Chapter 3
Choice & Control

Synopsis This chapter develops the central dynamical systems model for describing the behavior of cyber-physical systems with a programming language. It complements the previous understanding of continuous dynamics with an understanding of the discrete dynamics caused by choices and controls in cyber-physical systems. The chapter interfaces the continuous dynamics of differential equations with the discrete dynamics of conventional computer programs by directly integrating differential equations with discrete programming languages. This leverages well-established programming language constructs around elementary discrete and continuous statements to obtain *hybrid programs* as a core programming language for cyber-physical systems. In addition to embracing differential equations, semantical generalizations to mathematical reals as well as operators for nondeterminism are important to make hybrid programs appropriate for cyber-physical systems.

3.1 Introduction

Chapter 2 saw the beginning of cyber-physical systems, yet emphasized only their continuous part in the form of differential equations $x' = f(x)$. The sole interface between continuous physical capabilities and cyber capabilities was by way of their evolution domain. The evolution domain Q in a continuous program $x' = f(x) \,\&\, Q$ imposes restrictions on how far or how long the system can evolve along that differential equation. Suppose a continuous evolution has succeeded, and the system stops following its differential equation, e.g., because the state would otherwise leave the evolution domain Q if it kept going. Then what happens now? How does the cyber part take control? How do we describe what the cyber elements compute afterwards? What descriptions explain how cyber interacts with physics?

An overall understanding of a CPS ultimately requires an understanding of the joint model with both its discrete dynamics and its continuous dynamics. It takes both to understand, for example, what effect a discrete car controller has, via its engine and steering actuators, on the continuous physical motion of a car down the

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road. Continuous programs are powerful for modeling continuous processes, such as continuous motion. They cannot—on their own—model discrete changes of variables, however.[1] Such discrete state change is a good model for the impact of computer decisions on cyber-physical systems, in which computation decides to, say, stop speeding up and apply the brakes instead. During the evolution along a differential equation, such as $x' = v, v' = a$ for accelerated motion along a straight line, all variables change continuously over time, because the solution of a differential equation is (sufficiently) smooth. Discontinuous change of variables, such as a change of acceleration from $a = 2$ to $a = -6$ by applying the brakes instead arises from a discrete change of state resulting from how computers compute decisions one step at a time. Time passes while differential equations evolve, but no time passes during an immediate discrete change (it is easy to model computations that take time by mixing both). What could be a model for describing such discrete changes in a system?

Discrete change can be described by different models. The most prominent ones are conventional programming languages, in which everything takes effect one discrete step at a time, just like computer processors operate one clock cycle at a time.

CPSs combine cyber and physics, though. In CPS, we do not program computers, but program cyber-physical systems instead. We program the computers that control the physics, which requires programming languages for CPSs to involve physics, and integrate differential equations seamlessly with discrete computer operations. The basic idea is that discrete statements are executed by a computer processor, while continuous statements are handled by the physical elements, such as wheels, engines, or brakes. CPS programs need a mix of both to accurately describe the combined discrete and continuous dynamics.

Does it matter which discrete programming language we choose to enrich with the continuous statements from Chap. 2? It might be argued that the hybrid aspects are more important for CPS than the discrete language. After all, there are many conventional programming languages that are Turing-equivalent, i.e., that compute the same functions [3, 10, 26]. Yet there are numerous significant differences even among discrete programming languages that make some more desirable than others [7]. For the particular purposes of CPS, we will identify additional desired features. We will develop what we need as we go, culminating in the programming language of *hybrid programs* [16–21], which plays a fundamental rôle in this book.

Other areas such as automata theory and the theory of formal languages [10] or Petri nets [15] also provide models of discrete change. There are ways of augmenting these models with differential equations as well [1, 4, 13, 14]. But programming languages are uniquely positioned to extend their virtues of built-in compositionality. Just as the meaning and effect of a conventional program is a function of its pieces, the meaning and operation of a hybrid program is also a function of its parts.

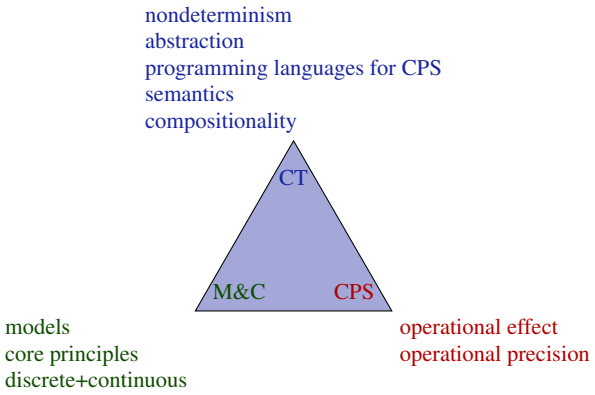
The most important learning goals of this chapter are:

[1] There is a much deeper sense [20] in which continuous dynamics and discrete dynamics have surprising similarities regardless. But even so, these similarities rest on the foundations of hybrid systems, which we need to understand first.

Modeling and Control: This chapter plays a pivotal rôle in understanding and designing models of CPSs. We develop an understanding of the core principles behind CPS by studying how discrete and continuous dynamics are combined and interact to model cyber and physics, respectively. We see the first example of how to develop models and controls for a simple CPS. Even if subsequent chapters will blur the overly simplistic categorization of cyber=discrete versus physics=continuous, it is useful to equate them for now, because cyber, computation, and decisions quickly lead to discrete dynamics, while physics naturally gives rise to continuous dynamics. Later chapters will show that some physical phenomena are better modeled with discrete dynamics, while some controller aspects also have a manifestation in the continuous dynamics.

Computational Thinking: We introduce and study the important phenomenon of nondeterminism, which is crucial for developing faithful models of a CPS's environment and helpful for developing effective models of the CPS itself. We emphasize the importance of abstraction, which is an essential modular organization principle in CPS as well as all other parts of computer science. We capture the core aspects of CPS in the programming language of hybrid programs.

CPS Skills: We develop an intuition for the operational effects of CPS. And we will develop an understanding for the semantics of the programming language of hybrid programs, which is the CPS model on which this textbook is based.



3.2 A Gradual Introduction to Hybrid Programs

This section gradually introduces the operations that hybrid programs provide, one step at a time. Its emphasis is on their motivation and an intuitive development before subsequent sections provide a comprehensive view. The motivating examples

we consider now are naïve but still provide a good introduction to the world of CPS programming. With more understanding, we will later be able to augment their designs.

3.2.1 Discrete Change in Hybrid Programs

Discrete change happens immediately in computer programs when a new value is assigned to a variable. The statement $x := e$ assigns the value of term e to variable x by evaluating the term e and assigning the result to the variable x. It leads to a discrete, discontinuous change, because the value of x does not vary smoothly over time but radically when the value of e is suddenly assigned to x.

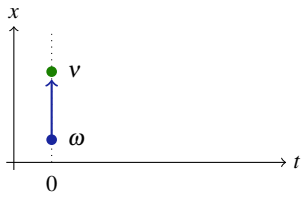


Fig. 3.1 An illustration of the behavior of an instantaneous discrete change at time = 0

This gives us a discrete model of change, $x := e$, in addition to the continuous model of change, $x' = f(x) \& Q$, from Chap. 2. We can now model systems that are *either* discrete *or* continuous. Yet, how can we possibly model proper CPSs that combine cyber and physics with one another and that, thus, simultaneously combine discrete and continuous dynamics? We need such hybrid behavior every time a system has both continuous dynamics (such as the continuous motion of a car down the street) in addition to discrete dynamics (such as shifting gears).

3.2.2 Compositions of Hybrid Programs

One way cyber and physics can interact is if a computer provides input to physics. Physics may mention variables such as a for acceleration, and the computer program sets its value depending on whether the computer program wants to accelerate or brake. That is, cyber may set the values of actuators that affect physics.

In this case, cyber and physics interact in such a way that first the cyber part does something and physics then follows. Such a behavior corresponds to a sequential composition $(\alpha; \beta)$ in which first the HP α on the left of the sequential composition operator (;) runs and, when it's done, the HP β on the right of operator ; runs. The

following HP[2]

$$a := a + 1;\ \{x' = v, v' = a\} \tag{3.1}$$

will first let cyber perform a discrete change of setting acceleration variable a to $a+1$ and then let physics follow the differential equation[3] $x'' = a$, which describes accelerated motion of the point x along a straight line. The overall effect is that cyber instantly increases the value of the acceleration variable a and physics then lets x evolve continuously with that acceleration a (increasing velocity v continuously with derivative a). HP (3.1) models a situation where the desired acceleration is commanded once to increase and the robot then moves with that fixed acceleration; see Fig. 3.2. The curve for position looks almost linear in Fig. 3.2, because the velocity difference is so small, which is a great example of how misleading visual representations can be compared to rigorous analysis methods. The sequential composition operator (;) has the same effect that it has in programming languages such as Java. It separates statements that are to be executed sequentially one after the other. If you look closely, you will find a minor subtle difference, because programming languages such as Java or C expect ; at the end of every statement, not just between sequentially composed statements. This syntactic difference is inconsequential, and a common trait of mathematical programming languages.

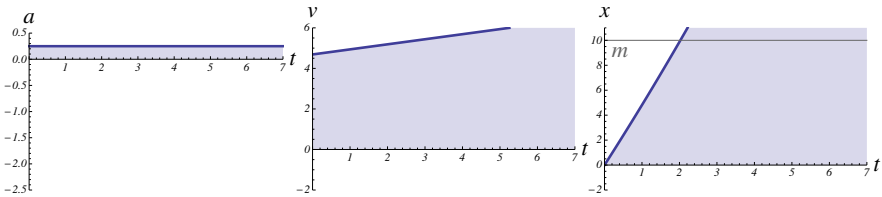


Fig. 3.2 Fixed acceleration a **(left)**, velocity v **(middle)**, and position x **(right)** change over time t

The HP in (3.1) executes control (it sets the acceleration for physics), but it has very little choice, or rather no choice at all. So only if the CPS is very lucky will an increase in acceleration be the right action to remain safe forever. Quite likely, the robot will have to change its mind ultimately, which is what we investigate next.

But first observe that the constructs we saw so far, assignments, sequential compositions, and differential equations, already suffice to exhibit typical hybrid systems dynamics. The behavior shown in Fig. 3.3 could be exhibited by this hybrid program:

[2] Note that the parentheses around the differential equation are redundant and will often be left out in the textbook or in scientific papers. HP (3.1) would be written $a := a + 1;\ x' = v, v' = a$. Round parentheses are often used in theoretical developments, while braces are useful for programs to disambiguate grouping in bigger CPS applications.

[3] We frequently use $x'' = a$ as an abbreviation for $x' = v, v' = a$, even if x'' is not officially permitted in the KeYmaera X theorem prover for hybrid systems.

$$
\begin{array}{ll}
a := -2; & \{x' = v, v' = a\}; \\
a := 0.25; & \{x' = v, v' = a\}; \\
a := -2; & \{x' = v, v' = a\}; \\
a := 0.25; & \{x' = v, v' = a\}; \\
a := -2; & \{x' = v, v' = a\}; \\
a := 0; & \{x' = v, v' = a\}
\end{array}
$$

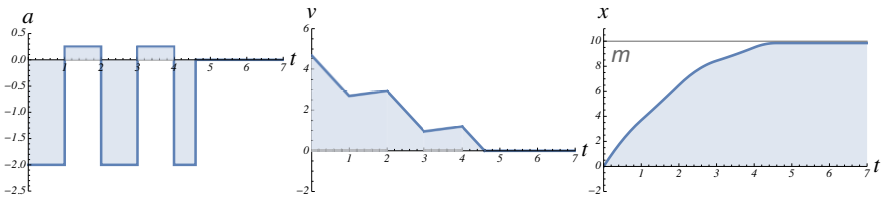


Fig. 3.3 Acceleration a **(left)**, velocity v **(middle)**, and position x **(right)** change over time t, with a piecewise constant acceleration changing discretely at instants of time while velocity and position change continuously over time

Can you already spot a question that comes up about how exactly we run this program? We will postpone the formulation of and answer to this question to Sect. 3.2.6.

3.2.3 Decisions in Hybrid Programs

In general, a CPS will have to check conditions on the state to see which action to take. Otherwise the CPS could not possibly be safe and, quite likely, will also not take the correct actions to get to its goal. One way of programming these conditions is the use of an if-then-else statement, as in classical discrete programs:

$$
\begin{array}{l}
\mathsf{if}(v < 4)\, a := a + 1 \,\mathsf{else}\, a := -b; \\
\{x' = v, v' = a\}
\end{array}
\tag{3.2}
$$

This HP will check the condition $v < 4$ to see whether the current velocity is still less than 4. If it is, then a will be increased by 1. Otherwise, a will be set to $-b$ for some braking deceleration constant $b > 0$. Afterwards, i.e., when the if-then-else statement in the first line has run to completion, the HP will again evolve x continuously with acceleration a along the differential equation in the second line.

The HP (3.2) takes only the current velocity into account to reach a decision on whether to accelerate or brake. That is usually not enough information to guarantee safety, because a robot doing that would be so fixated on achieving its desired speed that it would happily speed into any walls or other obstacles along the way. Consequently, programs that control robots also take other state information into account, for example sufficient distance $x - m$ to an obstacle m from the robot's position x:

$$\begin{array}{l}\mathsf{if}(x-m>s)\,a:=a+1\,\mathsf{else}\,a:=-b;\\ \{x'=v,v'=a\}\end{array} \tag{3.3}$$

Whether that is safe depends on the choice of the required safety distance s. Controllers could also take both distance *and* velocity into account for the decision:

$$\begin{array}{l}\mathsf{if}(x-m>s\wedge v<4)\,a:=a+1\,\mathsf{else}\,a:=-b;\\ \{x'=v,v'=a\}\end{array} \tag{3.4}$$

Note 9 (Iterative design) To design serious controllers, you will usually develop a series of increasingly more intelligent controllers for systems that face increasingly challenging environments. Designing controllers for robots or other CPSs is a serious challenge. You will want to start with simple controllers for simple circumstances and only move on to more advanced challenges when you have fully understood and mastered the previous controllers, what behavior they guarantee and what functionality they are still missing. If a controller is not even safe under simple circumstances (for example when it only knows how to brake), it will not be safe in more complex cases either.

3.2.4 Choices in Hybrid Programs

A common feature of CPS models is that they often include only some but not all detail about the system. This is for good reasons, because full detail about everything can be overwhelming and is often a distraction from the really important aspects of a system. A (somewhat) more complete model of (3.4) might have the following shape, with some further formula S as an extra condition for checking whether to actually accelerate based on battery efficiency or secondary considerations which are not safety-critical:

$$\begin{array}{l}\mathsf{if}(x-m>s\wedge v<4\wedge S)\,a:=a+1\,\mathsf{else}\,a:=-b;\\ \{x'=v,v'=a\}\end{array} \tag{3.5}$$

Consequently, (3.4) is not actually a faithful model of (3.5), because (3.4) insists that the acceleration would always be increased just because $x-m>s\wedge v<4$ holds, unlike (3.5), which also checks the additional condition S. Likewise, (3.3) certainly is no faithful model of (3.5). But it looks simpler.

How can we describe a model that is simpler than (3.5) because it ignores the details of S yet that is still faithful to the original system? What we want this model to do is characterize that the controller may either increase acceleration by 1 or brake. All acceleration should certainly only happen when certain safety-critical conditions are met. But the model should make less commitment than (3.3) about the precise circumstances under which braking is chosen. After all, braking may sometimes just be the right thing to do, for example when arriving at the goal. So we want a model

that allows braking under more circumstances than (3.3) without having to model precisely under what circumstances that is. If a system with more behavior is safe, then the actual implementation will be safe as well, because it will only ever exercise some of the verified behavior [12]. The extra behavior in the system might, in fact, occur in reality whenever there are minor lags or discrepancies. So it is good to have the extra assurance that some flexibility in the execution of the system will not break its safety guarantees.

Note 10 (Abstraction) Successful CPS models often include only the relevant aspects of the system and elide irrelevant detail. The benefit of doing so is that the model and its analysis become simpler, enabling us to focus on the critical parts without being bogged down in tangentials. This is the *power of abstraction*, probably the primary secret weapon of computer science. It does take considerable skill, however, to find the best level of abstraction for a system, a skill that you will continue to sharpen throughout your entire career.

Let us take the development of this model step by step. The first feature that the controller in the model has is a choice. The controller can choose to increase acceleration or to brake, instead. Such a choice between two actions is denoted by the choice operator $\cup$:

$$\begin{array}{l}(a:=a+1 \cup a:=-b);\\ \{x'=v, v'=a\}\end{array} \tag{3.6}$$

When running this hybrid program, the first thing that happens is that the first statement (before the ;) runs, which is a choice ($\cup$) between whether to run $a:=a+1$ or whether to run $a:=-b$. That is, the choice is whether to increase acceleration a by 1 or whether to reset a to $-b$ for braking. After this choice (i.e., after the ; sequential composition operator), the system follows the usual differential equation $x''=a$ describing accelerated motion along a line.

Now, wait. There was a choice. Who chooses? How is the choice resolved?

Note 11 (Nondeterministic $\cup$) The choice ($\cup$) is *nondeterministic*. That is, every time a choice $\alpha \cup \beta$ runs, exactly one of the two choices, α or β, is chosen to run. The choice is *nondeterministic*, i.e., there is no prior way of telling which of the two choices is going to be chosen. Both outcomes are perfectly possible and a safe system design needs to be prepared to handle either outcome.

The HP (3.6) is a *faithful abstraction* [12] of (3.5), because every way (3.5) can run can be mimicked by (3.6) so that the outcome of (3.6) corresponds to that of (3.5). Whenever (3.5) runs $a:=a+1$, which happens exactly if $x-m>s \land v<4 \land S$ is *true*, (3.6) only needs to choose to run the left choice $a:=a+1$. Whenever (3.5) runs $a:=-b$, which happens exactly if $x-m>s \land v<4 \land S$ is *false*, (3.6) needs to choose to run the right choice $a:=-b$. So all runs of (3.5) are possible runs of (3.6). Furthermore, (3.6) is much simpler than (3.5), because it contains less detail. It does not mention the complicated extra condition S. However, (3.6) is a little too permis-

sive, because it suddenly allows the controller to choose $a := a + 1$ even when it is already too fast or even at a small distance from the obstacle. That way, even if (3.5) was a safe controller, (3.6) is still unsafe, and, thus, not a very suitable abstraction.

3.2.5 Tests in Hybrid Programs

In order to build a faithful yet not overly permissive abstraction of (3.5), we need to restrict the permitted choices in (3.6) so that there is enough flexibility, but only so much that the acceleration choice $a := a + 1$ can only be chosen when it is currently safe to do so. The way to do that is to use tests on the current state of the system.

A *test* $?Q$ is a statement that checks the truth-value of a first-order formula Q of real arithmetic in the current state. If Q holds in the current state, then the test passes, nothing happens, and the HP continues to run normally. If, instead, Q does not hold in the current state, then the test fails, and the system execution is aborted and discarded. That is, when ω is the current state, then $?Q$ runs successfully without changing the state when $\omega \in [\![Q]\!]$. Otherwise, i.e., if $\omega \notin [\![Q]\!]$, the run of $?Q$ is aborted and not considered any further, because it did not play by the rules of the system.

Of course, it can be difficult to figure out which control choice is safe under what circumstances, and the answer also depends on whether the safety goal is to limit speed or to remain at a safe distance from other obstacles. For the model in this chapter, we simply pretend that $v < 4$ is the appropriate safety condition and revisit the question of how to design and explain such conditions in later chapters.

The test statement $?(v < 4)$ alias $?v < 4$ can be used to change (3.6) so that it allows acceleration only when $v < 4$, while braking is still allowed always:

$$\begin{array}{l} \big((?v < 4; a := a + 1) \cup a := -b\big); \\ \{x' = v, v' = a\} \end{array} \tag{3.7}$$

The first statement of (3.7) is a choice ($\cup$) between $(?v < 4; a := a + 1)$ and $a := -b$. All choices in hybrid programs are nondeterministic, so either outcome is always possible. In (3.7), this means that the left choice can always be chosen, just as well as the right one. The first statement that happens in the left choice, however, is the test $?v < 4$, which the system run has to pass in order to be able to continue. In particular, if $v < 4$ is indeed *true* in the current state, then the system passes that test $?v < 4$ and the execution proceeds to after the sequential composition (;) to run $a := a + 1$. If $v < 4$ is *false* in the current state, however, the system fails the test $?v < 4$ and that run is aborted and discarded. The right option to brake is always available, because it does not involve any tests to pass.

Note 12 (Discarding failed runs) System runs that fail tests $?Q$ are discarded and not considered any further, because a failed run did not play by the rules of the system. It is as if those failed system execution attempts had never happened. Even if one execution attempt fails, other runs may still be successful. Operationally, you can imagine finding them by backtracking through all the possible choices in the system run and taking alternative choices instead.

In principle, there are always two choices when running (3.7). However, which ones actually run successfully depends on the current state. If the car is currently slow (so the test $?v < 4$ will succeed), then both options of accelerating and braking are possible and can execute successfully. Otherwise, only the braking choice executes, because trying the left choice will fail its test $?v < 4$ and be discarded. Both choices formally exist but only one will succeed in that case.

Note 13 (Successful runs) Notice that only successfully executed runs of HPs will be considered, and all others will be discarded because they did not play by the rules. For example, $?v < 4; v := v + 1$ can only run in states where $v < 4$, otherwise there are no runs of this HP. Failed runs are discarded entirely, so the HP $v := v + 1; ?v < 4$ can also only run in states where $v < 3$. Operationally, you can imagine running the HP step by step and rolling all its changes back if any test ever fails. The velocity increases by $v := v + 1$, but this change is undone and the entire run discarded unless the subsequent test $?v < 4$ succeeds for the new value.

Comparing (3.7) with (3.5), we see that (3.7) is a faithful abstraction of the more complicated (3.5), because all runs of (3.5) can be mimicked by (3.7). Yet, unlike the intermediate guess (3.6), the improved HP (3.7) still retains the critical information that acceleration is only allowed by (3.5) when $v < 4$. Unlike (3.5), (3.7) does not restrict the cases where acceleration can be chosen to those that also satisfy $v < 4 \wedge S$. Hence, (3.7) is more permissive than (3.5). But (3.7) is also simpler and only contains crucial information about the controller. Hence, (3.7) is a more abstract faithful model of (3.5) that retains just the relevant detail. Studying the abstract (3.7) instead of the more concrete (3.5) has the advantage that only relevant details need to be understood while irrelevant aspects can be ignored. It also has the additional advantage that a safety analysis of the more abstract (3.7), which allows lots of behavior, will imply safety of the special concrete case (3.5) but also implies safety of other implementations of (3.7). For example, replacing S by a different condition in (3.5) still gives a special case of (3.7). So if all behaviors of (3.7) are safe, all behaviors of that different replacement will already be safe. With a single verification result about a more general, more abstract system, we can verify a whole class of systems rather than just one particular system. This important phenomenon [12] will be investigated in more detail in later parts of the book.

Of course, which details are relevant and which ones can be simplified depends on the analysis question at hand, a question that we will be better equipped to an-

swer in a later chapter. For now, suffice it to say that (3.7) has the relevant level of abstraction for our purposes.

Note 14 (Broader significance of nondeterminism) Nondeterminism comes up in the above cases for reasons of abstraction and to focus the system model on the most critical aspects of the system while suppressing irrelevant detail. This simplification is one important reason for introducing nondeterminism in system models, but there are other important reasons as well. Whenever a system includes models of its environment, nondeterministic models are crucial, because we often have only a partial understanding of what the environment will do. A car controller for example, will not always know for sure what other cars or pedestrians in its environment will do, exactly, so that nondeterministic models are the only faithful representations.

A pretty reasonable model of the controller of the acceleration c of another car in our environment is to nondeterministically either accelerate or brake, e.g., $c := 2 \cup c := -b$, because we cannot perfectly predict which one is going to happen, anyhow.

Note the notational convention that sequential composition ; binds more strongly than nondeterministic choice $\cup$ so we can leave parentheses out without changing (3.7):

$$\big(?v < 4; a := a+1 \cup a := -b\big); \quad \{x' = v, v' = a\} \tag{3.7*}$$

3.2.6 Repetitions in Hybrid Programs

The hybrid programs above were interesting, but only allowed the controller to choose what action to take at most once. All controllers so far inspected the state in a test or in an if-then-else condition and then chose what to do once, only to let physics take control subsequently by following a differential equation. That makes for rather short-lived controllers. They have a job only once in their lives. And most decisions they reach may end up being bad ones at some point. Say, one of those controllers, e.g., (3.7), inspects the state and finds it still okay to accelerate. If it chooses $a := a+1$ and then lets physics move with the differential equation $x'' = a$, there will probably come a time at which increased acceleration is no longer such a great idea. But the controller of (3.7) has no way to change its mind, because it has no more choices and cannot exercise any control anymore.

If the controller of (3.7) is supposed to be able to make a second control choice later after physics has followed the differential equation for a while, then (3.7) can simply be sequentially composed with itself:

$$\begin{array}{l}\big((?v<4;a:=a+1)\cup a:=-b\big);\\ \{x'=v,v'=a\};\\ \big((?v<4;a:=a+1)\cup a:=-b\big);\\ \{x'=v,v'=a\}\end{array} \tag{3.8}$$

In (3.8), the cyber controller can first choose to accelerate or brake (depending on whether $v<4$ is true in the present state), then physics evolves along differential equation $x''=a$ for some while, then the controller can again choose whether to accelerate or brake (depending on whether $v<4$ is true in the state reached then), and finally physics again evolves along $x''=a$.

For a controller that is supposed to be allowed to have a third control choice, copy and paste replication would again help:

$$\begin{array}{l}\big((?v<4;a:=a+1)\cup a:=-b\big);\\ \{x'=v,v'=a\};\\ \big((?v<4;a:=a+1)\cup a:=-b\big);\\ \{x'=v,v'=a\};\\ \big((?v<4;a:=a+1)\cup a:=-b\big);\\ \{x'=v,v'=a\}\end{array} \tag{3.9}$$

But this is neither a particularly concise nor a particularly useful modeling style. What if a controller might need 10 control decisions or 100? Or what if there is no way of telling ahead of time how many control decisions the cyber part will have to take to reach its goal? Think of how many control decisions you might need when driving in a car from Paris to Rome. Do you even know that ahead of time? Even if you do, do you want to model a system by explicitly replicating its controller that often?

Note 15 (Repetition) As a more concise and more general way of describing repeated control choices, hybrid programs allow for the repetition operator *, which works like the Kleene star operator in regular expressions, except that it applies to a hybrid program α as in α^*. It repeats α any number $n\in\mathbb{N}$ of times, including 0, by a nondeterministic choice.

The programmatic way of summarizing (3.7), (3.8), (3.9) and the infinitely many more n-fold replications of (3.7), for any $n\in\mathbb{N}$, is by using a repetition operator:

$$\Big(\big((?v<4;a:=a+1)\cup a:=-b\big);\\ \{x'=v,v'=a\}\Big)^* \tag{3.10}$$

This HP can repeat (3.7) any number of times (0,1,2,3,4,...). Of course, it would not be very meaningful to repeat a loop half a time or minus 5 times, so the repetition count $n\in\mathbb{N}$ still has to be some natural number.

But how often does a nondeterministic repetition like (3.10) repeat then? That choice is again nondeterministic.

Note 16 (Nondeterministic *) Repetition (*) is *nondeterministic*. That is, program α^* can repeat α *any* number ($n \in \mathbb{N}$) of times. The choice how often to run α is *nondeterministic*, i.e., there is no prior way of telling how often α will be repeated.

However, hold on, every time the loop in (3.10) is run, how long does the continuous evolution along $\{x' = v, v' = a\}$ in that loop iteration take? Or, actually, even in the loop-free (3.8), how long does the first $x'' = a$ take before the controller has its second control choice? How long did the continuous evolution take in (3.7) even?

There is a choice even in following a single differential equation! However deterministic the solution of the differential equation itself may be. Even if the solution of the differential equation is unique (which it is in the sufficiently smooth cases that we consider, according to Chap. 2), it is still a matter of choice how long to follow that solution. The choice is, as always in hybrid programs, nondeterministic.

Note 17 (Nondeterministic $x' = f(x)$) The duration of evolution of a differential equation ($x' = f(x) \,\&\, Q$) is *nondeterministic* (except that the evolution can never be so long that the state leaves Q). That is, $x' = f(x) \,\&\, Q$ can follow the solution of $x' = f(x)$ for any amount of time ($0 \leq r \in \mathbb{R}$) within the interval of existence of the solution within Q. The choice how long to follow $x' = f(x)$ is *nondeterministic*, i.e., there is no prior way of telling how long $x' = f(x)$ will evolve (except that it can never leave Q).

3.3 Hybrid Programs

Based on the above gradual motivation, this section formally defines the programming language of hybrid programs [18, 20], in which all of the operators motivated above are allowed.

3.3.1 Syntax of Hybrid Programs

Formal grammars have worked well to define the syntax of terms e and first-order logic formulas Q in Chap. 2, which is why we, of course, continue to use a grammar to define the syntax of hybrid programs.

Definition 3.1 (Hybrid program). *Hybrid programs* are defined by the following grammar (α, β are HPs, x is a variable, e is a term possibly containing x, e.g., a polynomial in x, and Q is a formula of first-order logic of real arithmetic):

$$\alpha, \beta \ ::= \ x := e \mid ?Q \mid x' = f(x) \,\&\, Q \mid \alpha \cup \beta \mid \alpha; \beta \mid \alpha^*$$

The first three cases are called *atomic HPs*, the last three *compound HPs*, because they are built out of smaller HPs. The *assignment* $x := e$ instantaneously changes the value of variable x to the value of term e with a discrete state change. The *differential equation* $x' = f(x) \,\&\, Q$ follows a continuous evolution from the present value of x along the differential equation $x' = f(x)$ for any amount of time but restricted to the domain of evolution Q, where x' denotes the time-derivative of x. It goes without saying that $x' = f(x) \,\&\, Q$ is an explicit differential equation, so no derivatives occur in $f(x)$ or Q. Recall that a differential equation $x' = f(x)$ without an *evolution domain constraint* is short for $x' = f(x) \,\&\, true$, since that imposes no restriction on the duration of the continuous evolution. The *test* action $?Q$ is used to define conditions. Its effect is that of a *no-op* if the formula Q is true in the current state; otherwise, like an *abort* statement would, it allows no transitions. That is, if the test succeeds because formula Q holds in the current state, then the state does not change (it was only a test), and the system execution continues normally. If the test fails because formula Q does not hold in the current state, however, then the system execution cannot continue, and is cut off, discarded, and not considered any further since it is a failed execution attempt that did not play by the rules of the HP.[4]

Nondeterministic choice $\alpha \cup \beta$, sequential composition $\alpha; \beta$, and nondeterministic repetition α^* of programs are as in regular expressions but generalized to the semantics of hybrid systems. *Nondeterministic choice* $\alpha \cup \beta$ expresses behavioral alternatives between the runs of α and β. That is, the HP $\alpha \cup \beta$ can choose nondeterministically to follow the runs of HP α, or, instead, to follow the runs of HP β. The *sequential composition* $\alpha; \beta$ models that the HP β starts running after HP α has finished (β never starts if α does not terminate successfully). In $\alpha; \beta$, the runs of α take effect first, until α terminates (if it does), and then β continues. Observe that, like repetitions, continuous evolutions within α can take more or less time, which causes uncountable nondeterminism. This nondeterminism occurs in hybrid systems because they can operate in so many different ways, which is reflected in HPs. *Nondeterministic repetition* α^* is used to express that the HP α repeats any number of times, including zero times. When following α^*, the runs of HP α can be repeated over and over again, any nondeterministic number of times (≥ 0).

[4] The effect of the test $?Q$ is the same as that of $\mathsf{if}(Q)\,\mathsf{skip}\,\mathsf{else}\,\mathsf{abort}$ where skip has no effect and abort aborts and discards the system run. Indeed, skip is equivalent to the trivial test $?true$ and abort is equivalent to the impossible test $?false$. But then we would have to add if-then-else, skip and abort, which HPs already provide for free.

Expedition 3.1 (Operator precedence for hybrid programs)

In practice, it is useful to save parentheses by agreeing on notational *operator precedences*. Unary operators (including repetition *) bind more strongly than binary operators and ; binds more strongly than $\cup$, so $\alpha;\beta \cup \gamma \equiv (\alpha;\beta) \cup \gamma$ and $\alpha \cup \beta;\gamma \equiv \alpha \cup (\beta;\gamma)$. Especially, $\alpha;\beta^* \equiv \alpha;(\beta^*)$.

3.3.2 Semantics of Hybrid Programs

After having developed a syntax for CPS and an operational intuition for its effects, we seek operational precision in its effects. That is, we will pursue one important leg of computational thinking and give an unambiguous meaning to all operators of HPs. We will do this in pursuit of the realization that the only way to be precise about an analysis of CPS is to first be precise about the meaning of the models of CPS. Furthermore, we will leverage another important leg of computational thinking rooted in logic by exploiting that the right way of understanding something is to understand it compositionally as a function of its pieces [6]. So we will give meaning to hybrid programs by giving a meaning to each of their operators. Thereby, a meaning of a large HP is merely a function of the meaning of its pieces. This is the style of denotational semantics for programming languages due to Scott and Strachey [25].

There is more than one way to define the meaning of a program, including defining a denotational semantics [24], an operational semantics [24], a structural operational semantics [22], or an axiomatic semantics [9, 23]. For our purposes, what is most relevant is how a hybrid program changes the state of the system. Consequently, the semantics of hybrid programs considers what (final) state ν is reachable by running an HP α from an (initial) state ω. Semantical models that expose more detail, e.g., about the internal states during the run of an HP, are possible [11] but can be ignored for most purposes in this book.

Recall that a *state* $\omega : \mathcal{V} \to \mathbb{R}$ is a mapping from variables to $\mathbb{R}$, which assigns a real value $\omega(x) \in \mathbb{R}$ to each variable $x \in \mathcal{V}$. The *set of states* is denoted $\mathcal{S}$. The meaning of an HP α is given by a reachability relation $[\![\alpha]\!] \subseteq \mathcal{S} \times \mathcal{S}$ on states. So $(\omega, \nu) \in [\![\alpha]\!]$ means that final state ν is reachable from initial state ω by running HP α. From any initial state ω, there might be many states ν that are reachable because the HP α may involve nondeterministic choices, repetitions, or differential equations, so there may be many different states ν for which $(\omega, \nu) \in [\![\alpha]\!]$. From other initial states ω, there might be no reachable states ν at all for which $(\omega, \nu) \in [\![\alpha]\!]$. So $[\![\alpha]\!]$ is a proper relation, not a function.

HPs have a compositional semantics [17–19]. Recall from Chap. 2 that the value of term e in state ω is denoted by $\omega[\![e]\!]$. Further, $\omega \in [\![Q]\!]$ denotes that first-order formula Q is true in state ω, where $[\![Q]\!] \subseteq \mathcal{S}$ is the set of all states in which formula Q is *true*. The semantics of an HP α is then defined by its reachability relation $[\![\alpha]\!] \subseteq \mathcal{S} \times \mathcal{S}$. The notation α^* for loops comes from the notation ρ^* for the re-

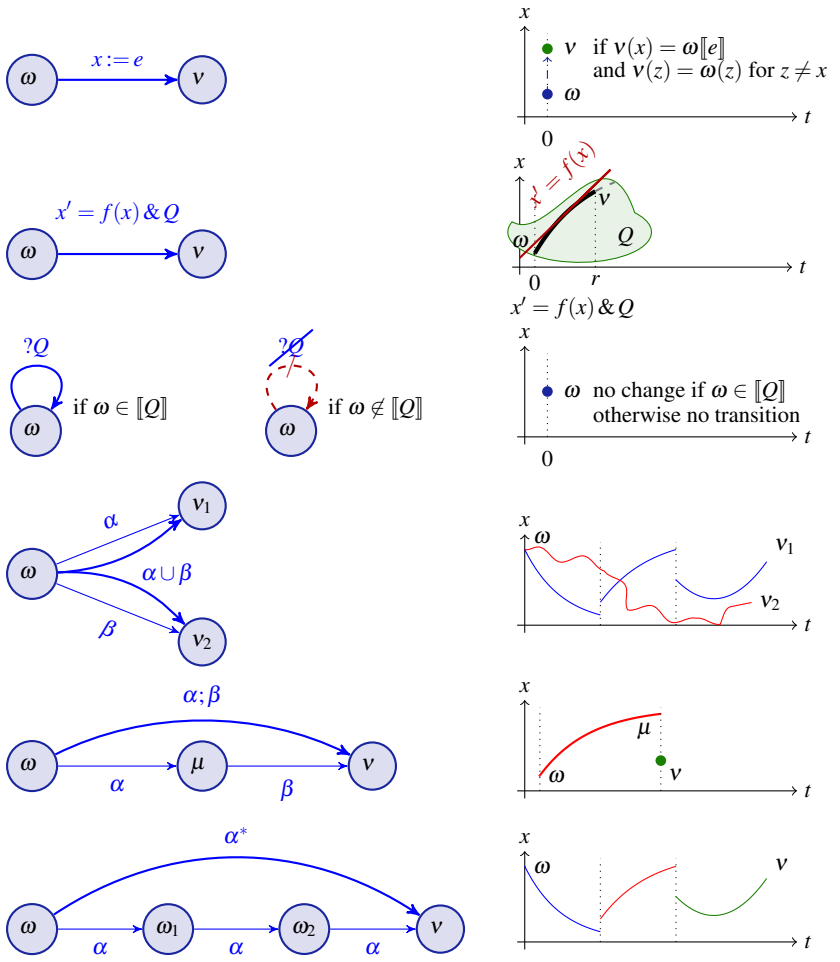


Fig. 3.4 Transition semantics **(left)** and example dynamics **(right)** of hybrid programs

flexive, transitive closure of a relation ρ. Graphical illustrations of the transition semantics of hybrid programs defined below and possible example dynamics are depicted in Fig. 3.4. The left of Fig. 3.4 illustrates the generic shape of the transition structure $[\![\alpha]\!]$ for transitions along various cases of hybrid programs α from state ω to state ν. The right of Fig. 3.4 shows examples of how the value of a variable x may evolve over time t when following the dynamics of the respective hybrid program α.

Definition 3.2 (Transition semantics of HPs). Each HP α is interpreted semantically as a binary reachability relation $[\![\alpha]\!] \subseteq \mathscr{S} \times \mathscr{S}$ over states, defined inductively by:

1. $[\![x := e]\!] = \{(\omega, \nu) : \nu = \omega \text{ except that } \nu[\![x]\!] = \omega[\![e]\!]\}$
   That is, final state ν differs from initial state ω only in its interpretation of the variable x, which ν changes to the value that the right-hand side term e has in the initial state ω.
2. $[\![?Q]\!] = \{(\omega, \omega) : \omega \in [\![Q]\!]\}$
   That is, the final state ω is the same as the initial state ω (no change) but there is such a transition only if test formula Q holds in ω, otherwise no transition is possible at all and the system is stuck because of a failed test.
3. $[\![x' = f(x) \,\&\, Q]\!] = \{(\omega, \nu) : \varphi(0) = \omega \text{ except at } x' \text{ and } \varphi(r) = \nu \text{ for a solution } \varphi : [0, r] \to \mathscr{S} \text{ of any duration } r \text{ satisfying } \varphi \models x' = f(x) \wedge Q\}$
   That is, the final state $\varphi(r)$ is connected to the initial state $\varphi(0)$ by a continuous function of some duration $r \geq 0$ that solves the differential equation and satisfies Q at all times; see Definition 3.3.
4. $[\![\alpha \cup \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!]$
   That is, $\alpha \cup \beta$ can do exactly any of the transitions that α can do as well as any of the transitions that β is capable of. Every run of $\alpha \cup \beta$ has to choose whether it follows α or β, but cannot follow both at once.
5. $[\![\alpha; \beta]\!] = [\![\alpha]\!] \circ [\![\beta]\!] = \{(\omega, \nu) : (\omega, \mu) \in [\![\alpha]\!], (\mu, \nu) \in [\![\beta]\!]\}$
   That is, the meaning of $\alpha; \beta$ is the composition[a] $[\![\alpha]\!] \circ [\![\beta]\!]$ of relation $[\![\beta]\!]$ after $[\![\alpha]\!]$. Thus, $\alpha; \beta$ can do any transitions that go through any intermediate state μ to which α can make a transition from the initial state ω and from which β can make a transition to the final state ν.
6. $[\![\alpha^*]\!] = [\![\alpha]\!]^* = \bigcup_{n \in \mathbb{N}} [\![\alpha^n]\!]$ with $\alpha^{n+1} \equiv \alpha^n; \alpha$ and $\alpha^0 \equiv ?true$.
   That is, α^* can repeat α any number of times, i.e., for any $n \in \mathbb{N}$, α^* can act like the n-fold sequential composition $\alpha^n \equiv \underbrace{\alpha; \alpha; \alpha; \ldots; \alpha}_{n \text{ times}}$ would.

[a] The notational convention for composition of relations is flipped compared to the composition of functions. For functions f and g, the function $f \circ g$ is the composition f after g that maps x to $f(g(x))$. For relations R and T, the relation $R \circ T$ is the composition of T after R, so first follow relation R to an intermediate state and then follow relation T to the final state.

To keep things simple, this definition uses simplifying abbreviations for differential equations. Chapter 2 provides full detail, including the definition for differential equation systems. The semantics of loops can also be rephrased equivalently as:

$$[\![\alpha^*]\!] = \bigcup_{n \in \mathbb{N}} \{(\omega_0, \omega_n) : \omega_0, \ldots, \omega_n \text{ are states such that } (\omega_i, \omega_{i+1}) \in [\![\alpha]\!] \text{ for all } i < n\}$$

For later reference, we repeat the definition of the semantics of differential equations separately:

Definition 3.3 (Transition semantics of ODEs).

$$[\![x' = f(x)\,\&\,Q]\!] = \big\{(\omega,\nu) : \varphi(0) = \omega \text{ except at } x' \text{ and } \varphi(r) = \nu \text{ for a solution } \varphi{:}[0,r] \to \mathcal{S} \text{ of any duration } r \text{ satisfying } \varphi \models x' = f(x) \wedge Q\big\}$$

where $\varphi \models x' = f(x) \wedge Q$, iff for all times $0 \leq z \leq r$: $\varphi(z) \in [\![x' = f(x) \wedge Q]\!]$ with $\varphi(z)(x') \overset{\text{def}}{=} \frac{\mathsf{d}\varphi(t)(x)}{\mathsf{d}t}(z)$ and $\varphi(z) = \varphi(0)$ except at x, x'.

The condition that $\varphi(0) = \omega$ *except at* x' is explicit about the fact that the initial state ω and the first state $\varphi(0)$ of the continuous evolution have to be identical (except for the value of x', for which Definition 3.3 only provides a value along φ). Part I of this book does not track the values of x' except during continuous evolutions. But that will change in Part II, for which Definition 3.3 is already prepared appropriately.

Observe that $?Q$ cannot run from an initial state ω with $\omega \notin [\![Q]\!]$, in particular $[\![?\mathit{false}]\!] = \emptyset$. Likewise, $x' = f(x)\,\&\,Q$ cannot run from an initial state ω with $\omega \notin [\![Q]\!]$, because no solution of any duration, not even duration 0, starting in ω will always stay in the evolution domain Q if it already starts outside Q. A nondeterministic choice $\alpha \cup \beta$ cannot run from an initial state from which neither α nor β can run. Similarly, $\alpha;\beta$ cannot run from an initial state from which α cannot run, nor from an initial state from which all final states after α make it impossible for β to run. Assignments and repetitions can always run, e.g., by repeating 0 times.

Example 3.1. When α denotes the HP in (3.8) on p. 74, its semantics $[\![\alpha]\!]$ is a relation on states connecting the initial to the final state along the differential equation with two control decisions according to the nondeterministic choice, one at the beginning and one after following the first differential equation. How long is that, exactly? Well, that's nondeterministic, because the semantics of differential equations is such that any final state after any permitted duration is reachable from a given initial state. So the duration for the first differential equation in (3.8) could be one second or two or 424 or half a second or zero or π or any other nonnegative real number. This would be very different for an HP whose differential equation has an evolution domain constraint, because that limits how long a continuous evolution can take. The exact duration is still nondeterministic, but it cannot ever evolve outside its evolution domain.

By plugging one transition structure pattern into another, Fig. 3.4 illustrates the generic shape of transition structures for more complex HPs. For example, Fig. 3.5 illustrates the transition structure of $(\alpha;\beta)^*$ and Fig. 3.6 illustrates $(\alpha \cup \beta)^*$. This plugging in is directly analogous to how the semantics of bigger programs is defined by recursively following their semantics in Definition 3.2 based on their respective top-level operator.

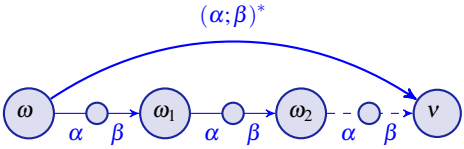


Fig. 3.5 Nested transition semantics pattern for $(\alpha;\beta)^*$

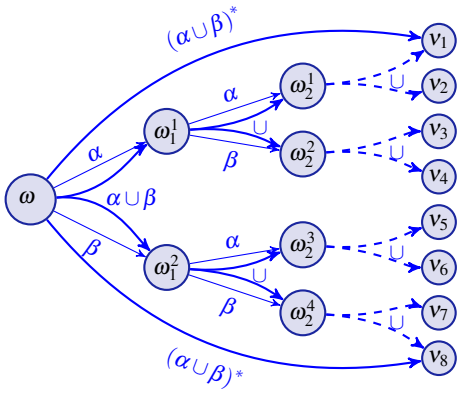


Fig. 3.6 Nested transition semantics pattern for $(\alpha\cup\beta)^*$

Expedition 3.2 (HP semantics $[\![\cdot]\!] : \text{HP} \to \wp(\mathcal{S}\times\mathcal{S})$)

The semantics of an HP α from Definition 3.2 directly defines the transition relation $[\![\alpha]\!] \subseteq \mathcal{S}\times\mathcal{S}$ of initial and final states inductively, for each HP α:

$$
\begin{array}{ll}
[\![x:=e]\!] & = \{(\omega,\nu) \;:\; \nu = \omega \text{ except that } \nu[\![x]\!] = \omega[\![e]\!]\} \\
[\![?Q]\!] & = \{(\omega,\omega) \;:\; \omega \in [\![Q]\!]\} \\
[\![x' = f(x)\,\&\,Q]\!] & = \{(\omega,\nu) : \varphi(0) = \omega \text{ except at } x' \text{ and } \varphi(r) = \nu \text{ for a solution} \\
 & \quad \varphi{:}[0,r] \to \mathcal{S} \text{ of any duration } r \text{ satisfying } \varphi \models x' = f(x) \wedge Q\} \\
[\![\alpha\cup\beta]\!] & = [\![\alpha]\!] \cup [\![\beta]\!] \\
[\![\alpha;\beta]\!] & = [\![\alpha]\!] \circ [\![\beta]\!] = \{(\omega,\nu) : (\omega,\mu) \in [\![\alpha]\!], (\mu,\nu) \in [\![\beta]\!]\} \\
[\![\alpha^*]\!] & = [\![\alpha]\!]^* = \bigcup_{n\in\mathbb{N}} [\![\alpha^n]\!] \text{ with } \alpha^{n+1} \equiv \alpha^n;\alpha \text{ and } \alpha^0 \equiv ?\mathit{true}
\end{array}
$$

When HP is the set of hybrid programs, the semantic brackets define an operator $[\![\cdot]\!] : \text{HP} \to \wp(\mathcal{S}\times\mathcal{S})$ that defines the meaning $[\![P]\!]$ for each hybrid program $\alpha \in \text{HP}$, which, in turn, defines the transition relation $[\![\alpha]\!] \subseteq \mathcal{S}\times\mathcal{S}$ where $(\omega,\nu) \in [\![\alpha]\!]$ indicates that final state ν is reachable from initial state ω when running HP α. The *powerset* $\wp(\mathcal{S}\times\mathcal{S})$ is the set of all subsets of the Cartesian product $\mathcal{S}\times\mathcal{S}$ of the set of states $\mathcal{S}$ with itself. The powerset $\wp(\mathcal{S}\times\mathcal{S})$, thus, is the set of binary relations on $\mathcal{S}$.

3.4 Hybrid Program Design

This section discusses some early lessons on good and bad modeling choices in hybrid systems. As our understanding of the subject matter advances throughout this textbook, we will find additional insights into tradeoffs and caveats. The aspects that can easily be understood on a pure modeling level will be discussed now.

3.4.1 To Brake, or Not to Brake, That Is the Question

As a canonical example for a system that has to make a choice, consider a ground robot at position x moving with velocity v and acceleration a along a straight line. This results in the differential equation $x' = v, v' = a$. When driving along a straight line, the ground robot has the control decision to either set the acceleration to a positive value $A > 0$ by the discrete assignment $a := A$ or set it to a negative value $-b < 0$ by the discrete assignment $a := -b$. The control question is when to brake and when not to brake (so accelerate since, for simplicity, this example does not allow coasting with a constant velocity). Let's call the condition under which acceleration can be chosen Q_A and the condition under which braking can be chosen Q_b:

$$\big((?Q_A; a := A \cup ?Q_b; a := -b);\ \{x' = v, v' = a\}\big)^* \tag{3.11}$$

What concrete formulas to best use for the tests $?Q_A$ and $?Q_b$ depends on the control objectives and is often quite nontrivial to determine. If the system can stay in its continuous evolution for an unbounded amount of time, then it is virtually never safe to accelerate, so Q_A will need to be *false*. Consequently, we assume that ε is the reaction time, so the maximal amount of time that a continuous evolution can take before it stops and gives the discrete controller a chance to react to the new situation again. The HP, thus, includes a clock t measuring the progress of time along $t' = 1$, which is reset with $t := 0$ before the differential equation and bounded by the evolution domain $t \leq \varepsilon$. Finally, if the controller chooses a negative acceleration with $a := -b$, then the differential equation system $x' = v, v' = a$ will ultimately move backwards when $v < 0$. In order to model that braking will not make the robot move backwards, we add $v \geq 0$ to the evolution domain constraint.

Taking these thoughts about braking and acceleration into account, we refine HP (3.11) with a clock and a lower velocity bound 0:

$$\big((?Q_A; a := A \cup ?Q_b; a := -b);\ t := 0;\ \{x' = v, v' = a, t' = 1 \,\&\, v \geq 0 \wedge t \leq \varepsilon\}\big)^* \tag{3.12}$$

The structure of the transition semantics for HP (3.12) according to the patterns in Fig. 3.4 is shown in Fig. 3.7. Any path through Fig. 3.7 that passes all tests along the way corresponds to an execution of HP (3.12), and vice versa. In particular, if a test fails, such as $?Q_A$, then only the other nondeterministic choices are available, which

is why it is important that at least one choice is always allowed. If both tests $?Q_A$ and $?Q_b$ pass in the current state, then both nondeterministic choices are available.

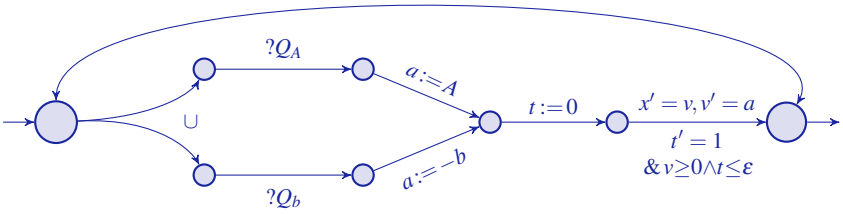


Fig. 3.7 Transition structure of the acceleration/braking example (3.12)

For verification purposes, it is often a good idea to design conditions Q_A and Q_b that overlap. Executing a controller may best be done in a deterministic way, which would argue for a disjoint design of Q_A and Q_b. But verification models benefit from nondeterminism, because if highly nondeterministic models have no unsafe behaviors then all more specific refinements have no unsafe behaviors either [12]. In particular, the controller is only guaranteed to remain responsive if there is always at least one option that can be chosen, which is obvious when choosing the trivial test $?true$ for $?Q_b$, which always allows the control to choose braking. Of course, when the control objective is to avoid collision with a moving obstacle in front of our robot, then it is always safe to brake, but only sometimes safe to accelerate. So, Q_A has to be some condition that ensures sufficient distance to the moving obstacle, which depends on how good the brakes are and several other parameters as well (Exercise 3.7).

3.4.2 A Matter of Choice

Let us change the HP from (3.8) and consider the following modification instead:

$$\begin{aligned} &?v<4; a:=a+1; \\ &\{x'=v, v'=a\}; \\ &?v<4; a:=a+1; \\ &\{x'=v, v'=a\} \end{aligned} \tag{3.13}$$

Then some behavior that was possible in (3.8) is no longer possible for (3.13). Let β denote the HP in (3.13), then the semantics $[\![\beta]\!]$ of β now only includes relations between initial and final states which can be reached by acceleration choices (because there are no braking choices in β). Note that the duration of the first differential equation in (3.13) is suddenly bounded, because if x keeps on accelerating for too

long during the first differential equation, the intermediate state reached then will violate the test $?v < 4$, which, according to the semantics of tests, will fail and be discarded.

That is what makes (3.13) a bad model, because it truncates and discards behavior that the real system still possesses. Even if the controller in the third line of (3.13) is not prepared to handle the situation where the test $?v < 4$ fails, it might fail in reality. In that case, the controller in (3.13) simply runs out of choices. A more realistic and permissive controller, thus, also handles the case if that test fails, at which point we are back at (3.8).

Similarly, $Q_b \overset{\text{def}}{\equiv} \mathit{false}$ is a bad controller design for (3.12), because it categorically disallows braking and, unrealistically, assumes $?Q_A$ to hold all the time.

Note 18 (Controllers cannot discard cases) While subsequent chapters discuss cases where hybrid programs use tests $?Q$ in crucial ways to discard non-permitted behaviors of the environment, great care needs to be exercised that controllers also handle the remaining cases. A bad controller

$$?(v < 4);\ \alpha$$

only handles the case where $v < 4$ and ignores all other circumstances, which renders the controller incapable of reacting and, thus, unsafe when $v \geq 4$. A better controller design always considers the case when a condition is not satisfied and handles it appropriately as well:

$$(?(v < 4); \alpha) \cup (?(v \geq 4); \ldots)$$

Liveness proofs can tell the two cases of controllers apart, but appropriate design principles of being prepared for both outcomes of each test go a long way in improving the controllers.
Similarly bad controller designs result from careless evolution domains:

$$a := -b;\ \{x' = v, v' = a \,\&\, v > 4\}$$

The differential equations in this controller silently assume the velocity will always stay above 4, which is clearly not always the case when braking. Accidental divisions by zero are another source of trouble in CPS controllers.

3.5 Summary

This chapter introduced hybrid programs as a model of cyber-physical systems, summarized in Table 3.1. Hybrid programs combine differential equations with conventional program constructs and discrete assignments. The programming language of hybrid programs embraces nondeterminism as a first-class citizen and features

differential equations that can be combined to form hybrid systems using the compositional operators of hybrid programs.

Table 3.1 Statements and effects of hybrid programs (HPs)

HP Notation	Operation	Effect
$x := e$	discrete assignment	assigns current value of term e to variable x
$x' = f(x) \,\&\, Q$	continuous evolution	follow differential equation $x' = f(x)$ within evolution domain Q for any duration
$?Q$	state test / check	test first-order formula Q at current state
$\alpha; \beta$	seq. composition	HP β starts after HP α finishes
$\alpha \cup \beta$	nondet. choice	choice between alternatives HP α and HP β
α^*	nondet. repetition	repeats HP α any $n \in \mathbb{N}$ times

Hybrid programs take advantage of the structuring principles of programming languages and emphasize the composition operators ; and $\cup$ and *, which combine smaller hybrid programs to make bigger hybrid programs and have a simple compositional semantics. Hybrid programs resemble regular expressions, except that they start from discrete assignments, tests, and differential equations as a basis instead of individual letters of a formal language. Regular expressions allow a language-theoretic study of formal languages and have an automata-theoretic counterpart called finite automata [10]. Similarly, hybrid programs have *hybrid automata* [2, 8] as an automata-theoretic counterpart, which are explored in Exercise 3.18.

3.6 Appendix: Modeling the Motion of a Robot Around a Bend

This appendix develops a hybrid program model describing how a robot can drive along a sequence of lines and circular curve segments in the two-dimensional plane. This dynamics is called Dubins car dynamics [5], because it also describes the high-level motion of a car in the plane or of an aircraft remaining at the same altitude (Fig. 3.8).

Suppose there is a robot at a point with coordinates (x, y) that is facing in direction (v, w). The robot moves into the direction (v, w), whose norm $\sqrt{v^2 + w^2}$, thus, simultaneously determines the constant linear speed: how quickly the robot is moving on the ground. Suppose that direction (v, w) is simultaneously rotating with an angular velocity ω as in Example 2.7 (Fig. 3.9). The differential equations describing the motion of this robot are

$$x' = v, y' = w, v' = \omega w, w' = -\omega v$$

The time-derivative of the x coordinate is the component v of the direction and the time-derivative of the y coordinate is the component w of the direction. The angular velocity ω determines how fast the direction (v, w) rotates. Bigger magnitudes of

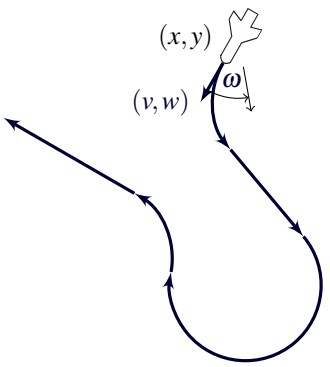


Fig. 3.8 Illustration of a Dubins path consisting of a sequence of lines and maximally curved circle segments

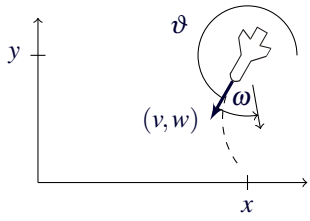


Fig. 3.9 Illustration of the Dubins dynamics of a point (x,y) moving in direction (v,w) along a dashed curve with angular velocity ω

ω give faster rotations of (v,w) so tighter curves for the position (x,y). Positive ω make the robot drive a right curve (when indeed plotting x as the x-axis and y as the y-axis as in Fig. 3.9). The robot will follow a straight line when $\omega = 0$, because the direction (v,w) then does not change.

Now if a robot can steer, its controller can change the angular velocity to make a left curve ($\omega < 0$), a right curve ($\omega > 0$), or drive straight ahead ($\omega = 0$). One might very well imagine a robot controller that additionally chooses sharp turns (ω with big magnitude) or gentle turns (ω with small magnitude), but let's not consider this yet. After all, Lester Dubins proved that the shortest curve connecting two points with such a dynamics consists of a sequence of line segments and maximally sharp turns [5].[5] If we simply assume that 1 and -1 are the extreme angular velocities, the hybrid program for the ground robot steering and moving in the plane is

$$\big((\omega := -1 \cup \omega := 1 \cup \omega := 0);\ \{x' = v, y' = w, v' = \omega w, w' = -\omega v\}\big)^* \qquad (3.14)$$

Repeatedly, in a loop, this HP allows a choice of extreme left curves ($\omega := -1$), extreme right curves ($\omega := 1$), or motion in a straight line ($\omega := 0$). After this discrete controller, the robot follows the continuous motion described by the differential equations for a nondeterministic period of time.

If the safety goal of the robot is never to collide with any obstacles, then HP (3.14) cannot possibly be safe, because it allows arbitrary left and right curves and

[5] If you build a self-driving car following such a path with straight lines and maximal curvature, don't be surprised if no passenger stays for a second ride. But robots are less fixated on comfort.

straight-line motion for any arbitrary amounts of time under any arbitrary conditions. HP (3.14) would even allow the controller to choose a left curve if that is the only direction that will immediately make the robot collide with an obstacle.

Consequently, each of the three control actions in HP (3.14) is only acceptable under certain conditions. After solving Exercise 3.9 you will have found logical formulas Q_{-1}, Q_1, Q_0 such that Q_ω indicates when it is safe to drive along the curve corresponding to angular velocity ω, which will transform the unsafe HP (3.14) into the following HP with a more constrained controller:

$$((?Q_{-1}; \omega := -1 \cup ?Q_1; \omega := 1 \cup ?Q_0; \omega := 0); \{x' = v, y' = w, v' = \omega w, w' = -\omega v\})^*$$

It is perfectly fine if multiple of the conditions Q_{-1}, Q_1, Q_0 are true in the same state, because that gives the controller a number of different control options to choose from. For example, there might be many states in which both a left curve and driving straight are safe. Of course, it would not be useful at all if there is a state in which all conditions Q_{-1}, Q_1, Q_0 are false, because the controller runs out of control choices and is then completely stuck, which is not very safe either. Grotesquely useless would be a controller that chose an impossible condition like $1 < 0$ for all three formulas Q_{-1}, Q_1, Q_0, because that robot can then never move anywhere, which is incredibly boring even for the most polite and patient robots. Yet, a robot that is initially stopped and never even begins to move at least does not bump into any walls. What is much worse is a controller that happily begins to drive and then fails to offer any acceptable control choices. That is why it is important that the disjunction $Q_{-1} \vee Q_1 \vee Q_0$ is true in every state, because the robot then always has at least one permitted choice.

Well, can this disjunction be true in every state? Since the conditions Q_ω are supposed to guarantee that the robot will never collide with an obstacle when following the trajectory with angular velocity ω, none of them can be true in a state where the robot has already collided to begin with. It is up to the robot controller to ensure that such a collision state is never reached. Hence, the disjunction $Q_{-1} \vee Q_1 \vee Q_0$ should be true in every collision-free state. How to design the Q_ω is your challenge for Exercise 3.9.

Exercises

3.1. The semantics of an HP α is its reachability relation $[\![\alpha]\!]$. For example,

$$[\![x := 2 \cdot x; x := x + 1]\!] = \{(\omega, \nu) : \nu(x) = 2 \cdot \omega(x) + 1 \text{ and } \nu(z) = \omega(z) \text{ for all } z \neq x\}$$

Describe the reachability relation of the following HPs in similarly explicit ways:

1. $x := x + 1;\ x := 2 \cdot x$
2. $x := 1 \cup x := -1$
3. $x := 1 \cup ?(x \leq 0)$

4. $x := 1;\ ?(x \leq 0)$
5. $?(x \leq 0)$
6. $x := 1 \cup x' = 1$
7. $x := 1;\ x' = 1$
8. $x := 1;\ \{x' = 1 \,\&\, x \leq 1\}$
9. $x := 1;\ \{x' = 1 \,\&\, x \leq 0\}$
10. $v := 1;\ x' = v$
11. $v := 1;\ \{x' = v\}^*$
12. $\{x' = v, v' = a \,\&\, x \geq 0\}$

3.2. The semantics of hybrid programs (Definition 3.2) requires evolution domain constraints Q to hold always throughout a continuous evolution. What exactly happens if the system starts in a state where Q does not hold to begin with?

3.3 (If-then-else). Sect. 3.2.3 considered if-then-else statements for hybrid programs. But they no longer showed up in the grammar of hybrid programs. Is this a mistake? Can you define $\mathsf{if}(P)\,\alpha\,\mathsf{else}\,\beta$ from the operators that HPs do provide?

3.4 (If-then-else). Suppose we add the if-then-else-statement $\mathsf{if}(P)\,\alpha\,\mathsf{else}\,\beta$ to the syntax of HPs. Define a semantics $[\![\mathsf{if}(P)\,\alpha\,\mathsf{else}\,\beta]\!]$ for if-then-else statements and explain how it relates to Exercise 3.3.

3.5 (Switch-case). Define a switch statement that runs the statement α_i if formula P_i is true, and chooses *nondeterministically* if multiple conditions are true:

$$
\begin{array}{l}
\mathsf{switch}\ (\\
\quad \mathsf{case}\ P_1 :\ \alpha_1 \\
\quad \mathsf{case}\ P_2 :\ \alpha_2 \\
\qquad \vdots \\
\quad \mathsf{case}\ P_n :\ \alpha_n \\
)
\end{array}
$$

What would need to be changed to make sure only the statement α_i with the first true condition P_i executes?

3.6 (While). Suppose we add the while loop $\mathsf{while}(P)\,\alpha$ to the syntax of HPs. As usual, $\mathsf{while}(P)\,\alpha$ is supposed to run α if P holds, and, whenever α finishes, repeat again if P holds. Define a semantics $[\![\mathsf{while}(P)\,\alpha]\!]$ for while loops. Can you define a program that is equivalent to $\mathsf{while}(P)\,\alpha$ from the original syntax of HPs?

3.7 (To brake, or not to brake, that is the question). Besides the positions of the robot and the position of the obstacle in front of it, what other parameters does the acceleration condition in (3.12) depend on if it is supposed to ensure that the robot does not collide with any obstacles on the straight line that it is driving along? Can you determine a corresponding formula Q_A that would guarantee safety?

3.8 (Two cars). Develop a model of the motion of two cars along a straight line, each of which has its own position, velocity, and acceleration. Develop a controller model that allows the leader car to accelerate or brake freely while limiting the choices of the follower car such that it will never collide with the car in front of it.

3.9 (Runaround robot). You are in control of a robot moving with constant ground speed in the two-dimensional plane, as in Sect. 3.6. It can follow a left curve ($\omega := -1$), a right curve ($\omega := 1$), or go straight ($\omega := 0$). Your job is to find logical formulas Q_{-1}, Q_1, Q_0 such that Q_ω indicates when it is safe to drive along the curve corresponding to angular velocity ω:

$$((?Q_{-1}; \omega := -1 \cup ?Q_1; \omega := 1 \cup ?Q_0; \omega := 0); \{x' = v, y' = w, v' = \omega w, w' = -\omega v\})^*$$

For the purpose of this exercise, fix one point (o_x, o_y) as an obstacle and consider this HP safe if it can never reach that obstacle. Does your HP always have at least one choice remaining or can it get stuck such that no choice is permitted? Having succeeded with these challenges, can you generalize your robot model and safety constraints to one where the robot can accelerate to speed up or brake to slow down?

3.10 (Other programming languages). Consider your favorite programming language and discuss in what ways it introduces discrete change and discrete dynamics. Can it model all behavior that hybrid programs can describe? Can your programming language model all behavior that hybrid programs without differential equations can describe? How about the other way around? And what would you need to add to your programming language to cover all of hybrid systems? How would you best do that?

3.11 (Choice vs. sequence). Can you find a discrete controller *ctrl* and a continuous program *plant* such that the following two HPs have different behaviors?

$$(ctrl; plant)^* \quad \text{versus} \quad (ctrl \cup plant)^*$$

3.12 (Nondeterministic assignments). Suppose we add a new statement $x := *$ for nondeterministic assignment to the syntax of HPs. The *nondeterministic assignment* $x := *$ assigns an *arbitrary* real number to the variable x. Define a semantics $[\![x := *]\!]$ for the $x := *$ statement.

3.13 (Nondeterministic choices from nondeterminism and if-then-else). Exercise 3.3 explored that if-then-else can be defined from nondeterministic choices. Once we add nondeterministic assignments, however, we could have defined it conversely. Using an auxiliary variable z, show that $\alpha \cup \beta$ has the same behavior as:

$$z := *; \ \mathsf{if}(z > 0)\, \alpha\, \mathsf{else}\, \beta$$

3.14 (Nondeterministic repetitions from nondeterminism and while). The Exercise 3.6 explored that while loops can be defined from nondeterministic repetitions. Once we add nondeterministic assignments, however, we could have defined it conversely. Using an auxiliary variable z, show that α^* has the same behavior as:

$$z := *;\ \mathsf{while}(z > 0)\,(z := *;\ \alpha)$$

3.15 (Set-valued semantics). The semantics of hybrid programs (Definition 3.2) is defined as a transition relation $[\![\alpha]\!] \subseteq \mathscr{S} \times \mathscr{S}$ on states. Define an equivalent semantics using functions $R(\alpha) : \mathscr{S} \to 2^{\mathscr{S}}$ from the initial state to the *set* of all final states, where $2^{\mathscr{S}}$ denotes the powerset of $\mathscr{S}$, i.e., the set of all subsets of $\mathscr{S}$. Define this set-valued semantics $R(\alpha)$ without referring to the transition relation semantics $[\![\alpha]\!]$, and prove that it is equivalent, i.e.,

$$\nu \in R(\alpha)(\omega) \quad \text{iff} \quad (\omega, \nu) \in [\![\alpha]\!]$$

Likewise, define an equivalent semantics based on functions $\varsigma(\alpha) : 2^{\mathscr{S}} \to 2^{\mathscr{S}}$ from the set of possible final states to the set of initial states that can end in the given set of final states. Prove that it is equivalent, i.e., for all sets of states $X \subseteq \mathscr{S}$

$$\omega \in \varsigma(\alpha)(X) \quad \text{iff} \quad \text{there is a state } \nu \in X \text{ such that } (\omega, \nu) \in [\![\alpha]\!]$$

3.16 (Switched systems). Hybrid programs come in different classes; see Table 3.2. A continuous program is an HP that only consists of one continuous evolution of the form $x' = f(x)\,\&\,Q$. A discrete system corresponds to an HP that has no differential equations. A switched continuous system corresponds to an HP that has no assignments, because it does not have any instant changes of state variables but merely switches mode (possibly after some tests) from one continuous mode into another.

Table 3.2 Classification of hybrid programs and correspondence to dynamical systems

HP class	Dynamical systems class
only ODE	continuous dynamical systems
no ODE	discrete dynamical systems
no assignment	switched continuous dynamical systems
general HP	hybrid dynamical systems

Consider an HP in which the variables are partitioned into state variables (x, v), sensor variables (m), and controller variables (a):

$$\Big(\big((?x < m - 5;\, a := A) \cup a := -b\big);\ \{x' = v, v' = a\}\Big)^*$$

Transform this HP into a switched program that has the same behavior on the observable state and sensor variables but is a switched system, so does not contain any assignments. The behavior of controller variables is considered irrelevant for the purpose of this transformation as long as the behavior of the other state variables x, v is unchanged.

3.17 (Program interpreter).** In a programming language of your choosing, fix a recursive data structure for hybrid programs from Definition 3.1 and fix some finite

representation for states where all variables have rational values instead of reals. Write a program interpreter as a computer program that, given an initial state ω and a program α, successively enumerates possible final states ν that can be reached by α from ω, that is $(\omega, \nu) \in [\![\alpha]\!]$, by implementing Definition 3.2. Resolve nondeterministic choices in the transition either by user input or by randomization. What makes the differential equation case particularly challenging?

3.18 (Hybrid automata).** The purpose of this exercise is to explore hybrid automata [2, 8], which are an automata-theoretic model for hybrid systems. Instead of the compositional language operators of hybrid programs, hybrid automata emphasize different continuous modes with discrete transitions between them. The system follows a continuous evolution while the automaton is in a node (called a location). Discrete jumps happen when following an edge of the automaton from one location to another. Hybrid automata augment finite automata with the specification of a differential equation and evolution domain in each location and with a description of the discrete transition (called a reset) for each edge in addition to a condition (called a guard) specifying when that edge can be taken.

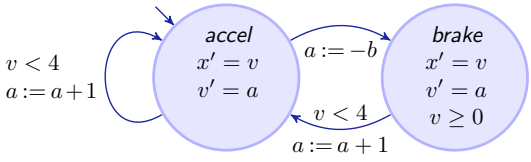


Fig. 3.10 Hybrid automaton for a car that can accelerate or brake

Figure 3.10 shows a hybrid automaton with two locations and three transitions between them, which starts in location *accel* as indicated by the initial arrow. While in location *accel*, the system follows the differential equation $x' = v, v' = a$. While in *brake* it follows $x' = v, v' = a \,\&\, v \geq 0$. When the automaton is in location *brake* and guard condition $v < 4$ is satisfied, then it can transition along the edge to location *accel*, which will change a by executing reset $a := a + 1$. When the automaton is in *accel*, then it can transition along the edge to *brake*, which will execute $a := -b$. This transition is always possible, because that edge has no guard. If $v < 4$, then the automaton can also transition from *accel* back to *accel*, which will execute $a := a + 1$.

1. Modify the hybrid automaton in Fig. 3.10 so that it directly corresponds to the hybrid program (3.10).
2. Draw a hybrid automaton for the hybrid program in Sect. 3.6.
3. Define the syntax of hybrid automata consisting of a (finite) set X of state variables and a (finite) set *Loc* of locations interconnected by a (finite) set *Edg* of edges where each location $\ell \in Loc$ has a differential equation $Flow(\ell)$ and evolution domain constraint $Inv(\ell)$ and where each edge $e \in Edg$ has a guard condition $Guard(e)$ and a reset $Reset(e)$ list of assignments. Also define the initial conditions by a formula $Init(\ell)$ per location $\ell \in Loc$ specifying in which region (if any) the hybrid automaton is allowed to start initially.

4. A state of a hybrid automaton is a pair (ℓ, ω) consisting of a location $\ell \in Loc$ and an assignment $\omega : X \to \mathbb{R}$ of real numbers to the variables X. Define the semantics of hybrid automata by defining which states (k, ν) are reachable from initial state (ℓ, ω) by running the hybrid automaton.
5. Every finite automaton can be implemented in imperative programming languages with the help of a variable q storing the present location of the automaton that is updated to reflect the transition from one location to another. By analogy, show how every hybrid automaton can be implemented as a hybrid program with one additional variable q for the location. If we assume that the locations Loc are a set of distinct real numbers, the fact that the hybrid automaton is in location ℓ will correspond to the location variable q having value ℓ in the hybrid program.
6. Explain how the states (ℓ, ω) reachable by a hybrid automaton correspond to the states ω_q^ℓ reachable by the corresponding hybrid program in which location variable q has value ℓ.

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