# Chapter 11
# Neutrino Oscillations and Neutrino Mass

*Today I have done something which you never should do in theoretical physics. I have explained something which is not understood by something which can never be observed!*

Wolfgang Pauli

The existence of neutrinos was proposed by Wolfgang Pauli in 1930, in order to explain the puzzling continuous energy spectrum of electrons in $\beta$-decay. If the neutron would decay only into a proton and an electron, the energy of the latter would be constant. In case of a 3-body decay, however, the third particle would carry away a certain amount of energy, and thereby generate a continuous energy spectrum for the electron. As we have seen in the last chapter, the interaction of neutrinos with other elementary particles is extremely weak, see (10.9). Therefore it was thought for a long time that the direct experimental verification of the existence of neutrinos was impossible. Only in 1956 Cowan and Reines finally succeeded in detecting electron antineutrinos originating from a nuclear reactor [8].

In the otherwise enormously successful standard model of particle physics (see Chap. 13), neutrinos are massless. However, in 1998 it was shown beyond doubt that neutrinos possess a non-vanishing rest mass. Till this date, this represents the only directly testable and in laboratories accessible physics beyond the standard model. This fact alone renders neutrinos highly interesting. In addition, neutrinos show some remarkable properties. For instance, they can transform from one flavour into another one, with a transition probability that changes periodically. These neutrino oscillations are a quantum mechanical interference effect on macroscopic distances, whose basic features and important experiments we will discuss in what follows. The precise value of the neutrino mass is a currently unresolved problem, we will discuss the most important approaches to answer this important question. Finally, neutrinos are the only known electrically neutral fermions and, therefore, have the option to be identical with their antiparticles. This would lead to processes that violate the conservation of total lepton number.



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## 11.1 Lepton Families

**The leptonic mixing matrix** We have outlined in Sect. 10.1 that leptons can be written in terms of three family doublets:

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}.$$

The flavour states $|\,\nu_e\,\rangle$, $|\,\nu_\mu\,\rangle$ and $|\,\nu_\tau\,\rangle$ are not identical to the states $|\,\nu_1\,\rangle$, $|\,\nu_2\,\rangle$ and $|\,\nu_3\,\rangle$, which possess a well-defined mass. However, in analogy to the quarks, we can write the flavour states as orthogonal linear combinations of the mass states:

$$\begin{pmatrix} |\,\nu_e\,\rangle \\ |\,\nu_\mu\,\rangle \\ |\,\nu_\tau\,\rangle \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \cdot \begin{pmatrix} |\,\nu_1\,\rangle \\ |\,\nu_2\,\rangle \\ |\,\nu_3\,\rangle \end{pmatrix}. \tag{11.1}$$

The $3 \times 3$ matrix $U$ is analogous to the CKM matrix $V$, which has been introduced in Sect. 10.4. In particular, it is unitary and contains three mixing angles and one phase (see also Sect. 15.4). The possibility of neutrino mixing was investigated theoretically very early. Pontecorvo [15] was the first to consider neutrino-antineutrino oscillations. Maki, Nakagawa and Sakata [14] have discussed flavour mixing of two neutrinos (interestingly already before the Cabibbo angle for quark mixing was introduced). Therefore $U$ is called the PMNS matrix. Recall that mass states are not constants of motion. The relative phases of these states change with time. If neutrinos were massless, this would not be the case. It would make no sense to distinguish between flavour and mass states and the PMNS matrix would not exist. The indirect proof that neutrinos possess a mass, in contrast to the prediction of the standard model, was possible by observing *neutrino oscillations*.

## 11.2 Neutrino Oscillations

To understand how the elements of $U$ can be determined, consider two generations of neutrinos, $|\,\nu_e\,\rangle$ and $|\,\nu_\mu\,\rangle$, which in analogy to (10.19) are written as

$$\begin{pmatrix} |\,\nu_e\,\rangle \\ |\,\nu_\mu\,\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} |\,\nu_1\,\rangle \\ |\,\nu_2\,\rangle \end{pmatrix}. \tag{11.2}$$

Neutrinos are produced as flavour states by the weak interaction, e.g. a $|\,\nu_e\,\rangle = \cos\theta\,|\,\nu_1\,\rangle + \sin\theta\,|\,\nu_2\,\rangle$ by a charged current electron-quark interaction. The time evolution of the mass states leads after a time $t$ to the following wave function of the electron neutrino:

$$|\nu_e(t)\rangle = \cos\theta\, e^{-iE_{\nu_1}t/\hbar}\,|\nu_1\rangle + \sin\theta\, e^{-iE_{\nu_2}t/\hbar}\,|\nu_2\rangle\,. \tag{11.3}$$

Neutrinos are ultra-relativistic, hence their energy is:

$$E_{\nu_i} = \sqrt{p^2c^2 + m_{\nu_i}^2c^4} \approx pc\left(1 + \frac{1}{2}\frac{m_{\nu_i}^2c^4}{p^2c^2}\right). \tag{11.4}$$

The probability to find an electron neutrino after the time $t$ is therefore

$$\begin{aligned} P_{\nu_e\to\nu_e} = |\langle\nu_e(t)|\nu_e\rangle|^2 &= \cos^4\theta + \sin^4\theta + 2\cos^2\theta\,\sin^2\theta\cos\left(\frac{1}{2}\frac{\Delta m_{21}^2c^4}{\hbar c}\frac{L}{pc}\right) \\ &= 1 - \sin^2 2\theta\,\sin^2\left(\frac{1}{4}\frac{\Delta m_{21}^2c^4}{\hbar c}\frac{L}{pc}\right). \end{aligned} \tag{11.5}$$

Here

$$\Delta m_{21}^2 = m_{\nu_2}^2 - m_{\nu_1}^2 \tag{11.6}$$

is the difference of the squares of the masses of the states $\nu_1$ and $\nu_2$, and $L = ct$ is the distance between production and detection travelled by the neutrino in the time $t$. We see that the survival probability $P_{\nu_e\to\nu_e}$ oscillates as a function of the ratio of $L$ and $p$. This is a known interference effect in quantum mechanics, and we will cover it once more later in this book, when we discuss oscillations of $K^0$ and $\overline{K}^0$ mesons in Sect. 15.4.

It follows that by measuring the survival probability one can determine the amplitude $\sin^2 2\theta$ (hence the elements of the mixing matrix) and the mass-squared difference $\Delta m_{21}^2$, which is proportional to the oscillation frequency. The transition probability, i.e., the probability that the electron neutrino becomes a muon neutrino follows from

$$P_{\nu_e\to\nu_\mu} = 1 - P_{\nu_e\to\nu_e} = \sin^2 2\theta\,\sin^2\left(\frac{1}{4}\frac{\Delta m_{21}^2\,c^4}{\hbar c}\frac{L}{pc}\right). \tag{11.7}$$

There is no oscillation in case neutrinos were massless, or when neutrinos had identical mass; the transition and survival probabilities would simply be $P_{\nu_e\to\nu_\mu} = 0$

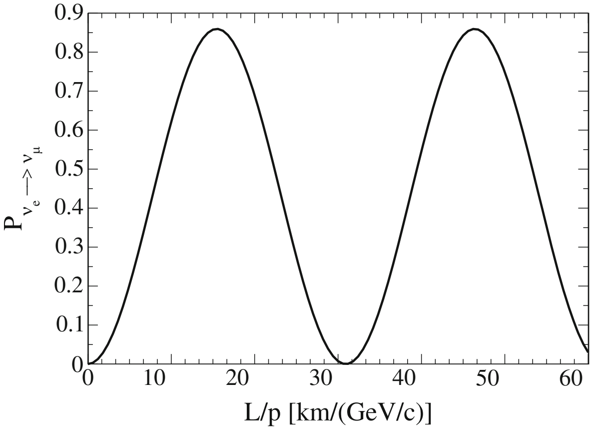


**Fig. 11.1** Typical oscillation curve for the transition probability of electron neutrinos in muon neutrinos, see (11.7). The chosen parameters are $\theta = 34°$ and $m_{\nu_2}^2 - m_{\nu_1}^2 = 8 \cdot 10^{-5}\,\text{eV}^2/c^4$. Hence the transition probability is zero for $L/p \approx 31\,\text{km}/(\text{MeV}/c)$ and maximal ($\sin^2 2\theta = 0.86$) for half this value. The oscillation length for a momentum of $3\,\text{MeV}/c$ is $L_{\text{osc}} \approx 93\,\text{km}$

and $P_{\nu_e \to \nu_e} = 1$. At the end we provide a very useful numerical form of the argument of the sine in the oscillation formula:

$$\frac{1}{4}\frac{\Delta m_{21}^2\, c^4}{\hbar c}\frac{L}{pc} = 1.27\left(\frac{\Delta m_{21}^2}{\text{eV}^2/c^4}\right)\left(\frac{\text{MeV}}{pc}\right)\left(\frac{L}{m}\right). \tag{11.8}$$

A simple example curve is shown in Fig. 11.1. The characteristic scale of oscillations is the distance between two minima or maxima, which is denoted as oscillation length:

$$L_{\text{osc}} = 4\pi\frac{\hbar pc^2}{\Delta m_{21}^2 c^4}\,. \tag{11.9}$$

An experiment is especially well suited to test oscillations when the argument (11.8) is of order 1. This rule of thumb allows to estimate the sensitivity on the mass-squared difference of an experiment. For instance, experiments that detect neutrinos at a distance of 1 km from nuclear power plants, which have an average momentum of $3\,\text{MeV}/c$, are sensitive to $\Delta m_{21}^2 \approx 10^{-3}\,\text{eV}^2/c^4$. Such considerations are confirmed in actual experiments. One should note from this example that we are talking about quantum mechanical interference effects on macroscopic distances.

Two extreme cases of the oscillation formula are of particular interest: if the argument of the sine is very small (small distances when compared to the oscillation length), then the oscillations have not yet taken place. If the argument is very large (large distances when compared to the oscillation length), then the oscillations take place on scales which are too small to be resolved by a detector.

## 11.3 Neutrino Oscillation Experiments

The existence of neutrino oscillations has been confirmed by various experiments. We will discuss in this section the basic physics behind this important area of modern particle physics. With three generations of neutrinos the expressions for the oscillation probabilities are lengthy and complicated (they can be found, e.g., in [12]). To a good approximation, however, one can describe all experiments with the 2-generation formulae (11.5) and (11.7), respectively, because the 3-generation probabilities simplify when the actual experimental parameters $pc$ and $L$ are inserted. The amplitude and mass-squared difference depends on the kind of experiment considered. Before we go into detail, let us stress again that only the mass-squared difference and not the masses themselves can be determined. Approaches to measure the neutrino mass will be discussed later. Oscillation experiments are often classified as "*appearance*" and "*disappearance*" experiments, depending on whether one looks for neutrino flavours that are not produced in the source, or whether one measures the expected flux of neutrinos.

**Solar neutrinos** Historically the first measurements that pointed towards oscillations were performed with solar neutrinos. The experimentally determined flux of solar $\nu_e$ was, depending on energy, about one third to half the value predicted in solar models. The interpretation is of course that the $\nu_e$ oscillate into $\nu_\mu$ and $\nu_\tau$.

Solar models describe in detail the Sun's energy production through a number of nuclear reactions. Effectively, the following fusion reaction takes place: (see Sect. 20.5):

$$4\,\mathrm{p} + 2\,\mathrm{e}^- \rightarrow {}^4\mathrm{He} + 2\,\nu_\mathrm{e} + 26.73\,\mathrm{MeV}\,. \tag{11.10}$$

It is realised by a complicated network of reactions. Of interest are here only the ones that generate neutrinos. The first step is the production of the deuteron:

$$\mathrm{p} + \mathrm{p} \rightarrow \mathrm{d} + \mathrm{e}^+ + \nu_\mathrm{e} \;\text{ and } \mathrm{p} + \mathrm{e}^- + \mathrm{p} \rightarrow \mathrm{d} + \nu_\mathrm{e}\,.$$

The first reaction leads to a continuous energy spectrum with a maximal energy of $E_\nu^{\mathrm{max}} = 0.42\,\mathrm{MeV}$, while for the second a fixed energy of $E_\nu = 1.44\,\mathrm{MeV}$ is predicted. The deuteron fuses with a proton to $^3$He. This isotope can either fuse with another $^3$He nucleus to $^4$He and two protons, or generate neutrinos via

$${}^3\mathrm{He} + \mathrm{p} \rightarrow {}^4\mathrm{He} + \nu_\mathrm{e} + \mathrm{e}^+\,,$$

which have $E_\nu^{\mathrm{max}} = 18.77\,\mathrm{MeV}$. Now we fuse $^4$He and $^3$He to $^7$Be, which reacts via

$${}^7\mathrm{Be} + \mathrm{e}^- \rightarrow {}^7\mathrm{Li} + \nu_\mathrm{e}$$

and again generates neutrinos. Because this reaction can end in the ground state (in about 90 % of the cases) or the excited state of $^7$Li, the neutrino energy is

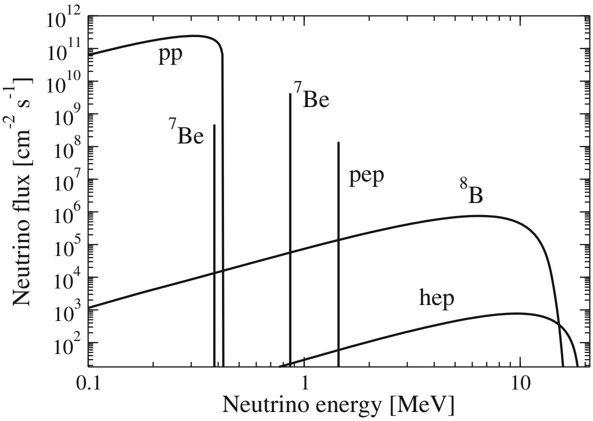


**Fig. 11.2** The solar neutrino spectrum from [7]. Plotted are the individual spectra of the five different neutrino sources

either $E_\nu = 0.862\,\text{MeV}$ or $E_\nu = 0.384\,\text{MeV}$.[1] Through proton capture $^7$Be is transformed into $^8$B, which is another source of neutrinos ($E_\nu^{\max} = 14.06\,\text{MeV}$) since it undergoes $\beta$-decay:

$$^8\text{B} \rightarrow {}^8\text{Be} + \text{e}^+ + \nu_\text{e}\,.$$

All in all there are five different neutrino sources with different spectra and calculable percentage of the total flux [7]. Their sum should give the total solar neutrino spectrum, see Fig. 11.2. We can estimate this flux once we know the so-called solar constant $\Phi$, which denotes the Sun's electromagnetic power reaching the Earth per area and time unit. Ignoring seasonal variations due to the Earth's elliptic orbit, it is given by $\Phi \approx 8.5 \cdot 10^{11}\,\text{MeV}\,\text{cm}^{-2}\,\text{s}^{-1}$. With two produced neutrinos per reaction in (11.10), one finds

$$\Phi(\nu_\text{e}) \approx 2\,\frac{\Phi}{27\,\text{MeV}} \approx 6 \cdot 10^{10}\,\text{cm}^{-2}\text{s}^{-1}\,,$$

almost $10^{11}$ neutrinos per square centimetre and second. Their energy is at most 18.77 MeV, its average value however only 0.3 MeV. Therefore the energy of solar neutrinos is too low to produce $\mu$ or $\tau$ leptons in charged-current reactions after the $\nu_\text{e}$ oscillate into $\nu_\mu$ or $\nu_\tau$.

Early experiments [16] were only sensitive on $\nu_\text{e}$, for instance via the reaction $\nu_\text{e} + {}^{37}\text{Cl} \rightarrow {}^{37}\text{Ar} + \text{e}^-$ that is mediated by charged currents. The radioactive Argon can be detected since it decays with a half-life of about 35 days:

[1]The direct detection of this small and low-energy flux was possible only in 2007 by the Borexino experiment [6].

$^{37}\text{Ar} \rightarrow {}^{37}\text{Cl} + e^- + \bar{\nu}_e$. To generate $^{37}$Ar one requires a threshold energy of 0.81 MeV, and therefore only part of the solar spectrum can be tested. Measurements with lower energy values were possible by constructing a similar experiment taking advantage of neutrino capture on $^{71}$Ga (threshold energy 0.23 MeV) and detection of the generated $^{71}$Ge, which is radioactive (see Exercise 11.1).

Another possibility to measure solar neutrinos is through elastic scattering on electrons, $\nu_e + e^- \rightarrow \nu_e + e^-$. In this reaction interference between charged and neutral currents occurs. In contrast to this, the reaction $\nu_{\mu,\tau} + e^- \rightarrow \nu_{\mu,\tau} + e^-$ can only be mediated by neutral currents of the $\nu_\mu$ and $\nu_\tau$, see Sect. 10.3. The result is

$$\sigma(\nu_e\, e^- \rightarrow \nu_e\, e^-) \approx 6.14 \cdot \sigma(\nu_{\mu,\tau}\, e^- \rightarrow \nu_{\mu,\tau}\, e^-)\,. \tag{11.11}$$

Consequently one has some sensitivity on the $\nu_{\mu,\tau}$. This reaction was examined mainly in the SuperKamiokande experiment [11], a Cherenkov detector filled with 50,000 tons of water and located 1,000 m below the surface of the Earth. The reaction is detected by the Cherenkov light of the scattered electrons. This radiation in the form of photons is generated when the electrons move within a medium (for SuperKamiokande this is water) with a velocity that is larger than the speed of light in that medium (see Sect. A.2). A light cone is produced whose opening angle is $\theta = \arccos \frac{1}{\beta_e\, n}$, where $n = 1.33$ is the index of refraction of water. Since electrons loose energy through bremsstrahlung, they move faster than light only for a short amount of time, and a so-called Cherenkov ring is formed. By determining the position of the original reaction and the opening angle of the Cherenkov cone one can measure the energy of the electron. At high energies, the scattering occurs mainly in forward direction.

The last doubts whether the solar models were really correct were removed by the SNO Experiment (Sudbury Neutrino Observatory) [2, 3]. This experiment determines the total neutrino flux by measuring also reactions which are mediated by neutral currents only. To those reactions the $\nu_\mu$ and $\nu_\tau$ are contributing as well, which implies that the total flux should come out in case the $\nu_e$ oscillate into $\nu_\mu$ and $\nu_\tau$. The Cherenkov detector is located in a depth of 2,000 m in a mine in Canada, and was filled with 1,000 tons of heavy water. Here the oxygen atom is bound to two deuterium atoms. The following reactions can now be measured:

$$\text{CC:} \quad \nu_e + d \rightarrow p + p + e^- \tag{11.12}$$

$$\text{NC:} \quad \nu_{e,\mu,\tau} + d \rightarrow p + n + \nu_{e,\mu,\tau} \tag{11.13}$$

$$\text{ES:} \quad \nu_{e,\mu,\tau} + e^- \rightarrow \nu_{e,\mu,\tau} + e^-\,. \tag{11.14}$$

The first one is only mediated by charged currents, and is measured by the Cherenkov light of the electrons.[2] It determines the incoming flux of electron

[2] The refractive index of heavy water is essentially identical to the one of normal water.

neutrinos, $\phi_{\mathrm{CC}} = \phi_{\mathrm{e}}$. The second reaction is mediated by neutral currents. It is independent of flavour and determines the total flux, $\phi_{\mathrm{NC}} = \phi_{\mathrm{e}} + \phi_{\mu\tau}$. The elastic scattering reaction (11.14) is sensitive on all three flavours, though slightly more on the $\nu_{\mathrm{e}}$, namely $\phi_{\mathrm{ES}} \approx \phi_{\mathrm{e}} + 0.16\,\phi_{\mu\tau}$, see (11.11). It can however also serve to measure the total flux.

All three reactions can experimentally be distinguished. Electrons from elastic scattering point, as mentioned above, in the same direction as the incoming neutrinos. Since in the charged current reaction the proton is much heavier than the electron, there is basically no direction dependence for the produced electron. The free neutron in the neutral current reaction is captured by a deuterium nucleus, whose de-excitation generates within typically 10 ms photons with a total energy of 6 MeV. Compton scattering of those photons with electrons results in Cherenkov light. The detector was furthermore spiced with NaCl, because $^{35}$Cl has a high capture rate for neutrons. In addition, special counters equipped with $^{3}$He were added to the experiment.

The result was that the total neutrino flux is about 3 times as large as the flux of the $\nu_{\mathrm{e}}$, and more importantly, consistent with the prediction of the solar models, see Fig. 11.3. The theoretical analysis of the solar models is complicated by so-called matter effects, which influence the oscillations of neutrinos in a medium such as the interior of the Sun, see Exercise 11.2.[3] The extracted survival probability $P_{\nu_{\mathrm{e}} \to \nu_{\mathrm{e}}}$

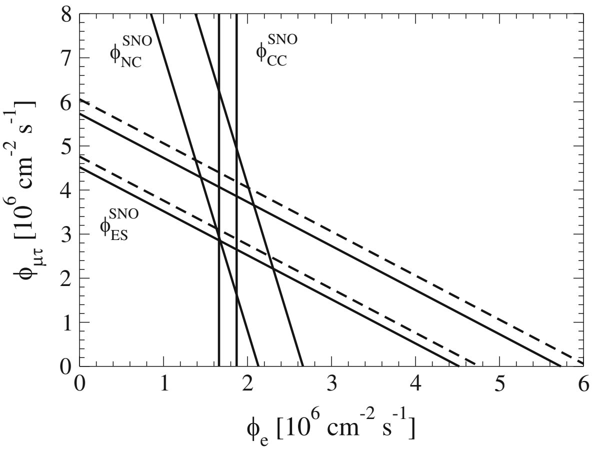


**Fig. 11.3** Results of the SNO experiment, see [3]. Plotted are the determined neutrino fluxes (including measuring errors) from elastic scattering $\phi_{\mathrm{ES}}$, charged current $\phi_{\mathrm{CC}}$ and neutral current $\phi_{\mathrm{NC}}$, see (11.12)–(11.14). The prediction of the solar standard model (shown here with theoretical uncertainty) lies within the *dashed lines*. Calculation and measurement agree excellently

[3]The reason lies in the fact that in a medium consisting of electrons, protons and neutrons, electron neutrinos can react through neutral and charged currents, whereas the other flavours only feel neutral currents.

gives for the mixing angle and the mass-squared difference that are relevant for solar neutrinos:

$$\theta_{12} \approx 34^\circ \text{ and } \Delta m^2_{21} = m^2_{\nu_2} - m^2_{\nu_1} \approx 8.0 \cdot 10^{-5}\,\text{eV}^2/c^4\,.$$

**Atmospheric neutrinos** Oscillations were also observed in the flux of atmospheric neutrinos. The atmosphere is constantly bombarded with protons and heavy nuclei from cosmic rays, and the reactions generate a large number of pions. Their decays produce the so-called atmospheric neutrinos

$$\pi^+ \to \mu^+ + \nu_\mu$$
$$\mu^+ \to \bar{\nu}_\mu + \text{e}^+ + \nu_\text{e}$$

and the appropriate antiparticles. The ratio of the two neutrino flavours is $[n(\nu_\mu) + n(\bar{\nu}_\mu)]/[n(\nu_\text{e}) + n(\bar{\nu}_\text{e})] = 2$, if effects coming from the finite lifetime of the muon are neglected. The energies of the neutrinos are determined again by the Cherenkov radiation of the scattered charged leptons. The most important measurement of atmospheric neutrinos was performed by the SuperKamiokande experiment in Kamioka, Japan. The electrons and muons, and therefore the incoming $\nu_\text{e}$ and $\nu_\mu$, are identified by their Cherenkov light. The Cherenkov ring of the electrons is smeared with respect to the one of the muons, since the lighter electrons scatter much more frequently in the water tank than the heavier muons. The neutrino energies that are of interest in our discussion are a few 100 MeV and more. It follows that the produced charged leptons point in the same direction as the neutrinos. This allows to determine if the neutrinos crossed only the atmosphere above the detector, or if they originate from the other side of the Earth. The important observable is the zenith angle $\theta$ of the charged leptons. For down-going particles this angle is $\theta = 0$, or $\cos\theta = 1$. The original neutrinos therefore were generated above the detector and have travelled about 20 km. Up-going neutrinos are characterised by $\theta = 180°$, or $\cos\theta = -1$. They stem from the other side of the Earth, and have therefore travelled about $10^4$ km.

A flux too low by a factor of 2 was measured [10] for neutrinos with energies above 1 GeV and travelled distances of $10^4$ km, see Fig. 11.4. Since the Earth is transparent to such neutrinos, there should be no attenuation of the flux. In contrast to muon neutrinos, electron neutrinos did not show any deviation from the expected flux; on length scales comparably to the radius of the Earth they do not develop appreciable oscillations. The decreased flux of the $\nu_\mu$ is therefore attributed to the oscillation of $\nu_\mu$ into $\nu_\tau$, which cannot be identified in the detector. Analysing the data with the transition probability $P_{\nu_\mu \to \nu_\tau}$ gives the parameters

$$\theta_{23} \approx 45^\circ \text{ and } |\Delta m^2_{31}| = |m^2_{\nu_3} - m^2_{\nu_1}| \approx |m^2_{\nu_3} - m^2_{\nu_2}| \approx 2.4 \cdot 10^{-3}\,\text{eV}^2/c^4\,.$$

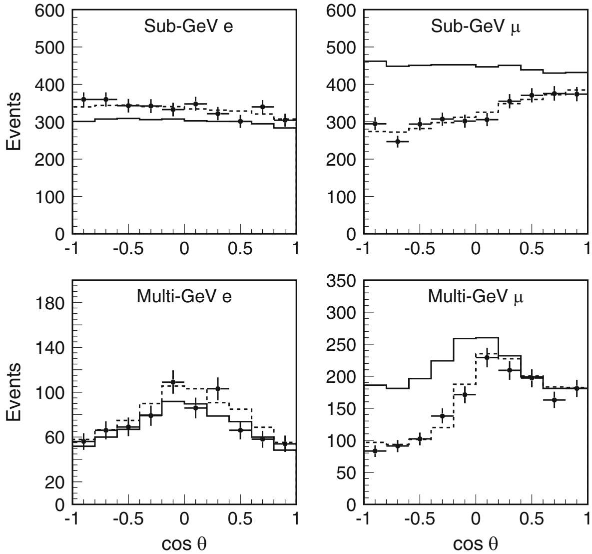


**Fig. 11.4** Measured fluxes of atmospheric neutrinos, see [10]. Shown are the event numbers including measurement errors for low (*upper plots*) and high (*lower plots*) energy neutrinos as function of the cosine of the zenith angle $\theta$. The *left plots* are events that have been identified as electrons, the *right plots* are muons. The *solid line* is the expectation in absence of neutrino oscillations, the *dashed line* is a fit to the data assuming oscillation of $\nu_\mu$ into $\nu_\tau$

The value of the mass-squared difference can be easily understood (using $L \approx 10^4$ km and $E \approx 1$ GeV) by our rule of thumb, which states that the argument of the sine in the oscillation formula should be one. Currently the sign of the larger mass-squared difference $\Delta m^2_{31}$ is not known. This is called the problem of the neutrino mass ordering, and can be solved by future neutrino oscillation experiments.

**Reactor neutrinos** Additional information comes from observing oscillations of antineutrinos produced in nuclear reactors. Here, we need the 3-generation oscillation formulae. For the survival probability we have an expression containing three different terms, which are proportional to $\Delta m^2_{21}$, $\Delta m^2_{32}$ and $\Delta m^2_{31}$, respectively. The results considered so far imply that $|\Delta m^2_{32}| \gg \Delta m^2_{21}$, which leads to $|\Delta m^2_{31}| \approx |\Delta m^2_{32}|$, since the relation $\Delta m^2_{21} + \Delta m^2_{32} - \Delta m^2_{31} = 0$ must hold. It follows

$$\begin{aligned} P_{\nu_e \to \nu_e} = & 1 - 4\, c^2_{12} s^2_{12}\, c^4_{13}\, \sin^2\left(\frac{1}{4}\frac{\Delta m^2_{21} c^4}{\hbar c}\frac{L}{pc}\right) \\ & -4\, c^2_{13}\, s^2_{13}\, \sin^2\left(\frac{1}{4}\frac{\Delta m^2_{32} c^4}{\hbar c}\frac{L}{pc}\right). \end{aligned} \tag{11.15}$$

We have abbreviated here $c_{12}^2 = \cos^2\theta_{12}$, $s_{12}^2 = \sin^2\theta_{12}$ etc.[4] The smaller one of the two oscillation lengths is proportional to $1/|\Delta m_{32}^2|$. If the distance of a detector is small when compared to this oscillation length, then we have $P_{\nu_e\to\nu_e} \approx 1$. If we increase the distance, the sine including $\Delta m_{32}^2$ becomes of order one, while the term including $\Delta m_{21}^2$ is negligible. This occurs at about $L \approx 1{,}000$ m. One is in this case sensitive to $\theta_{13}$. Increasing the distance further to $L \approx 100$ km, one sees that the sine including $\Delta m_{21}^2$ becomes of order one, while the fast oscillations of the other terms can no longer be resolved, and are negligible in the limit of small $\theta_{13}$. One expects to test in this case the parameters of solar neutrinos. Figure 11.5 plots (11.15) together with the results of experiments that were performed over many years at different distances.

In Kamioka (Japan) the KamLAND detector is located, which contains 1,000 tons of a liquid scintillator to detect charged particles. Nuclear reactors in Japan and South Korea generate $\bar{\nu}_e$, and have a typical distance of 200 km from the detector. As estimated above, KamLAND is sensitive to the same mass-squared difference as solar experiments. Indeed, the same neutrino parameters as with solar neutrino experiments could be measured [9].

The third mixing angle $\theta_{13}$ is determined, as estimated above, also in experiments with nuclear reactors, but with detectors which are located rather close (about 1 km) to the reactor core, e.g. Daya Bay [4] in China or Double Chooz [1] in France. The

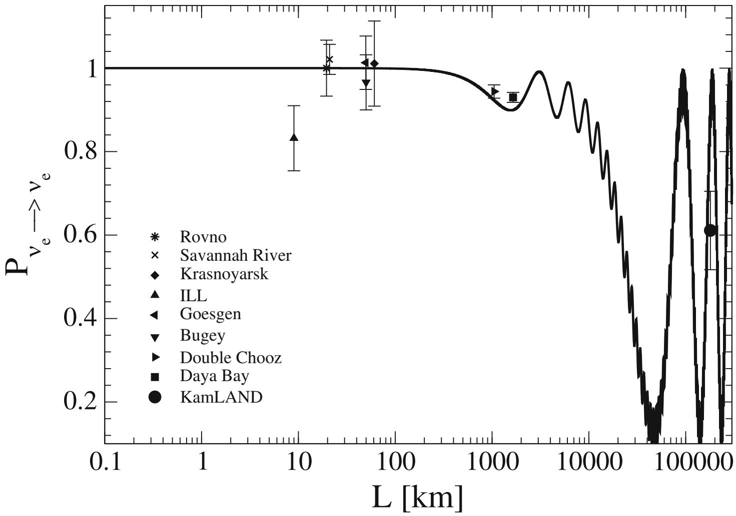


**Fig. 11.5** Oscillation curve $P_{\nu_e\to\nu_e}$ in the 3-generation case with a neutrino momentum 3 MeV/$c$. Plotted are the averaged results of several neutrino oscillation experiments at nuclear reactors

[4]One can show that $P_{\nu_e\to\nu_e} = P_{\bar{\nu}_e\to\bar{\nu}_e}$, and analogously for the survival probabilities of muon and tau neutrinos, see Exercise 11.3.

method to detect the $\bar{\nu}_e$ is inverse $\beta$-decay,

$$\bar{\nu}_e + p \to n + e^+ .$$

The produced positron annihilates with electrons. To determine the neutron, one adds Gadolinium to the detector, which has a very high capture rate for neutrons, and de-excites to its ground state after emitting photons with a total energy of about 8 MeV.

Comparing expected and measured flux gives $P_{\nu_e \to \nu_e}$, and one finds an angle $\theta_{13}$ of about 9°; the mass-squared difference is $\Delta m^2_{31}$, the same as for atmospheric neutrinos.

**Mixing matrix of neutrinos** The absolute values of the lepton mixing matrix elements are obtained from the results of all oscillation experiments. Their central values are

$$(\ |U_{\alpha i}|\ ) \approx \begin{pmatrix} 0.826 & 0.544 & 0.151 \\ 0.427 & 0.642 & 0.635 \\ 0.368 & 0.540 & 0.757 \end{pmatrix} . \tag{11.16}$$

The precision is not as high as for the CKM matrix (10.23). Possible effects of the CP phase in $U$ are not yet seen.

Let us finally summarise the main features of lepton mixing. First one notes that all elements of the mixing matrix are about the same size. It is therefore much different from the CKM matrix, see (10.23), for which the diagonal elements dominate. Such a drastically different mixing of quarks and leptons can be an important hint for the understanding of physics beyond the standard model (Sect. 20.4). In analogy to the electroweak unification (Sect. 12.2) one suspects a grander unification (Sect. 12.6) which unites also quarks and leptons.

**Neutrino mass** A second peculiarity of neutrinos is the smallness of their masses. Upper limits on $m_{\nu_i}$ are about $2\,\mathrm{eV}/c^2$, and neutrinos are therefore much lighter than all other fermions. The explanation which theorists consider the most plausible one is treated in Sect. 11.4.

Let us discuss here shortly the current information on neutrino masses. As mentioned above, oscillation experiments can only probe the differences of the squared masses. The sign of the larger mass-squared difference, $\Delta m^2_{31} = m_3^2 - m_1^2$, is unknown. The two possibilities are called normal and inverted ordering, Fig. 11.6 shows both cases. Per definition the largest mass in the normal ordering is $m_3$, whereas it is $m_2$ in the inverted ordering. Accordingly the smallest mass is $m_1$ or $m_3$, respectively. The smallest mass can be zero, for the normal ordering this case is called normal hierarchy:

$$m_2 = \sqrt{\Delta m^2_{21}} \approx 0.009\,\mathrm{eV}/c^2\,,\ m_3 = \sqrt{\Delta m^2_{31}} \approx 0.05\,\mathrm{eV}/c^2\,. \tag{11.17}$$

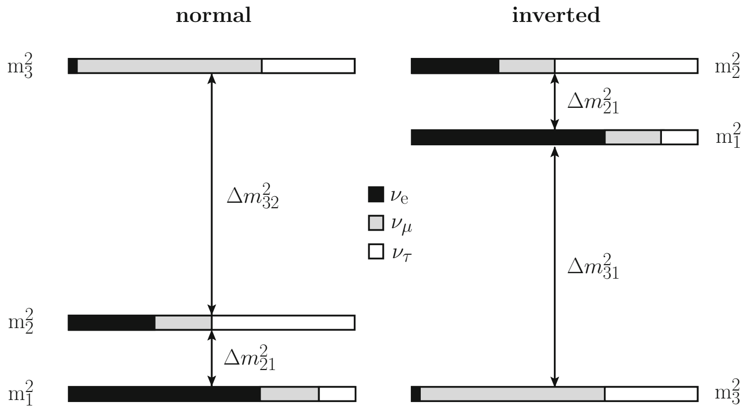


**Fig. 11.6** Possible ordering of neutrino masses. Shown are the normal (*left*) and the inverted (*right*) ordering. The size of the *shaded areas* of neutrino $\nu_i$ with mass $m_i$ corresponds to the size of $|U_{\alpha i}|^2$, i.e. the amount of the flavour state $\nu_\alpha$

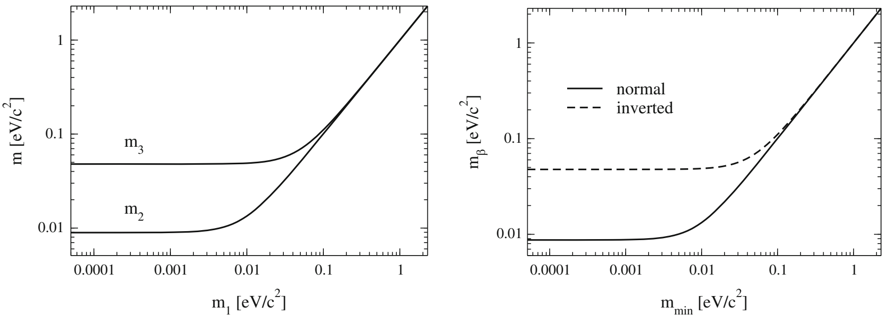


**Fig. 11.7** In the *left plot* the neutrino masses $m_2$ and $m_3$ are shown as functions of the smallest mass $m_1$ in the normal ordering. In the inverted ordering the according plot would be $m_2$ and $m_1$ as functions of $m_3$. Both curves would be indistinguishable. The *right plot* shows the quantity $m_\beta = \sqrt{\sum |U_{ei}|^2\, m_i^2}$, which is measurable in $\beta$-decays, as function of the smallest neutrino mass for both mass orderings

If in the inverted ordering the smallest mass is zero we talk about the inverted hierarchy:

$$m_2 \approx m_1 = \sqrt{|\Delta m_{31}^2|} \approx 0.05\,\mathrm{eV}/c^2\,. \tag{11.18}$$

The difference between $m_2$ and $m_1$ in the inverted hierarchy is $\sqrt{\Delta m_{21}^2}$. One notes that the ratio of neutrino masses is less extreme than for quarks or charged leptons, compare for instance $m_3/m_2$ with $m_t/m_c$ or $m_\tau/m_\mu$. In case the smallest mass is non-zero, the ratios of neutrino masses are even larger, see Fig. 11.7. If the smallest mass exceeds about $0.1\,\mathrm{eV}/c^2$, the differences in masses are negligible and one speaks of quasi-degenerate neutrinos. The largest possible neutrino mass value can be obtained from experiments on the energy spectrum of $\beta$-decays, which we will discuss in more detail in Sect. 18.6. Effectively one can measure or constrain the quantity $\sqrt{\sum |U_{\mathrm{e}i}|^2\, m_i^2}$, for which a current upper limit of $2.3\,\mathrm{eV}/c^2$ is quoted [13]. One can easily show that this is the largest possible value of $m_1$, $m_2$ and $m_3$. New experiments, one example is KATRIN, will be able to improve this number by a factor of 10 in the near future.

Another approach to determine neutrino mass is neutrinoless double beta decay, which we will discuss in Sect. 18.7. Here the observable is $\left|\sum U_{\mathrm{e}i}^2\, m_i\right|$. This method is however quite model-dependent, because one has to assume that neutrinos are Majorana particles (see the next section).

Yet another possibility to measure neutrino mass exists in cosmology, where observations of galaxy distributions can probe the influence of neutrinos in the hot early universe. This method is also very model-dependent.

## 11.4 Majorana Neutrinos?

Charged leptons and quarks are obviously different from their respective antiparticles, since those have opposite electric charge. Fermions which are different from their antiparticles are called *Dirac particles*. They can formally be described by four degrees of freedom, namely particle and antiparticle, each with positive and negative helicity. If an electron neutrino was a Dirac particle, we would write those four degrees of freedom as

$$\text{Dirac particle: } (\nu_{\mathrm{e}\uparrow},\ \nu_{\mathrm{e}\downarrow},\ \bar{\nu}_{\mathrm{e}\uparrow},\ \bar{\nu}_{\mathrm{e}\downarrow})\,.$$

The arrow $\uparrow$ denotes here positive helicity, $\downarrow$ accordingly negative helicity. However, since neutrinos are electrically neutral, they can be their own antiparticles. Such particles are called *Majorana particles*. They possess two degrees of freedom, namely particle = antiparticle with positive or negative helicity:

$$\text{Majorana particle: } (\nu_{\mathrm{e}\uparrow},\ \nu_{\mathrm{e}\downarrow})\,.$$

The distinction we have made so far, namely that in charged current reactions neutrinos generate electrons, and antineutrinos generate positrons, has to be discussed in a more subtle manner: As we have seen in Sect. 10.5, weak interactions couple only to left-handed electrons and right-handed positrons. Chirality is identical to helicity up to corrections of order mass divided by energy. The particle that is produced by a $\mathrm{W}^-$ together with a left-handed electron is now a right-handed Majorana fermion, which in case its mass is non-zero possesses mainly positive helicity, but also a small contribution of negative helicity: $\nu_{\mathrm{e}\uparrow} + \epsilon\, \nu_{\mathrm{e}\downarrow}$. Here $\epsilon$ is of the order $m_\nu c^2/E_\nu$.

This very tiny contribution can now interact with a second $W^-$ and generate in a charged current reaction a left-handed electron. All in all we have transformed two $W^-$ in two electrons:

$$\begin{aligned} W^- \to e^- + (\nu_{e\uparrow} + \epsilon\, \nu_{e\downarrow}) \\ \nu_{e\downarrow} + W^- \to e^- \,. \end{aligned} \tag{11.19}$$

The total reaction chain violates lepton number conservation by two units. It is only possible if neutrinos are Majorana particles. If neutrinos were Dirac particles, then the particle that is produced in the first step would be $\bar{\nu}_{e\uparrow} + \epsilon\, \bar{\nu}_{e\downarrow}$, and the small contribution with negative helicity $\bar{\nu}_{e\downarrow}$ would not interact with a $W^-$. The reaction chain in (11.19) would not take place.

In case of massive Majorana neutrinos the conservation of lepton number, as discussed at the end of Sect. 10.1 should not be obeyed exactly. However, the rates of lepton-number violating processes are strongly suppressed with the ratio of neutrino mass and their energy. As a numerical example, consider neutrinos from nuclear reactors, whose energy is $E_\nu \approx$ MeV. Assuming they have their largest allowed mass of $m_\nu \approx 1\,\text{eV}/c^2$, one finds that they have a small fraction $\epsilon$ of "wrong helicity" of about $m_\nu\, c^2/E_\nu \approx 10^{-6}$. The probability to absorb this part is then proportional to this small number squared. The dependence on neutrino mass implies that massless Majorana neutrinos cannot be distinguished from massless Dirac neutrinos.

The search for neutrinoless double beta decay, discussed in Sect. 18.7, is the most realistic possibility to prove the Majorana character of neutrinos. The factor that compensates the strong suppression $(m_\nu c^2/E_\nu)^2$ is the sheer number of atoms if one searches with several kg of the decaying isotope. In case neutrinos are Majorana particles the PMNS matrix contains two additional phases which however do not influence neutrino oscillation, and only become important in processes that violate lepton number.

**Seesaw mechanism** The idea of neutrinos being Majorana particles is appealing to most theorists, since it is realised in most theories that extend the standard model. In these models there are for each neutrino $\nu_1$, $\nu_2$ and $\nu_3$ (linear combinations of which form $\nu_e$, $\nu_\mu$ and $\nu_\tau$) additional neutrinos $N_1$, $N_2$ and $N_3$. The latter are Majorana particles with extremely large masses, whose magnitude is expected to correspond to the characteristic energy scale of the theory that extends the standard model. The different neutrinos interact with each other and thereby change their masses. For the sake of simplicity one can consider the case of one family, i.e. one neutrino $\nu$ and a heavy neutrino $N$. The initial mass $m_{SM}$ of $\nu$ is similar to the masses of the quarks and charged leptons, since one assumes that it is generated by the same mechanism that gives them masses. The interaction between $N$ and $\nu$ leads now to a suppression of the mass of the neutrinos, in the form of

$$m_\nu \approx \frac{m_{SM}^2}{M_N} = m_{SM}\frac{m_{SM}}{M_N}\,. \tag{11.20}$$

The mass is therefore much lighter than the one of the other fermions of the standard model, suppressed with a factor $m_{\mathrm{SM}}/M_{\mathrm{N}}$.[5] In addition the light neutrinos are, thanks to their interaction with their heavy partners, now Majorana fermions, too. As one can see from (11.20), $m_\nu$ becomes smaller when $M_{\mathrm{N}}$ becomes larger. Therefore this mechanism is called *seesaw mechanism*. Estimating $m_{\mathrm{SM}} \approx m_{\mathrm{t}}$ and $m_\nu \approx \sqrt{|m^2_{\nu_3} - m^2_{\nu_2}|}$ as largest standard model and neutrino mass, respectively, gives $M_{\mathrm{N}} \approx 10^{15}\,\mathrm{GeV}/c^2$. The corresponding energy scale of $10^{15}$ GeV is highly interesting for theorists, since it is the scale at which all three interactions are unified in Grand Unified Theories (Sect. 12.6). The fact that the same energy scale arises from considerations of Grand Unified Theories as well as from the neutrino masses makes the seesaw mechanism so plausible. As an additional bonus, the violation of lepton number and the possible CP violation in the decays of the heavy Majorana neutrinos help in understanding the generation of the matter-antimatter asymmetry in the early universe, see Sect. 20.4.

The distinction between Dirac and Majorana neutrino is for most practical purposes irrelevant, and we therefore return to the notation of neutrino and antineutrino.

## Problems

1. **Solar neutrinos**
   The GALLEX experiment measures solar neutrinos by the reaction ${}^{71}_{31}\mathrm{Ga} + \nu_{\mathrm{e}} \rightarrow {}^{71}_{32}\mathrm{Ge}$. The cross-section of this reaction at typical neutrino energies is about $2.5 \cdot 10^{-45}\,\mathrm{cm}^2$. One looks for radioactive ${}^{71}$Ge atoms (lifetime $\tau = 16$ days), which are produced in a tank containing 30 t Gallium (40 % ${}^{71}$Ga, 60 % ${}^{69}$Ga) as dissolved chloride [5]. About 50 % of all neutrinos have energies above the reaction threshold. All Germanium atoms are extracted from the tank. Estimate how many ${}^{71}$Ge atoms are generated per day. How many should be in the tank after 3 weeks? How many if one waits for an infinite amount of time?
2. **Matter effects**
   Convince yourself that the matrix

$$H = \frac{c^4 \Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

   is diagonalised by the matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

[5] A useful analogy exists with the effective 4-fermion description of weak interactions at low energies with the Fermi constant. The presence of the W bosons is indirect, and only apparent at high energies. In the same way the presence of the heavy Majorana neutrinos is felt only indirectly at low energies, namely by the smallness of neutrino masses.

i.e. $U^T H\, U = c^4\, \mathrm{diag}(-\Delta m^2/4E, \Delta m^2/4E)$. The effect of neutrino oscillations in matter can now be described by adding a term to the upper left entry of $H$:

$$H_\mathrm{M} = \frac{c^4 \Delta m^2}{4E} \begin{pmatrix} -\cos 2\theta + \frac{4\sqrt{2}\, G_\mathrm{F} N_\mathrm{e} E}{c^4 \Delta m^2} & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}.$$

Here $N_\mathrm{e}$ is the number density of electrons (assumed constant) in the medium through which the neutrinos travel. Diagonalising this matrix with $U_\mathrm{M}^T H_\mathrm{M}\, U_\mathrm{M}$ yields the mixing angle in matter $\theta_\mathrm{M}$. Show with

$$U_\mathrm{M} = \begin{pmatrix} \cos\theta_\mathrm{M} & \sin\theta_\mathrm{M} \\ -\sin\theta_\mathrm{M} & \cos\theta_\mathrm{M} \end{pmatrix}$$

that it is given as

$$\sin^2 2\theta_\mathrm{M} = \frac{(\frac{c^4 \Delta m^2}{2E})^2\, \sin^2 2\theta}{(\frac{c^4 \Delta m^2}{2E} \cos 2\theta - \sqrt{2}\, G_\mathrm{F} N_\mathrm{e})^2 + (\frac{c^4 \Delta m^2}{2E})^2\, \sin^2 2\theta}\,.$$

When is this angle maximal ($\theta_\mathrm{M} = 45°$)?

3. **CP and T violation in neutrino oscillations**
   Starting from the oscillation probability $P_{\nu_\alpha \to \nu_\beta}$ for arbitrary flavours $\alpha = \mathrm{e}, \mu, \tau$, find the CP- and T-transformed channels. When the combination CPT is conserved, what does this imply for the survival probabilities $P_{\nu_\alpha \to \nu_\alpha}$?
4. **The effective mass in neutrinoless double beta decay**
   The so-called effective mass, to which squared value the lifetime of neutrinoless double beta decay is proportional, can be written as:

$$m_{\beta\beta} = \left|\cos^2\theta_{12}\, \cos^2\theta_{13}\, m_1 + \sin^2\theta_{12}\, \cos^2\theta_{13}\, m_2\, e^{i\gamma} + \sin^2\theta_{13}\, m_3\, e^{i\delta}\right|.$$

Here $\gamma$ and $\delta$ are additional phases in the PMNS matrix which show up only for Majorana neutrinos. Show with the neutrino parameters given in the book that in case of an inverted mass ordering there is a lower limit on the effective mass. Argue how the Majorana character of the neutrinos can be ruled out.

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