Faraday's Law, Alternating Current, and Maxwell's Equations

27

Experimentally, M. Faraday and J. Henry show that a changing magnetic field can establish a current in a circuit that has no battery.

27.1 Faraday's Law of Induction

When we *move a magnet* toward a *stationary loop* that is connected to a galvanometer, see Fig. 27.1a, the galvanometer's needle deflects in one direction. When the magnet stops, as shown in Fig. 27.1b, no deflection is observed. Now, when we move the magnet away from the loop, as shown in Fig. 27.1c, the needle deflects in the opposite direction.

The current produced in this loop is called an *induced current* and the work done per unit charge in producing that current is called an *induced emf*. This emf is due to the change in magnetic flux through the loop, and this process is known as **Faraday's law of induction** and stated as:

> Faraday's law of induction
> The magnitude of the induced emf $|\mathcal{E}|$ in a conducting loop is equal to the rate of change of the magnetic flux Φ_B through the loop.

Lenz's Law

Soon after Faraday proposed his law, Lenz devised a rule—now known as **Lenz's law**, for determining the direction of an induced emf and the direction of an induced current in a loop. This law states that:

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_27,

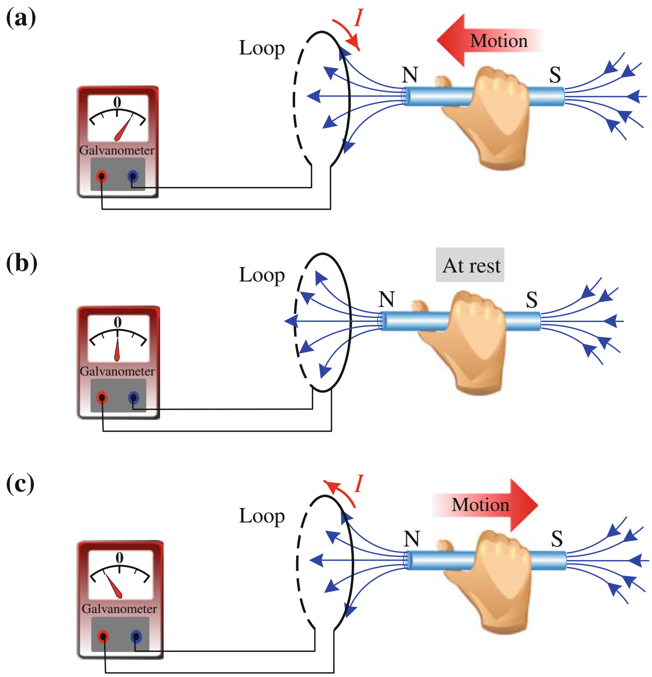


Fig. 27.1 A galvanometer registers an induced current in a loop when the magnet is moving with respect to the loop (parts a and c). In part b, the magnet is at rest, and no induced current is established

> Lenz's law
> An induced current in a loop is created such that the internal magnetic field of the loop opposes the changes in the external magnetic flux.

To get a sense of Lenz's law, let us consider the case of a bar magnet approaching the loop of Fig. 27.2a. During the motion toward the loop, the external magnetic field $\vec{B}_{ext}$ of the bar magnet increases the magnetic flux on the loop and thereby induces a current in the loop. The induced current produces its own internal magnetic field $\vec{B}_{int}$ that counteracts the increase in the external magnetic flux. In Fig. 27.2b, $\vec{B}_{int}$ opposes the decrease in the external flux.

Based on Faraday's law and Lenz's law, the induced emf in a coil of N loops of the same area is given by:

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} \quad \text{(Faraday's law)} \tag{27.1}$$

If a coil lies in a uniform magnetic field $\vec{B}$, then $\Phi_B = \vec{B} \bullet \vec{A} = B\,A\cos\theta$, where any combination of the quantities A, B, and θ can change with time. The induced emf in this case will take the form:

$$\mathcal{E} = -N\frac{d}{dt}(B\,A\cos\theta) \tag{27.2}$$

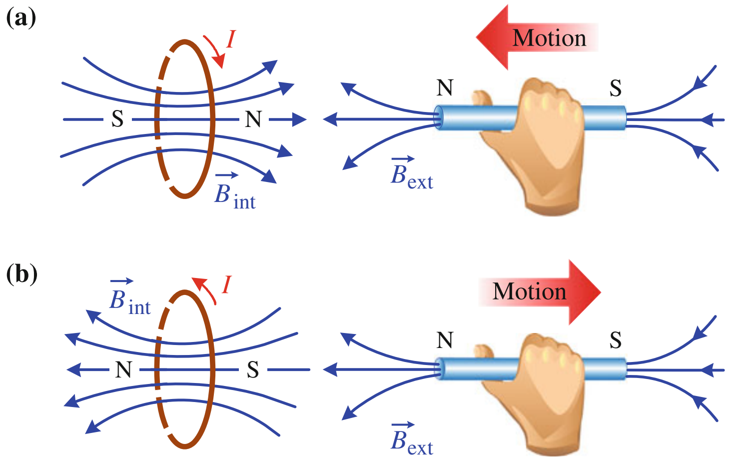


Fig. 27.2 The internal magnetic field $\vec{B}_{int}$: (a) opposes the *increase* in flux of $\vec{B}_{ext}$. (b) opposes the *decrease* in flux of $\vec{B}_{ext}$

Example 27.1

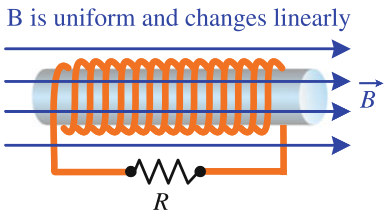
A coil of wire has $N = 15$ turns and each turn has an area $A = 0.04\,\text{m}^2$. The coil is placed in a uniform magnetic field directed perpendicular to the plane of the coil and connected to a resistor of resistance $R = 2\,\Omega$, see Fig. 27.3. The magnetic field changes linearly from $0.1\,\text{T}$ at time $t = 0$ to $0.6\,\text{T}$ at time $t = 0.5\,\text{s}$. (a) What is the magnitude of the induced emf in the coil during this time interval? (b) What is the magnitude and direction of the induced current?

Solution: (a) The flux Φ_B through each turn at $t = 0$ and $t = 0.5\,\text{s}$ is:

$$\Phi_B|_{t=0} = B\,A = (0.1\,\text{T})\,(0.04\,\text{m}^2) = 0.004\,\text{Wb}$$

$$\Phi_B|_{t=0.5\,\text{s}} = B\,A = (0.6\,\text{T})(0.04\,\text{m}^2) = 0.024\,\text{Wb}$$

Fig. 27.3



Therefore, from Eq. 27.1, the magnitude of the induced emf is:

$$|\mathcal{E}| = N\frac{\Delta\Phi_B}{\Delta t} = 15\frac{0.024\,\text{Wb} - 0.004\,\text{Wb}}{0.5\,\text{s} - 0} = 0.6\,\text{V}$$

(b) According to Lenz's law, since the magnetic flux increases, then the induced current I established in the circuit must be in a clockwise direction. The value of the induced current is:

$$I = \frac{|\mathcal{E}|}{R} = \frac{0.6\,\text{V}}{2\,\Omega} = 0.3\,\text{A}$$

27.2 Motional emf

The **Motional emf** is an induced emf in a conductor moving through a constant magnetic field.

Figure 27.4 shows a conducting bar of length L moving to the right with a velocity $\vec{v}$ perpendicular to a uniform magnetic field $\vec{B}$ into the page. Each conduction electron is subjected to a downward magnetic force $\vec{F}_B = -e\vec{v} \times \vec{B}$. Consequently, an accumulation of negative charges on the lower end is established, leaving a net positive charge on the upper end. Because of this accumulation, a downwards electric field $\vec{E}$ is produced inside the conducting bar and hence an upwards electric force $\vec{F}_e = -e\vec{E}$ is exerted on each electron. The accumulation process will continue until the magnitude of the downwards magnetic force $F_B = evB$ is balanced by the upwards electric force $F_e = eE$. This condition of equilibrium requires that:

$$eE = evB \quad \text{or} \quad E = vB \tag{27.3}$$

The established electric field is related to the induced potential difference ΔV between the ends of the conducting bar according to the relation $\Delta V = EL$ or $|\mathcal{E}| = EL$, see Eq. 22.17. Thus, from Eq. 27.3, the equilibrium condition requires that:

$$\Delta V = BLv \quad \text{or} \quad |\mathcal{E}| = BLv \tag{27.4}$$

Of course, the potential of the upper end is higher than the lower end, and this polarity is reversed when the direction of motion is reversed.

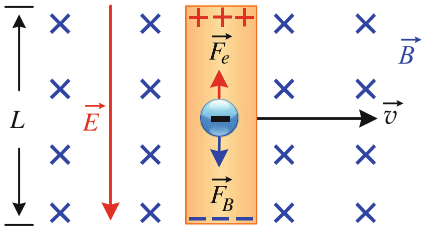


Fig. 27.4 A conductor of length L moving with velocity $\vec{v}$ across a uniform magnetic field $\vec{B}$. This establishes an electric field $\vec{E}$

Now, let us consider the sliding of this conducting bar on horizontal, frictionless, conducting rails connected to a resistor of a resistance R, as shown in Fig. 27.5 (top view). As the bar is pulled to the right with a velocity $\vec{v}$ under the influence of a force $\vec{F}_{\text{app}}$, a magnetic force acts on the free electrons, causing a counterclockwise induced conventional current I to pass in the circuit. At the same time, a magnetic force $\vec{F}_B = ILB$ will act on the bar opposite to $\vec{F}_{\text{app}}$. If both forces are equal, $F_B = F_{app}$, the bar will move with a constant speed v.

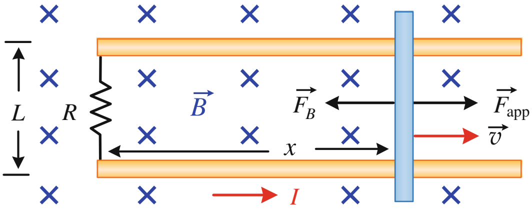


Fig. 27.5 A conducting bar of length L is connected to a resistor and moves on a horizontal conducting rails with velocity $\vec{v}$ across a uniform magnetic field $\vec{B}$. A counterclockwise current I is induced

The area of the circuit within the magnetic field is Lx, where x is the position of the bar from the resistor. Thus, the magnetic flux through this area is:

$$\Phi_B = BLx \tag{27.5}$$

Using Faraday's law, and noting that $v = dx/dt$, the induced emf is:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(BLx) = -BL\frac{dx}{dt}$$

Therefore,

$$\mathcal{E} = -BLv \tag{27.6}$$

and,

$$I = \frac{|\mathcal{E}|}{R} = \frac{BLv}{R} \tag{27.7}$$

The power delivered by the applied force is:

$$P = F_{\text{app}}\, v = (ILB)v = \frac{B^2L^2v^2}{R} = \frac{\mathcal{E}^2}{R} \tag{27.8}$$

This proves that the power input is equal to the rate at which energy is delivered to the resistor, which is confirmed by Eq. 24.24.

Example 27.2

The conducting bar in Fig. 27.5 is 0.1 m long and moves to the right with a speed of 10 m/s across a uniform magnetic field of 1.5 T.(a) Find the induced emf in the circuit. (b) Find the force per a unit charge in the conducting bar.

Solution: (a) From Eq. 27.6, the induced emf in the circuit is:

$$|\mathcal{E}| \ = \ BLv = (1.5\,\text{T})(0.1\,\text{m})(10\,\text{m/s}) = 1.5\,\text{V}$$

From Lenz's law, a counterclockwise emf is created in the circuit.

(b) Using Eq. 25.2, the force per unit charge is:

$$\frac{F_B}{|q|} = vB\sin 90^\circ = (10\,\text{m/s})(1.5\,\text{T}) = 15\,\text{N/C}$$

From $\vec{F}_B = q\,\vec{v} \times \vec{B}$, the force $\vec{F}_B$ must be along the bar and upwards.

Example 27.3

The conducting bar in Fig. 27.5 is 0.5 m long and moves to the right across a uniform magnetic field of 0.15 T. If the total resistance of the circuit is 3 Ω, calculate the force required to move the rod at a constant speed of 2 m/s. Find the power delivered.

Solution: From Eq. 27.6, the induced emf in the circuit is:

$$|\mathcal{E}| = BLv = (0.15\,\text{T})(0.5\,\text{m})(2\,\text{m/s}) = 0.15\,\text{V}$$

From Lenz's law, a counterclockwise emf is created in the circuit.

The current in the circuit is:

$$I = \frac{|\mathcal{E}|}{R} = \frac{0.15\,\text{V}}{3\,\Omega} = 0.05\,\text{A}$$

Equation 25.19 gives the magnitude of the magnetic force on the bar:

$$F_B = ILB\sin 90^\circ = (0.05\,\text{A})(0.5\,\text{m})(0.15\,\text{T}) = 3.75\times10^{-3}\,\text{N}$$

This magnetic force is directed to the left and must be equal in magnitude to the applied force F_{app}, but opposite in direction. The power delivered is calculated from Eq. 27.8 as:

$$P = F_{\text{app}}\,v = F_B\,v = (3.75\times10^{-3}\,\text{N})(2\,\text{m/s}) = 7.5\times10^{-3}\,\text{W}$$

Example 27.4

A rectangular conducting loop has a resistance R, width L, and length a. The loop is pulled at a constant speed v to the right while approaching, entering, and leaving a uniform magnetic field directed out of the page, which extends over a distance b along the x-axis, see Fig. 27.6 for different times. Plot as a function of x: (a) the magnetic flux through the loop, (b) the induced motional emf.

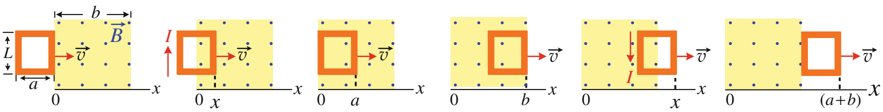


Fig. 27.6

Solution: (a) Figure 27.7 shows the flux Φ_B through the loop as a function of x for different times. The flux is zero when the loop is not in the field; it is BLx when the loop is entering the field; it is BLa when the loop is entirely in the field; it is $BL(a+b-x)$ when the loop is leaving the field; and finally zero when $x \geq a+b$.

(b) As the loop enters the field, the flux increases (with $\vec{B}$ out of the page). By Lenz's law, a clockwise current is created to produce a magnetic field into the page with emf $\mathcal{E} = BLv$. When the loop is entirely in the field, the change in flux is zero, and hence $\mathcal{E} = 0$. When the loop is leaving the field, the flux decreases and a counterclockwise current is created with $\mathcal{E} = -BLv$. When the loop leaves the field, the emf drops to zero, see Fig. 27.7. The current value is $I = BLv/R$.

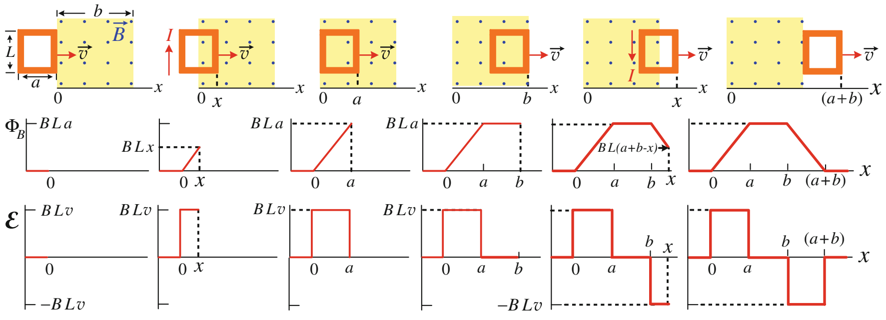


Fig. 27.7

27.3 Electric Generators

Electric generators are devices that convert rotational energy to electric energy.

A generator consists of a coil of wire wound on an *armature* that can rotate in a magnetic field between the poles of the magnet. The magnetic flux through the coil changes with time. Thus, according to Faraday's law, an induced emf and current will be created in the coil.

If θ is the angle between the magnetic field $\vec{B}$ and the normal to the plane of the coil, then the magnetic flux through each loop of the coil will be given by:

$$\Phi_B = \vec{B} \bullet \vec{A} = BA \cos\theta \tag{27.9}$$

If the shaft of the generator rotates with constant angular frequency ω (in rad/s), then the relation between the angular position θ (in rad) and the frequency ω is $\theta = \omega t$. Therefore, $\Phi_B = BA \cos\omega t$. Hence, according to Faraday's law, the induced emf of a coil of N loops will be:

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = -N\frac{d}{dt}(BA\cos\omega t) = N\,B\,A\,\omega \sin\omega t$$

which can be written in compact ($\mathcal{E} \equiv v$ and $\mathcal{E}_\circ \equiv V$) form as follows:

$$\boxed{v = V\sin\omega t} \quad \text{where} \quad \boxed{V = N\,B\,A\,\omega} \tag{27.10}$$

The output emf is sinusoidal with amplitude (or peak) V. From this relation we see that $v = 0$ when $\omega t = 0$ or $\omega t = \pi$, and this occurs when $\vec{B}$ is perpendicular to the plane of the coil. Furthermore, $v = V$ when $\omega t = \pi/2$ or $\omega t = 3\pi/2$, and this occurs when $\vec{B}$ is in the plane of the coil.

The Direct Current (dc) Generator

Figure 27.8 illustrates the simplest form of the **direct current** (**dc**) **generator**. The ends of the coil are connected to a split-ring commutator (as shown in Fig. 27.8a) that rotate with the coil. Those splits are in contact with two brushes that act as the output terminals of the generator. The output is always of the same polarity and varies with time as shown in Fig. 27.8b.

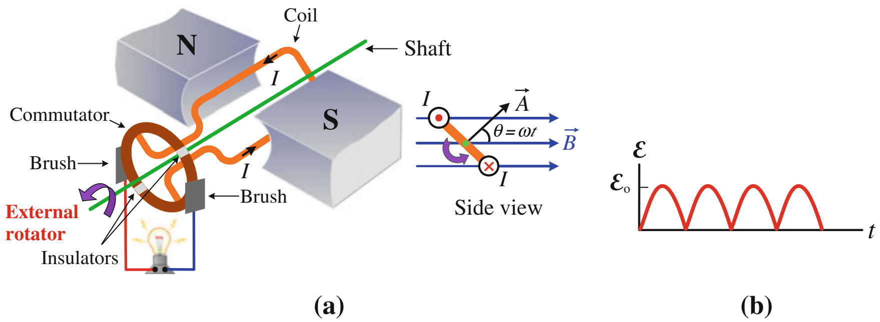


Fig. 27.8 (a) A sketch of a dc generator. An emf is induced in a coil when it rotates with constant angular frequency ω in a magnetic field $\vec{B}$. (b) The direct induced emf is plotted as a function of time

The Alternating Current (ac) Generator

Figure 27.9 illustrates the simplest form of an **alternating current** (**ac**) **generator**. The ends of the coil are connected to slip-rings (as shown in Fig. 27.9a) that rotate with the coil. Those slips are in contact with two brushes that act as the output terminals of the generator. The output varies sinusoidally with time. See Fig. 27.9b.

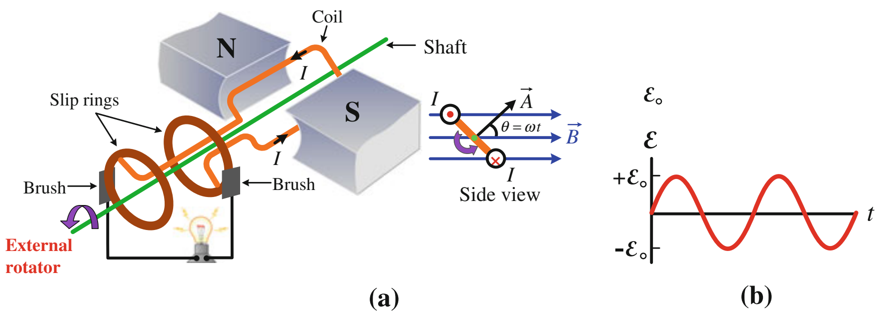


Fig. 27.9 (a) A sketch of an ac generator. An emf is induced in a coil when it rotates with constant angular frequency ω in a magnetic field. (b) The alternating, induced emf is plotted as a function of time

27.4 Alternating Current

When electric generators at electric power plants produce alternating emf we get **alternating current**, or **ac** (uppercase letters can be used) with the usual symbol . Alternating current reverses direction many times per second, which means that electrons in a wire will repeatedly move in one direction and then reverse their direction. Since the output emf of an ac generator is sinusoidal, as shown in Fig. 27.9b, then the current it produces is also sinusoidal.

Ohm's law, Eq. 24.8, is also valid for alternating voltage and current. Based on Eq. 27.10, when a sinusoidal voltage v exists across a resistance R, see Fig. 27.10a, then the alternating current i (we use the small letter i for ac) through the resistor is:

$$i = \frac{v}{R} = \frac{V}{R}\sin\omega t = I\sin\omega t \quad \text{where} \quad I = \frac{V}{R} \tag{27.11}$$

where I is the peak current, see Fig. 27.10b. From this figure we see that the average current is zero. This does not mean that no heat is produced in the resistor. On the contrary, electrons produce heat when they move back and forth in the resistor. The power delivered at time t to a resistor of resistance R, see Fig. 27.10c, is:

$$P(t) = i^2 R = I^2 R\sin^2\omega t \tag{27.12}$$

which indicates that the power is always positive because the current is squared. The quantity $\sin^2\omega t$ varies between 0 and 1 and we can prove that its mean (or average) value is $\frac{1}{2}$, i.e. $\overline{\sin^2\omega t} = 1/2$. Therefore, using $\overline{P} = \overline{i^2}\,R$ or $\overline{P} = \overline{v^2}/R^2$ as the average power delivered to the resistor, we get:

$$\overline{P} = I^2R/2 \quad \text{or} \quad \overline{P} = V^2/2R \tag{27.13}$$

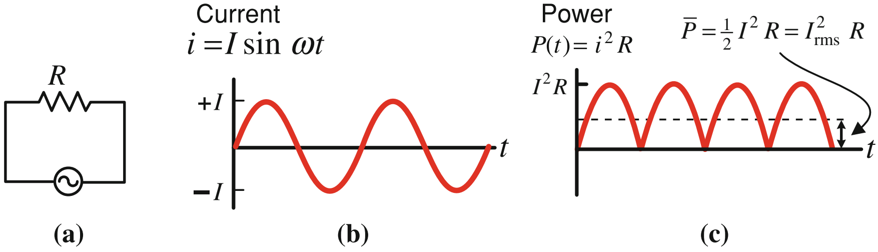


Fig. 27.10 (a) A resistor connected to an ac source. (b) Alternating current in a resistor as a function of time. (c) Power and average power delivered to a resistor as a function of time

As introduced in Sect. 13.1, the notation rms stands for *root-mean-square*, which here means *the square root of the mean value of the square of the current* $I_{rms} = \sqrt{\overline{i^2}}$ or the voltage $V_{rms} = \sqrt{\overline{\mathcal{E}^2}}$. Thus (remember $\mathcal{E} \equiv v$):

$$I_{rms} = \frac{I}{\sqrt{2}} = 0.707\ I \quad \text{and} \quad V_{rms} = \frac{V}{\sqrt{2}} = 0.707\ V \tag{27.14}$$

This means that an alternating current whose maximum value is 1 A delivers to a resistor the same power as a direct current of 0.707 A. Thus, Ohm's law and the average power delivered to a resistor give:

$$V_{rms} = I_{rms}\ R \qquad \text{and} \qquad \overline{P} = I_{rms}\ V_{rms} = I_{rms}^2\ R = V_{rms}^2/R \tag{27.15}$$

Alternating-current instruments are usually calibrated to read the rms values of the current defined by $I_{rms} = I/\sqrt{2}$ and voltage defined by $V_{rms} = V/\sqrt{2}$. More than 81% of countries around the globe use V_{rms} in the range from 220 to 240 V.

Example 27.5

Find the resistance and the peak current in a 1,000-W heater connected to a 220-V ac line.

Solution: Using Eq. 27.15, we can find the rms current:

$$I_{rms} = \frac{\overline{P}}{V_{rms}} = \frac{1{,}000\ \text{W}}{220\ \text{V}} = 4.55\ \text{A}$$

Then, the peak current and the resistance of the heater will be:

$$I = \sqrt{2}\ I_{rms} = 6.43\ \text{A}$$

$$R = \frac{V_{rms}}{I_{rms}} = \frac{220\ \text{V}}{4.55\ \text{A}} = 48.35\ \Omega$$

27.5 Transformers

A **transformer** is a device used to increase or decrease an ac voltage. Transformers are widely used in: reducing the high voltage from the electric power plant to a usable household electric ac outlet (120 or 220 V), in chargers for mobiles, laptops, and other electronic devices, in cars to increase the voltage to a high voltage needed to spark the plugs, in CRT monitors, and on many electrical applications.

An ideal transformer consists of two resistanceless coils known as the **primary** and **secondary** coils, wound around an iron core, see Fig. 27.11. We assume that the primary coil has N_P turns and the secondary coil has N_S turns. If all magnetic lines are confined to the iron core, then at any instant the magnetic flux per turn Φ_B produced by the current in the primary coil will pass through the secondary coil.

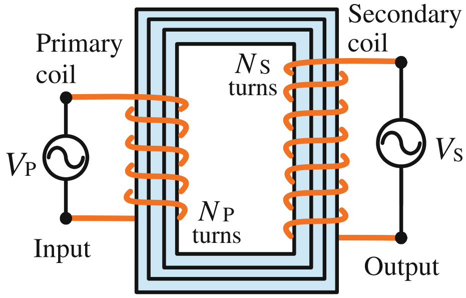


Fig. 27.11 An ac input of voltage v_P (with peak V_P) is connected to the primary coil of a transformer to get an ac output in the secondary coil. The figure shows a step-up transformer where $N_P = 5$ and $N_S = 7$

When an ac voltage v_P (with peak V_P) is applied to the primary coil, the magnetic flux change is the same in each turn of the primary and secondary coils. Thus, according to Faraday's law, the resulting induced emf in the primary and secondly coils will be:

$$\mathcal{E}_P = -N_P \frac{d\Phi_B}{dt} \quad \text{and} \quad \mathcal{E}_S = -N_S \frac{d\Phi_B}{dt} \tag{27.16}$$

where $\mathcal{E}_P$ and $\mathcal{E}_S$ have the same frequency as the ac input source v_P. Since the flux per turn Φ_B is the same, the ratio of the secondary emf to the primary emf at any instant is $\mathcal{E}_S/\mathcal{E}_P = N_S/N_P$. If the windings have zero resistance, the induced emf $\mathcal{E}_P$ and $\mathcal{E}_S$ are exactly balanced by the terminal voltage across the primary voltage v_P (with peak V_P) and the secondary voltage v_S (with peak V_S), respectively. Thus:

$$\frac{V_S}{V_P} = \frac{N_S}{N_P} \tag{27.17}$$

This **transformer equation** relates the output (the secondary) to the input (the primary) and can apply for the amplitude or the rms values.

If $N_S > N_P$, then $V_S > V_P$ and this kind of transformer is called a **step-up transformer**. Similarly, if $N_S < N_P$, then $V_S < V_P$ and this kind of the transformer is called a **step-down transformer**.

For step-up or step-down ideal transformers, the power output is equal to the power input. Using Eq. 24.24, we have:

$$I_P\, V_P = I_S\, V_S \quad \text{or} \quad \frac{I_S}{I_P} = \frac{N_P}{N_S} \tag{27.18}$$

Example 27.6

The transformer used to charge a laptop reduces 220-V ac to 19 V ac. Assume that the primary coil contains 400 turns and the charger supplies 5 A to the laptop. Find: (a) the number of turns in the secondary coil and the current in the primary coil, and (b) the power transformed.

Solution: (a) Using Eq. 27.17, we have:

$$\frac{V_S}{V_P} = \frac{N_S}{N_P} \quad \Rightarrow \quad N_S = N_P \frac{V_S}{V_P} = \frac{(400)(19\,\text{V})}{220\,\text{V}} = 35\,\text{turns}$$

Using Eq. 27.18, we have:

$$I_P\, V_P = I_S\, V_S \quad \Rightarrow \quad I_P = I_S\, \frac{V_S}{V_P} = \frac{(5\,\text{A})(19\,\text{V})}{220\,\text{V}} = 431.8\,\text{mA}$$

(b) The power transformed is:

$$P = I_S\, V_S = (5\,\text{A})(19\,\text{V}) = 95\,\text{W}$$

27.6 Induced Electric Fields

By Faraday's law a changing magnetic flux induces both an emf and a current in a conducting loop. But in Chap. 24, see Eq. 24.11, we related current to an electric field that applies electric forces on charged particles.

Similarly, we relate an induced current to an electric field such as:

Spotlight

A changing magnetic field in a conducting loop, or even in any hypothetical closed path, induces an electric field.

We can examine this statement by considering a circular copper loop of radius r placed in a uniform magnetic field $\vec{B}$ perpendicular to the loop, see Fig. 27.12. If the magnetic field *increases* with time, then according to Faraday's law, an induced emf and an induced current are created in the loop. But this induced current implies the existence of an induced electric field $\vec{E}$. The work done by $\vec{E}$ to move a test charge

$q_\circ$ around the loop is $W = q_\circ \mathcal{E}$. On the other hand, according to Eq. 22.3, this work is given for a closed loop by:

$$W = q_\circ \oint \vec{E} \bullet d\vec{s} \tag{27.19}$$

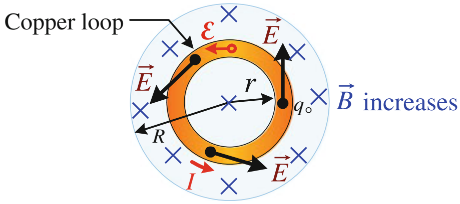


Fig. 27.12 A copper loop in a uniform magnetic field. If $\vec{B}$ changes with time, an induced electric field is produced tangent to the loop

Thus, equating the two expressions of work, we get:

$$q_\circ \oint \vec{E} \bullet d\vec{s} = q_\circ \mathcal{E} \tag{27.20}$$

Using $\mathcal{E} = -d\Phi_B/dt$, then Faraday's law of induction can be expressed in terms of the induced electric field as follows:

$$\oint \vec{E} \bullet d\vec{s} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law)} \tag{27.21}$$

The striking feature of Eq. 27.21 is that the electric field is induced even if there is no conducting loop of $r < R$. In addition, an induced electric field is established even if $r > R$, see Fig. 27.13.

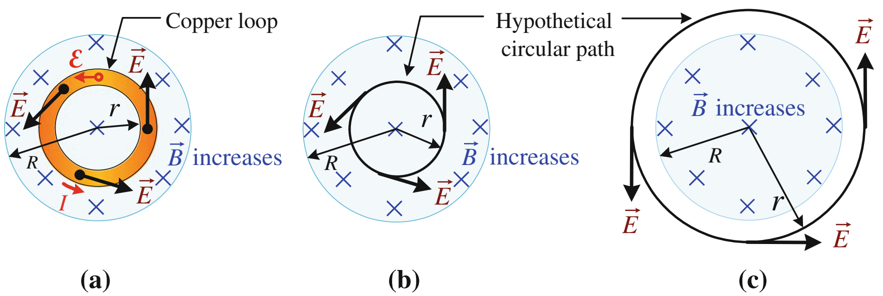


Fig. 27.13 (a) Same as Fig. 27.10 when the conducting loop is in place. (b) Induced electric field is established even for a hypothetical path of $r < R$. (c) Same as (b), but when $r > R$

Example 27.7

In Fig. 27.13, find an expression for the magnitude of the induced electric field E for $r < R$ and $r > R$. When $R = 8$ cm and the magnitude of the magnetic field increases at a rate given by $dB/dt = 0.2$ T/s, evaluate E for $r = 5$ cm and $r = 10$ cm

Solution: We evaluate the integral of Eq. 27.21 for any radius:

$$\oint \vec{E} \bullet d\vec{s} = \oint E\, ds = E \oint ds = E(2\pi r) = 2\pi r E$$

The flux Φ_B and its rate $d\Phi_B/dt$ through the circular path of $r < R$ are:

$$\Phi_B = BA = B(\pi r^2) \quad \text{and} \quad d\Phi_B/dt = \pi r^2 dB/dt$$

The flux Φ_B and its rate $d\Phi_B/dt$ through the circular path of $r > R$ are:

$$\Phi_B = BA = B(\pi R^2) \quad \text{and} \quad d\Phi_B/dt = \pi R^2 dB/dt$$

Thus, dropping the minus sign of Eq. 27.21 leads to:

$$E = \frac{r}{2}\frac{dB}{dt} \quad (\text{for } r < R) \quad \text{and} \quad E|_{r=5\,\text{cm}} = \frac{0.05\,\text{m}}{2} 0.2\,\text{T/s} = 5 \times 10^{-3}\,\text{V/m}$$

$$E = \frac{R^2}{2r}\frac{dB}{dt} \quad (\text{for } r > R) \quad \text{and} \quad E|_{r=10\,\text{cm}} = \frac{(0.08\,\text{m})^2}{2 \times (0.1\,\text{m})} 0.2\,\text{T/s} = 6.4 \times 10^{-3}\,\text{V/m}$$

27.7 Maxwell's Equations of Electromagnetism

From our previous studies, we can collect all the relationships between electric and magnetic fields and their sources. This collection consists of four equations, called **Maxwell's equations**. Maxwell used these equations to predict the existence of electromagnetic waves.

The first two equations involve a surface integral of $\vec{E}$ and $\vec{B}$ over a closed surface. The third and fourth two equations involve a line integral of $\vec{B}$ and $\vec{E}$ over a closed loop.

The first is simply Gauss's law for electric fields, Eq. 21.7, which states that "the net electric flux through any closed surface is equal to the net charge inside the surface divided by the permittivity of free space $\epsilon_\circ$". That is:

$$\oint \vec{E} \bullet \vec{A} = \frac{q_{\text{in}}}{\epsilon_\circ} \quad \text{(Gauss's law for } \vec{E}\text{)} \tag{27.22}$$

The second is the analogous relationship for magnetic field, Eq. 26.24, which states "The *net magnetic flux* throughout any closed surface is always zero". That is:

$$\oint \vec{B} \bullet d\vec{A} = 0 \quad \text{(Gauss's law for } \vec{B}\text{)} \tag{27.23}$$

The third equation is Ampere–Maxwell law, Eq. 26.20, which states "Magnetic fields are produced both by varying conduction currents i and by displacement currents i_d, created by a time varying electric flux". That is:

$$\oint \vec{B} \bullet d\vec{s} = \mu_\circ (i + i_d) = \mu_\circ \left(i + \epsilon_\circ \frac{d\Phi_E}{dt} \right) \quad \text{(Ampere–Maxwell law)} \tag{27.24}$$

The fourth equation is Faraday's, Eq. 27.21, which states that "A changing magnetic field in a conducting loop, or even in any hypothetical closed path, induces an electric field". That is:

$$\oint \vec{E} \bullet d\vec{s} = -\frac{d\Phi_B}{dt} \quad \text{(Faraday's law)} \tag{27.25}$$

In all Maxwell's equations, $\vec{E}$ is the total electric field, which comes from an electrostatic field $\vec{E}_s$ caused by static charges and magnetically-induced, non-electrostatic field $\vec{E}_{\text{ind}}$.

It is worth noting that in empty space, where there is no charge and conduction current, the first two equations are identical in form. In addition to that, replacing Φ_E by $\int \vec{E} \bullet d\vec{A}$ and Φ_B by $\int \vec{B} \bullet d\vec{A}$, we can write the third and fourth equations in different but equivalent forms. Thus, in empty space Maxell's equations reduce to:

$$\begin{aligned} &\oint \vec{E} \bullet d\vec{A} = 0, && \oint \vec{B} \bullet d\vec{A} = 0 \\ &\oint \vec{B} \bullet d\vec{s} = \mu_\circ \epsilon_\circ \frac{d}{dt} \int \vec{E} \bullet d\vec{A}, && \oint \vec{E} \bullet d\vec{s} = -\frac{d}{dt} \int \vec{B} \bullet d\vec{A}. \end{aligned} \tag{27.26}$$

The most remarkable feature about these Maxwell's equations is that time-varying of $\vec{E}$ induces a field $\vec{B}$ and time-varying of $\vec{B}$ induces a field $\vec{E}$ in neighboring regions of space. Maxwell recognized that Eq. 27.26 predict the existence of electromagnetic disturbance of electric and magnetic fields that propagate from one point to another, even if no matter is present. This disturbance is called an electromagnetic wave (EMW), and this provides the physical basis for light and all the rest of the electromagnetic spectrum.

The properties of electromagnetic waves can be deduced from Maxwell's equations, but the mathematical treatment is beyond the scope of this book. Instead, one can focus attention on an electromagnetic wave traveling in the x-direction. By doing so, we can show that the line integral of the last two forms of Eq. 27.26 lead to the following two differential equations:

$$\frac{\partial^2 E}{\partial x^2} - \mu_\circ \epsilon_\circ \frac{\partial^2 E}{\partial t^2} = 0 \quad \text{and} \quad \frac{\partial^2 B}{\partial x^2} - \mu_\circ \epsilon_\circ \frac{\partial^2 B}{\partial t^2} = 0 \tag{27.27}$$

These two differential equations have the identical form as the general wave equation introduced in Chap. 14, see Eq. 14.58, for one-dimensional wave motion, but with speed c, given by:

$$c = \frac{1}{\sqrt{\mu_\circ \epsilon_\circ}} \tag{27.28}$$

Taking $\mu_\circ = 4\pi \times 10^{-7}$ T.m/A and $\epsilon_\circ = 8.8542 \times 10^{-12}$ $C^2/N.m^2$, we find that $c = 2.99792 \times 10^8$ m/s, which is precisely the speed of light in empty space. In addition $\vec{E}$ and $\vec{B}$ are perpendicular to one another, and both are perpendicular to the wave velocity $\vec{c}$, see Fig. 27.14a.

The simplest solution to Eq. 27.27 is a sinusoidal wave, where E and B vary with x and t according to the two expressions:

$$E = E_\circ \cos(kx - \omega t) \quad \text{and} \quad B = B_\circ \cos(kx - \omega t) \tag{27.29}$$

where $E_\circ$ and $B_\circ$ are the peak values of the electric and magnetic waves, respectively, while k and ω are defined in Chap. 14. Figure 27.14b displays the various types of electromagnetic waves.

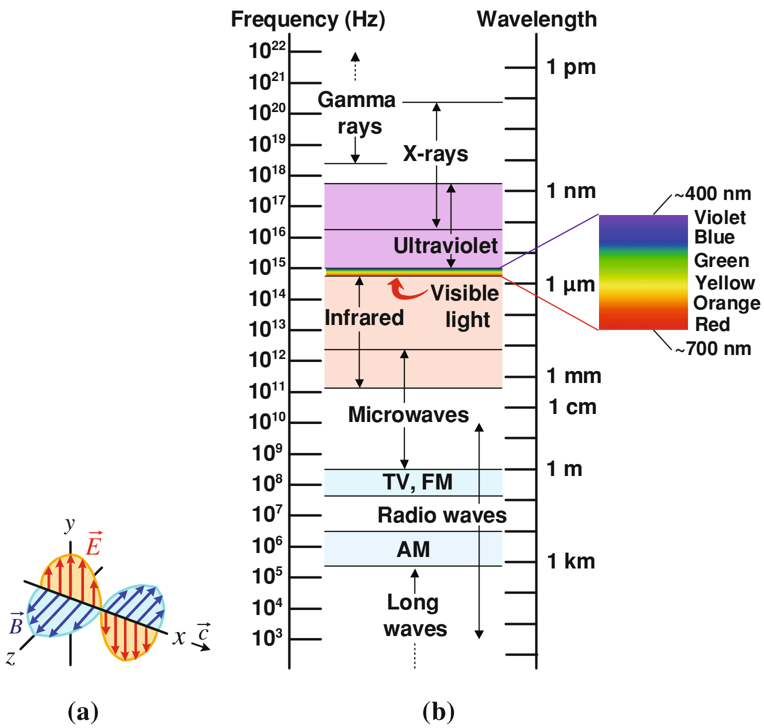


Fig. 27.14 (a) Propagation of EMW. (b) The EMW spectrum

27.8 Exercises

Section 27.1 Faraday's Law of Induction

(1) A loop of wire with a vector area $\vec{A} = (0.2\,\vec{i} + 0.3\,\vec{j} + 0.6\,\vec{k})\ \text{m}^2$ is placed in a uniform magnetic field $\vec{B} = 0.2\,\vec{j}$ (T). (a) Find the magnetic flux through the loop. (b) What is the angle between $\vec{B}$ and $\vec{A}$?

(2) If the magnetic field of Exercise 1 changes to $\vec{B} = 0.2\,\vec{k}$ (T) in a period of 0.5 s, find the magnitude of the average induced emf and current in the loop, assuming the loop has a resistance of $1.5\,\Omega$.

(3) In Fig. 27.15, the south pole of the magnet approaches the loop. What is the direction of the induced current in the resistor of the circuit?

(4) In Fig. 27.16, the north pole of the magnet recedes from the loop. What is the direction of the induced current in the resistor of the circuit?

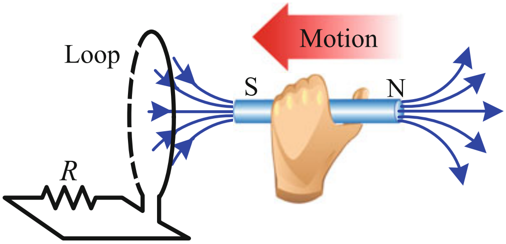


Fig. 27.15 See Exercise (3)

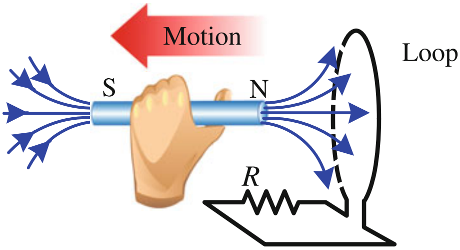


Fig. 27.16 See Exercise (4)

(5) The magnetic flux through a loop of wire changes from −20 to +20 Wb in 0.2 s. What is the average induced emf in the loop?

(6) Figure 27.17 shows a circuit containing a battery and a resistor whose resistance can vary. Two loops are located inside and outside the circuit. If the resistance is slowly decreased, what is the direction of the induced current in the two loops?

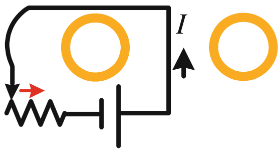


Fig. 27.17 See Exercise (6)

(7) A circular loop of radius $r = 5$ cm is perpendicular to a uniform magnetic field that is pointing out of the page and has an initial magnitude $B_i = 0.6$ T. During a time interval of 0.2 s the field is changed to a final magnitude $B_f = 0.2$ T and now points into the page. What is the average induced emf in the loop?

(8) A square loop of wire has a side $a = 5$ cm and is perpendicular to a uniform magnetic field of magnitude $B = 0.8$ T. The orientation of the loop changes in a period of 0.4 s until the surface of its plane is parallel to the field. What is the average induced emf in the loop?

(9) The plane of a circular loop of radius $r = 10$ cm is perpendicular to a uniform magnetic field of initial magnitude $B = 0.8$ T. The field's magnitude then decreases at a constant rate of $dB/dt = -10^{-3}$ T/s. (a) What is the magnitude of the field at any time? (b) What is the induced emf produced in the loop?

(10) For each situation in Fig. 27.18, show the direction of the induced current.

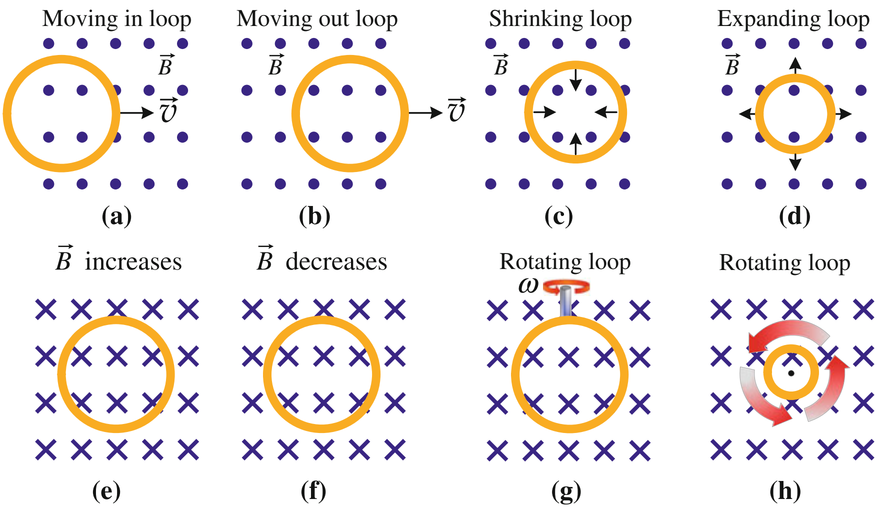


Fig. 27.18 See Exercise (10). (a) Moving in loop. (b) Moving out loop. (c) Shrinking loop. (d) Expanding loop. (e) $\vec{B}$ increases. (f) $\vec{B}$ decreases. (g) Rotating loop. (h) Rotating loop

(11) The diameter of the circular loop of Fig. 27.18c decreases from 20 to 5 cm in 0.6 s. The magnitude of the magnetic field in that figure has a value $B = 0.5$ T. (a) What is the direction of the induced current? (b) What is the average induced emf in the loop? (c) What is the average magnitude of the induced current if the loop's resistance is 2 Ω?

(12) The radius of the circular loop of Fig. 27.18d increases from 2 cm to 15 cm in 0.2 s. The magnitude of the magnetic field in that figure has a value $B = 0.25$ T. (a) What is the direction of the induced current? (b) What is the average induced emf in the loop? (c) What is the average magnitude of the induced current if the loop's resistance is 1.5 Ω?

(13) The plane of the circular loop of Fig. 27.18g has an area $A = 5\ \text{cm}^2$. In 0.2 s it rotates to make an angle $\theta = 60°$ with the field lines. The magnitude of the

magnetic field in that figure has a value $B = 0.75\,\text{T}$. (a) What is the direction of the induced current? (b) What is the average induced emf in the loop? (c) What is the average magnitude of the induced current if the loop's resistance is $2.5\,\Omega$?

(14) A solenoid of length $L = 0.25\,\text{m}$ and radius $r = 4\,\text{cm}$ has 100 turns. A coil of $N = 20$ turns and resistance $R = 4\,\Omega$ is wound tightly around the solenoid, see Fig. 27.19. The current in the direction shown in the figure increases uniformly from 0 to 2 A in 0.5 s. (a) What is the direction of the induced current in the coil during this period of time? (b) What is the magnitude of the induced emf in the coil? (c) What is the magnitude of the induced current in the coil? (d) Redo parts a, b, and c assuming this time that the solenoid's core is made entirely out of iron whose magnetic permeability μ_m is $1{,}000\mu_\circ$ and also assume that the direction of the solenoid's current is reversed.

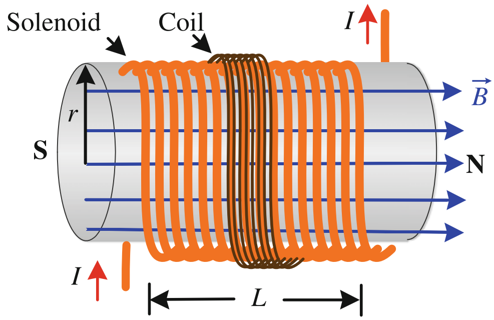


Fig. 27.19 See Exercise (14)

(15) A thin circular gold ring has a diameter of 2 cm, a resistance of $60\,\mu\Omega$, a mass of 20 g, and a specific heat of 129 J/kg.C°. In a period of 40 ms, the ring moves from a zero-field location to a location where a magnetic field has magnitude $B = 0.75\,\text{T}$ and points perpendicular to the ring. (a) What is the average magnitude of the induced emf in the ring? (b) Find the thermal energy dissipated in the ring. (c) Find the temperature rise in the ring if all thermal energy converts to heat.

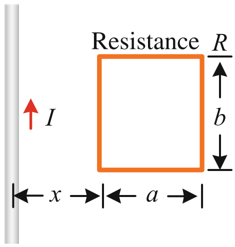
(16) A coil of radius $r_c = 10\,\text{cm}$ consists of $N = 30$ turns of copper wire. The wire of the coil has a radius of $r_w = 1.5\,\text{mm}$ and a resistivity $\rho_w = 1.68 \times 10^{-8}\ \Omega\text{.m}$. A uniform magnetic field perpendicular to the plane of the coil changes at a rate dB/dt of 8.5×10^{-3} T/s. (a) What is the induced emf produced in the coil? (b) What is the resistance of the wire of the coil? (c) What is the induced current

in the coil? (d) What is the rate at which the thermal energy is dissipated in the wire of the coil?

(17) A circular loop of wire has a radius $r = 15$ cm and resistance $R = 80\,\Omega$. The loop is initially in a uniform magnetic field perpendicular to the plane of the loop and has a magnitude $B = 0.5$ T. The loop is removed from the field in 150 ms. (a) What is the average induced emf produced in the loop? (b) Find the electric energy delivered by the process if it is equal to the energy dissipated in the loop.

(18) A vertical rectangular loop of width a and height b is at a distance x from a vertical long wire carrying a current I, see Fig. 27.20. (a) Find the magnetic flux through the loop. (b) If the rectangular loop is pulled away from the wire with a speed v, find the instantaneously-induced emf produced in the loop and the instantaneous force required.

Fig. 27.20 See Exercise (18)



Section 27.2 Motional emf

(19) The rod in Fig. 27.4 has a length $L = 12$ cm and moves with a speed $v = 1$ m/s in a uniform magnetic field of magnitude $B = 1.5$ T. Find the induced motional emf developed in the rod.

(20) An induced emf of 2 V is established by moving a rod 0.8 m long at a speed of 5 m/s perpendicular to a uniform magnetic field. Find the magnitude of that field.

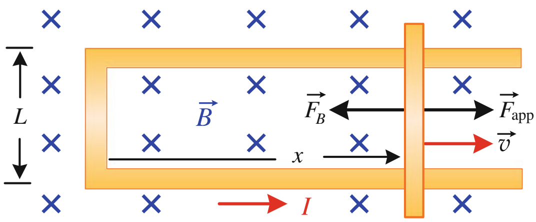
(21) A jet plane is flying horizontally at 250 m/s in a region where the vertical component of the Earth's magnetic field is 80 μT. The distance between the tips of the two wings of the plane is 30 m. What is the electric potential difference induced between the two wing tips?

(22) The rod in Fig. 27.5 has a length $L = 25$ cm and moves with a constant speed $v = 10$ m/s in a uniform magnetic field of magnitude $B = 1.5$ T. The resistor

has a resistance $R = 10\,\Omega$ and the rest of the circuit has a negligible resistance. (a) Find the induced motional emf developed in the circuit. (b) Find the force required to move the rod at this constant speed. (c) Find the power delivered to the resistor.

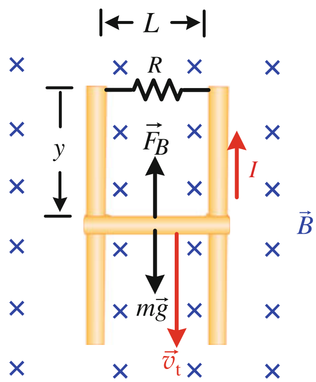
(23) Figure 27.21 shows a conducting rod that has a resistivity ρ and cross-sectional area A_{rod}. The rod makes contacts with horizontal conducting rails of the same type to complete the circuit. The circuit area is perpendicular to a uniform magnetic field of magnitude B. The rod starts from $x = 0$ at $t = 0$ and moves with constant speed v. (a) Find the induced current I as a function of time. (b) Find the power delivered by the applied force F_{app} as a function of time.

Fig. 27.21 See Exercise (23)



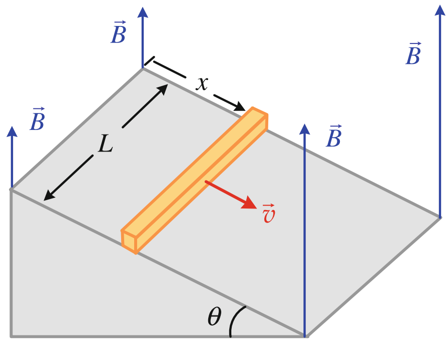
(24) A conducting rod of length $L = 25$ cm and mass $m = 3.5$ g slides along a pair of vertical metal guides connected to a resistor of resistance $R = 1.5 \times 10^{-3}\,\Omega$, see Fig. 27.22. The circuit is perpendicular to a magnetic field with $B = 0.05$ T. Friction and resistance of the rod and the guides are negligible. Find the terminal speed of the rod.

Fig. 27.22 See Exercise (24)



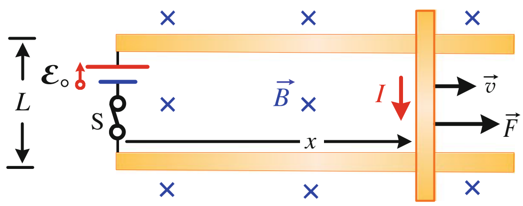
(25) A conducting rod of length L slides down from rest at the top of a frictionless incline of angle θ. Assume that a uniform vertical magnetic field $\vec{B}$ present throughout the motion of the rod, see Fig. 27.23. (a) Find the potential difference between the ends of the rod as a function of time. (b) Which side of the rod has a higher potential.

Fig. 27.23 See Exercise (25)



(26) Figure 27.24 shows a conducting rod of length L, mass m, and resistance R. The rod slides on two long horizontal frictionless and resistanceless parallel rails immersed in a uniform magnetic field $\vec{B}$. A battery of emf $\mathcal{E}_\circ$ and switch are connected to one end of the rails to complete the circuit. When the rod is at rest at $t = 0$, the switch S is closed. Find the speed v of the rod as a function of time, and find its terminal speed.

Fig. 27.24 See Exercise (26)



Section 27.3 Electric Generators

(27) When the rotating speed of the generator of a stationary car is 1,000 rpm its output is 12 V. What will the output of the generator be when its rotating speed is 2,500 rpm, assuming that the car is still stationary?

(28) If you plug an alternating-current voltmeter into a household electric outlet and the voltmeter reads 220 V, what is the peak value of the outlet voltage?

(29) The amplitude (or peak) of the sinusoidal output voltage of a generator is 311 V. The square coil of the generator has a side of $a = 10\,\text{cm}$ and rotates with an frequency $f = 50$ rev/s in a magnetic field with $B = 0.5\,\text{T}$. How many loops of wire should the coil consist of?

(30) A generator has a coil of $N = 500$ loops, each of area $A = 5 \times 10^{-2}\,\text{m}^2$. The coil can rotate freely between the poles of a permanent magnet of uniform magnetic field $B = 0.45\,\text{T}$. How fast must the coil rotate to produce a maximum output voltage of 311 V?

Section 27.4 Alternating Current

(31) Find the peak current when a 220-V rms source is connected to a resistor of resistance $R = 2\,\text{k}\Omega$.

(32) An ac power supply produces a peak voltage of 120 V and is connected to a 24-Ω resistor. What are the rms and peak currents in the resistor?

(33) Two light bulbs of 60-W and 100-W are connected in parallel to a 220-V rms source at your house. (a) What is the total resistance of the two bulbs as seen by the power company? (b) What is the resistance of each bulb?

(34) A heater rated as 1,000 W is connected to an ac source that allows a peak current of 12.86 A. What is the rms voltage of the source?

(35) A 1,100-W hair dryer is connected to 110-V ac source. Find the peak voltage applied to the dryer and the peak current passing through the dryer.

(36) A welding machine has a resistance $R = 22\,\Omega$ and is connected to a 220-V ac line. (a) What is the average electric power consumed in the welding machine? (b) What are the minimum and maximum values of the instantaneous consumed power?

Section 27.5 Transformers

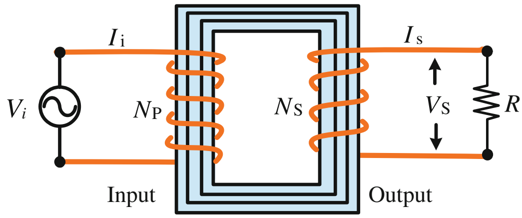
(37) A transformer has $N_P = 500$ turns in the primary coil and $N_S = 60$ turns in the secondary coil. (a) What kind of transformer is this? (b) By what factor does this transformer change the ac voltage and current?

(38) The transformer of a neon lamp operates from an alternating source of 220 V. The lamp requires 10 kV to operate. What is the ratio of the secondary to primary turns of the transformer coils?

(39) The input ac current of a 90-W transformer is 2 A and the output ac voltage is 12 V. (a) What kind of transformer is this? (b) By what factor does this transformer change the ac voltage?

(40) An ac source provides an output peak V_i and current peak I_i when connected to the primary coil of a transformer. The transformer has N_P turns in the primary coil and N_S turns in the secondary coil. A circuit of resistance R is connected to the transformer, see Fig. 27.25. What is the equivalent resistance of the circuit?

Fig. 27.25 See Exercise (40)



(41) Figure 27.26a shows the transmission of electric power from the generator of a power plant to a town, part b of the figure shows a simple equivalent circuit of part a, where the current and voltages are in rms values. The value of the ac voltage reaching the town is $V_{town} = 50$ kV with average power $\overline{P}_{town} = 80$ MW via a transmission line of resistance $R = 3.5\ \Omega$ from the generator. (a) Find the emf of the generator $\mathcal{E}_{gen}$. (b) What is the value of the average power generated by the power plant and the fraction of the lost generated power through the transmission line?

Section 27.6 Induced Electric Fields

(42) A positive charge $q = +20\ \mu$C is located on the right part of the circumference of the circle of Fig. 27.13b, where $R = 5$ cm. The magnetic field starts to decrease at a rate of -0.01 T/s. Find the initial force exerted on the charge.

(43) Repeat exercise 42 when the charge is $q = -20\ \mu$C and is located at $r = 25$ cm outside the region of the change of the magnetic field.

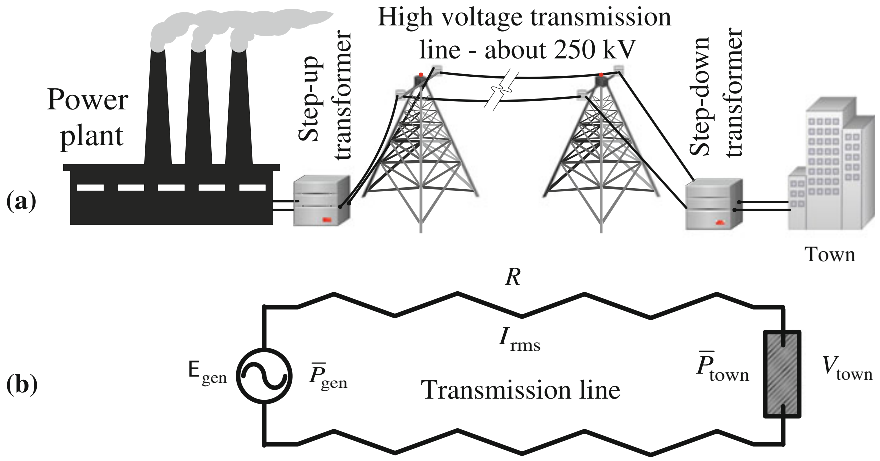


Fig. 27.26 See Exercise (41)

(44) A long solenoid has 500 turns per meter and a radius of 2 cm. The current in the solenoid is increasing at a rate of 2 A/s. What is the magnitude of the induced electric field at a point 1 cm from the axis of the solenoid?

(45) A long solenoid has a circular cross section of radius R. The magnetic field inside the solenoid is uniform and increasing at a rate dB/dt. The magnetic field outside the solenoid is essentially zero. (a) What is the rate of change of magnetic flux through a circle of radius $r < R$, normal to the axis of the solenoid and center coinciding with solenoid axis? (b) What is the magnitude of the induced electric field at a distance $r < R$ from the solenoid axis? (c) What is the magnitude of the induced electric field at a distance $r > R$ from the solenoid axis? (d) What is the magnitude of the induced emf in a circular loop of radius $r < R$, normal to the axis of the solenoid and center coinciding with the solenoid axis? (e) What is the magnitude of the induced emf if the radius in part d is R and $2R$?