9 Angular Momentum

In the previous chapter, we dealt with the kinematics and dynamics of the rotation of an extended object about a fixed axis. The rotational motion was analyzed in terms of Newton's second law for rotation as well as rotational kinetic energy.

In this chapter, we introduce the concept of ***angular momentum***, a quantity that plays a key role in *rotational dynamics*. Using classical physics, we saw how linear momentum was conserved. Similarly, we will see how the ***conservation of angular momentum*** is a fundamental law in rotational dynamics, and in further studies (not introduced in this book) can be proved to be equally valid for relativistic and quantum physics.

9.1 Angular Momentum of Rotating Systems

9.1.1 Angular Momentum of a Particle

Figure 9.1 depicts a particle of mass m that has a momentum $\vec{p} = m\vec{v}$ and a position vector $\vec{r}$ that is measured with respect to an origin O of an inertial frame. The angular momentum $\vec{L}$ of this particle about the origin O is defined by the vector product:

$$\vec{L} = \vec{r} \times \vec{p} \tag{9.1}$$

Following the right-hand rule introduced in Chap. 2, the direction of $\vec{L}$ is perpendicular to the plane containing the two vectors $\vec{r}$ and $\vec{p}$ as shown in Fig. 9.1 and its magnitude is given by:

$$L = r\,p \sin\theta = r\,p_{\perp} = r_{\perp}\,p \tag{9.2}$$

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_9,

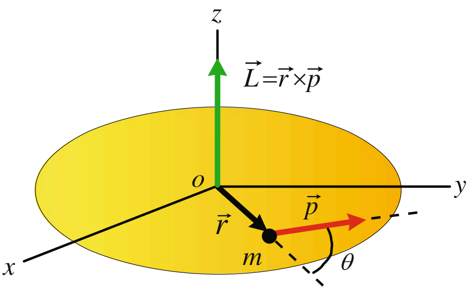


Fig. 9.1 The angular momentum $\vec{L}$ of a particle of mass m and momentum $\vec{p}$ located at position $\vec{r}$ is defined by $\vec{L} = \vec{r} \times \vec{p}$

where θ is the angle between $\vec{r}$ and $\vec{p}$. In addition, $p_\perp = p\sin\theta$ and $r_\perp = r\sin\theta$ are the components of $\vec{p}$ and $\vec{r}$ perpendicular to $\vec{r}$ and $\vec{p}$, respectively. It follows that $L = 0$ when $\vec{r}$ is parallel or antiparallel to $\vec{p}$ ($\theta = 0$ or $\theta = 180°$). The SI unit of L is kg.m^2/s (or J.s).

To find the relation between angular momentum and torque, we differentiate Eq. 9.1 with respect to time as follows:

$$\frac{d\vec{L}}{dt} = \frac{d\,(\vec{r} \times \vec{p})}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

Using Newton's second law $\Sigma\vec{F} = d\vec{p}/dt$ and the definition of net torque $\Sigma\vec{\tau} = \vec{r} \times \Sigma\vec{F}$, we see that the first term is just $\Sigma\vec{\tau}$. The second term is zero since $d\vec{r}/dt \times \vec{p} = \vec{v} \times (m\vec{v}) = m\vec{v} \times \vec{v} = 0$. Therefore:

$$\boxed{\Sigma\vec{\tau} = \frac{d\vec{L}}{dt}} \quad \text{(Single particle)} \tag{9.3}$$

That is, the net torque acting on a particle is equal to the time rate of change of the particle's angular momentum. The expression 9.3 is the rotational analogous to $\Sigma\vec{F} = d\vec{p}/dt$ in the case of linear motion, where $\Sigma\vec{F} \Leftrightarrow \Sigma\vec{\tau}$ and $\vec{p} \Leftrightarrow \vec{L}$.

For a particle of mass m moving with a constant speed v in a circular path of radius r, i.e. $v = r\omega$, the magnitude of the orbital angular momentum $\vec{L}$ is constant and given by:

$$L = rmv\sin 90° = mvr \quad \Rightarrow \quad L = I\omega \tag{9.4}$$

where $I = mr^2$ is the moment of inertia of the particle. Application of the right-hand rule shows that the direction of $\vec{L}$ is also constant and perpendicular to the plane of the circle, although the direction of $\vec{p} = m\vec{v}$ keeps changing.

9.1.2 Angular Momentum of a System of Particles

Consider a system made of n particles having angular momenta $\vec{L}_1, \vec{L}_2, \ldots, \vec{L}_n$. Regardless of whether these particles are loosely bound, or tightly bound together (as in a rigid body), or free, the total angular momentum $\vec{L}$ is always:

$$\vec{L} = \Sigma \vec{L}_i, \quad (i = 1, 2, \ldots, n) \tag{9.5}$$

If we differentiate this equation with respect to time, we get:

$$\frac{d\vec{L}}{dt} = \sum \frac{d\vec{L}_i}{dt} = \sum \vec{\tau}_i, \quad (i = 1, 2, \ldots, n) \tag{9.6}$$

Based on Newton's third law, the sum of all internal torques must add to zero due to the cancelation effect of all internal forces on the system. Therefore, the net torque on the system is only due to all external torques, and Eq. 9.6 reduces to:

$$\Sigma \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt} \quad \text{(System of particles)} \tag{9.7}$$

where $\vec{\tau}_{\text{ext}}$ and $\vec{L}$ are calculated with respect to a fixed point in an inertial frame. This is the rotational analogue to $\Sigma \vec{F}_{\text{ext}} = d\vec{P}/\,dt$ in the case of linear motion, where $\vec{F}_{\text{ext}} \Leftrightarrow \vec{\tau}_{\text{ext}}$ and $\vec{p} \Leftrightarrow \vec{L}$.

We can prove that Eq. 9.7 is valid for a fixed point on the center of mass of the system, even if the CM is accelerating. Thus:

$$\Sigma\, \vec{\tau}_{\text{CM}} = \frac{d\vec{L}_{\text{CM}}}{dt} \quad \text{(Even if CM is accelerating)} \tag{9.8}$$

9.1.3 Angular Momentum of a Rotating Rigid Body

Consider a rigid body rotating with an angular speed ω about a fixed axis; say the z-axis is as shown in Fig. 9.2. A typical mass element Δm_i of the rigid body moves with a speed v_i around the z-axis in a circular path of radius r_i, i.e. $v_i = r_i\ \omega$. If the position of this element is measured with respect to an origin O, then with the use of r_i we will be able to find the component of the angular momentum about the rotational axis, which is the z-axis in this case.

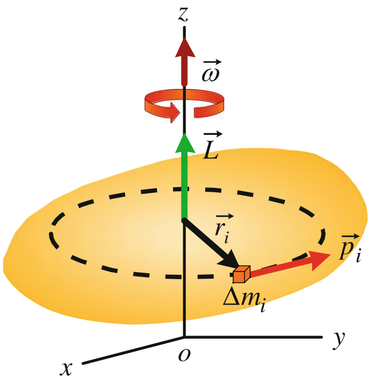


Fig. 9.2 A rigid body rotates about the z-axis with an angular speed ω. The component of the angular momentum will be along the z-axis

The z-component of the angular momentum of this element is:

$$L_i = r_i \Delta m_i v_i = \Delta m_i r_i^2\, \omega$$

where the vector $\vec{L_i}$ is directed along the z-axis just like the vector $\vec{\omega}$. The total component of the angular momentum about the rotational axis is the sum of all $\vec{L_i}$ and denoted by L_z. Thus:

$$L_z = \Sigma L_i = \Sigma\, \Delta m_i r_i^2 \omega = \left(\Sigma\, \Delta m_i r_i^2\right) \omega$$

Since $\sum \Delta m_i r_i^2 \rightarrow \int r^2 dm$, which is the moment of inertia of the body about the z-axis (see Chap. 8), then the above relation reduces to:

$$L_z = I\omega \quad \text{(Rigid body)} \tag{9.9}$$

Note that choosing any point on the z-axis and using that point as the origin O would yield the same Eq. 9.9. Accordingly, Eq. 9.7 will take on the following form for any rigid body:

$$\Sigma\, \tau_{\text{ext}} = \frac{dL_z}{dt} \quad \text{(Rigid body)} \tag{9.10}$$

If we differentiate Eq. 9.9 with respect to time, we get:

$$\frac{dL_z}{dt} = I \frac{d\omega}{dt} = I\, \alpha$$

Substituting this result into Eq. 9.10, we get:

$$\Sigma\tau_{\text{ext}} = I\alpha \quad \text{(Rigid body)} \tag{9.11}$$

This result is the same as in Eq. 8.32, which was derived using an approach that was based on the study of forces.

If the rigid body in Fig. 9.2 rotates about an axis of symmetry that passes through its center of mass, then L_z becomes the total angular momentum $\vec{L}$ of the body and Eqs. 9.9, 9.10, and 9.11 can be written in vector form as follows:

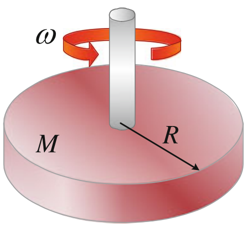
$$\left.\begin{aligned}\vec{L} &= I\vec{\omega}\\ \sum\vec{\tau}_{\text{ext}} &= \frac{d\vec{L}}{dt}\\ \sum\vec{\tau}_{\text{ext}} &= I\vec{\alpha}\end{aligned}\right. \quad \left(\begin{array}{l}\text{Rotation of rigid body}\\ \text{about its symmetry axis}\end{array}\right) \tag{9.12}$$

If the rigid object is *not symmetric*, then $\vec{L}$ and $\vec{\omega}$ may point in different directions and in this case $\vec{L}$ represents the component of the angular momentum along the axis of rotation.

Example 9.1

A disk of mass $M = 8$ kg and radius $R = 0.5$ m accelerates about its massless axle from rest to an angular speed $\omega = 8.5$ rad/s in a time $\Delta t = 2$ s, see Fig. 9.3. Find the angular momentum of the disk and the required constant torque used for this acceleration.

Fig. 9.3



Solution: According to Eq. 9.12, the angular momentum of the disk about its symmetry axis will be:

$$L = I\omega = \tfrac{1}{2}MR^2\,\omega = \tfrac{1}{2}(8\text{ kg})(0.5\text{ m})^2(8.5\text{ rad/s}) = 8.5\text{ J.s}$$

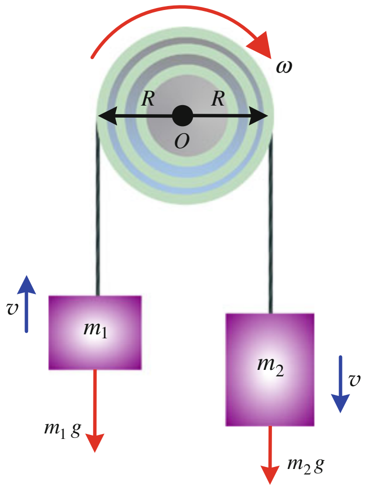
According to Eq. 9.12, the required constant torque that accelerates the disk from rest to 8.5 rad/s in 2 s is:

$$\tau_{ext} = \frac{\Delta L}{\Delta t} = \frac{L_f - L_i}{\Delta t} = \frac{8.5 \text{ J.s} - 0}{2 \text{ s}} = 4.25 \text{ m·N}$$

Example 9.2

An Atwood machine consists of two masses m_1 and m_2 ($m_2 > m_1$), which are connected by a light cord that passes over a freely rotating pulley, see Fig. 9.4. The pulley has a radius R and moment of inertia I about its axle. Find the acceleration of the two masses (consider $m_1 = 4$ kg, $m_2 = 6$ kg, $I = 2 \times 10^{-4}$ kg.m^2, and $R = 2$ cm).

Fig. 9.4



Solution: We can solve this problem by finding $\Sigma\vec{\tau}_{ext}$ and $\vec{L}_{net}$, and then by using Eq. 9.7, $\Sigma\vec{\tau}_{ext} = d\vec{L}_{net}/dt$, to find the acceleration $a = dv/dt$. Since the tensions in the two parts of the cord are internal forces, the net external torque of all external forces about the pulley's axel O (taking clockwise as positive since $m_2 > m_1$) is:

$$\begin{aligned} \Sigma\tau_{ext} &= m_2 gR - m_1 gR \\ &= (m_2 - m_1)gR \end{aligned}$$

At a given instant, when the speed of the two masses is v, the angular momenta of m_2 and m_1 are $Rm_2 v$ and $Rm_1 v$, respectively. In addition, the angular momentum of the pulley is $I\omega$, where $v = R\omega$. Thus, the total clockwise angular momentum about O is:

$$L = Rm_1v + Rm_2v + I\frac{v}{R} = \left(m_1 + m_2 + \frac{I}{R^2}\right)Rv$$

By applying $\Sigma\tau_{ext} = dL/dt$, we get:

$$(m_2 - m_1)gR = \left(m_1 + m_2 + \frac{I}{R^2}\right)R\frac{dv}{dt}$$

Solving for $a = dv/dt$, we get:

$$a = \frac{m_2 - m_1}{m_1 + m_2 + \dfrac{I}{R^2}}g = \frac{6\text{ kg} - 4\text{ kg}}{6\text{ kg} + 4\text{ kg} + \dfrac{2\times10^{-4}\text{ kg.m}^2}{(2\times10^{-2}\text{ m})^2}}(9.8\text{ m/s}^2) = 1.87\text{ m/s}^2$$

If I is ignored, we get $a = (m_2 - m_1)g/(m_1 + m_2) = 1.96\text{ m/s}^2$ as proved in Example 5.3 of Chap. 5. Since this value is larger than 1.87 m/s^2, then the moment of inertia actually slows down the system.

Example 9.3

A rod having a mass $M = 3$ kg and length $d = 2$ m is pivoted (without friction) at its center O. Then, two masses $m_1 = 4$ kg and $m_2 = 7$ kg are treated as points and placed on the ends of that rod such that they are equidistant from O. At a particular moment in time the rod makes angle θ with the horizontal and the system is rotating in a vertical plane with an angular speed ω, see Fig. 9.5. (a) Find the system's angular momentum L and angular acceleration α. (b) How far away from the pivot O should m_2 be placed in order to acquire a balanced system having zero angular acceleration? Take $g = 10$ m/s^2.

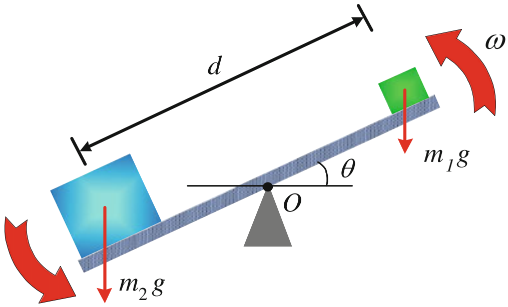
Solution: (a) The moment of inertia of the system about O is:

$$\begin{aligned} I &= m_1(\tfrac{1}{2}d)^2 + m_2(\tfrac{1}{2}d)^2 + \tfrac{1}{12}Md^2 = \tfrac{1}{4}(m_1 + m_2 + \tfrac{1}{3}M)d^2 \\ &= \tfrac{1}{4}(4\text{ kg} + 7\text{ kg} + \tfrac{1}{3}\times 3\text{ kg})(2\text{ m})^2 = 12\text{ kg.m}^2 \end{aligned}$$

Then, the magnitude of the total angular momentum of the system is:

$$L = I\omega = 12\,\omega \quad \text{(Out of the page with units of J.s)}$$

Fig. 9.5



To find the angular acceleration at any angle θ, we use $\Sigma\tau_{ext} = I\alpha$. To achieve this, we first find the magnitude of the two torques about O due to the forces m_1g and m_2g as follows:

$$\tau_1 = (d/2)m_1g\cos(90^\circ + \theta) = \tfrac{1}{2}m_1gd\cos\theta \quad \text{(Into page)}$$
$$\tau_2 = (d/2)m_2g\cos(90^\circ - \theta) = \tfrac{1}{2}m_2gd\cos\theta \quad \text{(Out of the page)}$$

Since $m_2 > m_1$, then the net external torque on the system about O is:

$$\begin{aligned}\sum\tau_{ext} &= \tau_2 - \tau_1 = \tfrac{1}{2}(m_2 - m_1)gd\cos\theta \\ &= \tfrac{1}{2}(7\text{ kg} - 4\text{ kg})(10\text{ m/s}^2)(2\text{ m})\cos\theta \\ &= 30\cos\theta\ \text{(m.N)} \quad \text{(Out of the page)}\end{aligned}$$

The angular acceleration at the instant shown in Fig. 9.5 is thus:

$$\alpha = \frac{\Sigma\tau_{ext}}{I} = \frac{\frac{1}{2}(m_2 - m_1)gd\cos\theta}{\frac{1}{4}(m_1 + m_2 + \frac{1}{3}M)d^2} = \frac{30\cos\theta\ \text{(m.N)}}{12\text{ kg.m}^2} = 2.5\cos\theta\text{ rad/s}^2$$

(b) Notice that as m_2 slides towards the pivot, the value of $\Sigma\tau_{ext}$ decreases and the system tends to be more balanced. In the case where the system is balanced we have $\Sigma\tau_{ext} = 0$ and $\alpha = 0$. If we assume the balance occurs when the distance between m_2 and the pivot is x, then:

$$\Sigma\tau_{ext} = \tau_2 - \tau_1 = m_2gx\cos\theta - \tfrac{1}{2}m_1gd\cos\theta = 0$$

Thus: $$x = \tfrac{1}{2}\frac{m_1}{m_2}d = \tfrac{1}{2}\frac{4\text{ kg}}{7\text{ kg}} \times (2\text{ m}) = \tfrac{4}{7}\text{ m}$$

9.2 Conservation of Angular Momentum

In Chap. 7, we found that the general form of Newton's second law for the translational motion, Eq. 7.43, is given by:

$$\sum \vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt}$$

where $\sum \vec{F}_{\text{ext}}$ is the net external force acting on a system of particles (including rigid objects) and $\vec{P}$ is the total linear momentum of the system. If the system has a total mass M and its CM is moving with velocity $\vec{v}_{\text{CM}}$, then $\vec{P} = M\,\vec{v}_{\text{CM}}$. In addition, if the net external force is zero, then the total momentum $\vec{P}$ is conserved (which is the law of conservation of momentum) and $\vec{v}_{\text{CM}} = \text{constant}$.

In this chapter, we found an analogous relationship, Eq. 9.7, which describes the general rotational motion of a system of particles (including rigid objects). This was given by:

$$\Sigma \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$$

where $\Sigma \vec{\tau}_{\text{ext}}$ is the net external torque acting on a system of particles (including rigid objects) and $\vec{L}$ is the total angular momentum of the system. This relation is valid when $\Sigma \vec{\tau}_{\text{ext}}$ and $\vec{L}$ are evaluated either about a point fixed in an inertial reference frame, or about the CM of the system (even if the CM is accelerating). In addition, for isolated systems, the last relation leads to the following conclusion:

$$\text{If } \Sigma \vec{\tau}_{\text{ext}} = 0, \text{ then} \frac{d\vec{L}}{dt} = 0 \quad \text{and} \quad \vec{L} = \text{constant}$$

Therefore:

$$\boxed{\vec{L}_i = \vec{L}_f} \quad \text{(For an isolated system)} \tag{9.13}$$

This is the law of **conservation of angular momentum**, where i refers to some initial time, and f refers to a later time. In other words:

> Conservation of angular momentum:
> If the net external torque acting on a system is zero (i.e. an isolated system), the total angular momentum of the system remains constant in both magnitude and direction.

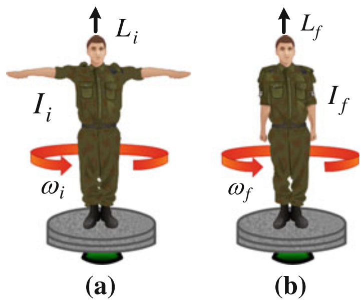
We can now state that the total energy, total linear momentum, and total angular momentum of an isolated system all remain constant:

$$\begin{aligned} E_f &= E_i \\ \vec{P}_f &= \vec{P}_i \\ \vec{L}_f &= \vec{L}_i \end{aligned} \quad \text{(For an isolated system)} \tag{9.14}$$

Example 9.4

A soldier stands with his arms stretched out at the center of a platform that rotates without friction with an angular speed $\omega_i = 1.8$ rev/s, see Fig. 9.6a. The rotational inertia of the soldier and platform is $I_i = 6\,\text{kg.m}^2$. When the soldier pulls his arms close to his body, as shown in Fig. 9.6b, he decreases the rotational inertia of the system to $I_f = 4\,\text{kg.m}^2$. (a) What is the resulting final angular speed of the system? (b) Is there a gain or a loss in the rotational kinetic energy of the system; and which of the objects, soldier or platform, gained or lost this energy? (c) If the platform is a disk of mass $M = 10$ kg and radius $R = 40$ cm, what is the moment of inertia of the soldier when his arms are close to his body?

Fig. 9.6



Solution: (a) Because there is no net external torque acting on the system about the axis of rotation, we can apply the law of conservation of angular momentum as follows:

$$L_f = L_i \quad \Rightarrow \quad I_f\omega_f = I_i\omega_i \quad \Rightarrow \quad \omega_f = \frac{I_i}{I_f}\omega_i = \frac{6\,\text{kg.m}^2}{4\,\text{kg.m}^2}1.8\,\text{rev/s} = 2.7\,\text{rev/s}$$

(b) The ratio of the final to the initial rotational kinetic energy is:

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}I_f\omega_f^2}{\frac{1}{2}I_i\omega_i^2} = \frac{(4\,\text{kg.m}^2)(2.7\,\text{rev/s})^2}{(6\,\text{kg.m}^2)(1.8\,\text{rev/s})^2} = 1.5$$

This gain in rotational kinetic energy to both the solider and platform is due to the work done by the soldier by moving his arms inwards.

(c) Since $I_f = I_s + I_{\text{disk}}$, then the moment of inertia of the soldier I_s is:

$$I_s = I_f - \tfrac{1}{2}MR^2 = 4\,\text{kg.m} - \tfrac{1}{2}(10\,\text{kg})(0.4\,\text{m})^2 = 3.2\,\text{kg.m}^2$$

Example 9.5

A small mass m attached to one end of a light cord is constrained to rotate in a circular path over a frictionless table. The other end of the cord passes through a small hole O in the table, see Fig. 9.7. For an initial tension T_i and radius R_i, the initial angular speed of the mass is $\omega_i = 0.5$ rad/s, see Fig. 9.7a. The tension is then increased gradually to T_f when the cord is pulled until the radius is reduced to $R_f = R_i/3$, see Fig. 9.7b. (a) Find the final angular speed of the mass. (b) Find the ratio of the tensions T_f/T_i.

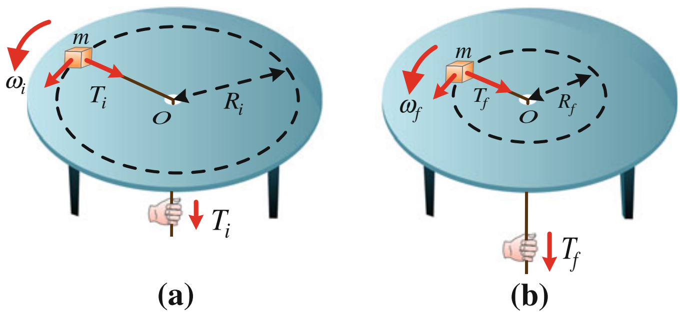


Fig. 9.7

Solution: (a) There is no torque about O since the force is central. Therefore, angular momentum is conserved. Thus:

$$L_f = L_i \quad \Rightarrow \quad I_f\omega_f = I_i\omega_i$$

If we treat the small mass as a particle with a moment of inertia $I = mr^2$, then we have:

$$mR_f^2\omega_f = mR_i^2\omega_i$$

Thus: $\omega_f = \left(\frac{R_i}{R_f}\right)^2 \omega_i = \left(\frac{R_i}{R_i/3}\right)^2 \omega_i = 9\omega_i = 9 \times 0.5\,\text{rad/s} = 4.5\,\text{rad/s}$

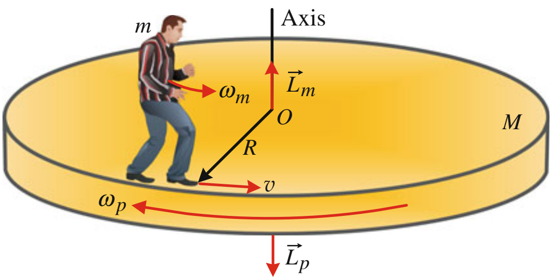
(b) The tension supplies the centripetal force which is needed to constrain the mass to move in a circle. So, $T = ma_r = mr\omega^2$ and we have:

$$\frac{T_f}{T_i} = \frac{mR_f\omega_f^2}{mR_i\omega_i^2} = \left(\frac{R_f}{R_i}\right)\left(\frac{\omega_f}{\omega_i}\right)^2 = (\tfrac{1}{3})(9)^2 = 27$$

Example 9.6

A man of mass $m = 60\,\text{kg}$ stands at the edge of a stationary circular platform of mass $M = 400\,\text{kg}$ and radius $R = 3\,\text{m}$. The platform is mounted on a frictionless bearing. When the man begins running at a speed $v = 4\,\text{m/s}$ around the platform's edge, the platform begins to rotate in the opposite direction as shown in Fig. 9.8. What is the angular speed and the period of the platform?

Fig. 9.8



Solution: Initially, the total angular momentum is zero, i.e. $\vec{L} = 0$. Since there is no net external torque on the system while the man is running on the platform, $\vec{L}_f$ of the system will remain zero. Thus:

$$\vec{L}_f = \vec{L}_i \quad \Rightarrow \quad \vec{L}_m + \vec{L}_p = 0 \quad \Rightarrow \quad L_m - L_p = 0 \quad \Rightarrow \quad L_p = L_m$$

where L_m and L_p are the magnitudes of the man's and platform's angular momentum, respectively. Modeling the man as a particle, we can write his moment of inertia as $I_m = mR^2$ and his angular speed about the axis of rotation as $\omega_m = v/R$. Then, treating the platform as a disk with a moment of inertia $I_p = \frac{1}{2}MR^2$, we can use the previous result of conservation of angular momentum $I_p\omega_p = I_m\omega_m$ to find:

$$\omega_p = \frac{I_m}{I_p}\omega_m = \frac{mR^2}{\frac{1}{2}MR^2}\frac{v}{R} = \frac{2mv}{MR} = \frac{2(60\,\text{kg})(4\,\text{m/s})}{(400\,\text{kg})(3\,\text{m})} = 0.4\,\text{rad/s}$$

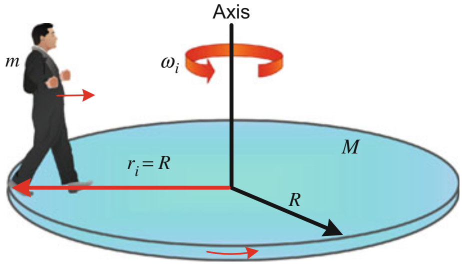
The rotational period of the platform is thus:

$$T_p = \frac{2\pi}{\omega_p} = \frac{2\pi}{0.4\,\text{rad/s}} = 15.7\text{ s per revolution}$$

Example 9.7

A man of mass $m = 60$ kg stands at the edge of a rotating circular platform of mass $M = 220$ kg and radius R. The platform is mounted on a frictionless bearing. Initially, the angular speed of the system is $\omega_i = 0.5$ rad/s. The man starts to walk slowly and radially towards the center from the edge at $r_i = R$, see Fig. 9.9. What is the angular speed of the system when the man reaches a radius of $r_f = R/2$?

Fig. 9.9



Solution: The angular speed changes due to the change in the moment of inertia of the system during the walk. We model the man as a particle in this example. Since there is no net external torque on the system while the man is walking on the platform, the angular momentum of the system will remain constant. Thus:

$$L_f = L_i \quad \Rightarrow \quad I_f\ \omega_f = I_i\ \omega_i \quad \Rightarrow \quad (I_p + I_{mf})\ \omega_f = (I_p + I_{mi})\ \omega_i$$

where the moment of inertia of the platform about the rotational axis is constant during the man's walk and given by $I_p = \frac{1}{2}MR^2$. In addition, the initial and final moment of inertia of the man about this axis are $I_{mi} = mr_i^2 = mR^2$ and $I_{mf} = mr_f^2 = mR^2/4$, respectively. Therefore:

$$(\tfrac{1}{2}MR^2 + \tfrac{1}{4}mR^2)\omega_f = (\tfrac{1}{2}MR^2 + mR^2)\omega_i$$

$$(\tfrac{1}{2}M + \tfrac{1}{4}m)\omega_f = (\tfrac{1}{2}M + m)\omega_i$$

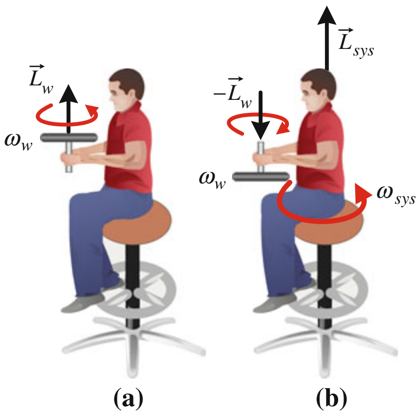
$$\omega_f = \frac{\frac{1}{2}M + m}{\frac{1}{2}M + \frac{1}{4}m}\omega_i = \frac{\frac{1}{2} \times 220\,\text{kg} + 60\,\text{kg}}{\frac{1}{2} \times 220\,\text{kg} + \frac{1}{4} \times 60\,\text{kg}} 0.5\,\text{rad/s} = 0.68\,\text{rad/s}$$

Note that ω_f is independent of R and that $\omega_f > \omega_i$ as expected.

Example 9.8

A student is sitting on a stationary stool that can rotate freely. This student is holding the axle of a rotating wheel whose moment of inertia about its axle is $I_w = 1.5\,\text{kg.m}^2$, see Fig. 9.10a. The rotating wheel has an angular speed $\omega_i = 4\,\text{rev/s}$ and its angular momentum $\vec{L}_w$ points upward. When the student inverts the wheel, its angular momentum becomes $-\vec{L}_w$, and the system (student+stool+wheel) starts rotating about the stool's axle, see Fig. 9.10b. The moment of inertia of the system about the stool's axle is $I_{sys} = 7.5\,\text{kg.m}^2$. What is the angular speed of the system after the inversion?

Fig. 9.10



Solution: The torque applied by the student to invert the wheel is internal to the system. Since there is no net external torque on the system, the angular momentum about any vertical axis is conserved. Initially, the total angular momentum of the system $\vec{L}_i$ comes entirely from the wheel. Thus:

$$\vec{L_i} = \vec{L_w}$$

After inverting the wheel, its angular momentum becomes $-\vec{L_w}$. For the total angular momentum to be conserved, the system must start rotating in the opposite direction with an angular momentum $\vec{L}_{sys}$, so:

$$\vec{L_f} = \vec{L}_{sys} + (-\vec{L_w})$$

Conservation of the angular momentum before and after the inversions of the wheel gives:

$$\vec{L_f} = \vec{L_i}$$
$$\vec{L}_{sys} - \vec{L_w} = \vec{L_w}$$

$$\vec{L}_{sys} = 2\vec{L_w} \quad \Rightarrow \quad L_{sys} = 2L_w \quad \Rightarrow \quad I_{sys}\,\omega_{sys} = 2I_w\omega_w$$

This yields: $$\omega_{sys} = \frac{2I_w}{I_{sys}}\omega_w = \frac{2 \times 1.5\,\text{kg.m}^2}{7.5\,\text{kg.m}^2} 4\,\text{rev/s} = 1.6\,\text{rev/s}$$

Example 9.9

Figure 9.11 shows a simple clutch which consists of two cylindrical disks that can be pressed together to connect two sections of an axle in a machine. The two disks have masses $M_1 = 5$ kg and $M_2 = 7$ kg, and have equal radii $R = 0.5$ m. Disk M_1 is accelerated from rest to an angular speed $\omega_1 = 6$ rad/s in a time interval $\Delta t = 2.5$ s. (a) Find the angular momentum of disk M_1. (b) Find the average torque required to accelerate M_1 to $\omega_1 = 6$ rad/s. (c) When disk M_2 (initially at rest) is coupled to disk M_1 such that they rotate as one unit, what is their angular speed after coupling?

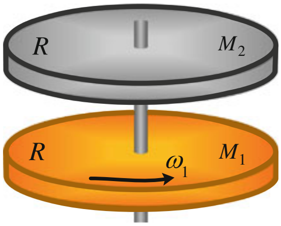
Solution: (a) The angular momentum of disk M_1 is:

$$L_1 = I_1\omega_1 = \tfrac{1}{2}M_1R^2\omega_1 = \tfrac{1}{2}(5\,\text{kg})(0.5\,\text{m})^2(6\,\text{rad/s}) = 3.75\,\text{J.s}$$

(b) The average torque required to accelerate M_1 is:

$$\overline{\tau}_{\text{ext}} = \frac{\Delta L}{\Delta t} = \frac{L_{\text{f}} - L_{\text{i}}}{\Delta t} = \frac{3.75\,\text{J.s} - 0}{2.5\,\text{s}} = 1.5\,\text{m.N}$$

Fig. 9.11



(c) When the stationary disk M_2 is coupled with M_1, each exerts a torque on the other, and there are no external torques in effect. Thus, conservation of angular momentum leads to:

$$\vec{L}_f = \vec{L}_i \quad \Rightarrow \quad (I_1 + I_2)\omega_2 = I_1\omega_1$$

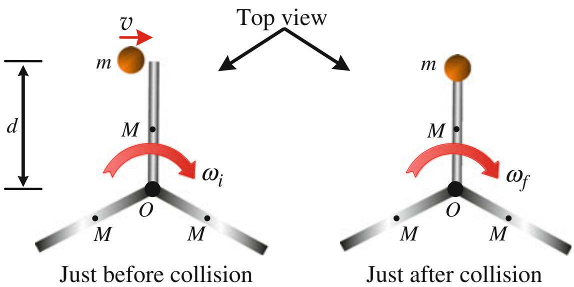
Thus:

$$\omega_2 = \frac{I_1}{I_1 + I_2}\omega_1 = \frac{M_1}{M_1 + M_2}\omega_1 = \frac{5\,\text{kg}}{5\,\text{kg} + 7\,\text{kg}}6\,\text{rad/s} = 2.5\,\text{rad/s}$$

Example 9.10

Figure 9.12 shows a top view of three identical rods that are rigidly connected at one end at O and make an angle of 120° with each other. Each rod has a mass M and a length d, and the entire assembly is rotating horizontally with an initial angular speed ω_i about a vertical axle passing through O. A ball of clay of mass m moving horizontally with a speed v collides perpendicularly with the tip of one of the rods and sticks to it (i.e. the collision is completely inelastic). What is the final angular speed of the system?

Fig. 9.12



Solution: Just before the collision, the initial angular momentum of the clay about O is clockwise with magnitude $L_{c,i} = mvd$ and the initial angular momentum of the rod assembly is also clockwise with magnitude $L_{r,i} = I_r \omega_i$. We have $I_r = 3(Md^2/3) = Md^2$ as obtained from Fig 9.12. Thus, $L_{r,i} = Md^2\omega_i$ and the total initial angular momentum of the system about the axle is:

$$L_i = L_{c,i} + L_{r,i} = mvd + Md^2\omega_i$$

Just after collision, the system is composed of the clay with moment of inertia md^2 attached to the assembly having moment of inertia $I_r = Md^2$. Thus, the system has moment of inertia $I_{sys} = md^2 + Md^2$ and the total angular momentum of the system about the axle is:

$$L_f = I_{sys}\omega_f = (m + M)d^2\omega_f$$

During the impact (internal forces cancel), no external forces acting on the system have a torque about the rotational axis. Thus, conservation of angular momentum before and after the collision gives:

$$L_f = L_i \quad \Rightarrow \quad \omega_f = \frac{mvd + Md^2\omega_i}{(m + M)d^2}$$

9.3 The Spinning Top and Gyroscope

In all our previous studies, the axis of rotation either stayed fixed or was moving and kept moving in the same direction. However, a variety of new physical phenomena can occur when the axis of rotation changes its direction.

It is quite natural to wonder why a top spinning rapidly about its axis of symmetry does not fall over, even when its center of mass is not directly above its tip, see Fig. 9.13. In this figure, the top rotates rapidly about its axis of symmetry with angular speed ω. At the same time, this axis rotates slowly about the vertical direction (the z-axis) with angular speed Ω (capital Greek omega), where usually $\Omega \ll \omega$. The rotation of the top's axis about the vertical is called *precession*.

The essential features of the two rotations can be understood by examining the effect of the net torque $\vec{\tau}_{\text{ext}}$ on the top's angular momentum $\vec{L}$. During the rotational processes, the only two effective forces on the top are the weight $M\vec{g}$ acting at the CM and the normal force $\vec{N}$ acting upward on the tip O. The normal force produces zero torque about the tip, while the weight produced a torque $\vec{\tau}_{\text{ext}} = \vec{r} \times M\vec{g}$ about O. The direction of $\vec{\tau}_{\text{ext}}$ is perpendicular to the plane containing $\vec{r}$ and $M\vec{g}$. Also, $\vec{\tau}_{\text{ext}}$ is perpendicular to $\vec{L}$, since $\vec{r}$ and $\vec{L}$ are pointing in the same direction. In addition, $\vec{\tau}_{\text{ext}}$ always lies in the $x\,y$-plane.

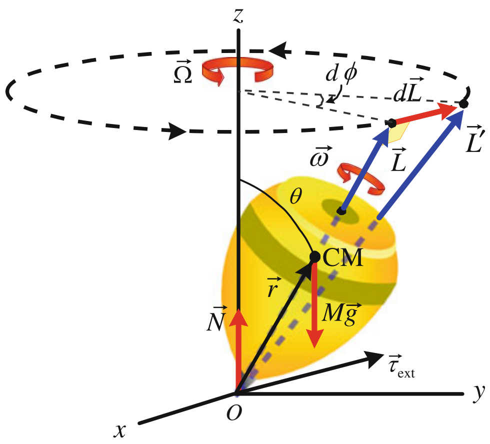


Fig. 9.13 A top rotating with angular velocity $\vec{\omega}$ about its symmetry axis and experiencing precession about the vertical axis with angular velocity $\vec{\Omega}$

Based on Eq. 9.7, the applied torque and angular momentum on the top are related through $\vec{\tau}_{\text{ext}} = d\vec{L}/dt$. Accordingly, during time interval dt, the change in angular momentum $d\vec{L}$ will be as follows:

$$d\vec{L} = \vec{L}' - \vec{L} = \vec{\tau}_{\text{ext}}\, dt \tag{9.15}$$

This relation indicates that the change in momentum $d\vec{L}$ has the same direction as $\vec{\tau}_{\text{ext}}$. But since $\vec{\tau}_{\text{ext}}$ is perpendicular to $\vec{L}$, then $d\vec{L}$ is also perpendicular to $\vec{L}$. Therefore, the magnitude of $\vec{L}$ does not change ($|\vec{L}'| = |\vec{L}|$) but only its direction changes perpendicular to $d\vec{L}$, as shown in Fig. 9.13. That is, the upper end of the top's axis moves in a horizontal circle. In other words, $\vec{\tau}_{\text{ext}}$ and $d\vec{L}$ rotate so as to be horizontal and perpendicular to $\vec{L}$.

To determine the angular velocity of precession, Ω, we notice that dL in Fig. 9.13 is related to the angle $d\phi$ by the relation:

$$dL = L \sin\theta \, d\phi \tag{9.16}$$

Substituting with $d\phi$ from this relation into the angular velocity of precession $\Omega = d\phi/dt$ and using $\tau_{\text{ext}} = dL/dt$, we get:

$$\Omega = \frac{d\phi}{dt} = \frac{1}{L\sin\theta}\frac{dL}{dt} = \frac{\tau_{\text{ext}}}{L\sin\theta} \tag{9.17}$$

But $|\vec{\tau}_{\text{ext}}| = |\vec{r} \times M\vec{g}\,| = rMg\sin(180° - \theta) = rMg\sin\theta$, then Ω becomes:

$$\Omega = \frac{Mgr}{L} \tag{9.18}$$

Using Eq. 9.12, we can write $L = I\omega$, where I and ω are the moment of inertia and angular speed of the spinning top about its axis of symmetry. Then the top's precessional angular speed becomes:

$$\boxed{\Omega = \frac{Mgr}{I\omega}} \tag{9.19}$$

This relation is valid only when $\Omega \ll \omega$, and this condition is satisfied if ω is large. If this condition is not fulfilled, the motion of the top becomes much more complicated.

Using $\Omega = 2\pi/T_p$ and $\omega = 2\pi/T_s$, where T_p is the precession period and T_s is the spinning period, we find that the period of precession T_p is given by:

$$\boxed{T_p = \frac{4\pi^2 I}{MgrT_s}} \tag{9.20}$$

In fact, Eqs. 9.19 and 9.20 also apply to gyroscopes. A gyroscope is a device for measuring or maintaining orientation, based on the concept of conservation of angular momentum.

The toy gyroscope shown in Fig. 9.14 has one end of its axle resting on a support (assumed to be a frictionless pivot), while the other end is free and precessing horizontally with angular speed Ω. A symmetric wheel attached to this axle spins rapidly about its axis with a large angular speed ω (like the top of Fig. 9.13).

Based on our findings for the spinning top, let us analyze the behavior of the toy gyroscope of Fig. 9.14. The wheel is rotating about its axis of symmetry with an angular speed ω and has an initial angular momentum $\vec{L}$ along the x-axis. Since $\vec{\tau}_{\text{ext}}$ and $d\vec{L}$ are along the y-axis and perpendicular to $\vec{L}$, this causes the direction of $\vec{L}$ to change, but not its magnitude. Therefore, the changes $d\vec{L}$ are always in the horizontal xy-plane. Consequently, the angular momentum $\vec{L}$ and the wheel axis with which the wheel moves are always horizontal. This means that the axis of the wheel does not fall, but will precess with the angular speed Ω given by Eq. 9.19.

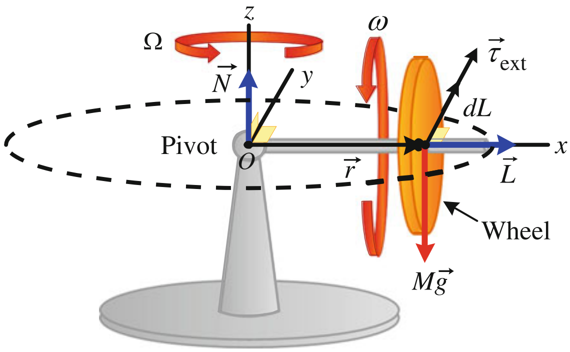


Fig. 9.14 A toy gyroscope is a wheel rotating with an angular speed ω about an axis supported at one end while the other is free. During time dt, the torque $\vec{\tau}_{ext}$ and the change in angular momentum $d\vec{L}$ are perpendicular to $\vec{L}$, which rotates in the xy-plane with a precessional angular speed Ω

Example 9.11

Assume that the cylindrical wheel of the gyroscope of Fig. 9.14 has a radius $R = 4\,\text{cm}$ and a center of mass located 3 cm from the pivot O. If the gyroscope takes 5 s for completing one revolution of precession, what is the spinning angular speed of the wheel and its period?

Solution: The precessional angular speed about the z-axis is:

$$\Omega = \frac{1\ \text{rev}}{5\ \text{s}} = \frac{2\pi\ \text{rad}}{5\ \text{s}} = 0.4\,\pi\ \text{rad/s} = 1.257\ \text{rad/s}$$

The moment of inertia of a cylindrical wheel about its axis of symmetry is $I = \frac{1}{2}MR^2$ and its weight is Mg. From Eq. 9.19:

$$\omega = \frac{Mgr}{I\Omega} = \frac{Mgr}{(\frac{1}{2}MR^2)\Omega} = \frac{2gr}{R^2\Omega}$$
$$= \frac{2(9.8\ \text{m/s}^2)(0.03\ \text{m})}{(0.04\ \text{m})^2(1.257\ \text{rad/s})} = 292.4\ \text{rad/s} = 46.5\ \text{rev/s}$$

Thus, the spinning period is: $T_s = \dfrac{2\pi}{\omega} = \dfrac{2\pi\ \text{rad}}{292.4\ \text{rad/s}} = 2.15 \times 10^{-2}\ \text{s}$

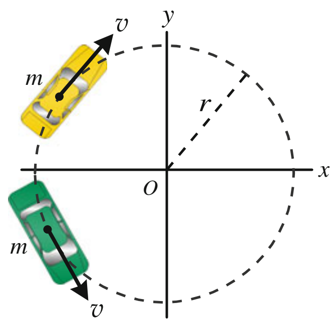
9.4 Exercises

Section 9.1 Angular Momentum of Rotating Systems

(1) Calculate the angular momentum of a particle of mass $m = 2\,\text{kg}$ that has a velocity $\vec{v} = (2\vec{i} + 3\vec{j})$ m/s when its position vector is $\vec{r} = (3\vec{i} - 4\vec{j})$ m.

(2) Two cars, each having a mass $m = 1{,}500\,\text{kg}$, are moving in a horizontal circle of radius $r = 10$ m with the same speed $v = 10$ m/s. The circle is centered at the origin O in the xy-plane, and the positive z-axis is directed upwards. If one of them is moving clockwise and the other counterclockwise, see Fig. 9.15, find the angular momentum of each car about O.

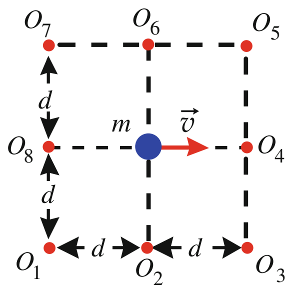
Fig. 9.15 See Exercise (2)



(3) A particle of mass $m = 2\,\text{kg}$ has a position vector that depends on time t and is given by $\vec{r} = (3t\vec{i} - 4t^2\vec{j})$ m. Find the angular momentum of the particle as a function of time.

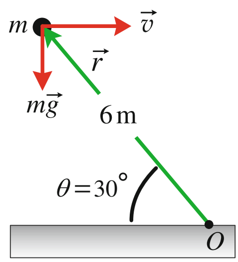
(4) A particle of mass m is moving horizontally with constant velocity $\vec{v}$ as shown in Fig. 9.16. Find the magnitude and direction of the angular momentum of the particle, $\vec{L}_i$, $(i = 1, 2, \ldots, 8)$, respectively about the eight points O_i, $(i = 1, 2, \ldots, 8)$.

Fig. 9.16 See Exercise (4)



(5) A ball of mass $m = 0.5$ kg is moving horizontally with a speed $v = 10$ m/s at the instant when its position is identified in Fig. 9.17. (a) What is the angular momentum of the ball about O at this instant? (b) Neglecting air resistance, find the rate of change of its angular momentum about O at this instant.

Fig. 9.17 See Exercise (5)

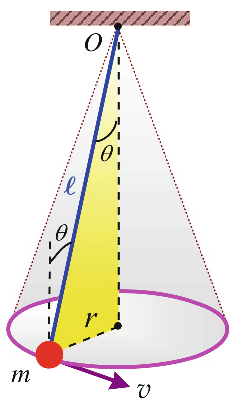


(6) By definition, kinetic energy $K = \frac{1}{2}mv^2$, where m and v are the mass and speed of a particle, respectively. Show that the kinetic energy of a particle moving in a circular path is $K = L^2/2I$, where L and I are, respectively, the angular momentum and moment of inertia of the particle about the center of the circle.

(7) A canonical pendulum consists of a bob of mass m attached to the end of a cord of length ℓ. The bob whirls around in a horizontal circle of radius r at a constant speed v while the cord always makes an angle θ with the vertical, see Fig. 9.18. Show that the magnitude of the angular momentum of the bob about its point of support O is given by:

$$L = \sqrt{m^2 g \ell^3 \sin\theta \tan\theta}$$

Fig. 9.18 See Exercise (7)



(8) Two identical particles 1 and 2 have respective position vectors $\vec{r}_1$ and $\vec{r}_2$ with respect to an arbitrary origin O. The two particles have equal and opposite linear momenta $\vec{p}$ and $-\vec{p}$ as shown in Fig. 9.19. Show that the total angular momentum of this system is independent of the choice of the origin and independent of where the traveling particles are located.

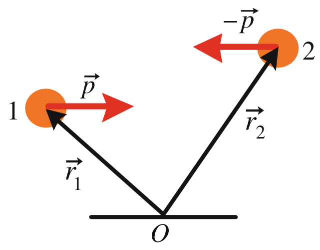


Fig. 9.19 See Exercise (8)

(9) Two particles of masses $m_1 = 2$ kg and $m_2 = 3$ kg are joined by a rod of mass $M = 0.5$ kg and length $d = 0.75$ m. The assembly rotates freely in the xy-plane about a pivot through the center of the rod, as shown in Fig. 9.20. Find the angular momentum of the system when the speed of each particle is $v = 6$ m/s.

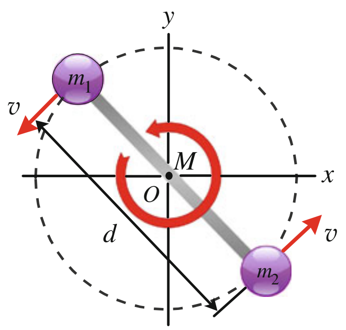
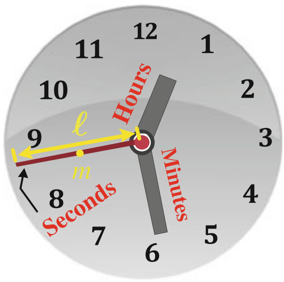


Fig. 9.20 See Exercise (9)

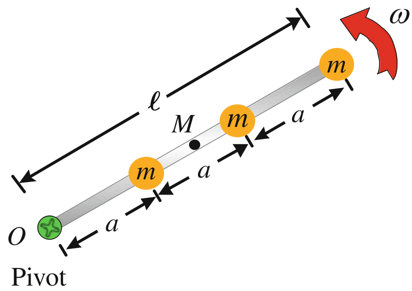
(10) Consider the seconds hand of a particular analog clock to be a thin rod of length $\ell = 14$ cm and mass $m = 5$ g, see Fig. 9.21. (a) If the seconds hand rotates constantly, what is its angular speed? (b) Find the magnitude of the angular momentum of the seconds hand about an axis perpendicular to the center of the clock's face.

Fig. 9.21 See Exercise (10)



(11) Three identical particles, each of mass $m = 0.5$ kg, are attached at equal distances from one end of a rod of length $\ell = 2$ m and mass $M = 3$ kg, see Fig. 9.22. The system is rotating with angular speed $\omega = 2$ rad/s about an axis perpendicular to the rod through the free end at O. (a) What is the moment of inertia of the system about O? (b) What is the angular momentum of the system about O?

Fig. 9.22 See Exercise (11)



(12) Each of two identical particles of mass $m = 0.5$ kg attached at one end of two identical rods each of length $a = 0.2$ m and mass $M = 0.3$ kg. The other ends of the two rods are mounted perpendicular to a lightweight axle such that the distance between the rods is $d = 0.6$ m, see Fig. 9.23. The axle rotates at $\omega = 4$ rad/s. (a) What is the total angular momentum of the two particles about the CM of the system? (b) What is the total angular momentum of the two rods about the axle? (c) What angle does the total angular momentum of the whole system make with the axle?

(13) Three identical thin rods, each of mass m and length R, are fastened together to form the letter H. A circular hoop, of mass m and radius R, is fastened to the rods to form the rigid structure shown in Fig. 9.24. The rigid structure rotates

with a constant angular speed about a vertical axis with a period of rotation T. (a) Find an expression for the structure's moment of inertia and angular momentum about the axis of rotation. (b) Evaluate the two expressions of part (a) when $m = 0.5$ kg, $R = 0.1$ m, and $T = 2$ s.

Fig. 9.23 See Exercise (12)

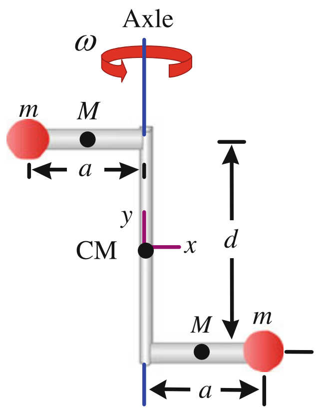
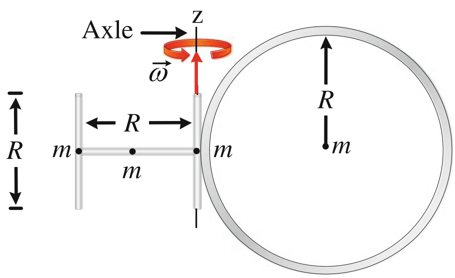


Fig. 9.24 See Exercise (13)



(14) A block of mass m_1 located on a smooth horizontal surface is connected by a light non-stretchable cord that passes over a pulley to a second block of mass m_2, which hangs vertically, see Fig. 9.25. The pulley is a uniform cylinder of mass M and radius R, and it rotates freely about its axle. (a) Find an expression for the net external torque about the pulley's axle. (b) Find an expression for the net angular momentum about the pulley's axle. (c) Find an expression for the magnitude of the acceleration of the two blocks and its value if $m_1 = 6$ kg, $m_2 = 3$ kg, $M = 2$ kg, $R = 0.1$ m, and $g = 10$ m/s^2.

(15) A disk has a moment of inertia $I = 2$ kg.m^2 about its axis of symmetry. The angular speed of the disk depends on the time t by $\omega = (12\,\text{rad/s}^3)\,t^2$. (a) Find

the angular acceleration α and the angular momentum L of the disk as a function of time, and find their values at $t = 2$ s. (b) Show that using the expressions for α and L leads to the same expression for the net torque on the disk as a function of time, and find its value at $t = 2$ s.

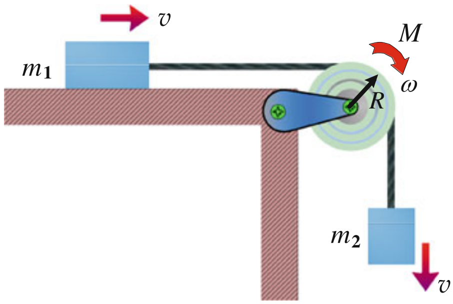


Fig. 9.25 See Exercise (14)

(16) A uniform solid sphere of mass $M = 10$ kg and radius $R = 10$ cm turns counterclockwise with an angular speed $\omega = 5$ rad/s about a vertical axis that touches its surface, see Fig. 9.26. What is the magnitude and direction of its angular momentum about this axis?

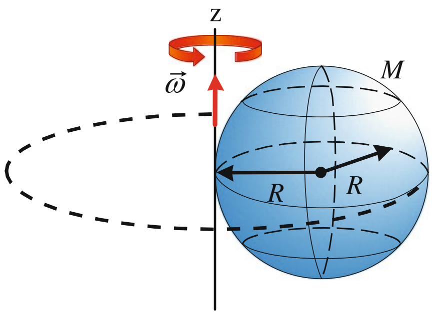


Fig. 9.26 See Exercise (16)

(17) A boy of mass $m = 40$ kg is standing on the rim of a merry-go-round that is rotating with angular speed $\omega = 0.5$ rev/s about an axis through its center. The merry-go-round is a uniform disk of mass $M = 120$ kg and radius $R = 3.5$ m. Find the total angular momentum of the boy-disk system by treating the boy as a point.

(18) Two wheels of radii R_a and R_b are connected by a non-stretchable belt that does not slip on their circumferences, see Fig. 9.27. The radius R_a is four times

the radius R_b. Find the ratio of the moment of inertia I_a/I_b and mass M_a/M_b if both wheels have: (a) the same angular momentum about their central axis, and (b) the same rotational kinetic energy.

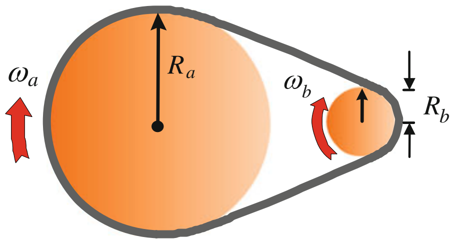


Fig. 9.27 See Exercise (18)

(19) If an impulsive force $F(t)$ with moment arm R acts on a rigid body of moment of inertia I for a short time Δt, then show that the angular speed of the body will change from an initial value ω_i to a final value ω_f according to the *angular impulse* formula:

$$J_R = \int \tau \, dt = \bar{F} R \Delta t = I(\omega_f - \omega_i)$$

where $\bar{F}$ is the average value of the force during the time it acts on the body. [Hint: It is the rotational analogy of Eq. 7.9].

(20) A wheel of radius R_a and moment of inertia I_a is rotating about its central axle with angular speed ω_a. Another small wheel is stationary and has a radius R_b and moment of inertia I_b about its central axle. The smaller wheel is moved until it touches the larger wheel and rotates due to the friction between them, as in the upper part of Fig. 9.28. After the initial slipping period is over, the two wheels rotate at constant angular speeds ω'_a and ω'_b, see the lower part of Fig. 9.28. By applying the angular impulse relationship of Exercise 19, find the final angular speed ω'_b of the small wheel.

(21) A block of mass m_1 located on a rough horizontal surface is connected by a light non-stretchable cord that passes over a pulley to a second block of mass m_2, which is allowed to move on a rough inclined plane of angle θ, as shown in Fig. 9.29. The pulley is a uniform cylinder of mass M and radius R, and rotates freely about its axle. The coefficients of kinetic friction for the two blocks on the horizontal and inclined planes are $\mu_{k1} = 0.35$ and $\mu_{k2} = 0.5$, respectively. (a) Draw free-body diagrams of the two blocks and the pulley. (b) Find the acceleration of the two blocks and the tensions in the two sections

of the cord when $m_1 = 2$ kg, $m_2 = 5$ kg, $M = 10$ kg, $R = 0.1$ m, $\sin\theta = 4/5$, $\cos\theta = 3/5$, and $g = 10\ \text{m/s}^2$. (c) If the system starts from rest, find the angular momentum of the pulley about its axis as a function of time.

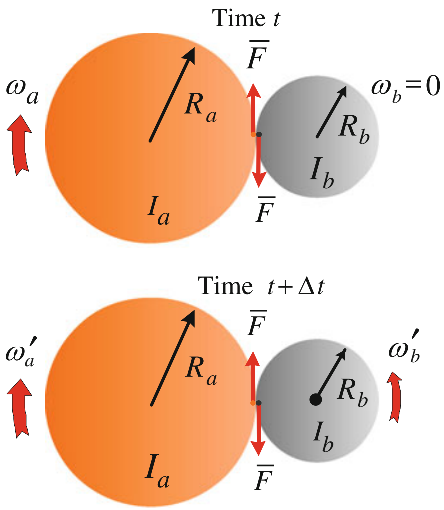


Fig. 9.28 See Exercise (20)

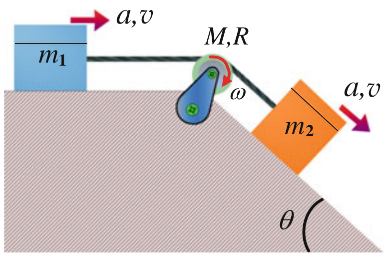


Fig. 9.29 See Exercise (21)

(22) Determine the angular momentum of the Earth: (a) about its rotational axis (assume that Earth is a uniform sphere of mass $M = 6.0 \times 10^{24}$ kg and radius $R = 6.4 \times 10^6$ m), and (b) about the Sun (assume Earth to be a particle at 1.5×10^{11} m from the Sun).

(23) Two blocks having masses m_1 and m_2 ($m_2 > m_1$) are connected to each other by a light non-stretchable cord that passes over two identical pulleys; each pulley is a uniform cylinder with a mass M and radius R, which rotates freely about its axle, as shown in Fig. 9.30. Assume no slipping happens between the cord and

the pulleys. (a) Find an expression for the net external torque of each pulley about its axle; then find the total net external torque of the system. (b) Find an expression for the net angular momentum of each pulley about its axle; and then find the total net angular momentum of the system. (c) Apply $\Sigma\tau_{ext} = dL/dt$ onto the whole system to find the acceleration of each block and the tensions T_1, T_2, and T_3 in the cord.

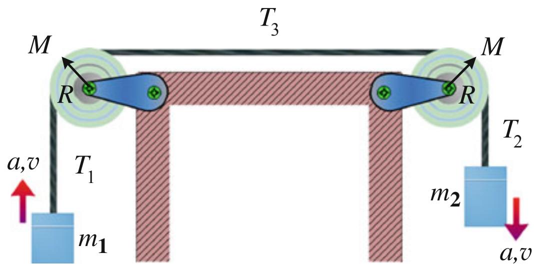


Fig. 9.30 See Exercise (23)

Section 9.2 Conservation of Angular Momentum

(24) A person is rotating on a frictionless surface at a rate of 1.5 rev/s with his arms at his sides. When he raises his arms to the horizontal position, the speed of rotation decreases to a rate of 0.75 rev/s. What is the percentage increase in moment of inertia of the person?

(25) A skater has a moment of inertia 4.5 kg.m^2 when rotating on a frictionless surface at a rate of 1 rev/s. What is her final moment of inertia if she increases her spin to the maximum value of 2.5 rev/s? How can she accomplish this change?

(26) A diver pushes a swimming pool board to jump into the air and acquires an initial angular momentum about her center of mass. Then she curls her body about her center of mass (by tucking in her arms and legs) to reduce her moment of inertia by a factor of 3.25. If she is able to make 3 revolutions in 2.25 s while she is in that tucked position, what was her initial angular speed?

(27) A uniform horizontal rod of mass M and length d rotates initially with angular speed ω_i about a vertical frictionless axle running through its center. Then two stationary small balls of clay, each of mass m, are made to stick to each end of the rod. What is the final angular speed of the system?

(28) A merry-go-round of radius $R = 2.5$ m and moment of inertia $I = 300$ kg.m^2 is rotating at 10 rev/min. A boy of mass $m = 40$ kg jumps onto the merry-go-round and manages to sit down quickly on its rim. What is the final angular speed of the system?

(29) A merry-go-round of a mass $M = 210$ kg and radius $R = 5.5$ m is mounted on a frictionless bearing. While a man of mass $m = 90$ kg is standing on its outer edge, the system is rotating with an angular speed $\omega_i = 0.2$ rev/s. Then, slowly, the man walks 3 m towards the center of the merry-go-round and stops. How fast will the merry-go-round be rotating after he stops?

(30) Rather than walking inwards, suppose the man in Exercise 29 decided to jump radially outwards relative to the merry-go-round. What will be the angular speed of the merry-go-round?

(31) A boy of mass $m = 30$ kg stands on the edge of a stationary small merry-go-round of moment of inertia $I_m = 150$ kg.m^2 and radius $R = 2$ m. The merry-go-round can rotate freely without friction about its axis. The boy jumps off the merry-go-round in a tangential direction with a linear speed $v = 2$ m/s. What is the angular speed of the merry-go-round after the boy leaves it?

(32) A person of mass $m = 80$ kg (treated as a point) stands at the center of a freely rotating cylindrical platform of a mass $M = 120$ kg and radius $R = 4$ m. The platform is mounted on a frictionless bearing and rotates with an angular speed $\omega_i = 1.5$ rad/s. The person walks radially and slowly to the edge of the platform and stops. (a) What is the final angular speed of the system? (b) Find the initial and final total rotational energy of the system.

(33) A uniform disk of radius R and a uniform rod of length $2R$ have the same mass M. The disk is rotating freely without friction about its axle with angular speed $\omega_i = 3$ rev/s while the rod is at rest and has its center coinciding with the disk's axle, see Fig. 9.31. The rod is dropped onto the disk and sticks to it such that their centers coincide (i.e. the collision is completely inelastic). What is the final angular speed of the system?

(34) Two disks have a common frictionless axle and moments of inertia $I_1 = 5$ kg.m^2 and $I_2 = 10$ kg.m^2. Initially, disk I_1 is rotating with an angular speed $\omega_i = 6$ rev/s about the axle, while disk I_2 is not rotating. Disk I_2 then drops onto disk I_1, see Fig. 9.32. Due to the friction between their surfaces, the two disks eventually reach the same angular speed ω_f. (a) Find the final angular frequency ω_f. (b) Find the percentage decrease in the rotational kinetic energy.

Fig. 9.31 See Exercise (33)

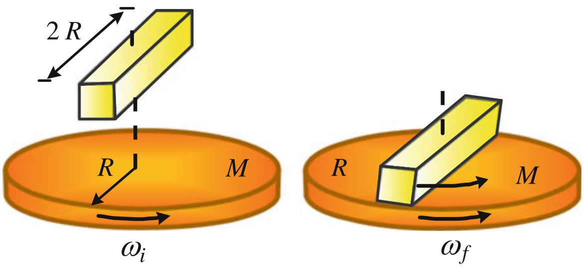
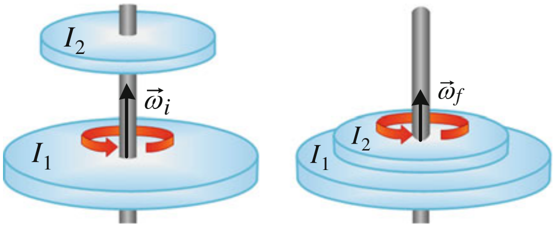
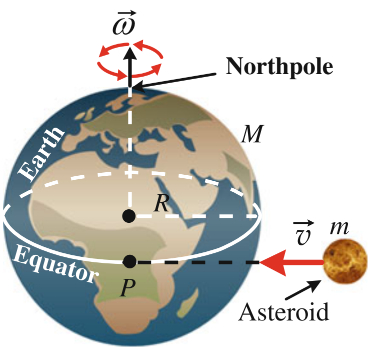


Fig. 9.32 See Exercise (34)



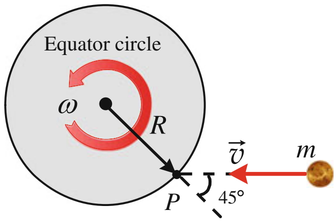
(35) An asteroid of mass $m = 10^6$ kg is traveling at a speed of $v = 4 \times 10^4$ m/s relative to the Earth. The asteroid hits the Earth tangentially at the equator in the direction opposite to its rotation and gets embedded at its surface, see Fig. 9.33. Assume that Earth is a uniform sphere of mass $M = 6.0 \times 10^{24}$ kg and radius $R = 6.4 \times 10^6$ m. Since no net external forces acting on this system can produce a torque about the axis of the Earth (all forces and torques are internal), then the angular momentum of the system is conserved about the axis of the Earth. As a result of this collision, find the percentage change in the Earth's angular speed.

Fig. 9.33 See Exercise (35)



(36) Suppose the asteroid in Exercise 35 hits the equator circle with an incident angle $\theta = 45°$, see Fig. 9.34. By what factor does this completely inelastic collision affect the angular speed of the Earth?

Fig. 9.34 See Exercise (36)



(37) A thin vertical rod of mass M and length d can rotate about a frictionless pivot at its upper end, see Fig. 9.35. A small clay ball of mass m traveling horizontally with a speed v hits the rod at its center and sticks to it. (a) Find the angular speed of the system just after the collision. (b) How high does the lower end rise?

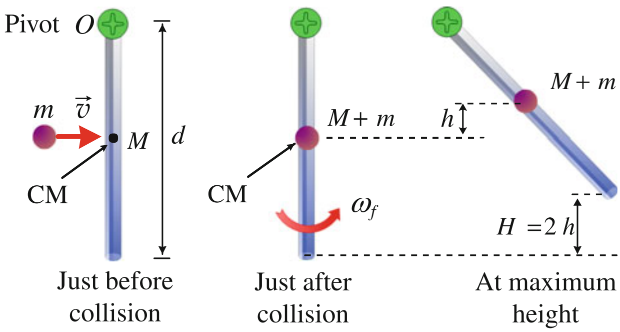


Fig. 9.35 See Exercise (37)

(38) A stationary thin horizontal rod of mass M and length d can rotate about a frictionless vertical axle through its center at O, see Fig. 9.36. A small clay ball of mass m traveling horizontally with a speed v hits the rod at one of its ends and sticks to it. (a) Find the angular speed of the system just after the collision. (b) What is the fractional loss in mechanical energy due to the collision?

(39) A stationary horizontal wooden stick of length $d = 75$ cm and mass $M = 0.4$ kg can rotate about a frictionless vertical axle through its center at O, see Fig. 9.37.

A bullet of mass $m = 5 \times 10^{-3}$ kg and horizontal speed $v_i = 200$ m/s is shot into the stick midway between the axle and one end. The bullet penetrates the stick in a very short time and leaves with a speed $v_f = 100$ m/s. (a) Find the angular speed of the stick after the collision. (b) Find the percentage decrease in total energy.

Fig. 9.36 See Exercise (38)

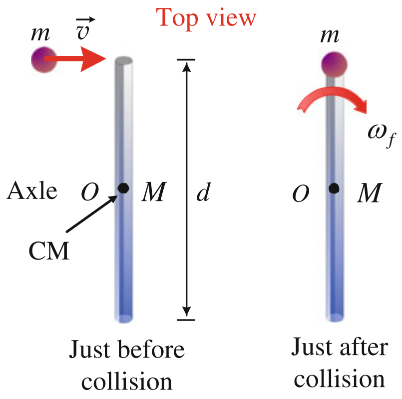
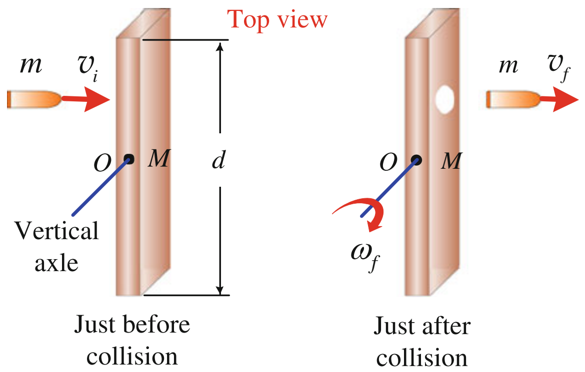


Fig. 9.37 See Exercise (39)



(40) A stationary thin rod of mass M and length d rests on a frictional table. A small clay ball of mass m traveling horizontally with a speed v hits the rod perpendicularly at a point $d/4$ from its center and sticks to it, see Fig. 9.38. Determine the translational and rotational motion of the rod after the collision.

(41) A student stands at the center of a turntable with his arms outstretched. In each hand, he holds a 10 kg-dumbbell at 1 m from the axis of the turntable. The turntable is rotating about a vertical frictionless axle with angular speed $\omega_i = 0.5$ rev/s. (a) Find his final angular speed if he pulls each dumbbell to

his stomach at 0.2 m from the axis of the turntable. The moment of inertia of the student with his arms outstretched is 4 kg.m^2, but it is 3.2 kg.m^2 with his hands at his stomach. (b) Find the initial and final kinetic energy of the system. Explain the meaning if they are different.

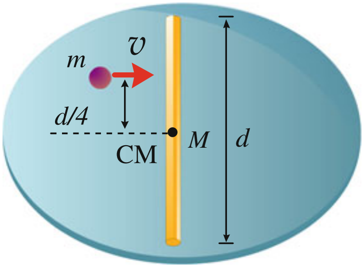


Fig. 9.38 See Exercise (40)

Section 9.3 The Spinning Top and Gyroscope

(42) To form a top, a uniform disk of mass $M = 50$ g and radius $R = 2$ cm is rigidly attached to an axial rod of negligible mass. The top spins on a frictionless surface about its axis of symmetry with angular speed $\omega = 6{,}000$ rev/min. How much work was done to get the top to spin at that rate?

(43) The center of the disk in exercise 42 is at $r = 3$ cm from the tip of the top at the surface of contact to the disk. What is the angular speed of precession of the top about the vertical axis?

(44) To form a toy gyroscope, a disk of mass $M = 150$ g and radius $R = 6$ cm is mounted at the center of a thin axle of 20 cm length. The disk spins at $\omega = 50$ rev/s when one end of the axle rests on a stand and the other end precesses horizontally. What is the angular speed of precession of the top about the vertical axis?

(45) A top of mass $M = 200$ g spins about its axis of symmetry with angular speed $\omega = 18$ rev/s and makes an angle $\theta = 25°$ with the vertical. It experiences precession at a rate of 1 rev every 5 s. The center of mass of the top is 4 cm from its tip. (a) What is the moment of inertia of the top? (b) Find the torque on the top.