Gauss's Law

21

Although Coulomb's law is the governing law in electrostatics, its form does not always simplify calculations in situations involving symmetry. In this chapter, we introduce Gauss's law as an alternative method for calculating electric fields of certain highly symmetrical charge distribution systems. In addition to being simpler than Coulomb's law, Gauss's law permits us to use qualitative reasoning.

21.1 Electric Flux

Consider a uniform electric field $\vec{E}$ penetrating a small area A oriented perpendicularly to the field as shown in Fig. 21.1. Recall from Sect. 20.5 that the number of electric field lines per unit area (measured in a plane perpendicular to the lines) is proportional to the magnitude of $\vec{E}$. Therefore, the total number of lines penetrating the surface is proportional to EA. This product is called the **electric flux**[1] Φ_E. Thus:

$$\Phi_E = EA \tag{21.1}$$

The SI units for Φ_E is newton-meters square per coulomb (N.m^2/C).

> Spotlight
> Electric flux is proportional to the number of electric field lines penetrating a certain area.

If the area in Fig. 21.1 is tilted by an angle θ with respect to $\vec{E}$, the flux through it (the number of electric lines) will decrease. To visualize this, Fig. 21.2 shows an

[1] The word "flux" comes from the Latin word meaning "to flow".

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area A that makes an angle θ with the field. The number of lines that cross the area A is equal to the number that cross the area A', which is *perpendicular* to $\vec{E}$ and hence $A' = A\cos\theta$. Thus, the flux through A, $\Phi_E(A)$, is equal to the flux through A', $\Phi_E(A')$. But according to Eq. 21.1, the flux through A' is defined as $\Phi_E(A') = EA'$. Therefore, the flux through A is:

$$\Phi_E(A) = \Phi_E(A') = EA' = EA\cos\theta \tag{21.2}$$

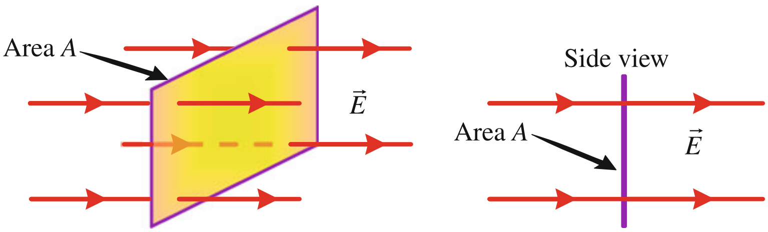


Fig. 21.1 Electric field lines representing a uniform electric field $\vec{E}$ that penetrates an area A perpendicularly (shown both in 3D and as a side view). The electric flux Φ_E through this area is EA

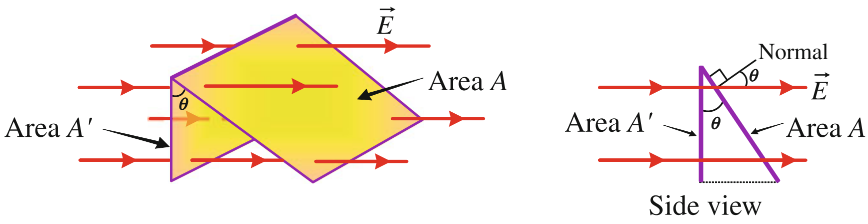


Fig. 21.2 Electric field lines representing a uniform electric field $\vec{E}$ penetrating an area A that is at an angle θ with the field (both three dimensional and side views are displayed). Since the flux through A is the same as through A', the flux through A is $\Phi_E = EA\cos\theta$

If we define a vector area $\vec{A}$ whose magnitude represents the surface area and whose direction is defined to be perpendicular to the surface area as in Fig. 21.3, then Eq. 21.2 can be written as:

$$\Phi_E = \vec{E} \bullet \vec{A} = EA\cos\theta \tag{21.3}$$

The flux through a surface of area A has a maximum value EA when the surface is perpendicular to the field (i.e. when $\theta = 0°$), and is zero when the surface is parallel to the field (i.e. when $\theta = 90°$).

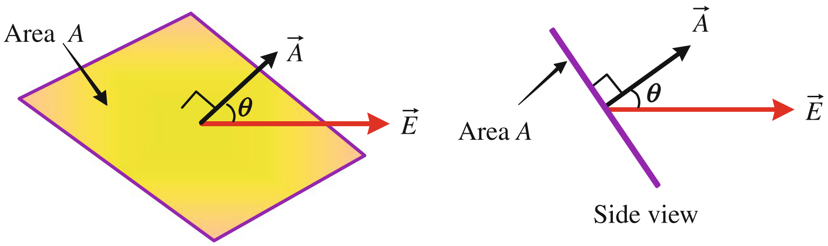


Fig. 21.3 The definition of a vector area $\vec{A}$ whose magnitude represents the surface area and whose direction is defined to be perpendicular to the surface area

Generally, the electric field may vary over the surface of any shape. Let us consider the general surface depicted by the shape in Fig. 21.4 and calculate the electric flux over the whole surface.

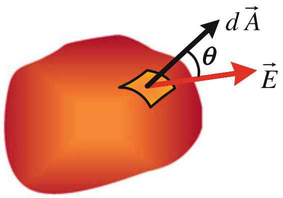


Fig. 21.4 The differential surface vector area $d\vec{A}$ of magnitude dA and direction perpendicular to the differential surface area and pointing outwards. When the electric field $\vec{E}$ makes an angle θ with that differential surface area, the differential flux $d\Phi_E$ is $\vec{E} \bullet d\vec{A}$

We start by considering a differential vector surface area $d\vec{A}$ to be normal to the surface and to point outwards at a specific location. If the electric field vector at this location is $\vec{E}$, then the differential electric flux $d\Phi_E$ through this differential area will be:

$$d\Phi_E = \vec{E} \bullet d\vec{A} \tag{21.4}$$

We integrate this relation over a surface S to get the electric flux as:

$$\Phi_E = \int_S \vec{E} \bullet d\vec{A} \tag{21.5}$$

Generally, Φ_E depends on the field pattern and the surface shape.

According to the definition of the vector area $d\vec{A}$ which always points outwards, the sign of the flux depends on the angle between $\vec{E}$ and $d\vec{A}$ as follows:

(1) If $\theta < 90°$, then $\vec{E}$ crosses the surface from the inside to the outside and hence $d\Phi_E = \vec{E} \bullet d\vec{A}$ is positive.

(2) If $\theta = 90°$, then $\vec{E}$ grazes the surface and hence $d\Phi_E = \vec{E} \bullet d\vec{A}$ is zero.

(3) If $90° < \theta < 180°$, then $\vec{E}$ crosses the surface from the outside to the inside and hence $d\Phi_E = \vec{E} \bullet d\vec{A}$ is negative.

The *net flux* through a surface is proportional to the *net number* of electric field lines leaving the surface. If more lines are entering than leaving, then the net flux is negative. If more lines are leaving than entering, then the net flux is positive.

We can write the net flux through a closed surface as:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} \tag{21.6}$$

where the symbol $\oint$ represents an integral over a closed surface.

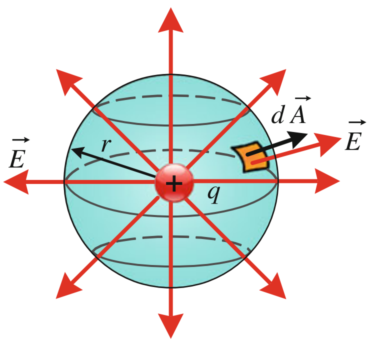
Example 21.1

Find the electric flux through a sphere of radius r enclosing at its center: (a) a positive charge q, and (b) a negative charge $-q$.

Solution: (a) When a positive charge q is at the center of such a sphere, the electric field would be directed outwards and be normal to the surface. It would also have a constant magnitude, $E = q/(4\pi\epsilon_\circ r^2)$, see Fig. 21.5. Therefore:

$$\vec{E} \bullet d\vec{A} = E\,dA\cos 0° = E\,dA = \frac{q}{4\pi\epsilon_\circ r^2}dA$$

Fig. 21.5



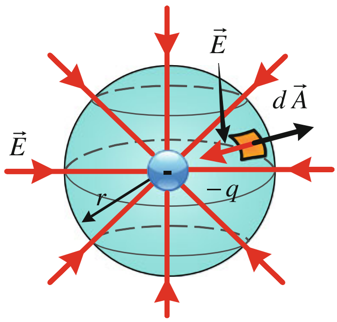
From Eq. 21.6 and the fact that $\oint dA$ over a spherical surface gives the area of the sphere, i.e. $\oint dA = A = 4\pi r^2$, we find the net flux through the sphere that encloses the positive charge q to be:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q}{4\pi\epsilon_\circ r^2}\oint dA = \frac{q}{4\pi\epsilon_\circ r^2}4\pi r^2 = \frac{q}{\epsilon_\circ}$$

(b) When a negative charge $-q$ is at the center of such a sphere, the electric field would be directed inwards and be normal to the surface. It would also have a constant magnitude, $E = q/(4\pi\epsilon_\circ r^2)$, see Fig. 21.6. Therefore:

$$\vec{E} \bullet d\vec{A} = E\, dA \cos 180° = -E\, dA = -\frac{q}{4\pi\epsilon_\circ r^2}dA$$

Fig. 21.6



Performing similar steps as in part (a), we find the net flux to be:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = -\frac{q}{4\pi\epsilon_\circ r^2}\oint dA = -\frac{q}{4\pi\epsilon_\circ r^2}4\pi r^2 = -\frac{q}{\epsilon_\circ}$$

Negative electric flux means electric lines are entering the surface.

21.2 Gauss's Law

In this section, we introduce a new foundation of Coulomb's law, called **Gauss's law**. This law can be used to take advantage of symmetry in the problem under consideration. Central to Gauss's law is a *hypothetical* closed surface called a **Gaussian surface**.

In Example 21.1 we noticed that the flux over a sphere of radius r was equal to the charge q inside the sphere divided by the permittivity of free space $\epsilon_\circ$. Now consider several closed Gaussian surfaces surrounding the charge as shown in Fig. 21.7. The number of electric field lines passing through the spherical surface S_1 is the same as the number of lines passing through the non-spherical surfaces S_2 and S_3. Therefore, we conclude that the flux through any closed Gaussian surface surrounding the point charge q is $q/\epsilon_\circ$.

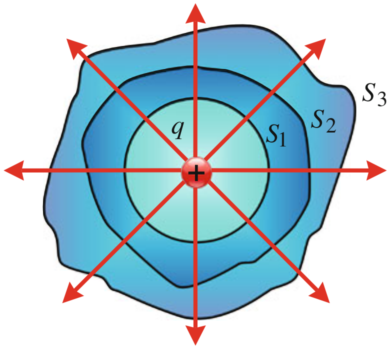


Fig. 21.7 Different closed Gaussian surfaces enclosing a point charge q. The net electric flux is the same through all surfaces

Gauss's law is a generalization to what we just described.

Gauss's Law
The net electric flux through any closed surface is equal to the net charge inside the surface divided by the permittivity of free space $\epsilon_\circ$.

That is:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{in}}{\epsilon_\circ} \tag{21.7}$$

where q_{in} represents the *net charge inside* the surface and $\vec{E}$ represents the *total electric field* at any point on the surface, which includes contributions from charges inside and/or outside (if any).

Note that Gauss's law is very useful in calculating electric fields in situations where the charge distributions have planar, cylindrical, or spherical symmetry. In these charge distribution systems, one must carefully construct the imaginary Gaussian surface such that it simplifies the integral in Eq. 21.7. This can be done by trying to satisfy one or more of the following conditions:

(1) The value of the field over the surface is constant, $E =$ constant.
(2) The dot product $\vec{E} \bullet d\vec{A}$ is $E\,dA$ because $\vec{E} \;//\; d\vec{A}$.
(3) The dot product $\vec{E} \bullet d\vec{A}$ is zero because $\vec{E} \perp d\vec{A}$.
(4) The value of the field over the surface is zero, $E = 0$.

21.3 Applications of Gauss's Law

Example 21.2

Using Gauss's law, find the electric field at a distance r from a positive point charge q, and compare it with Coulomb's law.

Solution: We apply Gauss's law to the spherical Gaussian surface of radius r in Fig. 21.5. From the symmetry of the problem, we know that at any point, the electric field $\vec{E}$ is perpendicular to the surface and directed outwards from the spherical center. Thus, $\vec{E} \,//\, d\vec{A}$ and $\vec{E} \bullet d\vec{A} = E\,dA$. Then, with $q_{in} = q$, we can write Gauss's law as:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{in}}{\epsilon_\circ} \quad \Rightarrow \quad \oint \vec{E} \bullet d\vec{A} = \oint E\,dA = E \oint dA = 4\pi r^2 E = \frac{q}{\epsilon_\circ}$$

This leads to:

$$E = \frac{1}{4\pi\epsilon_\circ}\frac{q}{r^2} = k\frac{q}{r^2}$$

which is simply Coulomb's law. This proves that Gauss's law and Coulomb's law are equivalent.

Example 21.3

Prove that any excess positive charge q on the isolated conductor of Fig. 21.8 will lie entirely on its outer surface.

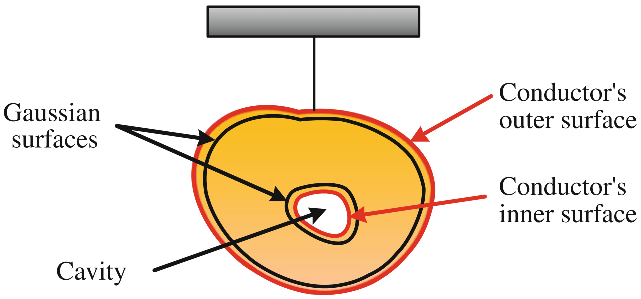


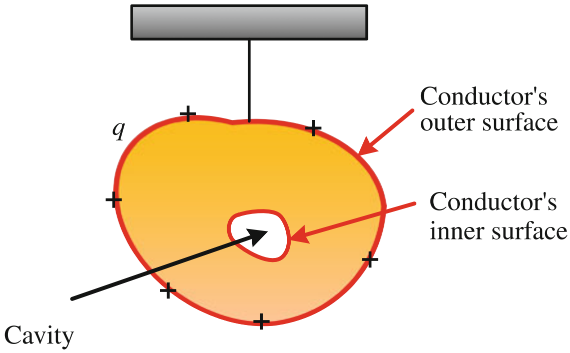
Fig. 21.8

Solution: The electric field inside the conductor must be zero. If this is not the case, the field would exert forces on the free electrons and a current would flow within the conductor. Of course, there are no such currents in an isolated conductor in electrostatic equilibrium.

First, we draw a Gaussian surface surrounding the conductor's cavity, close to its surface, as seen in Fig. 21.8. Since $\vec{E} = 0$ inside the conductor, then $\Phi_E = 0$ and hence according to Gauss's law, no net charge would exist on the inner walls of the cavity.

Then we draw a Gaussian surface just inside the outer surface of the conductor. Since $\vec{E} = 0$ inside the conductor, then $\Phi_E = 0$ and according to Gauss's law, no net charge will exist inside the Gaussian surface. If the excess charge is not inside the Gaussian surface, it must then be outside that surface, on the conductor's surface; see Fig. 21.9.

Fig. 21.9



Example 21.4

Using Gauss's law, find the electric field just outside the surface of a conductor carrying a positive surface charge density σ.

Solution: Consider a small section of the conductor's surface so as to neglect curvature. Then construct a cylindrical Gaussian surface normal to the conductor as shown in Fig. 21.10, where one end of the cylinder is inside the conductor while the other end is outside. Each end has an area A. The electric field $\vec{E}$ inside the conductor is zero, and the electric field $\vec{E}$ just outside the conductor's surface must be perpendicular to the surface. If this were not true, the component of the field along the surface of the conductor would force the free electrons to move, violating the conductor's electrostatic equilibrium.

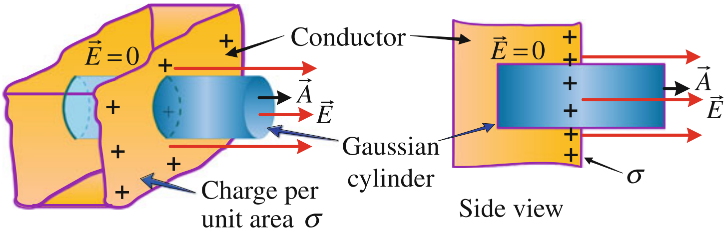


Fig. 21.10

To find the net flux through this cylindrical Gaussian surface, let us consider each of the four faces of the cylinder. (1) Because $\vec{E}=0$ inside the conductor, then the flux through the end of the cylinder inside the conductor is $\Phi_E(1)=0$. (2) For the same reason, the flux through the curved surface of the cylinder inside the conductor is $\Phi_E(2)=0$. (3) Since $\vec{E}\perp d\vec{A}$ along the curved surface of the cylinder outside the conductor, the flux there is also $\Phi_E(3)=0$. (4) Since $\vec{E}\,//\,d\vec{A}$ along the end of the cylinder that is outside the conductor, the flux there is $\Phi_E(4)=EA$. Thus, the net flux through the cylindrical Gaussian surface is $\Phi_E=EA$. Since $q_{in}=\sigma A$, we can then find electric field just outside the surface of the conductor as follows:

$$\Phi_E=\oint \vec{E}\bullet d\vec{A}=\frac{q_{in}}{\epsilon_\circ} \quad\Rightarrow\quad EA=\frac{\sigma A}{\epsilon_\circ} \quad\Rightarrow\quad E=\frac{\sigma}{\epsilon_\circ}$$

Example 21.5

Find the electric field due to an infinite plane sheet of charge with a uniform positive surface charge density σ.

Solution: By symmetry, the electric field $\vec{E}$ outside the infinite plane sheet must be: (1) uniform, (2) perpendicular to the sheet, (3) of the same magnitude at all points equidistant from the sheet, and (4) in opposite direction on the other side of the sheet, see Fig. 21.11. The choice that reflects that symmetry is a cylindrical

Gaussian surface normal to the sheet as shown in Fig. 21.11, where one end of the cylinder (of area A) is to the right of the sheet while the other end is to the left of it.

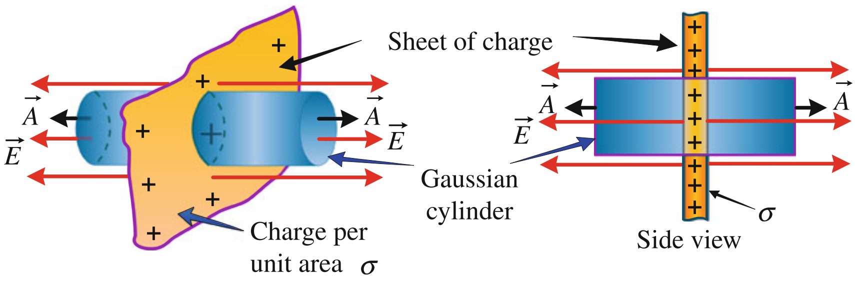


Fig. 21.11

As in Example 21.4, to find the net flux through this cylindrical Gaussian surface, let us consider each of the four faces of the cylinder. (1) Because $\vec{E}\ // \ d\vec{A}$ through the left end of the cylinder, then the flux there is $\Phi_E(1) = EA$. (2) Because $\vec{E} \perp d\vec{A}$ through the left curved surface of the cylinder then the flux there is $\Phi_E(2) = 0$. (3) For the same reason ($\vec{E} \perp d\vec{A}$), and the flux through the right curved surface of the cylinder is $\Phi_E(3) = 0$. (4) Because $\vec{E}\ // \ d\vec{A}$ through the right end of the cylinder, then the flux there is $\Phi_E(4) = EA$. Thus, the net flux through the Gaussian surface is $\Phi_E = 2EA$. Since $q_{in} = \sigma A$, we then can find electric field as follows:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{in}}{\epsilon_\circ} \quad \Rightarrow \quad 2\,EA = \frac{\sigma\,A}{\epsilon_\circ} \quad \Rightarrow \quad E = \frac{\sigma}{2\epsilon_\circ}$$

Example 21.6

Two infinite conducting plates have a uniform surface charge density of magnitude σ' on their faces. The excess charge is positive on one plate and negative on the other, see Fig. 7.12a, b. The two plates are brought together as shown in Fig. 21.12c. Find the electric field to the left and right of the plates in each part of the figure.

Solution: The charge on the positively charged conductor in Fig. 21.21a will spread over its two faces each with a surface density of magnitude σ'. From

Example 21.4, the electric field just outside each of these surfaces would be directed away from the two faces and would have a magnitude of:

$$E_+ = \frac{\sigma'}{\epsilon_\circ}$$

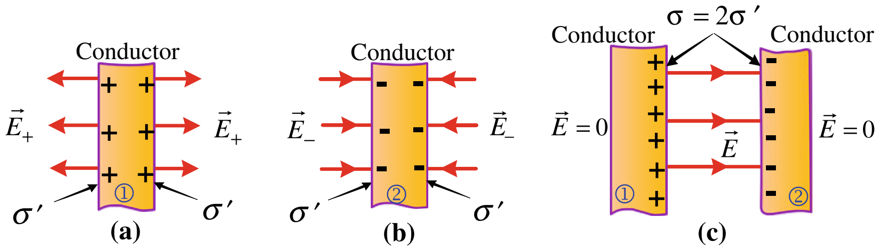


Fig. 21.12

We have a similar situation in Fig. 21.12b, except that the electric field is directed *toward* the two faces and has a magnitude given by:

$$E_- = \frac{\sigma'}{\epsilon_\circ}$$

When we bring the two plates together, the excess charge on one plate attracts the excess of charge on the other, and all the excess charge moves onto the inner surfaces of the plates, see Fig. 21.12c. The magnitude of the new surface charge density of the inner surfaces is $\sigma = 2\sigma'$. Thus, the electric field between the plates is to the right with a magnitude:

$$E = \frac{\sigma}{\epsilon_\circ} \qquad \text{(Between the plates)}$$

The electric field on the outer sides of the two plates of Fig. 21.12c is zero since no charge is left on those sides.

Example 21.7

Two infinitely long nonconductive sheets are aligned in parallel, see Fig. 21.13a. Each sheet has a fixed uniform charge. One sheet is positively charged and has a surface charge density of magnitude $\sigma_+ = 6.5\,\mu\text{C/m}^2$. The other sheet is negatively charged with $|\sigma_-| = 4.5\,\mu\text{C/m}^2$. Find the electric field: (a) to the left (L) of the sheets, (b) between (B) the sheets, and (c) to the right (R) of the sheets.

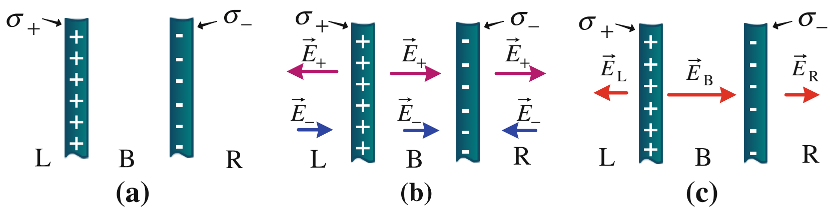


Fig. 21.13

Solution: For items of fixed charge, we can calculate the electric field of the items in the same manner as if each item were isolated, i.e. by adding the fields algebraically using the superposition principle. Thus, the magnitude of the electric field due to the positive sheet is:

$$E_+ = \frac{\sigma_+}{2\epsilon_\circ} = \frac{6.5 \times 10^{-6}\,\text{C/m}^2}{2(8.85 \times 10^{-12}\,\text{C}^2/\text{N.m}^2)} = 3.67 \times 10^5\,\text{N/C}$$

Also, the magnitude of the electric field due to the negative sheet is:

$$E_- = \frac{\sigma_-}{2\epsilon_\circ} = \frac{4.5 \times 10^{-6}\,\text{C/m}^2}{2(8.85 \times 10^{-12}\,\text{C}^2/\text{N.m}^2)} = 2.54 \times 10^5\,\text{N/C}$$

The directions of the fields to the left (L), between (B), and right (R) of the sheets are shown in Fig. 21.13b. The resultant field depends on the values of E_+ and E_-. Since $E_+ > E_-$, then E_L is:

$$E_L = E_+ - E_- = 3.67 \times 10^5\,\text{N/C} - 2.54 \times 10^5\,\text{N/C} = 1.13 \times 10^5\,\text{N/C} \quad \text{(to the left)}$$

The field to the right of the sheets E_R has the same magnitude but is directed to the right. Between the sheets, we have:

$$E_B = E_+ + E_- = 3.67 \times 10^5\,\text{N/C} + 2.54 \times 10^5\,\text{N/C} = 6.21 \times 10^5\,\text{N/C} \quad \text{(to the right)}$$

Example 21.8

Using Gauss's law, find the electric field at a distance r from a long thin rod that has a uniform charge per unit length λ.

Solution: By symmetry, the electric fields outside the rod are radial and lie in a plane perpendicular to the rod. Additionally, the field has the same magnitude at all points at the same radial distance from the rod. This suggests that we can construct a cylindrical Gaussian surface of an arbitrary radius r and height ℓ. Such a cylinder would have its ends perpendicular to the rod as shown in Fig. 21.14.

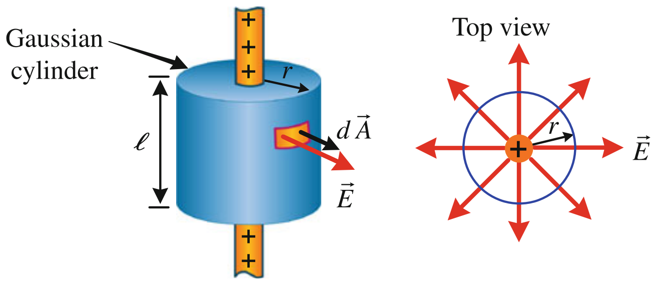


Fig. 21.14

We divide the flux into two cases: (1) The flux through the two ends of the Gaussian cylinder is zero because $\vec{E}$ is parallel to these surfaces, i.e. $\vec{E} \perp \vec{A}$. (2) The flux through the curved surface of the Gaussian cylinder can be obtained by taking into account that $E =$ constant and $\vec{E}$ is parallel to $d\vec{A}$, i.e. $\vec{E} \bullet d\vec{A} = E\,dA$. Therefore, $\oint \vec{E} \bullet d\vec{A} = \oint E\,dA = E \oint dA = EA$, where A is the area of the curved cylinder and is given by $A = 2\pi r\ell$. The net charge inside the Gaussian cylinder is $q_{in} = \lambda\ell$. We can now use Gauss's law to find the electric field as follows:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{in}}{\epsilon_\circ} \quad \Rightarrow \quad \oint \vec{E} \bullet d\vec{A} = \oint E\,dA = E \oint dA$$

$$= E(2\pi\, r\ell) = \frac{\lambda\ell}{\epsilon_\circ}$$

Then: $$E = \frac{1}{2\pi\epsilon_\circ}\frac{\lambda}{r} = 2k\frac{\lambda}{r}$$

This relation was derived in Chap. 20 using Coulomb's law (Eq. 20.36).

Example 21.9

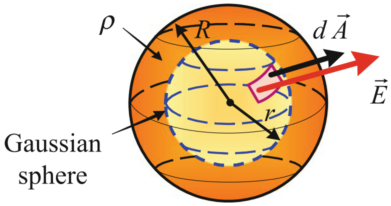
A solid sphere of radius R has a uniform volume charge density ρ and carries a total positive charge Q. Find and sketch the electric field at any distance r away from the sphere's center.

Solution: We divide the solution to $0 \leq r \leq R$ and $r \geq R$.

(1) For $0 \leq r \leq R$

When dealing with a spherically symmetric charge distribution, we chose a spherical Gaussian surface of radius $r < R$ concentric with the charged sphere as shown in Fig. 21.15.

Fig. 21.15



By symmetry, the magnitude of the electric field is constant everywhere on the spherical Gaussian surface and normal to the surface at any point, i.e. $\vec{E} \,//\, d\vec{A}$. Thus:

$$\oint \vec{E} \bullet d\vec{A} = \oint E\, dA = E \oint dA = E(4\pi r^2)$$

It is important to notice that the volume, say V', of the Gaussian sphere encloses a net charge $q_{\text{in}} = \rho V'$; that is:

$$q_{\text{in}} = \rho V' = \rho(\tfrac{4}{3}\pi r^3)$$

We can now use Gauss's law to find electric field as follows:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{\text{in}}}{\epsilon_\circ} \quad \Rightarrow \quad E(4\pi r^2) = \frac{\rho(\frac{4}{3}\pi r^3)}{\epsilon_\circ}$$

Then: $$E = \frac{\rho}{3\epsilon_\circ} r \qquad (0 \leq r \leq R)$$

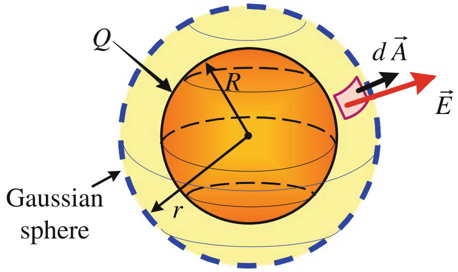
Using the definition $\rho = Q/(\frac{4}{3}\pi R^3)$ and $k = 1/(4\pi\epsilon_\circ)$, we get:

$$E = \frac{Q}{4\pi\epsilon_\circ R^3}\, r = k\frac{Q}{R^3}\, r \qquad (0 \leq r \leq R)$$

(2) For $r \geq R$

Again, because the charge distribution is spherically symmetric, we can construct a Gaussian sphere of radius $r > R$ concentric with the charged sphere, as shown in Fig. 21.16.

Fig. 21.16



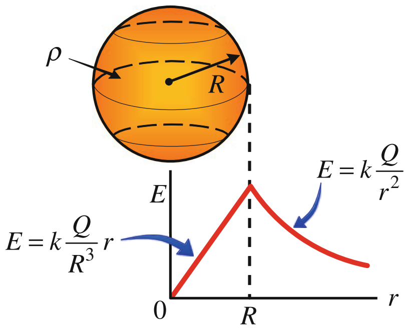
Just as when $r < R$, $\oint \vec{E} \bullet d\vec{A} = E(4\pi r^2)$, but $q_{in} = Q$. Thus, we can use Gauss's law to find the electric field as follows:

$$\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{q_{\text{in}}}{\epsilon_\circ} \quad \Rightarrow \quad E(4\pi r^2) = \frac{Q}{\epsilon_\circ}$$

i.e.:
$$E = \frac{1}{4\pi\epsilon_\circ}\frac{Q}{r^2} = k\frac{Q}{r^2} \qquad (r \geq R)$$

Notice that this is identical to the result obtained for a point charge. Therefore, we conclude that the electric field outside any uniformly charged sphere is equivalent to that of a point charge located at the center of the sphere. At $r = R$, the two cases give identical results $E = kQ/R^2$. A plot of E versus r is shown in Fig. 21.17. This figure shows the continuation of E and its maximum at $r = R$.

Fig. 21.17

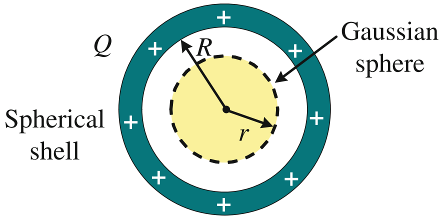


Example 21.10

A thin spherical shell of radius R has a total positive charge Q distributed uniformly over its surface. Find the electric field inside and outside the shell.

Solution: By symmetry, if any field exists inside the shell, it must be radial. Let us construct a spherical Gaussian surface of radius $r < R$ concentric with the shell, see the cross sectional view in Fig. 21.18.

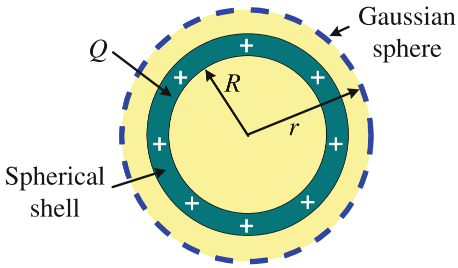
Fig. 21.18



Based on Gauss's law, the lack of charge inside the surface indicates that $\oint \vec{E} \bullet d\vec{A} = E(4\pi r^2) = 0$, or $E = 0$. Accordingly, we conclude that *there is no electric field inside a uniformly charged spherical shell*.

Outside the shell, we construct a spherical Gaussian surface of radius $r > R$ concentric with the charged shell as shown in Fig. 21.19.

Fig. 21.19



Symmetry suggests that $E =$ constant on that surface and $\vec{E}$ is parallel to $d\vec{A}$, i.e. $\oint \vec{E} \bullet d\vec{A} = E(4\pi r^2)$. Since the net charge q_{in} inside the Gaussian surface is equal to the total charge Q on the shell, the *shell is equivalent to a point charge located at the center*. That is:

$$E = \frac{1}{4\pi\epsilon_\circ}\frac{Q}{r^2} = k\frac{Q}{r^2} \qquad (r > R).$$

21.4 Conductors in Electrostatic Equilibrium

Conductors contain free electrons that can move freely. When there is no net motion of electrons within the conductor, the conductor is in electrostatic equilibrium and has the following four properties:

(1) The electric field inside the conductor is zero (see Example 21.3).

(2) The excess charge on an isolated conductor lies on its outer surface (see Example 21.3).

(3) The electric field just outside a charged conductor at any point is perpendicular to its surface and has a magnitude $E = \sigma/\epsilon_\circ$, where σ is the surface charge density at that point (see Example 21.4).

(4) The surface charge density is greatest at locations where the radius of curvature of the surface is smallest (see Chap. 22).

We can elaborate more about the first property by considering the conducting slab in electrostatic equilibrium on the left of Fig. 21.20, where the free electrons are uniformly distributed throughout the slab, i.e. $\vec{E}_{int} = 0$. When we place the slab in an external electric field $\vec{E}_{ext}$ as in the right part of Fig. 21.20, the free electrons move to the left. In time, more negative and positive charges accumulate on the left and right surfaces, respectively. These two planes of charge create an increasing internal electric field $\vec{E}_{int}$ inside the conductor. After awhile, $\vec{E}_{int}$ will compensate $\vec{E}_{ext}$, resulting in a zero net electric field inside the conductor, i.e. $\vec{E}_{net} = \vec{E}_{ext} - \vec{E}_{int} = 0$. The time to reach this new electrostatic equilibrium is of the order 10^{-6} s.

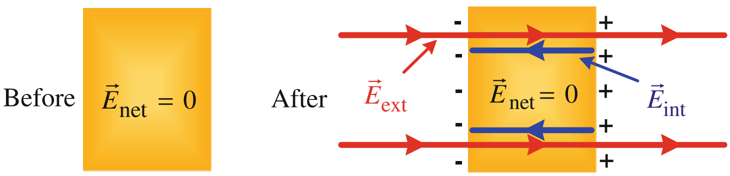


Fig. 21.20 An external electric field $\vec{E}_{ext}$ creates an internal electric field $\vec{E}_{int}$ in the conductor such that the net electric field $\vec{E}_{net}$ is zero

Example 21.11

A conducting sphere of radius R carries a net positive charge $2Q$. A conducting spherical shell of inner radius R_1 ($R_1 > R$) and outer radius R_2 carries a net negative charge $-Q$. This shell is concentric with the conducting sphere. Find the magnitude of the electric field at a distance r away from the common center when: (a) $r < R$, (b) $R < r < R_1$, (c) $R_1 < r < R_2$, and (d) $r > R_2$.

Solution: The charge distributions under consideration are characterized by being spherically symmetrical around the common center c. This suggests that a spherical Gaussian surface of radius r is to be constructed in each case such as S_1, S_2, S_3, and S_4 that are displayed in Fig. 21.21. In addition, we use the fact that the electric field inside a conductor is zero and all the excess charge will lie entirely on the outer surface of the isolated conductor.

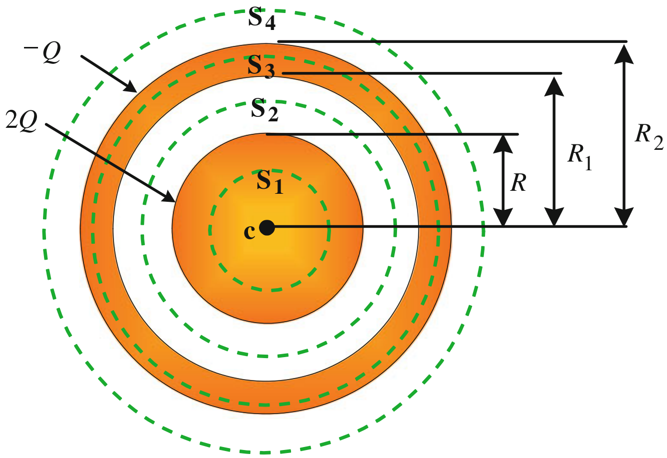


Fig. 21.21

(a) In this region the Gaussian sphere S_1 of Fig. 21.21 satisfies the condition $r < R$. Because there is no charge inside the conductor in this region, i.e. $q_{\text{in}} = 0$; then, $E_1 = 0$.

(b) In this region the Gaussian sphere S_2 of Fig. 21.21 satisfies the condition $R < r < R_1$. Because $q_{\text{in}} = 2Q$ inside this surface and because $\oint \vec{E}_2 \bullet d\vec{A} = E_2(4\pi r^2)$, we can use Gauss's law to find:

$$E_2 = \frac{1}{4\pi\epsilon_\circ}\frac{2Q}{r^2} = k\frac{2Q}{r^2} \qquad (R < r < R_1)$$

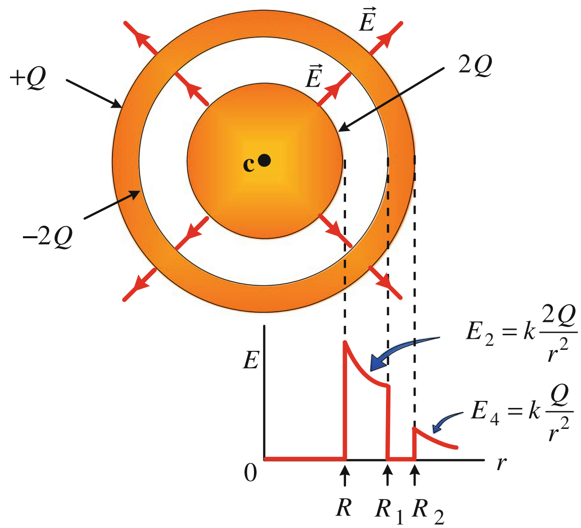
(c) In this region, the Gaussian sphere S_3 of Fig. 21.21 satisfies the condition $R_1 < r < R_2$. Because the electric field inside an equilibrium conductor is zero, i.e. $E_3 = 0$; then, based on Gauss's law, the net charge q_{in} must be zero. From this argument, we find that an induced charge $-2Q$ must be established on the inner surface of the shell to cancel the charge $+2Q$ on the solid sphere. In addition, because the net charge on the whole shell is $-Q$, we conclude that its outer surface must carry an induced charge $+Q$.

(d) In this region, the Gaussian sphere S_4 of Fig. 21.21 satisfies the condition $r > R_2$. Because $q_{in} = 2Q - Q = Q$ inside this surface and because $\oint \vec{E}_4 \bullet d\vec{A} = E_4(4\pi r^2)$, we can use Gauss's law to find:

$$E_4 = \frac{1}{4\pi\epsilon_\circ}\frac{Q}{r^2} = k\frac{Q}{r^2} \qquad (r > R_2)$$

Figure 21.22 shows a graphical representation of the variation of the electric field E with r. In addition, the figure shows the final distribution of the charge on the two conductors.

Fig. 21.22



The expressions that we have arrived at for the electric fields established by simple charge distributions are presented in Table 21.1.

Table 21.1 Electric fields due to simple charge distributions

Charge distribution	Electric field
Single point charge q	$E = k\dfrac{q}{r^2} \qquad r > 0$
Charge q uniformly distributed on the surface of a conducting sphere of radius R	$E = \begin{cases} k\dfrac{q}{r^2} & r \geq R \\ 0 & r < R \end{cases}$
Charge q uniformly distributed with uniform charge density ρ over an insulating sphere of radius R	$E = \begin{cases} k\dfrac{q}{r^2} & r \geq R \\ k\dfrac{q}{R^3}r & r \leq R \end{cases}$
Infinitely long thin rod of a uniform charge per unit length λ	$E = 2k\dfrac{\lambda}{r}$ r outside the line
Infinite plane sheet of charge of uniform surface charge density σ	$E = \dfrac{\sigma}{2\epsilon_\circ}$ Everywhere outside the plane
Conductor having surface charge density σ	$E = \begin{cases} \dfrac{\sigma}{\epsilon_\circ} & \text{Just outside the conductor} \\ 0 & \text{Inside the conductor} \end{cases}$
Two oppositely charged conducting plates with surface charge density of magnitude σ	$E = \begin{cases} \dfrac{\sigma}{\epsilon_\circ} & \text{Any point betwee the plates} \\ 0 & \text{Outside the plates} \end{cases}$

21.5 Exercises

Section 21.1 Electric Flux

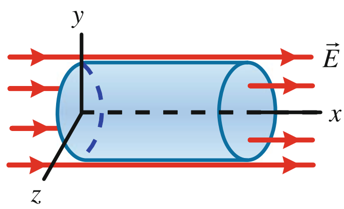
(1) A uniform electric field is directed from left to right along the positive x-axis. If the magnitude of the field is 2×10^5 N/C, what flux passes through a circular loop of area $0.5\,\text{m}^2$ if the normal to loop is: (a) in the positive x-direction, (b) in the negative x-direction, (c) in the positive y-direction, (d) in the negative y-direction, and (e) in a direction that makes an angle 60° from the x-axis.

(2) The maximum flux through a rectangle of area $0.2\,\text{m}^2$ is $5 \times 10^5\,\text{N.m}^2/\text{C}$. Find the magnitude of the electric field.

(3) A cylinder of length $L = 40$ cm and radius $R = 10$ cm has its axis along the x-axis, see Fig. 21.23. The electric field in this region is $\vec{E} = (10^5\,\vec{i}\,)$ N/C.

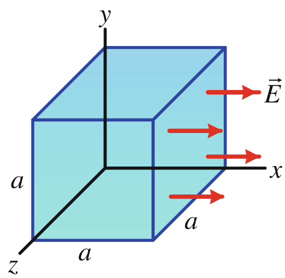
Find the flux through: (a) the cylindrical wall, (b) the cap at the left end of the cylinder, (c) the cap at the right end of the cylinder.

Fig. 21.23 See Exercise (3)



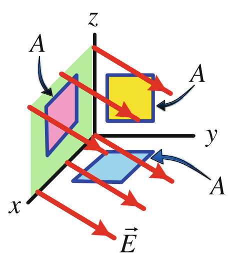
(4) An electric field $\vec{E} = (\alpha + \beta x)\ \vec{\text{i}}$ passes through a cube of side a, as shown in Fig. 21.24. (a) Find the net electric flux through the cube. (b) Calculate this flux given that $\alpha = 2\,\text{N/C}$, and $a = 0.2\,\text{m}$ for $\beta = 5\,\text{N.m/C}$ and $\beta = 0$.

Fig. 21.24 See Exercise (4)



(5) A uniform electric field $\vec{E} = \alpha\,\vec{\text{i}} + \beta\,\vec{\text{j}}$ passes through a square surface of area A. What is the flux through this area if the surface lies: (a) in the xy-plane? (b) in the xz-plane? and (c) in the yz-plane? see Fig. 21.25.

Fig. 21.25 See Exercise (5)



(6) A pyramid has a horizontal square base of side $a = 20\,\text{cm}$. The pyramid is placed in a uniform electric field E of $70\,\text{N/C}$ that is directed upwards, see Fig. 21.26. (a) Find the electric flux through the pyramid's base. (b) Find the electric flux through the pyramid's four slanted surfaces.

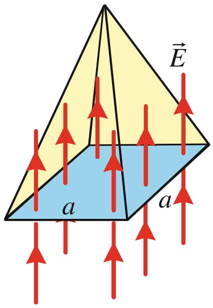


Fig. 21.26 See Exercise (6)

(7) A horizontal uniform electric field E penetrates a vertical cone of base radius r and height h, see Fig. 21.27. (a) Find the electric flux through the left-hand side of the cone. (b) Find the electric flux through the right-hand side of the cone. (c) Find the electric flux through the base of the cone.

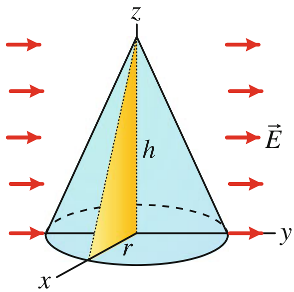


Fig. 21.27 See Exercise (7)

Section 21.2 Gauss's Law

(8) A point charge q is located at the center of a charged ring of radius R. The ring has a linear charge density λ, see Fig. 21.28. (a) Find the total electric flux through the Gaussian sphere S_1 of radius $r < R$. (b) Find the total electric flux through the Gaussian sphere S_2 of radius $r > R$.

(9) Figure 21.29 shows four closed surfaces S_1, S_2, S_3, and S_4 and four point charges q, $-q$, $2q$, and $-2q$. (a) Find the electric flux through each surface. (b) Would the electric field lines produced by the point charge $-2q$ have an effect on the calculated fluxes? (c) Explain the reasoning behind your answer for (b).

Fig. 21.28 See Exercise (8)

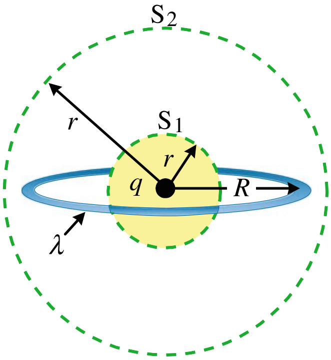
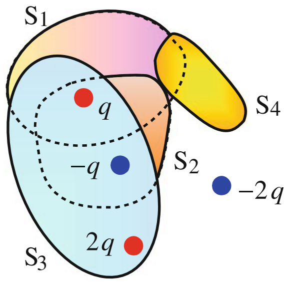
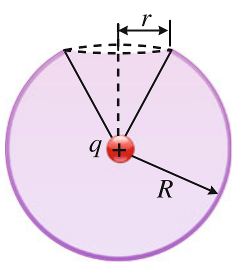


Fig. 21.29 See Exercise (9)



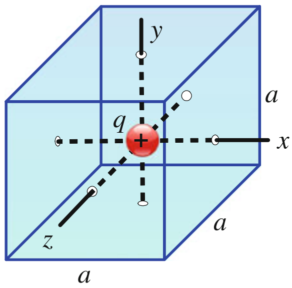
(10) A point charge $q = 25\,\mu\text{C}$ is located at the center of a sphere of radius $R = 25$ cm. A circular cut of radius $r = 5$ cm is removed from the surface of the sphere, see Fig. 21.30. (a) Find the electric flux that passes through that cut. (b) Repeat when the cut has a radius $r = 25$ cm. Is the answer $q/2\epsilon_\circ$?

Fig. 21.30 See Exercise (10)



(11) A point charge $q = 53.1$ nC is located at the center of a cube of side $a = 5$ cm, see Fig. 21.31. (a) Find the electric flux through each face of the cube. (b) Find the flux through the four slanted surfaces of a pyramid formed from a vertex on the center of the cube and one of its six square faces.

Fig. 21.31 See Exercise (11)



(12) At an altitude $h_1 = 700$ m above the ground, the electric field in a particular region is $E_1 = 95$ N/C downwards. At an altitude $h_2 = 800$ m, the electric field is $E_2 = 80$ N/C downwards. Construct a Gaussian surface as a box of horizontal area A and height lying between h_1 and h_2, to find the average volume-charge density in the layer of air between these two elevations.

(13) A point P is at a distance $a = 10$ cm from an infinite rod, charged with a uniform charge per unit length $\lambda = 5$ nC/m. (a) Find the electric flux through a sphere of radius $r = 5$ cm centered at P, see left of Fig. 21.32. (b) Find the electric flux through a sphere of radius $r = 15$ cm centered at P, see right of Fig. 21.32.

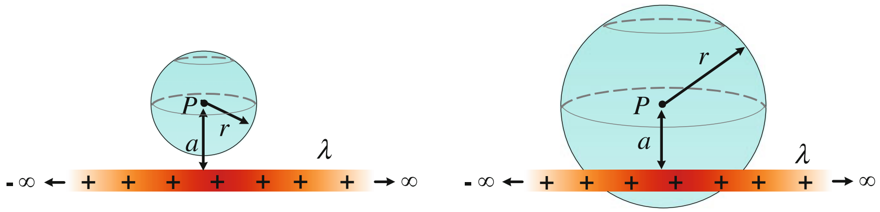
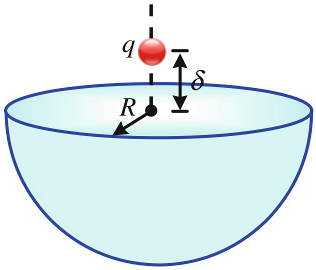


Fig. 21.32 See Exercise (13)

(14) A point charge q is located at a distance δ just above the center of the flat face of a hemisphere of radius R as shown in Fig. 21.33. (a) When δ is very small, use the argument of symmetry to find an approximate value for the electric flux Φ_{curved} through the curved surface of the hemisphere. (b) When δ is very small, use Gauss's law to find an approximate value of the electric flux Φ_{flat} through the flat surface of the hemisphere.

Fig. 21.33 See Exercise (14)



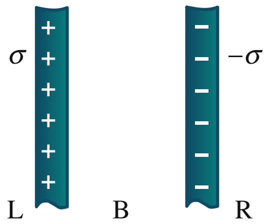
Section 21.3 Applications of Gauss's Law

(15) An infinite horizontal sheet of charge has a charge per unit area $\sigma = 8.85\ \mu C/m^2$. Find the electric field just above the sheet.

(16) A nonconductive wall carries a uniform charge density $\sigma = 8.85\ \mu C/cm^2$. Find the electric field 7 cm away from the wall. Does your result change as the distance from the wall increases such that it is much less than the wall's dimensions?

(17) Two infinitely long, nonconductive charged sheets are parallel to each other. Each sheet has a fixed uniform charge. The surface charge density on the left sheet is σ while on the right sheet is $-\sigma$, see Fig. 21.34. Use the superposition principle to find the electric field: (a) to the left of the sheets, (b) between the sheets, and (c) to the right of the sheets.

Fig. 21.34 See Exercise (17)

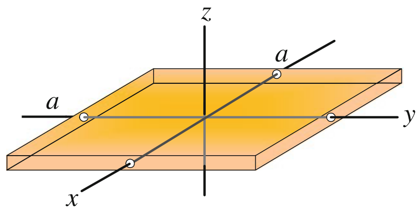


(18) Repeat the calculations for Exercise 17 when: (i) both the sheets have positive uniform surface charge densities σ, and (ii) both the sheets have negative uniform surface charge densities $-\sigma$.

(19) A thin neutral conducting square plate of side $a = 80$ cm lies in the xy-plane, see Fig. 21.35. A total charge $q = 5$ nC is placed on the plate. Assuming that

the charge density is uniform, find: (a) the surface charge density on the plate, (b) the electric field just above the plate, and (c) the electric field just below the plate.

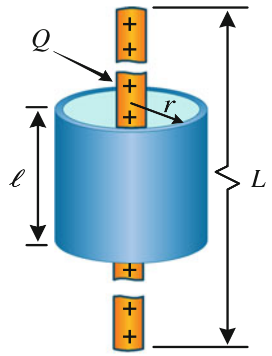
Fig. 21.35 See Exercise (19)



(20) A long filament has a charge per unit length $\lambda = -80\,\mu\text{C/m}$. Find the electric field at: (a) 10 cm, (b) 20 cm, and (c) 100 cm from the filament, where distances are measured perpendicular to the length of the filament.

(21) A uniformly charged straight wire of length $L = 1.5$ m has a total charge $Q = 5\,\mu\text{C}$. A thin uncharged nonconductive cylinder of height $\ell = 2$ cm and radius $r = 10$ cm surrounds the wire at its central axis, see Fig. 21.36. Using reasonable approximations, find: (a) the electric field at the surface of the cylinder and (b) the total electric flux through the cylinder.

Fig. 21.36 See Exercise (21)



(22) A thin nonconductive cylindrical shell of radius $R = 10$ cm and length $L = 2.5$ m has a uniform charge Q distributed on its curved surface, see Fig. 21.37. The radial outward electric field has a magnitude 4×10^4 N/C at a distance $r = 20$ cm from its axis (measured from the midpoint of the shell).

Find: (a) the net charge on the shell, and (b) the electric field at a point $r = 5$ cm from its axis.

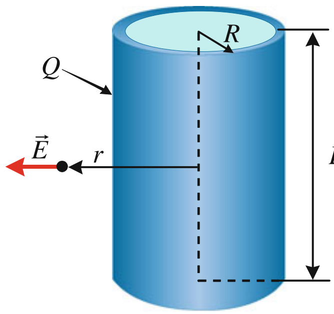


Fig. 21.37 See Exercise (22)

(23) A long non-conducting cylinder of radius R has a uniform charge distribution of density ρ throughout its volume. Find the electric field at a distance r from its axis where $r < R$?

(24) A thin spherical shell of radius $R = 15$ cm has a total positive charge $Q = 30\,\mu$C distributed uniformly over its surface, see Fig. 21.38. Find the electric field at: (a) 10 cm and (b) 20 cm from the center of the charge distribution.

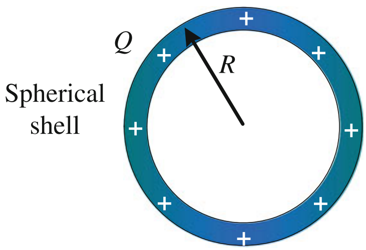


Fig. 21.38 See Exercise (24)

(25) Two concentric thin spherical shells have radii $R_1 = 5$ cm and $R_2 = 10$ cm. The two shells have charges of the same magnitude $Q = 3\,\mu$C, but different in sign, see Fig. 21.39. Use the shown three Gaussian surfaces S_1, S_2, and S_3 to find the electric field in the three regions: (a) $r < R_1$, (b) $R_1 < r < R_2$, and (c) $r > R_2$.

(26) A particle with a charge $q = -60$ nC is located at the center of a non-conducting spherical shell of volume $V = 3.19 \times 10^{-2}\,\text{m}^3$, see Fig. 21.40. The spherical shell carries over its interior volume a uniform negative charge Q of volume density $\rho = -1.33\,\mu\text{C/m}^3$. A proton moves outside the spherical shell in a circular orbit of radius $r = 25$ cm. Calculate the speed of the proton.

Fig. 21.39 See Exercise (25)

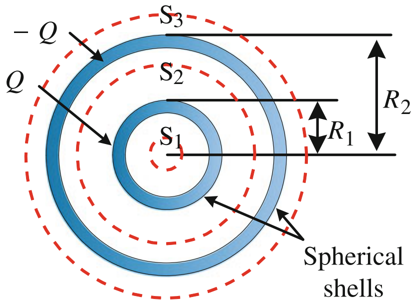
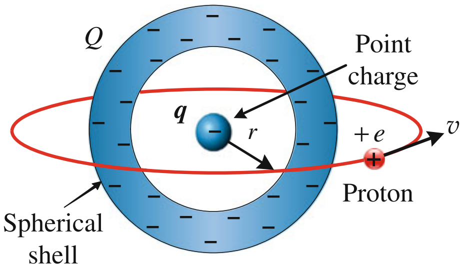


Fig. 21.40 See Exercise (26)



(27) A solid non-conducting sphere is 4 cm in radius and carries a 7.5 μC charge that is uniformly distributed throughout its interior volume. Calculate the charge enclosed by a spherical surface, concentric with the sphere, of radius (a) $r = 2$ cm and (b) $r = 8$ cm.

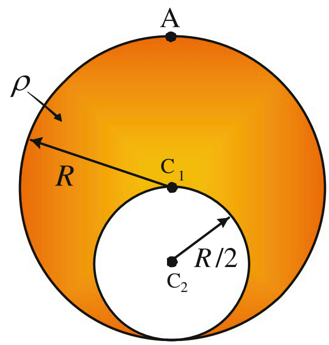
(28) A solid non-conducting sphere of radius $R = 20$ cm has a total positive charge $Q = 30$ μC that is uniformly distributed throughout its volume. Calculate the magnitude of the electric field at: (a) 0 cm, (b) 10 cm, (c) 20 cm, (d) 30 cm, and (e) 60 cm from the center of the sphere.

(29) If the electric field in air exceeds the threshold value $E_{\text{thre}} = 3 \times 10^6$ N/C, sparks will occur. What is the largest charge Q can a metal sphere of radius 0.5 cm hold without sparks occurring?

(30) The charge density inside a non-conducting sphere of radius R varies as $\rho = \alpha\ r$ (C/m^3), where r is the radial distance from the center of the sphere. Use Gauss's law to find the electric field inside and outside the sphere.

(31) A solid sphere of radius R with a center at point C_1 has a uniform volume charge density ρ. A spherical cavity of radius $R/2$ with a center at point C_2 is then scooped out and left empty, see Fig. 21.41. Point A is at the surface of the big sphere and collinear with points C_1 and C_2. What is the magnitude and direction of the electric field at points C_1 and A?

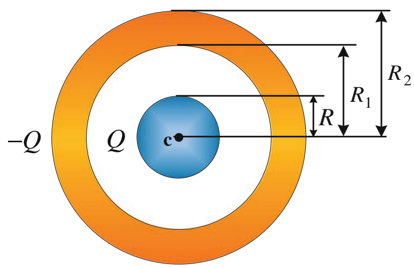
Fig. 21.41 See Exercise (31)



Section 21.4 Conductors in Electrostatic Equilibrium

(32) A non-conducting sphere of radius R and charge $+Q$ uniformly distributed throughout its volume is concentric with a spherical conducting shell of inner radius R_1 and outer radius R_2. The shell has a net charge $-Q$, see Fig. 21.42. Find an expression for the electric field as a function of the radius r when: (a) $r < R$ (within the sphere). (b) $R < r < R_1$ (between the sphere and the shell). (c) $R_1 < r < R_2$ (inside the shell). (d) $r > R_2$ (outside the shell). (e) What are the charges on inner and outer surfaces of the conducting shell?

Fig. 21.42 See Exercise (32)



(33) A large, thin, copper plate of area A has a total charge Q uniformly distributed over its surfaces. The same charge Q is uniformly distributed over the upper surface of a glass plate, which is identical to the copper plate, see Fig. 21.43. (a) Find the surface charge density on each face of the two plates. (b) Compare the electric fields just above the center of the upper surface of each plate.

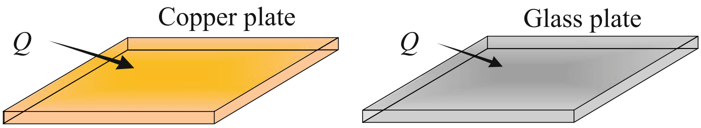
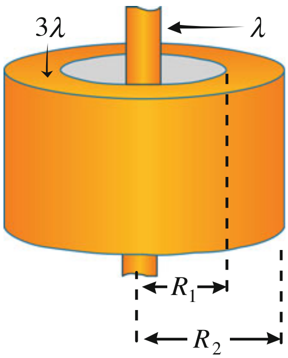


Fig. 21.43 See Exercise (33)

(34) A thin, long, straight wire carries a charge per unit length λ. The wire lies along the axis of a long conducting cylinder carrying a charge per unit length 3λ. The cylinder has an inner radius R_1 and an outer radius R_2, see Fig. 21.44. (a) Use a Gaussian surface inside the conducting cylinder to find the charge per unit length on its inner and outer surfaces. (b) Use Gauss's law to find the electric field E outside the wire. (c) Sketch the electric field E as a function of the distance r from the wire's axis.

Fig. 21.44 See Exercise (34)



(35) An uncharged solid conducting sphere of radius R contains two cavities. A point charge q_1 is placed within the first cavity, and a point charge q_2 is placed within the second one. Find the magnitude of the electric field for $r > R$, where r is measured from the center of the sphere.