition among the veto players. The D-core is equal to the core for simple games except in one case where there is exactly one $k \in N$ with $v(k) = 1$ and k is not a veto player. See also Example 16.13 below.

Theorem 16.12 *Let (N, v) be a simple game. Then:*

(1) $C(v) = conv\{\mathbf{e}^i \in \mathbb{R}^n \mid i \in veto(v)\}$.
(2) *If $veto(v) = \emptyset$ and $\{i \in N \mid v(i) = 1\} = \{k\}$, then $C(v) = \emptyset$ and $DC(v) = \{\mathbf{e}^k\}$. Otherwise, $DC(v) = C(v)$.*

Proof

(a) Suppose $i \in veto(v)$. Let $S \in 2^N \setminus \{\emptyset\}$. If $i \in S$ then $e^i(S) = 1 \geq v(S)$, otherwise $e^i(S) = 0 = v(S)$. Obviously, $e^i(N) = 1 = v(N)$. So $\mathbf{e}^i \in C(v)$. This proves the inclusion $\supseteq$ in (1) because $C(v)$ is a convex set.
(b) To prove the inclusion $\subseteq$ in (1), let $\mathbf{x} \in C(v)$. It is sufficient to prove: $i \notin veto(v) \Rightarrow x_i = 0$. Suppose, to the contrary, that $x_i > 0$ for some non-veto player i. Take S with $v(S) = 1$ and $i \notin S$ (such an S exists otherwise i would be a veto player). Then $x(S) = x(N) - x(N \setminus S) \leq 1 - x_i < 1$, contradicting the fact that $\mathbf{x}$ is a core element. This concludes the proof of (1).
(c) If $veto(v) = \emptyset$ and k is the only player in the set $\{i \in N \mid v(i) = 1\}$, then $C(v) = \emptyset$ by part (1), whereas $I(v) = \{\mathbf{e}^k\}$, hence $DC(v) = \{\mathbf{e}^k\}$. If $veto(v) = \emptyset$ and $\{i \in N \mid v(i) = 1\} = \emptyset$ then (16.1) is satisfied, so that core and D-core are equal by Theorem 16.8. If $veto(v) = \emptyset$ and $|\{i \in N \mid v(i) = 1\}| \geq 2$ then $I(v) = \emptyset$ so that $C(v) = DC(v) = \emptyset$.
(d) To complete the proof of (2), suppose $veto(v) \neq \emptyset$. Then $C(v) \neq \emptyset$ by part (1). Hence $C(v) = DC(v)$ by Remark 16.9.   ∎

Example 16.13 Let $N = \{1, 2, 3\}$, $v(1) = v(2, 3) = v(1, 2, 3) = 1$ and $v(S) = 0$ for the other coalitions. Then $veto(v) = \emptyset$, $C(v) = \emptyset$, $DC(v) = \{\mathbf{e}^1\}$. Note that this simple game is not super-additive, and does not satisfy (16.1).   □

16.4   Stable Sets

The definition of a stable set is again based on the concept of domination. By way of example, let v be the three-person game with all worths equal to 1 except for the one-person coalitions, which have worth equal to 0. Observe that the three vectors $(\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{1}{2}, 0, \frac{1}{2})$, and $(0, \frac{1}{2}, \frac{1}{2})$ are imputations that do not dominate each other. Moreover, each imputation other than one of these three is dominated by one of these three (see Problem 16.5). For this reason, von Neumann and Morgenstern called the set of these three imputations a 'solution' of the game.

Definition 16.14 Let v be a game and let $A \subseteq I(v)$. The set A is called a *stable set* if

(1) if $\mathbf{x}, \mathbf{y} \in A$ then $\mathbf{x}$ does not dominate $\mathbf{y}$,
(2) if $\mathbf{x} \in I(v) \setminus A$ then there is a $\mathbf{y} \in A$ that dominates $\mathbf{x}$. $\qquad\square$

The first property in Definition 16.14 is called *internal stability* and the second one *external stability*.

   The three-person game described above has many stable sets: see Problem 16.5. But even if a game has only one stable set then still a selection would have to be made, for practical purposes; stability, however, is a property of sets, not of single payoff distributions. The core does not suffer from this problem and, moreover, in that case there exist some plausible choices (like the nucleolus, see Chap. 19). Moreover, games with non-empty cores have been exactly characterized (see Sect. 16.5), whereas the problem of existence of stable sets is only partially solved.

   Some partial existence results are given now. First, essential simple games always have stable sets:

Theorem 16.15 *Let v be a simple game and let S be a minimal winning coalition. Let Δ^S be the set of those imputations $\mathbf{x}$ with $x_i = 0$ for every $i \notin S$. Then, if $\Delta^S \neq \emptyset$, it is a stable set.*

Proof Problem 16.8. $\qquad\blacksquare$

A game (N, v) is called a *zero-one game* if all one-person coalitions have worth 0 and the grand coalition N has worth 1. In the following example symmetric three-person zero-one games are considered.

Example 16.16 Let (N, v) be a game with $N = \{1, 2, 3\}$ and $v(i) = 0$ for all $i \in N$, $v(N) = 1$, and $v(S) = \alpha$ for every two-person coalition S, where $0 \leq \alpha \leq 1$. Then:

(a) Let $\alpha \geq \frac{2}{3}$. Then

$$\{(x, x, 1 - 2x), (x, 1 - 2x, x), (1 - 2x, x, x) \mid \frac{\alpha}{2} \leq x \leq \frac{1}{2}\} \qquad (16.3)$$

   is a stable set.
(b) For $\alpha < \frac{2}{3}$, the set in (16.3) is internally but not externally stable. The union of this set with the core of the game is a stable set.
(c) For $\alpha \leq \frac{1}{2}$ the core is a (the unique) stable set.

For the proofs of these statements see Problem 16.9. $\qquad\square$

The next theorem gives the relation between the domination core and stable sets.

Theorem 16.17 *Let (N, v) be a game. Then:*

(a) *The D-core of v is a subset of any stable set.*
(b) *If A and B are stable sets and $A \neq B$, then $A \nsubseteq B$.*
(c) *Suppose the D-core of v is a stable set. Then it is the unique stable set of the game.*

Proof Problem 16.10.   ∎

16.5   Balanced Games and the Core

In this section we derive the Bondareva-Shapley Theorem, which characterizes games with non-empty cores in terms of balancedness. First, the concepts of balanced maps, collections, and games are introduced.

Let $N = \{1, 2, \ldots, n\}$. A map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+ := \{t \in \mathbb{R} \mid t \geq 0\}$ is called a *balanced map* if

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) \mathbf{e}^S = \mathbf{e}^N .$$

Here $\mathbf{e}^S \in \mathbb{R}^N$ is the *characteristic vector* for coalition S with

$$e_i^S = 1 \text{ if } i \in S \text{ and } e_i^S = 0 \text{ if } i \in N \setminus S .$$

A collection B of nonempty coalitions is called a *balanced collection* if there is a balanced map λ such that

$$B = \{S \in 2^N \mid \lambda(S) > 0\} .$$

Example 16.18

(1) Let the nonempty coalitions $N_1, N_2, \ldots, N_k$ form a partition of N, i.e., $N = \bigcup_{r=1}^{k} N_r$ and $N_s \cap N_t = \emptyset$ if $s \neq t$. Then $\{N_1, N_2, \ldots, N_k\}$ is a balanced collection, corresponding to the balanced map λ with $\lambda(S) = 1$ if $S \in \{N_1, N_2, \ldots, N_k\}$ and $\lambda(S) = 0$ otherwise.
(2) For $N = \{1, 2, 3\}$ the set $B = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ is balanced and corresponds to the balanced map λ with $\lambda(S) = 0$ if $|S| \in \{1, 3\}$ and $\lambda(S) = \frac{1}{2}$ if $|S| = 2$. □

In order to have an interpretation of a balanced map, one can think of each player having one unit of time (or energy, labor, …) to spend. Each player can distribute his time over the various coalitions of which he is a member. Such a distribution is 'balanced' if it corresponds to a balanced map λ, where $\lambda(S)$ is interpreted as the length of time that the coalition S exists ('cooperates'); balancedness of λ means that each player spends exactly his one unit of time over the various coalitions.

Definition 16.19   A game (N, v) is called a *balanced game* if for each balanced map $\lambda : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ we have

$$\sum_S \lambda(S) v(S) \leq v(N) . \tag{16.4}$$

$\square$

Extending the interpretation of a balanced map in terms of a distribution of time to a game, balancedness of a game could be interpreted as saying that it is at least as productive to have the grand coalition operate during one unit of time as to have a balanced distribution of time over various smaller coalitions—worths of coalitions being interpreted as productivities. Thus, in a balanced game, it seems advantageous to form the grand coalition. Indeed, technically the importance of the notion of balancedness follows from Theorem 16.22. This theorem characterizes games with a nonempty core. Its proof is based on the following duality theorem.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \cdot \mathbf{y}$ denotes the usual inner product: $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$.

Theorem 16.20   *Let A be an $n \times p$-matrix, $\mathbf{b} \in \mathbb{R}^p$ and $\mathbf{c} \in \mathbb{R}^n$, and let $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}A \geq \mathbf{b}\} \neq \emptyset$ and $\{\mathbf{y} \in \mathbb{R}^p \mid A\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0}\} \neq \emptyset$. Then*

$$\min\{\mathbf{x} \cdot \mathbf{c} \mid \mathbf{x}A \geq \mathbf{b}\} = \max\{\mathbf{b} \cdot \mathbf{y} \mid A\mathbf{y} = \mathbf{c}, \mathbf{y} \geq \mathbf{0}\} .$$

Proof   Problem 16.13.   ■

Remark 16.21   In Theorem 16.20 also the following holds: if one of the programs is infeasible (i.e., one of the two sets in the theorem is empty), then both programs do not have an optimal solution (i.e., neither the minimum nor the maximum are attained). See Problem 16.14 for a proof. $\square$

Theorem 16.22   *Let (N, v) be a TU-game. Then the following two assertions are equivalent:*

(1) $C(v) \neq \emptyset$,
(2) (N, v) *is a balanced game.*

Proof   First note that $C(v) \neq \emptyset$ if and only if

$$v(N) = \min\{\sum_{i=1}^n x_i \mid \mathbf{x} \in \mathbb{R}^N, \ x(S) \geq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\}\} . \tag{16.5}$$

By the duality theorem, Theorem 16.20, equality (16.5) holds if and only if

$$v(N) = \max\{\sum \lambda(S) v(S) \mid \sum \lambda(S) \mathbf{e}^S = \mathbf{e}^N, \lambda \geq \mathbf{0}\} . \tag{16.6}$$

(Take for A the matrix with the characteristic vectors $\mathbf{e}^S$ as columns, let $\mathbf{c} := \mathbf{e}^N$ and let $\mathbf{b}$ be the vector of coalitional worths. Obviously, the non-emptiness conditions in Theorem 16.20 are satisfied.) Now (16.6) holds if and only if (16.4) holds. Hence (1) and (2) are equivalent.   ∎

An alternative proof of Theorem 16.22 can be based directly on Farkas' Lemma (Lemma 22.5): see Problem 16.15.

16.6   Problems

16.1. *Imputation Set of an Essential Game*
Prove that for an essential game v, $I(v)$ is the convex hull of the points $\mathbf{f}^1, \mathbf{f}^2, \ldots, \mathbf{f}^n$, as claimed in Sect. 16.1.

16.2. *Convexity of the Domination Core*
Prove that for each game the domination core is a convex set.

16.3. *Dominated Sets of Imputations*

(a) Prove that for each game (N, v), $D(S) = \emptyset$ if $|S| \in \{1, n\}$.
(b) Determine for each S the set $D(S)$ for the cost savings game (three communities game) in Chap. 1. Answer the same question for the glove game in Chap. 1.

16.4. *The Domination Relation*

(a) Prove that dom and dom_S are irreflexive relations and that dom_S is transitive and antisymmetric.[1]
(b) Construct a game (N, v) and imputations $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x} \, \text{dom} \, \mathbf{y}$ and $\mathbf{y} \, \text{dom} \, \mathbf{x}$.
(c) Construct a game (N, v) and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in I(v)$ with $\mathbf{x} \, \text{dom} \, \mathbf{y}$ and $\mathbf{y} \, \text{dom} \, \mathbf{z}$ and not $\mathbf{x} \, \text{dom} \, \mathbf{z}$.

16.5. *Stable Sets in a Three-Person Game*
Let $(\{1, 2, 3\}, v)$ be the game with all worths equal to 1 except for the one-person and the empty coalitions, which have worth equal to 0.

(a) Prove that each element of the imputation set of this game is dominated by another element.
(b) Prove that in this game each $\mathbf{x} \in I(v) \setminus A$ is dominated by an element of $A := \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$.
(c) If $c \in [0, \frac{1}{2})$ and $B := \{\mathbf{x} \in I(v) \mid x_3 = c\}$, then each element of $I(v) \setminus B$ is dominated by an element of B. Show this.

16.6. *Singleton Stable Set*

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[1] See Sect. 11.1 for definitions.

Prove that if a game (N, v) has a one-element stable set then $v(N) = \sum_{i \in N} v(i)$.

16.7. *A Glove Game*
Consider the three-person simple game v defined by

$$v(S) := \begin{cases} 1 & \text{if } S = \{1, 2\} \text{ or } \{2, 3\} \text{ or } \{1, 2, 3\} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that any imputation (x_1, x_2, x_3) that is not equal to $\mathbf{e}^2$ is dominated by another imputation.
(b) Compute the core and the domination core.
(c) Show that the domination core is not a stable set.
(d) Show that

$$B := \{(\lambda, 1 - 2\lambda, \lambda) \mid 0 \le \lambda \le \frac{1}{2}\}$$

is a stable set.

16.8. *Proof of Theorem 16.15*
Prove Theorem 16.15.

16.9. *Example 16.16*
Prove the statements in Example 16.16.

16.10. *Proof of Theorem 16.17*
Prove Theorem 16.17. Does this theorem also hold for the core instead of the D-core?

16.11. *Core and D-Core*
Is (16.1) also a necessary condition for equality of the core and the D-core? (Cf. Theorem 16.8.)

16.12. *Strategic Equivalence*
Let (N, w) be *strategically equivalent* to (N, v), that is, there are $k \in \mathbb{R}$, $k > 0$ and $\mathbf{a} \in \mathbb{R}^N$ such that for each coalition S: $w(S) = kv(S) + a(S)$. Show that

(i) $C(w) = kC(v) + \mathbf{a}$ $(:= \{\mathbf{x} \in \mathbb{R}^N \mid \mathbf{x} = k\mathbf{y} + \mathbf{a} \text{ for some } \mathbf{y} \in C(v)\})$
(ii) $DC(w) = kDC(v) + \mathbf{a}$.

[The equalities (i) and (ii) express that the core and the D-core are covariant w.r.t. strategic equivalence.]

16.13. *Proof of Theorem 16.20*
Prove Theorem 16.20. [Hint: use Theorem 22.6.]

16.14. *Infeasible Programs in Theorem 16.20*
Prove the claim made in Remark 16.21. [Hint: Suppose, say, that there is no $\mathbf{y} \ge \mathbf{0}$ with $A\mathbf{y} = \mathbf{c}$. Then, certainly, the max-program does not have an optimal solution.

Use Farkas' Lemma (Lemma 22.5) to conclude that there exists a vector $\mathbf{z}$ with $\mathbf{z}A \geq \mathbf{0}$ and $\mathbf{z} \cdot \mathbf{c} < 0$. Suppose the min-program is feasible, i.e., there is an $\mathbf{x}$ with $\mathbf{x}A \geq \mathbf{b}$. Then, show that the min-program does not have an optimal solution by considering the vectors $\mathbf{x} + t\mathbf{z}$ for $t \in \mathbb{R}$, $t > 0$.]

16.15. *Proof of Theorem 16.22 Using Lemma 22.5*
Prove Theorem 16.22 with the aid of Lemma 22.5. [Hint: List the nonempty coalitions $S \subseteq N$ as $S_1, \ldots, S_p$ ($p = 2^n - 1$) with $S_p = N$. Define the $(n + n + p) \times p$ matrix A as follows. Column $k < p$ is $(\mathbf{e}^{S_k}, -\mathbf{e}^{S_k}, -\mathbf{e}^k)$ where: $\mathbf{e}^{S_k} \in \mathbb{R}^n$, $e_i^{S_k} = 1$ if $i \in S_k$, $e_i^{S_k} = 0$ if $i \notin S_k$. Column p is $(\mathbf{e}^N, -\mathbf{e}^N, \mathbf{0})$. Then $C(N, v) \neq \emptyset$ iff there exists $(\mathbf{z}, \mathbf{z}', \mathbf{w}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$ with $(\mathbf{z}, \mathbf{z}', \mathbf{w}) \geq \mathbf{0}$ and $(\mathbf{z}, \mathbf{z}', \mathbf{w})A = \mathbf{b}$, where $\mathbf{b} = (v(S_k))_{k=1}^p$. This has the form as in (a) of Lemma 22.5.]

16.16. *Balanced Maps and Collections*

(a) Show that for any balanced map λ one has $\sum_S \lambda(S) \geq 1$, with equality if and only if the corresponding balanced collection equals $\{N\}$.
(b) If B is a balanced collection unequal to $\{N\}$, then

$$B^c := \{S \in 2^N \setminus \{\emptyset\} \mid N \setminus S \in B\}$$

   is also a balanced collection. Give the corresponding balanced map.
(c) Let $S \in 2^N \setminus \{\emptyset, N\}$. Prove that $\{S, (N \setminus \{i\})_{i \in S}\}$ is balanced collection.
(d) Prove that the balanced maps form a convex set Λ^n.

16.17. *Minimum of Balanced Games*
Show that the minimum of two balanced games is again balanced.

16.18. *Balanced Simple Games*
A simple game has a non-empty core if and only if it has veto players, cf. Theorem 16.12(1). Derive this result from Theorem 16.22.

16.7 Notes

The concepts of domination, imputation, and stable set were introduced by von Neumann and Morgenstern (1944/1947). The core was introduced by Gillies (1953).

Lucas (1969) gives an example of a(n essential) game that does not have a stable set; see also Owen (1995), p. 253. Theorem 16.22 is due to Bondareva (1962) and Shapley (1967).

Problem 16.6 is taken from Morris (1994).

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