Dimensions and Units

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The laws of physics are expressed in terms of basic quantities that require a clear definition for the purpose of measurements. Among these measured quantities are length, time, mass, temperature, etc.

In order to describe any physical quantity, we first have to define a **unit** of measurement (which was among the earliest tools invented by humans), i.e. a measure that is defined to be exactly 1.0. After that, we define a **standard** for this quantity, i.e. a reference to compare all other examples of the same physical quantity.

1.1 The International System of Units

Seven physical quantities have been selected as base quantities in the 14th General Conference on Weights and Measurements, held in France in 1971. These quantities form the basis of the International System of Units, abbreviated **SI** (from its French name *Système International*) and popularly known as the *metric system*. Table 1.1 depicts these quantities, their unit names, and their unit symbols.

Many SI derived units are defined in terms of the first three quantities of Table 1.1. For example, the SI unit for force, called the newton (abbreviated N), is defined in terms of the base units of mass, length, and time. Thus, as we will see from the study of Newton's second law, the unit of force is given by:

$$1\ \text{N} = 1\ \text{kg.m/s}^2 \tag{1.1}$$

When dealing with very large or very small numbers in physics, we use the so-called scientific notation which employs powers of 10, such as:

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Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_1,

$$3\ 210\ 000\ 000\ \text{m} = 3.21 \times 10^9\ \text{m} \tag{1.2}$$

$$0.000\ 000\ 789\ \text{s} = 7.89 \times 10^{-7}\ \text{s} \tag{1.3}$$

Table 1.1 The seven independent SI base units

Quantity	Unit name	Unit symbol
Length	Meter	m
Time	Second	s
Mass	Kilogram	kg
Temperature	Kelvin	K
Electric current	Ampere	A
Amount of substance	Mole	mol
Luminous intensity	Candela	cd

An additional convenient way to deal with very large or very small numbers in physics is to use the prefixes listed in Table 1.2. Each one of these prefixes represents a certain power of 10.

Table 1.2 Prefixes for SI units[a]

Factor	Prefix	Symbol	Factor	Prefix	Symbol
10^{24}	yotta-	Y	10^{-24}	yocto-	y
10^{21}	zeta-	Z	10^{-21}	zepto-	z
10^{18}	exa-	E	10^{-18}	atto-	a
10^{15}	peta-	P	10^{-15}	**femto-**	**f**
10^{12}	**tera-**	**T**	10^{-12}	**pico-**	**p**
10^{9}	**giga-**	**G**	10^{-9}	**nano-**	**n**
10^{6}	**mega-**	**M**	10^{-6}	**micro-**	**μ**
10^{3}	**kilo-**	**k**	10^{-3}	**milli-**	**m**
10^{2}	hecta-	h	10^{-2}	**centi-**	**c**
10^{1}	deca-	da	10^{-1}	deci-	d

[a] The most commonly used prefixes are shown in bold face type

Accordingly, we can express a particular magnitude of force as:

$$\begin{aligned} 1.23 \times 10^6\ \text{N} &= 1.23\ \text{mega newtons} \\ &= 1.23\ \text{MN} \end{aligned} \tag{1.4}$$

or a particular time interval as:

$$\begin{aligned} 1.23 \times 10^{-9}\ \text{s} &= 1.23 \text{ nano seconds} \\ &= 1.23 \text{ ns} \end{aligned} \tag{1.5}$$

We often need to change units in which a physical quantity is expressed. We do that by using a method called *chain-link conversion*, in which we multiply by a conversion factor that equals unity. For example, because 1 minute and 60 seconds are identical time intervals, then we can write:

$$\boxed{\frac{1 \text{ min}}{60 \text{ s}} = 1 \quad \text{and} \quad \frac{60 \text{ s}}{1 \text{ min}} = 1} \tag{1.6}$$

This does not mean that $\frac{1}{60} = 1$ or $60 = 1$, because the number and its unit must be treated together.

Example 1.1

Convert the following: (a) 1 kilometer per hour to meter per second, (b) 1 mile per hour to meter per second, and (c) 1 mile per hour to kilometer per hour [to a good approximation 1mi = 1.609 km].

Solution: (a) To convert the speed from the kilometers per hour unit to meters per second unit, we write:

$$1 \text{ km/h} = \left(1\ \frac{\cancel{\text{km}}}{\cancel{\text{h}}}\right) \times \left(\frac{10^3 \text{ m}}{1\ \cancel{\text{km}}}\right) \times \left(\frac{1\ \cancel{\text{h}}}{60 \times 60\ \text{s}}\right) = 0.2777...\ \frac{\text{m}}{\text{s}} = 0.278 \text{ m/s}$$

(b) To convert from miles per hour to meters per second, we write:

$$1 \text{ mi/h} = \left(1\ \frac{\cancel{\text{mi}}}{\cancel{\text{h}}}\right) \times \left(\frac{1609\ \text{m}}{1\ \cancel{\text{mi}}}\right) \times \left(\frac{1\ \cancel{\text{h}}}{60 \times 60\ \text{s}}\right) = 0.447\ \frac{\text{m}}{\text{s}} = 0.447 \text{ m/s}$$

(c) To convert from miles per hour to kilometers per hour, we write:

$$1\text{mi/h} = \left(1\ \frac{\cancel{\text{mi}}}{\text{h}}\right) \times \left(\frac{1.609\ \text{km}}{1\ \cancel{\text{mi}}}\right) = 1.609\ \frac{\text{km}}{\text{h}} = 1.609 \text{ km/h}$$

1.2 Standards of Length, Time, and Mass

Definitions of the units of length, time, and mass are under constant review and are changed from time to time. We only present in this section the latest definitions of those quantities.

Length (L)

In 1983, the precision of the meter was redefined as the distance traveled by a light wave in vacuum in a specified time interval. The reason is that the measurement of the speed of light has become extremely precise, so it made sense to adopt the speed of light as a defined quantity and to use it to redefine the meter. In the words of the 17th General Conference on Weights and Measurements:

> One Meter
> One meter is the distance traveled by light in vacuum during the time interval of 1/299 792 458 of a second.

This time interval number was chosen so that the speed of light in vacuum c will be exactly given by:

$$c = 299\ 792\ 458 \text{ m/s} \tag{1.7}$$

For educational purposes we usually consider the value $c = 3 \times 10^8$ m/s.

Table 1.3 lists some approximate interesting lengths.

Table 1.3 Some approximate lengths

Length	Meters
Distance to farthest known galaxy	4×10^{25}
Distance to nearest star	4×10^{16}
Distance from Earth to Sun	1.5×10^{11}
Distance from Earth to Moon	4×10^{8}
Mean radius of Earth	6×10^{6}
Wave length of light	5×10^{-7}
Radius of hydrogen atom	5×10^{-11}
Radius of proton	1×10^{-15}

Time (T)

Recently, the standard of time was redefined to take advantage of the high-precision measurements that could be obtained by using a device known as an *atomic clock*. Cesium is most common element that is typically used in the construction of atomic clocks because it allows us to attain high accuracy.

Since 1967, the International System of Measurements has been basing its unit of time, the **second**, on the properties of the isotope cesium-133 ($^{133}_{55}$Cs). One of the transitions between two energy levels of the ground state of $^{133}_{55}$Cs has an oscillation frequency of 9 192 631 770 Hz, which is used to define the second in SI units. Using this characteristic frequency, Fig. 1.1 shows the cesium clock at the National Institute of Standards and Technology. The uncertainty is about 5×10^{-16} (as of 2005). Or about 1 part in 2×10^{15}. This means that it would neither gain nor lose a second in 64 million years.

> One Second
>
> One second is the time taken for the cesium atom $^{133}_{55}$Cs to perform 9 192 631 770 oscillations to emit radiation of a specific wavelength

Fig. 1.1 The cesium atomic clock at the National Institute of Standards and Technology (NIST) in Boulder, Colorado (photo with permission)

Table 1.4 lists some approximate interesting time intervals.

Table 1.4 Some approximate time intervals

Time intervals	Seconds
Lifetime of proton (predicted)	1×10^{39}
Age of the universe	5×10^{17}
Age of the Earth	1.3×10^{17}
Period of one year	3.2×10^{7}
Time between human heartbeats	8×10^{-1}
Period of audible sound waves	1×10^{-3}
Period of visible light waves	2×10^{-15}
Time for light to cross a proton	3.3×10^{-24}

Mass (M)

The Standard Kilogram

A cylindrical mass of 3.9 cm in diameter and of 3.9 cm in height and made of an unusually stable platinum-iridium alloy is kept at the International Bureau of Weights and Measures near Paris and assigned in the SI units a mass of 1 kilogram by international agreement, see Fig. 1.2.

> One Kilogram
> The SI unit of mass, 1 kilogram, is defined as the mass of a platinum-iridium alloy cylinder kept at the International Bureau of Weights and Measures in France.

Fig. 1.2 The standard 1 kilogram of mass is a platinum-iridium cylinder 3.9 cm in height and diameter and kept under a double bell jar at the International Bureau of Weights and Measures in France

Accurate copies of this standard 1 kilogram have been sent to standardizing laboratories in other countries. Table 1.5 lists some approximate mass values of various interesting objects.

A Second Standard Mass

Atomic masses can be compared with each other more precisely than the kilogram. By international agreement, the carbon-12 atom, ${}^{12}_{6}C$, has been assigned a mass of 12 **atomic mass units** (u), where:

$$1\ \text{u} = (1.660\ 540\ 2 \pm 0.000\ 001\ 0) \times 10^{-27}\ \text{kg} \tag{1.8}$$

Experimentally, with reasonable precision, all masses of other atoms can be measured relative to the mass of carbon-12.

Table 1.5 Mass of various objects (approximate values)

Object	Kilogram
Known universe (predicted)	1×10^{53}
Our galaxy the milky way (predicted)	2×10^{41}
Sun	2×10^{30}
Earth	6×10^{24}
Moon	7×10^{22}
Small mountain	1×10^{12}
Elephant	5×10^{3}
Human	7×10^{1}
Mosquito	1×10^{-5}
Bacterium	1×10^{-15}
Uranium atom	4×10^{-25}
Proton	2×10^{-27}
Electron	9×10^{-31}

1.3 Dimensional Analysis

Throughout your experience, you have been exposed to a variety of units of length; the SI meter, kilometer, and millimeter; the English units of inches, feet, yards, and miles, etc. All of these derived units are said to have **dimensions** of length, symbolized by L. Likewise, all time units, such as seconds, minutes, hours, days, years, and centuries are said to have **dimensions** of time, symbolized by T. The kilogram and all other mass units have **dimensions** of mass, symbolized by M. In general, we may take the dimension (length, time, and mass) as the concept of the physical quantity.

From the three fundamental physical quantities of length L, time T, and mass M, we can derive a variety of useful quantities. Derived quantities have different dimensions from the fundamental quantities. For example, the area obtained by multiplying one length by another has the dimension L^2. Volume has the dimension L^3. Mass density is defined as mass per unit volume and has the dimension M/L^3. The SI unit of speed is meters per second (m/s) with the dimension L/T.

The concept of dimensionality is important in understanding physics and in solving physics problems. For example, the addition or subtraction of quantities with different dimensions makes no sense, i.e. 2 kg plus 8 s is meaningless. Actually, physical equations must be dimensionally consistent. For example, the equation giving the position of a freely falling body (see Chap. 3) is giving by:

$$x = v_\circ\, t + \frac{1}{2} g\, t^2 \tag{1.9}$$

where x is the position (length), $v_\circ$ the initial speed (length/time), g is the acceleration due to gravity (length/time^2), and t is time. If we analyze the equation dimensionally, we have:

$$\mathrm{L} = \frac{\mathrm{L}}{\cancel{\mathrm{T}}} \times \cancel{\mathrm{T}} + \frac{\mathrm{L}}{\cancel{\mathrm{T}^2}} \times \cancel{\mathrm{T}^2} = \mathrm{L} + \mathrm{L} \quad \text{(Dimensional analysis)}$$

Note that every term of this equation has the dimension of length L. Also note that numerical factors, such as $\frac{1}{2}$ in Eq. 1.9, are ignored in dimensional analysis because they have no dimension. Dimensional analysis is useful since it can be used to catch careless errors in any physical equation. On the other hand, Eq. 1.9 may be correct with respect to dimensional analysis, but could still be wrong with respect to dimensionless numerical factors.

If we had incorrectly written Eq. 1.9 as follows:

$$x = v_\circ\, t^2 + \frac{1}{2} g\, t \tag{1.10}$$

Then, by analyzing this equation dimensionally, we have:

$$\begin{aligned} \mathrm{L} &= \frac{\mathrm{L}}{\mathrm{T}} \times \mathrm{T}^2 + \frac{\mathrm{L}}{\mathrm{T}^2} \times \mathrm{T} \\ &= \frac{\mathrm{L}}{\cancel{\mathrm{T}}} \times \mathrm{T} \times \cancel{\mathrm{T}} + \frac{\mathrm{L}}{\mathrm{T} \times \cancel{\mathrm{T}}} \times \cancel{\mathrm{T}} \end{aligned} \qquad \text{(Dimensional analysis)}$$

and finally we get:

$$\xcancel{\mathrm{L} = \mathrm{LT} + \frac{\mathrm{L}}{\mathrm{T}}} \qquad \text{(Dimensional analysis)}$$

Dimensionally, Eq 1.10 is meaningless, and thus cannot be correct.

Example 1.2

Use dimensional analysis to show that the expression $v = v_\circ + at$ is dimensionally correct, where v and $v_\circ$ represent velocities, a is acceleration, and t is a time interval.

Solution: Since L/T is the dimension of v and $v_\circ$, and the dimension of a is L/T^2, then when we analyze the equation $v = v_\circ + at$ dimensionally, we have:

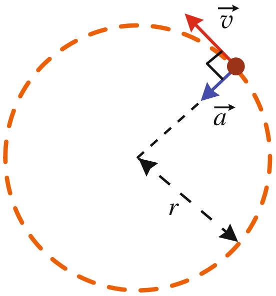
$$\begin{aligned} \frac{\mathrm{L}}{\mathrm{T}} &= \frac{\mathrm{L}}{\mathrm{T}} + \frac{\mathrm{L}}{\mathrm{T}^2} \times \mathrm{T} \\ &= \frac{\mathrm{L}}{\mathrm{T}} + \frac{\mathrm{L}}{\mathrm{T} \times \cancel{\mathrm{T}}} \times \cancel{\mathrm{T}} \end{aligned} \qquad \text{(Dimensional analysis)}$$

and finally we get: $$\frac{\mathrm{L}}{\mathrm{T}} = \frac{\mathrm{L}}{\mathrm{T}} + \frac{\mathrm{L}}{\mathrm{T}} \qquad \text{(Dimensional analysis)}$$

Thus, the expression $v = v_\circ + at$ is dimensionally correct.

Example 1.3

A particle moves with a constant speed v in a circular orbit of radius r, see the figure below. Given that the magnitude of the acceleration a is proportional to some power of r, say r^m, and some power of v, say v^n, then determine the powers of r and v.



Solution: Assume that the variables of the problem can be expressed mathematically by the following relation:

$$a = k\, r^m v^n,$$

where k is a dimensionless proportionality constant. With the known dimensions of r, v, and a we analyze the dimensions of the above relation as follows:

$$\frac{\mathrm{L}}{\mathrm{T}^2} = \mathrm{L}^m \times \left(\frac{\mathrm{L}}{\mathrm{T}}\right)^n = \frac{\mathrm{L}^{m+n}}{\mathrm{T}^n} \qquad \text{(Dimensional analysis)}$$

This dimensional equation would be balanced, i.e. the dimensions of the right hand side equal the dimensions of the left hand side only when the following two conditions are satisfied:

$$m + n = 1,$$

and $$n = 2.$$

Thus: $$m = -1.$$

Therefore, we can rewrite the acceleration as follows:

$$a = k\, r^{-1} v^2 = k\, \frac{v^2}{r}$$

When we later introduce uniform circular motion in Chap. 4, we shall see that $k = 1$ if SI units are used. However, if for example we choose a to be in m/s^2 and v to be in km/h, then k would not be equal to one.

1.4 Exercises

Section 1.1 The International System of Units

(1) Use the prefixes introduced in Table 1.2 to express the following: (a) 10^3 lambs, (b) 10^6 bytes, (c) 10^9 cars, (d) 10^{12} stars, (e) 10^{-1} Kelvin, (f) 10^{-2} meter, (g) 10^{-3} ampere, (h) 10^{-6} newton, (i) 10^{-9} kilogram, (j) 10^{-15} second.

Section 1.2 Standards of Length, Time, and Mass

Length

(2) The original definition of the meter was based on distance from the North pole to the Earth's equator (measures along the surface) and was taken to be 10^7 m. (a) What is the circumference of the Earth in meters? (b) What is the radius of the Earth in meters, (c) Give your answer to part (a) and part (b) in miles. (d) What is the circumference of the Earth in meters assuming it to be a sphere of radius 6.4×10^6 m? Compare your answer to part (a)

(3) The time of flight of a laser pulse sent from the Earth to the Moon was measured in order to calculate the Earth-Moon distance, and it was found to be 3.8×10^5 km. (a) Express this distance in miles, meters, centimeters, and millimeters.

(4) A unit of area, often used in measuring land areas, is the *hectare*, defined as 10^4 m^2. An open-pit coal mine excavates 75 hectares of land, down to a depth of 26 m, each year. What volume of Earth, in cubic kilometers, is removed during this time?

(5) The units used by astronomers are appropriate for the quantities they usually measure. As an example, for planetary distances they use the astronomical

unit (AU), which is equal to the mean Earth-Sun distance (1.5×10^{11} m). For stellar distances they use the light-year (1 ly $= 9.461 \times 10^{12}$ km), which is the distance that light travels in 1 yr (1 yr = 365.25 days $= 3.156 \times 10^7$ s) with a speed of 299 792 458 m/s. They use also the parsec (pc), which is equal to 3.26 light-years. Intergalactic distances might be described with a more appropriate unit called the megaparsec. Convert the following to meters and express each with an appropriate metric prefix: (a) The astronomical unit, (b) The light-year, (c) The parsec, and (d) The megaparsecs.

(6) When you observe a total solar eclipse, your view of the Sun is obstructed by the Moon. Assume the distance from you to the Sun (d_s) is about 400 times the distance from you to the Moon (d_m). (a) Find the ratio of the Sun's radius to the Moon's radius. (b) What is the ratio of their volumes? (c) Hold up a small coin so that it would just eclipse the full Moon, and measure the distance between the coin and your eye. From this experimental result and the given distance between the Moon and the Earth (3.8×10^5 km), estimate the diameter of the Moon.

(7) Assume a spherical atom with a spherical nucleus where the ratio of the radii is about 10^5. The Earth's radius is 6.4×10^6 m. Suppose the ratio of the radius of the Moon's orbit to the Earth's radius (3.8×10^5 km) were also 10^5. (a) How far would the Moon be from the Earth's surface? (b) How does this distance compare with the actual Earth-Moon distance given in exercise 6?

Time

(8) Using the day as a unit, express the following: (a) The predicted life time of proton, (b) The age of universe, (c) The age of the Earth, (d) The age of a 50-year-old tree.

(9) Compare the duration of the following: (a) A microyear and a 1-minute TV commercial, and (b) A microcentury and a 60-min TV program.

(10) Convert the following approximate *maximum* speeds from km/h to mi/h: (a) snail (5×10^{-2} km/h), (b) spider (2 km/h), (c) human (37 km/h),(d) car (220 km/h), and (e) airplane (1,000 km/h).

(11) A 12-hour-dial clock happens to gain 0.5 min each day. After setting the clock to the correct time at 12:00 noon, how many days must one wait until it again indicates the correct time?

(12) Is a cesium clock sufficiently precise to determine your age (assuming it is exactly 19 years, not a leap year) within 10^{-6} s ? How about within 10^{-3} s ?

(13) The slowing of the Earth's rotation is measured by observing the occurrences of solar eclipses during a specific period. Assume that the length of a day is increasing uniformly by 0.001 s per century. (a) Over a span of 10 centuries, compare the length of the last and first days, and find the average difference. (b) Find the cumulative difference on the measure of a day over this period.

Mass

(14) A person on a diet loses 2 kg per week. Find the average rate of mass loss in milligrams every: day, hour, minute, and second.

(15) Density is defined as mass per unit volume. If a crude estimation of the average density of the Earth was 5.5×10^3 kg/m^3 and the Earth is considered to be a sphere of radius 6.37×10^6 m, then calculate the mass of the Earth.

(16) A carbon-12 atom ($^{12}_{6}$C) is found to have a mass of $1.992\,64 \times 10^{-26}$ kg. How many atoms of $^{12}_{6}$C are there in: (a) 1 kg? (b) 12 kg? (This number is Avogadro's number in the SI units.)

(17) A water molecule (H_2O) contains two atoms of hydrogen ($^{1}_{1}$H), each of which has a mass of 1 u, and one atom of oxygen ($^{16}_{8}$O), that has a mass 16 u, approximately. (a) What is the mass of one molecule of water in units of kilograms? (b) Find how many molecules of water are there in the world's oceans, which have an estimated mass of 1.5×10^{21} kg?

(18) Density is defined as mass per unit volume. The density of iron is 7.87 kg/m^3, and the mass of an iron atom is 9.27×10^{-26} kg. If atoms are cubical and tightly packed, (a) What is the volume of an iron atom, and (b) What is the distance between the centers of two adjacent atoms.

Section 1.3 Dimensional Analysis

(19) A simple pendulum has periodic time T given by the relation:

$$T = 2\pi\sqrt{L/g}$$

where L is the length of the pendulum and g is the acceleration due to gravity in units of length divided by the square of time. Show that this equation is dimensionally correct.

(20) Suppose the displacement s of an object moving in a straight line under uniform acceleration a is giving as a function of time by the relation $s = ka^m t^n$, where k is a dimensionless constant. Use dimensional analysis to find the values of the powers m and n.

(21) Using dimensional analysis, determine if the following equations are dimensionally correct or incorrect: (a) $v^2 = v_\circ^2 + 2a\,s$, (b) $s = s_\circ + v_\circ\,t + \frac{1}{2}a\,t^2$, (c) $s = s_\circ \cos kt$, where k is a constant that has the dimension of the inverse of time.

(22) Newton's second law states that the acceleration of an object is directly proportional to the force applied and inversely proportional to the mass of the object. Find the dimensions of force and show that it has units of kg·m/s^2 in terms of SI units.

(23) Newton's law of universal gravitation is given by $F = Gm_1 m_2/r^2$, where F is the force of attraction of one mass, m_1, upon another mass, m_2, at a distance r. Find the SI units of the constant G.