Chapter 17
The Nuclear Force

Unfortunately, nuclear physics has not profited as much from analogy as has atomic physics. The reason seems to be that the nucleus is the domain of new and unfamiliar forces, for which men have not yet developed an intuitive feeling.

V. L. Telegdi [15]

The enormous richness of complex structures that we see all around us (molecules, crystals, amorphous materials) is due to chemical interactions. The short-distance forces through which electrically neutral atoms interact can and do produce large scale structures.

The interatomic potential can generally be determined from spectroscopic data about molecular excited states and from measuring the binding energies with which atoms are tied together in chemical substances. These potentials can be quantitatively explained in non-relativistic quantum mechanics. We thus nowadays have a consistent picture of chemical binding based upon atomic structure.

The nuclear force is responsible for holding the nucleus together. This is an interaction between colourless nucleons and its range is of the same order of magnitude as the nucleon diameter. The obvious analogy to the atomic force is, however, limited. In contrast to the situation in atomic physics, it is not possible to obtain detailed information about the nuclear force by studying the structure of the nucleus. The nucleons in the nucleus are in a state that may be described as a degenerate Fermi gas. To a first approximation the nucleus may be viewed as a collection of nucleons in a potential well. The behaviour of the individual nucleons is thus more or less independent of the exact character of the nucleon-nucleon force. It is therefore not possible to extract the nucleon-nucleon potential directly from the properties of the nucleus. The potential must rather be obtained by analysing two-body systems such as nucleon-nucleon scattering and the proton-neutron bound state, i.e., the deuteron.

There are also considerably greater theoretical difficulties in elucidating the connection between the nuclear forces and the structure of the nucleon than for the atomic case. This is primarily a consequence of the strong coupling constant α_s being two orders of magnitude larger than α, its electromagnetic equivalent. We will therefore content ourselves with an essentially qualitative explanation of the nuclear force.

B. Povh et al., *Particles and Nuclei*, Graduate Texts in Physics,
DOI 10.1007/978-3-662-46321-5_17

17.1 Nucleon-Nucleon Scattering

Nucleon-nucleon scattering at low energies, below the pion production threshold, is purely elastic. At such energies the scattering may be described by non-relativistic quantum mechanics. The nucleons are then understood as point-like structureless objects that nonetheless possess spin and isospin. The physics of the interaction can then be understood in terms of a potential. It is found that the nuclear force depends upon the total spin and isospin of the two nucleons. A thorough understanding therefore requires experiments with polarised beams and targets, so that the spins of the particles involved in the reaction can be specified, and both protons and neutrons must be employed.

If we consider nucleon-nucleon scattering and perform measurements for both parallel and antiparallel spins perpendicular to the scattering plane, then we can single out the spin triplet and singlet parts of the interaction. If the nucleon spins are parallel, then the total spin must be 1, while for opposite spins there are equally large (total) spin 0 and 1 components.

The algebra of angular momentum can also be applied to isospin. In proton-proton scattering we always have a state with isospin 1 (an isospin triplet) since the proton has $I_3 = +1/2$. In proton-neutron scattering there are both isospin singlet and triplet contributions.

Scattering phases Consider a nucleon coming in "from infinity" with kinetic energy E and momentum $\boldsymbol{p}$ which scatters off the potential of another nucleon. The incoming nucleon may be described by a plane wave and the outgoing nucleon as a spherical wave. The cross-section depends upon the phase shift between these two waves.

For states with well defined spin and isospin the cross-section of nucleon-nucleon scattering into a solid angle element $\mathrm{d}\Omega$ is given by the scattering amplitude $f(\theta)$ of the reaction

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta)|^2 . \tag{17.1}$$

For scattering off a short ranged potential a *partial wave decomposition* is used to describe the scattering amplitude. The scattered waves are expanded in terms with fixed angular momentum ℓ. In the case of elastic scattering the following relation holds at large distances r from the centre of the scattering:

$$f(\theta) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell + 1)\, \mathrm{e}^{i\delta_\ell} \;\sin\delta_\ell \; P_\ell(\cos\theta) , \tag{17.2}$$

where

$$k = \frac{1}{\lambda\!\!\!^{-}} = \frac{|\,\boldsymbol{p}|}{\hbar} = \frac{\sqrt{2ME}}{\hbar} \tag{17.3}$$

is the wave number of the scattered nucleon, δ_ℓ a phase shift angle and P_ℓ, the angular momentum eigenfunction, an ℓ-th order Legendre polynomial. The phase shifts δ_ℓ describe the phase difference between the scattered and unscattered waves. They contain the information about the shape and strength of the potential and the energy dependence of the cross-section. The fact that δ_ℓ appears not only as a phase factor but also in the amplitude ($\sin \delta_\ell$) follows from the conservation of the particle current in elastic scattering. This is also known as *unitarity*. The partial wave decomposition is especially convenient at low energies since only a few terms enter the expansion. This is because for a potential with range a we have

$$\ell \leq \frac{|\boldsymbol{p}| \cdot a}{\hbar} . \tag{17.4}$$

The phase shift δ_0 of the partial waves with $\ell = 0$ (i.e., s waves) is decisive for nuclear binding. From (17.4) we see that the s waves dominate proton-proton scattering (potential range 2 fm) for relative momenta less than 100 MeV/c. The Legendre polynomial P_0 is just 1, i.e., independent of θ. The phase shifts δ_0 as measured in nucleon-nucleon scattering are separately plotted for spin triplet and singlet states against the momentum in the centre-of-mass frame in Fig. 17.1. For momenta larger than 400 MeV/c δ_0 is negative, below this it is positive. We learn from this that the nuclear force has a repulsive character at short distances and an attractive nature at larger separations. This may be simply seen as follows.

Consider a, by definition, spherically symmetric s wave $\psi(\boldsymbol{x})$. We may define a new radial function $u(r)$ by $u(r) = \psi(r) \cdot r$ which obeys the Schrödinger equation

$$\frac{\mathrm{d}^2 u(r)}{\mathrm{d}r^2} + \frac{2m(E-V)}{\hbar^2} u(r) = 0 . \tag{17.5}$$

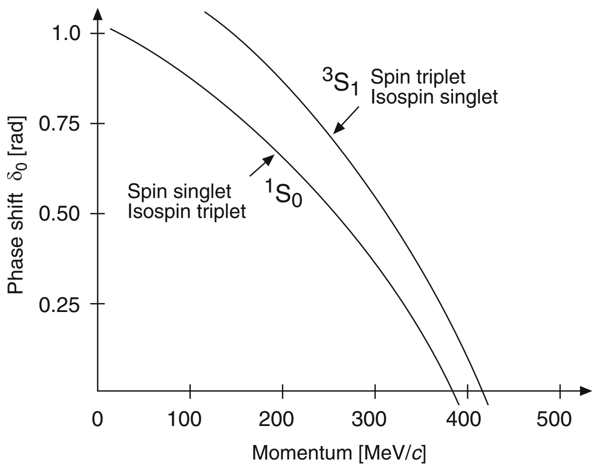

Fig. 17.1 The phase shift δ_0 as determined from experiment both for the spin triplet-isospin singlet $^3S_1$ and for the spin singlet-isospin triplet $^1S_0$ systems plotted against the relative momenta of the nucleons. The rapid variation of the phases at small momenta is not plotted since the scale of the diagram is too small

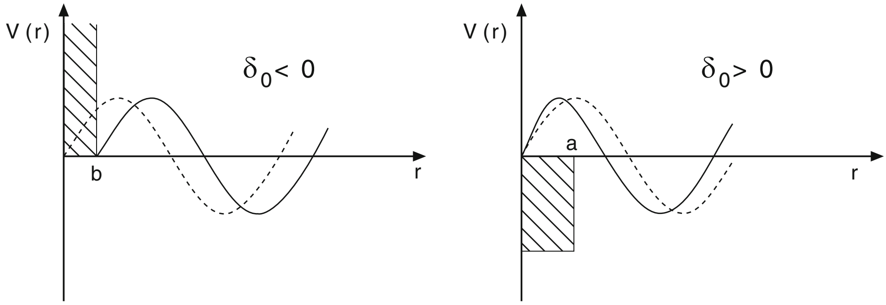

Fig. 17.2 Sketch of the scattering phase for a repulsive (*left*) and an attractive (*right*) potential. The *dashed curves* denote unscattered waves, the continuous ones the scattered waves

If we now solve this equation for a repulsive rectangular potential V with radius b and $V \to \infty$ (Fig. 17.2 (left)), we find

$$\delta_0 = -kb\,. \tag{17.6}$$

The scattering phase is negative and proportional to the range of the potential. A negative scattering phase means that the scattered wave lags behind the unscattered one.

For an attractive potential the scattered wave runs ahead of the unscattered one and δ_0 is positive (Fig. 17.2 (right)). The size of the phase shift is the difference between the phase of the wave scattered off the edge of the potential a and that of the unscattered wave:

$$\delta_0 = \arctan\left(\sqrt{\frac{E}{E+|V|}}\tan\frac{\sqrt{2mc^2(E+|V|)}\cdot a}{\hbar c}\right) - \frac{\sqrt{2mc^2E}\;\cdot a}{\hbar c}\,. \tag{17.7}$$

The phase shift δ_0 is then positive and decreases at higher momenta. If we superimpose the phase shifts associated with a short ranged repulsive potential and a longer ranged attractive one we obtain Fig. 17.3, where the effective phase shift changes sign just as the observed one does.

The relationship between the scattering phase δ_0 and the scattering potential V is contained, in principle, in (17.6) and (17.7) since the wave number k in the region of the potential depends both upon the latter's size and shape and upon the initial energy E of the projectile. A complete scattering phase analysis leads to the nuclear potential shown in Fig. 17.4 which has – as remarked above – a short ranged repulsive and a longer ranged attractive nature. Since the repulsive part of the potential increases rapidly at small r it is known as the *hard core*.

The nucleon-nucleon potential We may obtain a general form of the nucleon-nucleon potential from a consideration of the relevant dynamical quantities. We will, however, neglect the internal structure of the nucleons, which means that

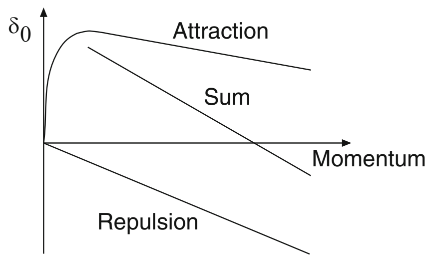

Fig. 17.3 Superposition of negative and positive scattering phases δ_0 plotted against the relative momenta of the scattered particles. The resulting effective δ_0 is generated by a short distance repulsive and a longer range attractive nucleon-nucleon potential

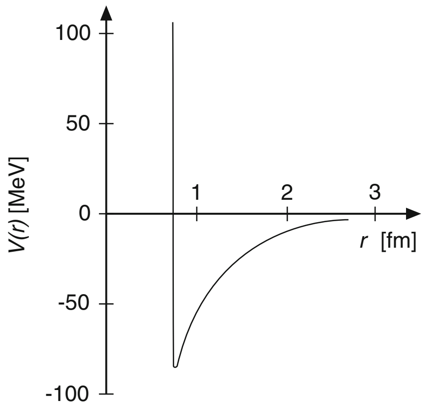

Fig. 17.4 Sketch of the radial dependence of the nucleon-nucleon potential for $\ell = 0$. Note that the spin and isospin dependence of the potential is not shown

this potential will only be valid for nucleon-nucleon bound states and low energy nucleon-nucleon scattering.

The quantities which determine the interaction are the separation of the nucleons $\boldsymbol{x}$, their relative momenta $\boldsymbol{p}$, the total orbital angular momentum $\boldsymbol{L}$ and the relative orientations of the spins of the two nucleons, $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. The potential is a scalar and must at the very least be invariant under translations and rotations. Furthermore it should be symmetric under exchange of the two nucleons. These preconditions necessarily follow from various properties, such as parity conservation, of the underlying theory of the strong force and they limit the scalars which may appear in the potential. At the end of the day the potential, for fixed isospin, has the form [10]:

$$\begin{aligned} V(r) = \quad & V_0(r) \\ & + V_{\mathrm{ss}}(r)\, \boldsymbol{s}_1 \cdot \boldsymbol{s}_2/\hbar^2 \\ & + V_{\mathrm{T}}(r)\, \left(3(\boldsymbol{s}_1 \cdot \boldsymbol{x})(\boldsymbol{s}_2 \cdot \boldsymbol{x})/r^2 - \boldsymbol{s}_1 \boldsymbol{s}_2\right)/\hbar^2 \end{aligned}$$

$$+ V_{\mathrm{LS}}(r)\,(\boldsymbol{s}_1 + \boldsymbol{s}_2)\cdot \boldsymbol{L}/\hbar^2$$
$$+ V_{\mathrm{Ls}}(r)\,(\boldsymbol{s}_1 \cdot \boldsymbol{L})(\boldsymbol{s}_2 \cdot \boldsymbol{L})/\hbar^4$$
$$+ V_{\mathrm{ps}}(r)\,(\boldsymbol{s}_2 \cdot \boldsymbol{p})(\boldsymbol{s}_1 \cdot \boldsymbol{p})/(\hbar^2 m^2 c^2)\,. \tag{17.8}$$

V_0 is a standard central potential. The second term describes a pure spin-spin interaction, while the third term is called the *tensor potential* and describes a non-central force. These two terms have the same spin dependence as the interaction between two magnetic dipoles in electromagnetism. The tensor term is particularly interesting, since it alone can mix orbital angular momentum states. The fourth term originates from a spin-orbit force, which is generated by the strong interaction (the analogous force in atomic physics is of magnetic origin). The final two terms in (17.8) are included on formal grounds, since symmetry arguments do not exclude them. They are, however, both quadratic in momentum and thus mostly negligible in comparison to the LS-term.

The significance of this ansatz for the potential is not that the various terms can be merely formally written down, but rather that, as we will see in Sect. 17.3, the spin and isospin dependence of the nuclear force can be explained in meson exchange models. Attempts to fit the potential terms to the experimental data have not fixed it exactly, but a general agreement exists for the first four terms. It should be also noted that many-body forces need to be taken into account for conglomerations of nucleons.

The central potential for the $S = 0$ case is applicable to the low energy proton-proton and neutron-neutron interactions. The attractive part is, however, not strong enough to create a bound state. For $S = 1$ on the other hand this potential together with the tensor force and the spin-spin interaction is strong enough to present us with a bound state, the deuteron.

17.2 The Deuteron

The deuteron is the simplest of all the nucleon bound states i.e., the atomic nuclei. It is therefore particularly suitable for studying the nucleon-nucleon interaction. Experiments have yielded the following data about the deuteron ground state:

Binding energy	$B = 2.225$ MeV
Spin and parity	$J^P = 1^+$
Isospin	$I = 0$
Magnetic moment	$\mu = 0.857\ \mu_{\mathrm{N}}$
Elec. quadrupole moment	$Q = 0.282\ e{\cdot}\mathrm{fm}^2$.

The proton-neutron system is mostly made up of an $\ell = 0$ state. If it were a pure $\ell = 0$ state then the wave function would be spherically symmetric, the quadrupole moment would vanish and the magnetic dipole moment would be just the sum of the proton and neutron magnetic moments (supposing that the nucleonic magnetic moments are not altered by the binding interaction). This prediction for the deuteron magnetic moment

$$\mu_\mathrm{p} + \mu_\mathrm{n} = 2.793\,\mu_\mathrm{N} - 1.913\,\mu_\mathrm{N} = 0.880\,\mu_\mathrm{N} \tag{17.9}$$

differs slightly from the measured value of $0.857\,\mu_\mathrm{N}$. Both the magnetic dipole moment and the electric quadrupole moment can be explained by the admixture of a state with the same J^P quantum numbers

$$|\psi_\mathrm{d}\rangle = 0.98 \cdot |\,^3\mathrm{S}_1\rangle + 0.20 \cdot |\,^3\mathrm{D}_1\rangle\,. \tag{17.10}$$

In other words there is a 4 % chance of finding the deuteron in a $^3\mathrm{D}_1$ state. This admixture can be explained from the tensor components of the nucleon-nucleon interaction.

We now want to calculate the nucleon wave function inside a deuteron. Since the system is more or less in an $\ell = 0$ state, the wave function will be spherically symmetric. We will need the depth V of the potential well (averaged over the attractive and repulsive parts) and its range, a. The binding energy of the deuteron alone gives us one parameter – the "volume" of the potential well, i.e., Va^2. The solutions of the Schrödinger equation (17.5) are

$$\begin{aligned} &\text{if } r<a: \ u_\mathrm{I}(r) = A\sin kr \quad \text{where } k = \sqrt{2m(E-V)}/\hbar,\ (V<0)\,,\\ &\text{if } r>a: \ u_\mathrm{II}(r) = C\mathrm{e}^{-\kappa r} \quad \text{where } \kappa = \sqrt{-2mE}/\hbar, \quad (E<0)\,, \end{aligned} \tag{17.11}$$

and $m \approx M_\mathrm{p}/2$ is the reduced mass of the proton-neutron system.

Continuity of $u(r)$ and $\mathrm{d}u(r)/\mathrm{d}r$ at the edge of the well, i.e., $r = a$, implies that [13]

$$k\cot ka = -\kappa \qquad\qquad ak \approx \frac{\pi}{2} \tag{17.12}$$

and

$$Va^2 \approx Ba^2 + \frac{\pi^2}{8}\frac{(\hbar c)^2}{mc^2} \approx 100\ \mathrm{MeV\,fm}^2\,. \tag{17.13}$$

Current values for the range of the nuclear force, and hence the effective extension of the potential $a \approx 1.2 \ldots 1.4\,\mathrm{fm}$, imply that the depth of the potential is $V \approx 50\,\mathrm{MeV}$. This is much greater than the deuteron binding energy B (just

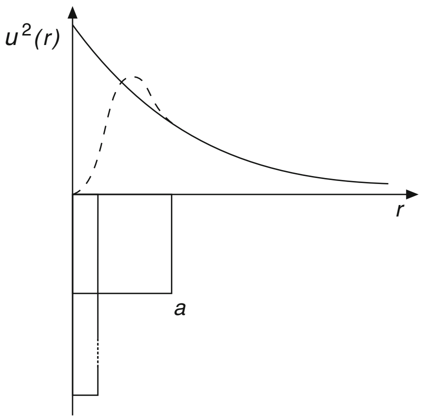

Fig. 17.5 Radial probability distribution $u^2(r) = r^2|\psi|^2$ of the nucleons in the deuteron for an attractive potential with range a (*dashed curve*) and for the range $a \to 0$ with a fixed volume Va^2 for the potential well (*continuous curve*)

2.25 MeV). The tail of the wave function, which is characterised by $1/\kappa \approx 4.3$ fm, is large compared to the range of the nuclear force.

The radial probability distribution of the nucleons is sketched in Fig. 17.5 for two values of a but keeping the volume of the potential well Va^2 constant. Since deuterium is a very weakly bound system the two calculations differ only slightly, especially at larger separations.

A more detailed calculation which takes the repulsive part of the potential into account only changes the above wave function at separations smaller than 1 fm (cf. Fig. 17.5). In Fig. 17.6 the probability distribution of nucleons in deuterium and of hydrogen atoms in a hydrogen molecule are given for comparison. The separations are in both cases plotted in units of the spatial extension *RMb* of the relevant hard core. The hard-core sizes are about $0.4 \cdot 10^{-10}$ m for the hydrogen molecule and roughly $0.5 \cdot 10^{-15}$ m for the deuteron. The atoms in the molecule are well localised – the uncertainty in their separation ΔR is only about 10 % of the separation (cf. Fig. 17.6). The nuclear binding in the deuteron is relatively "weak" and the bound state is much more spread out. This means that *the average kinetic energy is comparable to the average depth of the potential* and so the binding energy, which is just the sum of the kinetic and potential energies, must be very small.

The binding energy of the nucleons in larger nuclei is somewhat greater than that in the deuteron and the density is accordingly larger. Qualitatively we still have the same situation: a relatively weak effective force is just strong enough to hold nuclei together. The properties of the nuclei bear witness to this fact: it is a precondition both for the description of the nucleus as a degenerate Fermi gas and for the great mobility of the nucleons in nuclear matter.

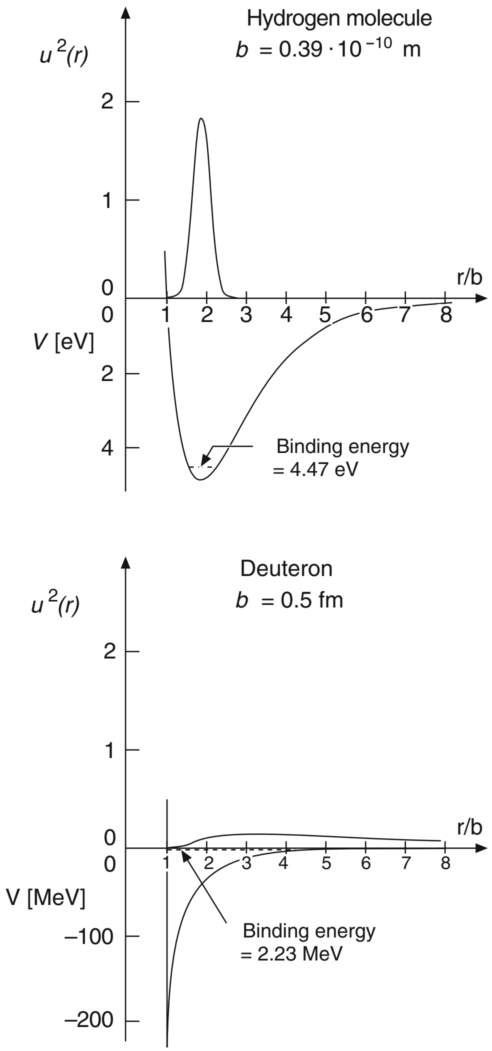

Fig. 17.6 The radial probability distribution $u^2(r)$ of the hydrogen atoms in a hydrogen molecule (*top*) [7] and of nucleons in a deuteron (*bottom*) in units of the relevant hard cores (From [1]). The covalent bond strongly localises the H atoms, since the binding energy is comparable to the depth of the potential. The weak nuclear bond, since the potential energy is comparable in size to the kinetic energy, means that the nucleons are delocalised

17.3 Nature of the Nuclear Force

We now turn to the task of understanding the strength and the radial dependence of the nuclear force. A sketch of the radial shape of the nucleon-nucleon potential that has been derived from a huge amount of elastic proton-proton and neutron-proton scattering data is shown in Fig. 17.7. It resembles very much the potential between two atoms that is repulsive when the two atoms overlap and attractive at larger distances. Therefore, we will first attempt to describe it by analogies to the atomic case and will employ mainly qualitative arguments.

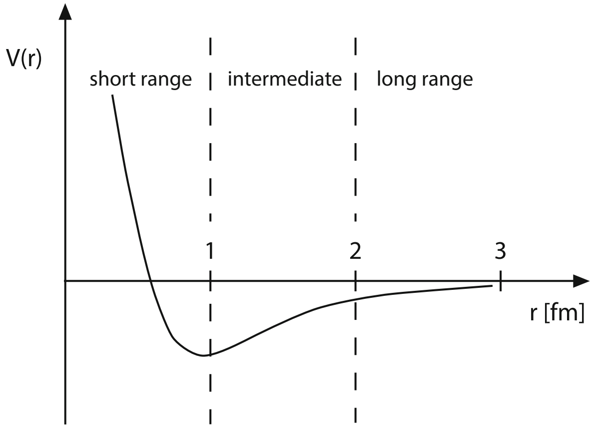

Fig. 17.7 Sketch of the nucleon-nucleon potential derived from phase shift analyses of low-energy elastic nucleon-nucleon scattering data

Ideally, we would like to derive the nuclear force between nucleons from QCD, the field theory of the strong interaction between quarks mediated by the exchange of gluons. Such a derivation is, however, not yet possible, one of the reasons being that nucleons are colour-neutral. At distances larger than the confinement scale only colour-neutral objects can be exchanged between them, i.e., only the exchange of two or more gluons, of quark-antiquark pairs or of mesons is possible.

Consequently, one has to rely on simplifications and approximations. Below, we will shortly present some aspects of two different approaches for the description of the nuclear force: quark models and meson-exchange models.

The nuclear force in the quark model A consistent theory of the nuclear force, based upon the interaction of quarks and gluons, does not yet exist. But we can explain qualitatively at least part of the nuclear potential in this model. Let us begin with the short-distance repulsive part of the nuclear force and try to construct some analogies to better understood phenomena. That atoms repel each other at short distances is a consequence of the Pauli principle. The electron clouds of both atoms occupy the lowest possible energy levels and if the clouds overlap then some electrons must be elevated into excited states using the kinetic energy of the colliding atoms. Hence we observe a repulsive force at short distances.

The quarks in a system of two nucleons also obey the Pauli principle, i.e., the 6-quark wave function must be totally antisymmetric. It is, however, possible to put as many as 12 quarks into the lowest $\ell = 0$ state without violating the Pauli principle, since the quarks come in three colours and have two possible spin ($\uparrow$, $\downarrow$) and isospin (u-quark, d-quark) directions. The spin-isospin part of the complete wave function must be symmetric since the colour part is antisymmetric and, for $\ell = 0$, the spatial part is symmetric. We thus see that the Pauli principle does not limit the occupation of the lowest quark energy levels in the spatial wave function, and so the fundamental reason for the repulsive core must be sought elsewhere.

The real reason is the spin-spin interaction between the quarks [4]. We have already seen how this makes itself noticeable in the baryon spectrum: the Δ baryon,

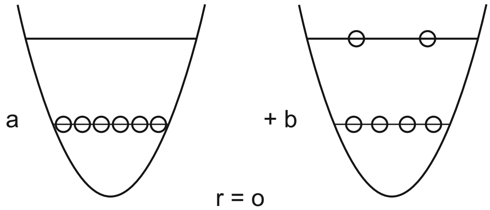

Fig. 17.8 The quark state for overlapping nucleons. This is composed of a configuration with six quarks in the $\ell = 0$ state (*left*) and a configuration with two quarks in the $\ell = 1$ state (*right*). In a non-adiabatic approximation it is found that the latter state dominates at separation $r = 0$ (probability $8/9$) [5, 14]. For larger distances this state becomes less important and disappears as $r \to \infty$

where the three quark spins are parallel to one another, is about 300 MeV/c^2 heavier than the nucleon. The potential energy then increases if two nucleons overlap and all six quarks remain in the $\ell = 0$ state since the number of quark pairs with parallel spins is greater than for separated nucleons. For each and every quark pair with parallel spins the potential energy increases by half the Δ-nucleon energy difference (16.11).

Of course, the nucleon-nucleon system tries to minimise its "chromomagnetic" energy by maximising the number of antiparallel quark spin pairs. But this is incompatible with remaining in an $\ell = 0$ state since the spin-flavour part of the wave function must be completely symmetric. The colourmagnetic energy can be reduced if at least two quarks are put into the $\ell = 1$ state. The necessary excitation energy is comparable to the decrease in the chromomagnetic energy, so the total energy will in any case increase if the nucleons strongly overlap. Hence the effective repulsion at short distances is in equal parts a consequence of an increase in the chromomagnetic and the excitation energies (Fig. 17.8). If the nucleons approach each other very closely ($r = 0$) one finds in a non-adiabatic approximation that there is an 8/9 probability of two of the quarks being in a p state [5, 14]. This configuration expresses itself in the relative wave function of the nucleons through a node at 0.4 fm. This together with the chromomagnetic energy causes a strong, short-range repulsion. The nuclear force may be described by a nucleon-nucleon potential which rises sharply at separations less than approximately 0.8 fm.

While the quark model can provide a rather plausible explanation of the repulsive core, the situation is much less satisfactory for the attractive part of the nuclear force. Again we will pursue analogies from atomic physics. As we know, the bonds between atoms are connected to a change in their internal structure and we expect something similar from the nucleons bound in the nucleus. Indeed a change in the quark structure of bound nucleons compared to that of their free brethren has been observed in deep-inelastic scattering off nuclei (EMC effect, see Sect. 8.5).

It is clear upon a moments reflection that the nuclear force is not going to be well described by an *ionic bond*: the confining forces are so strong that it is not possible to lend a quark from one nucleon to another if they do not overlap substantially.

It is often argued that the nuclear forces that are responsible for nuclear binding are residual colour forces, much like the *Van der Waals force* in atomic and molecular physics where the atoms polarise each other and then stick to each other via the resulting dipole-dipole interaction (two-photon exchange) which leads to a potential that decreases with the distance r between the centres of the atoms like $1/r^6$. The analogy in nuclear physics would be a Van der Waals force transmitted by the exchange of two gluons. Explicit calculations have shown, however, that such a force would be too weak to explain the nuclear force at intermediate distances [12].

The only analogy left to us to explain the nuclear force is a *covalent bond*, such as that which is, e.g., responsible for holding the H_2 molecule together. Here the electrons of the two H atoms are continually swapped around and can be ascribed to both atoms. The attractive part of the nuclear force is strongest at distances of around 1 fm, corresponding approximately to the mean square charge radius of the proton. At this distance there is still substantial overlap between the two nucleons. The force could be expressed by the exchange of "single" quarks and indeed reminds us of the atomic covalent bond. Again, model calculations [12] show that the depth of the corresponding potential is much smaller than the experimental value. In fact, quark exchange is less effective than its atomic counterpart of electron exchange. This is partly because to be exchanged the quarks must have the same colour, and there is only a $1/3$ probability of this. Thus the covalent bond concept, if it is directly transferred from molecules to nuclei, does not give us a good quantitative description of what is going on in nuclei.

Up to now we have neglected the fact that as well as the three quarks in the nucleon there are additional quark-antiquark pairs which are continually being created from gluons and annihilated back into them again. An effective quark-quark exchange may be produced by colour neutral quark-antiquark pairs. This quark-antiquark exchange actually plays a larger role in the nucleon-nucleon interaction than does the simple swapping of two quarks.

The nuclear force in meson-exchange models Ever since Yukawa in 1935 first postulated the existence of the pion [16], there have been attempts to describe the inter-nuclear forces in terms of meson exchange [2]. The exchange of mesons with mass m leads to a potential of the form

$$V = g \cdot \frac{\mathrm{e}^{-\frac{mc}{\hbar} r}}{r}, \tag{17.14}$$

where g is a charge-like constant. This is known as the *Yukawa potential.*

■ To derive the Yukawa potential we first assume that the nucleon acts as a source of virtual mesons in the same way as an electric charge may be viewed as a source of virtual photons.

We start with the wave equation of a free, relativistic particle with mass m. If we replace the energy E and momentum $\boldsymbol{p}$ in the energy momentum relationship $E^2 = \boldsymbol{p}^2c^2 + m^2c^4$ by the operators $i\hbar\,\partial/\partial t$ and $-i\hbar\boldsymbol{\nabla}$, as is done in the Schrödinger equation, we obtain the *Klein-Gordon equation*:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi(\boldsymbol{x},t) = \left(\boldsymbol{\nabla}^2 - \mu^2\right)\Psi(\boldsymbol{x},t)\,, \qquad \text{where} \quad \mu = \frac{mc}{\hbar}\,. \tag{17.15}$$

For a massless particle ($\mu = 0$) this equation describes a wave travelling at the speed of light. If we replace Ψ by the electromagnetic four-potential $A = (\phi/c, \boldsymbol{A})$ we obtain the equation for electromagnetic waves in vacuo at a large distance from the source. One may thus interpret $\Psi(\boldsymbol{x},t)$ as the wave function of the photon.

Consider now the static field limit where (17.15) reduces to

$$\left(\boldsymbol{\nabla}^2 - \mu^2\right)\psi(\boldsymbol{x}) = 0\,. \tag{17.16}$$

If we demand a spherically symmetric solution, i.e., one that solely depends upon $r = |\boldsymbol{x}|$ we find

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\psi(r)}{\mathrm{d}r}\right) - \mu^2\psi(r) = 0\,. \tag{17.17}$$

A particularly simple ansatz for the potential V that results from exchanging the particle is $V(r) = g\cdot\psi(r)$, where g is an arbitrary constant. It is clear that this ansatz can make sense if we consider the electromagnetic case: in the limit $\mu \to 0$ we obtain the Poisson equation for a space without charges from (17.16), and we obtain from (17.17) the Coulomb potential $V_\mathrm{C} \propto 1/r$, i.e., the potential of a charged particle at a large separation where the charge density is zero. If we now solve (17.17) for the massive case, we obtain the Yukawa potential (17.14). This potential initially decreases roughly as $1/r$ and then much more rapidly. The range is of the order of $1/\mu = \hbar/mc$, which is also what one would expect from the uncertainty relation. The interaction due to pion exchange has a range of about 1.4 fm.

The above remarks are somewhat naive and not an exact derivation. We have ignored the spin of the particle: the Klein-Gordon equation holds for spinless particles (luckily this is true for the pion). Additionally a virtual meson does not automatically have the rest mass of a free particle. Furthermore these interactions take place in the immediate vicinity of the nucleons and the mesons can strongly interact with them. The wave equation of a free particle can at best be an approximation.

A detailed derivation and discussion of the contributions to the nucleon-nucleon potential by the exchange of the various meson types and their spin and isospin dependences can be found in many reviews, for instance in [8]. Here, we will only summarize the main qualitative features. Nucleon-nucleon interaction by meson exchange is sketched in Fig. 17.9. The range of the corresponding potential decreases as the meson mass increases. At large distances the exchange of single pions dominates. The pion is a pseudoscalar particle and contributes weakly to the central attractive part V_0 and the spin-dependent part V_ss of the nucleon-nucleon potential (17.8), but strongly to the tensor part V_T that is necessary to describe the deuteron properties. To explain the magnitude of the potential in the intermediate range, one needs to introduce the exchange of a scalar object with quantum numbers $J^P(I) = 0^+(0)$ and a mass of several times the pion mass. The exchange of this scalar object leads to a strong attractive central force and a contribution to the spin-orbit potential V_LS. This elusive scalar object, originally named "σ" meson, might possibly be identified with the $\mathrm{f}_0(500)$, a state with a mass of 400–550 MeV/c^2 and

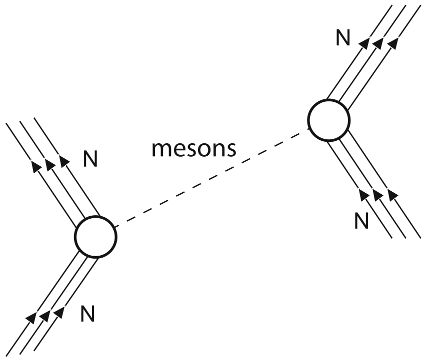

Fig. 17.9 At distances between neighboured nucleons larger than 0.5 fm the nuclear force is mediated by the exchange of mesons. At large distances one-pion exchange dominates, the repulsion at small distances is due to the exchange of vector mesons. In the intermediate range the exchange of a scalar object (two-pion exchange) plays an important role

a large width of 400–700 MeV/c^2 decaying dominantly into two pions [11]. For a long time the existence of this particle was very uncertain. Therefore, the exchange of this fictitious particle has been considered to effectively be the exchange of two pions combining to the correct quantum numbers. At distances below approximately 0.8 fm the potential is governed by the exchange of the two lightest vector mesons ϱ and ω with masses of $\sim$780 MeV/c^2. For vector particles, one finds a strong repulsive central force (analogous to the repulsion between two like charges caused by one-photon exchange) and a spin-orbit force which has the same sign as in the scalar case but is (for similar masses of scalar and vector particles) by a factor of three stronger. The tensor force has the opposite sign compared to the pseudoscalar case and damps the strong tensor force of one-pion exchange at small distances.

Various potentials based on meson exchange have been developed in the last decades that reproduce the measured cross-sections and phase-shifts with high precision. These models are, however, not fundamental. One neglects, for example, the internal structure of nucleons and mesons and assumes that they are point-particles. The meson-nucleon coupling constants that emerge from experiment must be adapted to take this into account. Altogether more than 30 parameters are needed in these models to describe the potential.

Since mesons are really colour neutral quark-antiquark pairs their exchange and that of colour neutral $q\overline{q}$ pairs give us, in principle, two equivalent ways of describing the nucleon-nucleon interaction. At short distances, however, where the structure of the nucleons must definitely play a part, a description in terms of meson exchange appears to be inadequate. The coupling constant for the exchange of ω mesons, which is responsible for the repulsive part of the potential, has to be given an unrealistically high value – about two or three times the size one would accept from a comparison with the other meson-nucleon couplings. The repulsive part of the potential is better described in a quark picture. On the other hand one-pion exchange models give an excellent fit to the data at larger separations.

The most recent approach to describe the nucleon-nucleon potential and the nuclear force is an effective field theory based on *chiral perturbation theory* [6]. This theory has pions, the nucleon and eventually the Δ resonance (all point-like) as effective fundamental degrees of freedom rather than quarks and gluons. It is nowadays considered by its advocates as the state-of-the-art method for microscopic calculations of interactions at low energies of hadrons and light mesons, of two-nucleon and many-nucleon forces and many aspects of nuclear structure. It is beyond the scope of this book to go into any detail. The interested reader may find derivations and detailed discussions in several reviews like, e.g., [3, 9].

Problems

1. **The nuclear force**
   The nuclear force is transmitted by exchanging mesons. What are the ranges of the forces generated by exchanging the following: a π, two π's, a ϱ, an ω? Which properties of the nuclear force are determined by the exchange particles?
2. **Neutron-proton scattering**
   How large would the total cross-section for neutron-proton scattering be if only the short range repulsion (range, $b = 0.7\,\mathrm{fm}$) contributed? Consider the energy regime in which $\ell = 0$ dominates.

References

1. A. Bohr, B.R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1969)
2. D.M. Brink, *Nuclear Forces* (Pergamon Press, Oxford, 1965)
3. E. Epelbaum et al., Rev. Mod. Phys. **81**, 1773 (2009)
4. A. Faessler, Prog. Part. Nucl. Phys. **20**, 151 (1988)
5. A. Faessler et al., Nucl. Phys. **A402**, 555 (1982)
6. H. Georgi, *Weak Interactions and Modern Particle Theory* (Dover, Mineola, 2009)
7. G. Herzberg, *Spectra of Diatomic Molecules* (Van Nostrand, New York, 1950)
8. R. Machleidt, Adv. Nucl. Phys. **19**, 189 (1989)
9. R. Machleidt, D.R. Entem, Phys. Rep. **503**, 1 (2011)
10. M.A. Preston, *Physics of the Nucleus,* 2nd edn. (Addison-Wesley, Wokingham, 1962)
11. Particle Data Group, J. Beringer et al., *Review of Particle Properties*. Phys. Rev. D **86**, 010001 (2012)
12. M. Rosina, B. Povh, Nucl. Phys. **A572**, 48 (1994)
13. F. Schwabl, *Quantum Mechanics,* 4th edn. (Springer, Berlin/Heidelberg/New York, 2007)
14. U. Straub et al., Nucl. Phys. **A483**, 686 (1988)
15. V.A. Telegdi, Sci. Am. **206**, 50 (1962)
16. H. Yukawa, Proc. Phys. Math. Soc. Jpn. **17**, 48 (1935)