Inductance, Oscillating Circuits, and AC Circuits

28

An emf produced by a physical source (like a battery) is quite different from that produced by changing magnetic flux. In this chapter, we study how an emf is induced as a result of a changing magnetic flux produced by the circuit itself or by a nearby circuit.

28.1 Self-Inductance

First, consider the loop shown in Fig. 28.1, which contains a battery of emf $\mathcal{E}$, a resistor of resistance R, and a switch S. When the switch is closed, the current does not jump immediately from 0 to its final value $\mathcal{E}/R$. Actually, Faraday's law and Lenz's law can be used as follows to describe what happens in this loop:

- *As the current I in the loop increases with time, the magnetic flux through the loop also increases*
- *The increasing magnetic flux creates an induced emf $\mathcal{E}_L$ in the loop*
- *The induced emf and induced current are opposing the battery's emf $\mathcal{E}$ and its current I*
- *This process causes a gradual increase in the battery's current to its final value $\mathcal{E}/R$*

This effect is called **self-induction** because it arises from the loop itself. The induced emf $\mathcal{E}_L$ is called a **self-induced emf,** or ***back emf***.

Consider a coil wound on a cylindrical core with a current passing through the coil, as shown in Fig. 28.2a. As a result, a magnetic field directed to the right is set up inside the coil. Faraday's law can be used to describe the effect of increasing or decreasing the current.

H. A. Radi and J. O. Rasmussen, *Principles of Physics*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-3-642-23026-4_28,

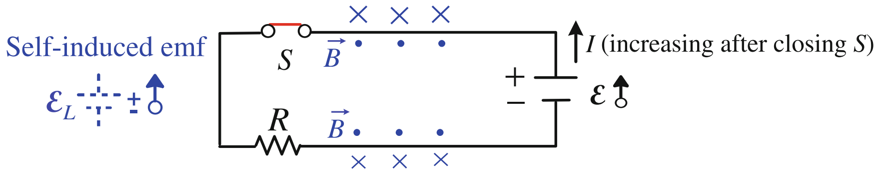


Fig. 28.1 After closing the switch S, the current increases and so does the magnetic flux through the loop. A self-induced emf $\mathcal{E}_L$ (the dashed battery) is created in the loop opposite to the battery's emf $\mathcal{E}$

When the current I in the coil *increases* with time as in Fig. 28.2b:

- *The magnetic flux through the coil also increases*
- *The increasing magnetic flux creates an induced emf* $\mathcal{E}_L$ *in the coil*
- *The induced emf and its induced current are opposing the emf and current generated by the source*

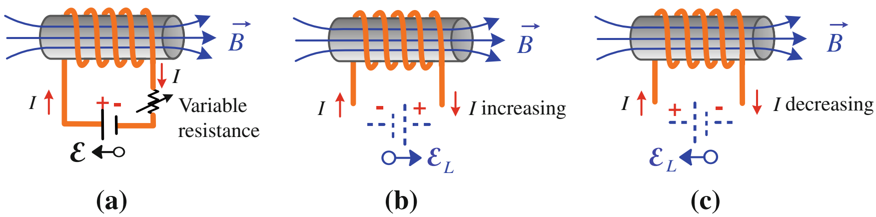


Fig. 28.2 (a) A current in the coil creates a magnetic field to the right. (b) When the current increases, the increasing magnetic flux creates a self-induced emf $\mathcal{E}_L$ (the dashed battery) opposite to the emf of the source. (c) The polarity of the self-induced emf $\mathcal{E}_L$ reverses if the current decreases

When the current I in the coil *decreases* with time as in Fig. 28.2c:

- *The magnetic flux through the coil also decreases*
- *The decreasing magnetic flux creates an induced emf* $\mathcal{E}_L$ *in the coil*
- *The induced emf and its induced current are in the same direction as the emf and current I generated by the source*

> Spotlight
> In conclusion, the self-induction of a coil prevents the current in the coil from increasing or decreasing instantaneously.

The magnetic flux Φ_B in a loop is proportional to the magnetic field $\vec{B}$, which in turn is proportional to the current I that produced this magnetic field in the loop. Therefore, $\Phi_B \propto I$. The proportionality constant between Φ_B and I is called the **self-inductance** (or inductance) of the loop, and is denoted by the symbol L. Thus:

$$\Phi_B = LI \tag{28.1}$$

According to both Faraday's law and Lenz's law, $\mathcal{E} = -d\Phi_B/dt$, the self-induced emf $\mathcal{E}_L$ is given by:

$$\mathcal{E}_L = -L\frac{dI}{dt} \tag{28.2}$$

The negative sign indicates that the self-induced emf $\mathcal{E}_L$ is a back emf that opposes the change in current. The SI unit of self-inductance is the henry (abbreviated by H). Thus:

$$1\,\text{H} = 1\frac{\text{V·s}}{\text{A}} = 1\,\Omega\text{.s} \quad \text{or} \quad 1\,\text{H} = 1\frac{\text{Wb}}{\text{A}} = 1\,\text{T.m}^2/\text{A}$$

Comparing Eq. 28.2 with Faraday's law for N loops, we find:

$$L = \frac{N\Phi_B}{I} \tag{28.3}$$

When used in circuits, elements with large values of L are referred to as **inductors** and denoted by the circuit symbol .

Example 28.1

The current in a coil changes according to the formula: $I = 0.5 - 0.2\,t$ where t is in seconds and I is in amperes. Experimental measurements show that a self-induced emf of 0.5 mV is produced across the terminals of the coil. What is the self-inductance of the coil?

Solution: From the current-time relation, we have:

$$\frac{dI}{dt} = \frac{d}{dt}(0.5 - 0.2\,t) = -0.2\,\text{A/s}$$

which means that I decreases with time. Given that $\mathcal{E}_L = 0.5\,\text{mV}$, then:

$$L = -\frac{\mathcal{E}_L}{dI/dt} = -\frac{0.5\times10^{-3}\,\text{V}}{-0.2\,\text{A/s}} = 2.5\times10^{-3}\,\text{H} = 2.5\,\text{mH}$$

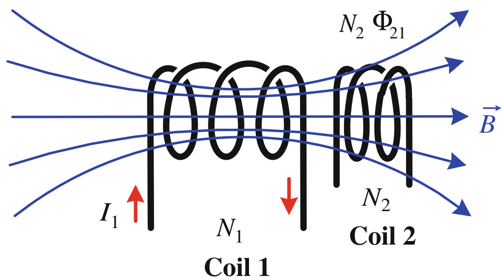
28.2 Mutual Inductance

We have seen that a changing current in a coil causes a changing magnetic flux, which in turn creates a self-induced emf $\mathcal{E}_L = -L dI/dt$ in the coil. If two coils exist in close proximity, then a changing current in one coil will result in a changing flux through the second coil; hence, there will be an induced emf in this second coil.

Figure 28.3 shows two coils with a common axis near each other. Coil 1 has N_1 turns and carries a current I_1 and coil 2 has N_2 turns. Part of the flux established by I_1 in coil 1 passes each turn of coil 2 and is represented by Φ_{21}. The total linkage flux through coil 2 is thus $N_2\Phi_{21}$. Since this total flux is directly proportional to the current I_1, then in analogy to Eq. 28.3, we define the **mutual inductance** M_{21} of coil 2 as:

$$M_{21} = \frac{N_2\Phi_{21}}{I_1} \tag{28.4}$$

Fig. 28.3 If the current I_1 in coil 1 changes, a mutual induced emf $\mathcal{E}_2 = -M dI_1/dt$ will be established in a nearby coil 2



It is clear that the mutual inductance depends on the geometry of both coils. We can rearrange the last equation as:

$$M_{21} I_1 = N_2\Phi_{21} \tag{28.5}$$

If the current I_1 varies with time, then:

$$M_{21}\frac{dI_1}{dt} = N_2\frac{d\Phi_{21}}{dt} \tag{28.6}$$

According to Faraday's law, apart from a sign, the right side is just the emf $\mathcal{E}_{M2}$ established in coil 2. Thus, the mutual emf in coil 2 is:

$$\mathcal{E}_{M2} = -M_{21}\frac{dI_1}{dt} \tag{28.7}$$

The preceding steps can be repeated to show that if a current I_2 in coil 2 varies with time, then the mutual emf in coil 1 will be:

$$\mathcal{E}_{M1} = -M_{12}\frac{dI_2}{dt} \tag{28.8}$$

Thus, the emf produced in either coil is proportional to the rate of change of current in the other coil. It can be shown that $M_{12} = M_{21} = M$, i.e. the two coils have a single *mutual inductance M*. Then, we have:

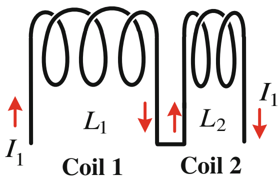
$$\boxed{\mathcal{E}_{M2} = -M\frac{dI_1}{dt}} \quad \text{and} \quad \boxed{\mathcal{E}_{M1} = -M\frac{dI_2}{dt}} \tag{28.9}$$

It is obvious that the unit of mutual inductance is the henry.

Example 28.2

Two nearby coils 1 and 2 have self-inductances $L_1 = 0.2\,\text{mH}$ and $L_2 = 0.1\,\text{mH}$, respectively. When the current in coil 1 changes at a rate of 4 A/s, it is found that a mutual emf of 10 mV is induced in coil 2. (a) What is the mutual inductance of the combination? (b) If the two coils are joined as shown in Fig. 28.4, find the total induced emf of the combination.

Fig. 28.4



Solution: (a) Using the magnitude values of Eq. 28.9, we get:

$$\mathcal{E}_{M2} = M\frac{dI_1}{dt} \quad \Rightarrow \quad M = \frac{\mathcal{E}_{M2}}{dI_1/dt} = \frac{10\times10^{-3}\,\text{V}}{4\,\text{A/s}} = 2.5\times10^{-3}\,\text{H}$$

(b) The mutually-induced and self-induced emfs in the two coils are in the same direction. Since the current is the same in both coils, then:

$$\begin{aligned}\mathcal{E}_{\text{tot}} &= \mathcal{E}_{L1} + \mathcal{E}_{L2} + \mathcal{E}_{M1} + \mathcal{E}_{M2} = -(L_1 + L_2 + 2M)\frac{dI_1}{dt}\\ &= -(0.2\times10^{-3}\,\text{H} + 0.1\times10^{-3}\,\text{H} + 2\times2.5\times10^{-3}\,\text{H})(4\,\text{A/s})\\ &= -21.2\times10^{-3}\,\text{V}\end{aligned}$$

28.3 Energy Stored in an Inductor

It is necessary to do work in order to overcome the back emf in any conductor when the current is changing. Because energy is not dissipated by an ohm-less inductor, we may consider any work done as energy stored in the inductor in the form of a magnetic field.

Consider the circuit of Fig. 28.5a in which a battery of emf $\mathcal{E}$ is connected to an ohm-less inductor in series with an open switch S.

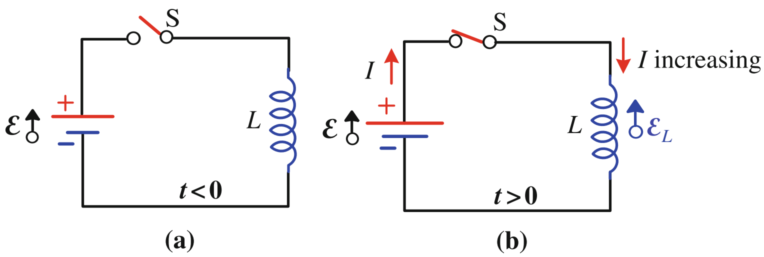


Fig. 28.5 (a) A battery, inductor, and open switch at $t < 0$. (b) The circuit at time $t > 0$ when I is increasing after S is closed at $t = 0$

When S is closed at time $t = 0$, the current I begins to increase, see Fig. 28.5b, and a back emf that opposes I is induced in the inductor; thus the induced emf is against $\mathcal{E}$. The loop theorem yields:

$$\mathcal{E} - L\frac{dI}{dt} = 0 \quad \Rightarrow \quad \mathcal{E} = L\frac{dI}{dt} \quad \text{(At time } t > 0\text{)} \tag{28.10}$$

Multiplying by the instantaneous value of the current, we get:

$$I\mathcal{E} = LI\frac{dI}{dt} \tag{28.11}$$

Since $I\mathcal{E}$ is the rate at which energy is being supplied by the battery, then $P = LI\,dI/dt$ must represent the rate at which energy is being stored in the inductor. The energy U_L stored is the conductor is thus:

$$U_L = \int_0^t P\,dt = \int_0^t LI\frac{dI}{dt}dt = L\int_0^I I\,dI = \frac{1}{2}LI^2$$

$$\boxed{U_L = \frac{1}{2}LI^2} \tag{28.12}$$

This is the energy stored in the magnetic field of an inductor when the current is I. We can prove that $u_B = B^2/2\mu_\circ$ is the energy density, see Eq. 23.38.

28.4 The $L-R$ Circuit

We can place inductors in circuits to prevent the current in these circuits from increasing or decreasing instantaneously.

Figure 28.6 displays a circuit containing a battery of emf $\mathcal{E}$, a resistor of resistance R, an inductor of inductance L, and a switch S. Assume that the switch S is open for $t < 0$, as shown in Fig. 28.6a.

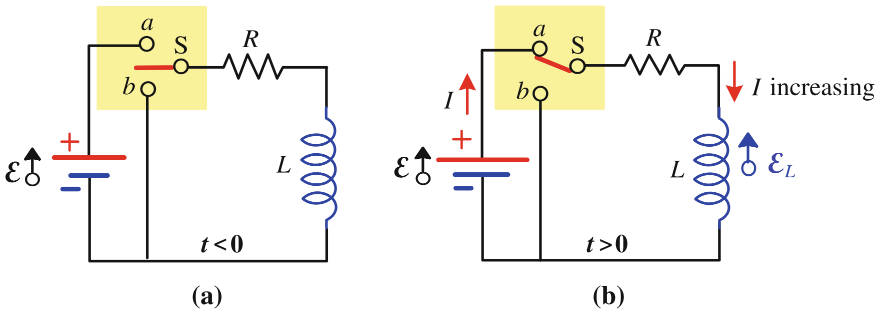


Fig. 28.6 (a) The circuit diagram of an inductor in series with a resistor, an open switch, and a battery. (b) The circuit diagram at time $t > 0$ when I is increasing after the switch S is connected to a at $t = 0$

Connecting Switch S to Position a

Once the switch S is connected to position a at time $t = 0$, the current begins to increase, and a back emf that opposes the increasing current is induced in the inductor; thus $\mathcal{E}_L$, is against the battery's emf.

Assume that the current in the circuit at time $t > 0$ is I, as shown in Fig. 28.6b. Applying Kirchhoff's loop rule and traversing the circuit clockwise, we get:

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0 \quad (\text{At time } t > 0) \tag{28.13}$$

Using the condition $I = 0$ at $t = 0$ and changing the variables by letting $x = \mathcal{E}/R - I$, it is left as a problem to show that the solution of (28.13) is:

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}), \quad \tau = \frac{L}{R} \tag{28.14}$$

This relation shows that $I = 0$ at $t = 0$ and $I = \mathcal{E}/R$ at $t = \infty$, as expected.

We can generalize these results as follows:

Spotlight
Initially, an inductor acts to oppose the increase in the current, but after a long time it acts like an ordinary conductive connecting wire.

If we take the first time derivative of Eq. 28.14, we get:

$$\frac{dI}{dt} = \frac{\mathcal{E}}{L}e^{-t/\tau}, \quad \tau = \frac{L}{R} \tag{28.15}$$

Thus, dI/dt is a maximum and is equal to $\mathcal{E}/L$ at $t = 0$ and falls off exponentially to zero as t approaches infinity.

The quantity L/R in the exponents of Eqs. 28.14 and 28.15 is called the **time constant** τ of the circuit. Therefore, the quantity $\tau = L/R$ represents the time interval during which the current in the circuit increases to $(1-e^{-1}) = 0.632 \equiv 63.2 \sim 63\%$ of its final value $\mathcal{E}/R$. Similarly, after a time interval τ, the current rate dI/dt decreases to $e^{-1} = 0.368 \equiv 36.8 \sim 37\%$ of its initial value $\mathcal{E}/L$.

Figure 28.7 shows the variation of the circuit current I and the current rate dI/dt as a function of time.

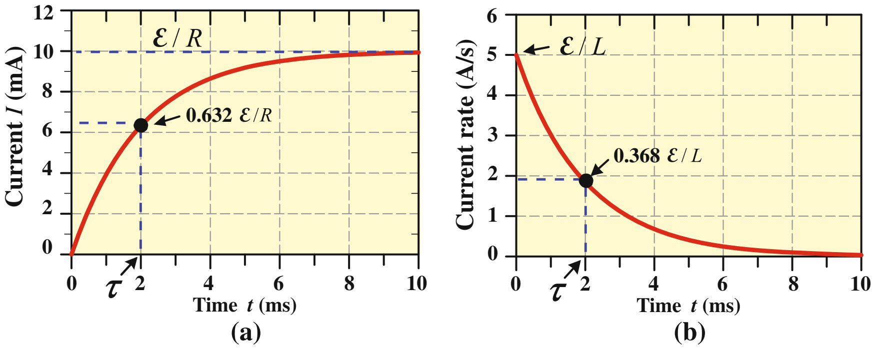


Fig. 28.7 (a) A *plot* of the current I in the circuit of Fig. 28.6 versus time t. (b) A *plot* of current rate dI/dt in the circuit of Fig. 28.6 versus time t. The two *curves* of parts (a) and (b) are based on the values $R = 200\,\Omega$, $L = 0.4\,\text{H}$, and $\mathcal{E} = 2\,\text{V}$

Connecting *S* is Changed to Position *b* After Being Connected to *a*

Suppose that the switch S has been connected to position a first for a long period to allow the current to reach to its equilibrium value $\mathcal{E}/R$, as shown in Fig. 28.8a.

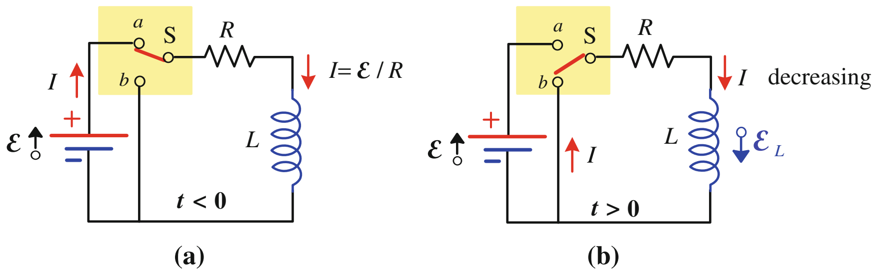


Fig. 28.8 (a) The circuit diagram with a saturated current of constant value $I = \mathcal{E}/R$. (b) The circuit diagram at time $t > 0$ when I is decreasing after the switch S is connected to position b at $t = 0$

At $t = 0$, the switch S is disconnected from position a and instantaneously connected to position b. At this moment, the current begins to decrease, and a self-induced emf that opposes the decreasing current is induced in the inductor; thus $\mathcal{E}_L$ is clockwise.

Assume that the current in the circuit at time $t > 0$ is I, as show in Fig. 28.8b. With the switch in position b, the battery's emf $\mathcal{E}$ is removed and Eq. 28.13 reduces to:

$$0 - IR - L\frac{dI}{dt} = 0 \quad (\text{At time } t > 0) \tag{28.16}$$

It is left as an exercise to show that the solution of Eq. 28.16 is:

$$I = \frac{\mathcal{E}}{R} e^{-t/\tau}, \quad \tau = \frac{L}{R} \tag{28.17}$$

This current falls exponentially from $\mathcal{E}/R$ to zero. In a time interval $\tau = L/R$, the current in the circuit declines to $e^{-1} = 0.368 \sim 37\%$ of its initial value $\mathcal{E}/R$. Note that the direction of the current is the same when the switch is connected to position a or position b.

Example 28.3

In Fig. 28.6, let $R = 12\,\Omega$, $\mathcal{E} = 24$ V, and $L = 60$ mH. (a) Find the time constant of the circuit. (b) After closing S at $t = 0$, find the current in the circuit at $t = 2$ ms. (c) Find the energy stored in the inductor when the current is 1.5 A.

Solution: (a) The time constant of the R–L circuit is given by:

$$\tau = \frac{L}{R} = \frac{60 \times 10^{-3}\,\mathrm{H}}{12\,\Omega} = 5 \times 10^{-3}\,\mathrm{s} = 5\,\mathrm{ms}$$

(b) Using Eq. 28.14, we find the current in the circuit at $t = 2$ ms:

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) = \frac{24\,\mathrm{V}}{12\,\Omega}(1 - e^{-0.4}) = 0.659\,\mathrm{A}$$

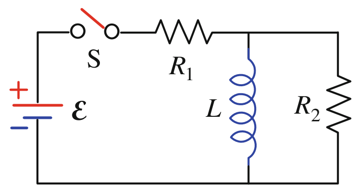
(c) Using Eq. 28.12, the energy stored when $I = 1.5$ A is:

$$U_B = \frac{1}{2}LI^2 = 0.5(60 \times 10^{-3}\,\mathrm{H})(1.5\,\mathrm{A})^2 = 67.5 \times 10^{-3}\,\mathrm{J} = 67.5\,\mathrm{mJ}$$

Example 28.4

In Fig. 28.9, determine the initial current at $t = 0$ (when the switch is closed) and the final current at $t \to \infty$ (when the switch is closed for a long time).

Fig. 28.9



Solution: When the switch is closed at $t = 0$, the current in the inductance coil cannot change instantaneously. Therefore, at $t = 0$ the current from the battery must flow through R_1 and R_2 only. Hence:

$$I(\text{at}\,t{=}0) = \frac{\mathcal{E}}{R_1 + R_2}$$

When the switch is closed for a long time, the current in the inductor is not changing and therefore the induced emf is zero. In this case the inductor (which has zero resistance) is short circuited with R_2. Thus, there is no current in R_2 and the same current will flow through R_1 and L. Hence:

$$I(\text{at}\, t \to \infty) = \frac{\mathcal{E}}{R_1}$$

28.5 The Oscillating $L-C$ Circuit

In Fig. 28.10a, assume that the switch S is open when the capacitor has an initial charge Q (the maximum charge), and hence the total energy stored in the capacitor is $U = Q^2/2C$. In addition, we assume a resistance-free, non-radiating LC circuit.

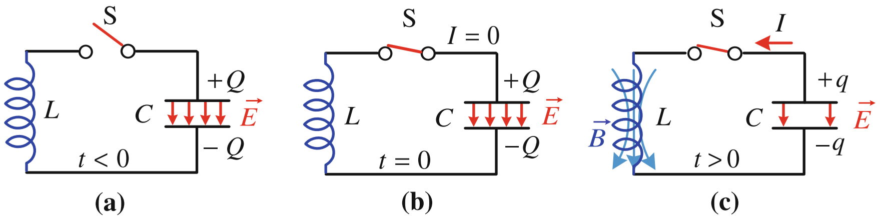


Fig. 28.10 (a) Before starting ($t < 0$), switch S is open and the capacitor has an initial maximum charge Q. (b) When the switch is closed at $t = 0$, the current in the circuit is zero and the charge begins decreasing. (c) For $t > 0$, the charge has decreased to $q(t)$ and the current in the circuit $I = -dq(t)/dt$ establishes a magnetic field $\vec{B}(t)$ in the inductor

When the switch is closed at $t = 0$, the current I in the circuit is zero, and the capacitor starts to discharge through the inductor, see Fig. 28.10b.

At $t > 0$, represented in Fig. 28.10c, the charge on the capacitor decreases to q (where $q < Q$) and the rate at which the charges leave (or enter) the capacitor is equal to the current I in the circuit. This current establishes a magnetic field $\vec{B}$ in the inductor.

When the capacitor is fully discharged, the current at this time reaches its maximum value I_{max}, and all of the energy is now stored in the inductor. The current continues in the same direction, but it is now decreasing in magnitude and the capacitor is being charged with polarity opposite to the initial polarity. This is followed by another discharge until the circuit returns to its original state. In a system with zero resistance the energy continues to oscillate between the capacitor and inductor indefinitely. We refer to this as an "*oscillating circuit*".

At an arbitrary time t, the current in the circuit is related to the decreasing charge q by $I = -dq/dt$. In addition, at time t the sum of the stored energy in the capacitor U_C and the inductor U_L must equal the initial energy stored in the capacitor U at $t = 0$. Thus:

$$\begin{aligned} U_C + U_L &= U \\ \frac{q^2}{2C} + \frac{1}{2}LI^2 &= \frac{Q^2}{2C} \end{aligned} \tag{28.18}$$

Differentiating this equation with respect to the time t and noting that $dI/dt = -d^2q/dt^2$, we can reach the following differential equations:

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

or:

$$\frac{d^2q}{dt^2} + \omega^2 q = 0 \tag{28.19}$$

where:

$$\omega = \frac{1}{\sqrt{LC}} \tag{28.20}$$

This equation is analogous to a block-spring system given by Eq. 14.8. By consideration of the initial conditions, $q = Q$ and $I = 0$ at $t = 0$, we find that Eq. 28.19 has a solution given by:

$$q = Q\cos(\omega t) \tag{28.21}$$

where ω is the angular frequency of the oscillations, which is a frequency solely depends on the capacitance C and inductance L of the circuit. The **undamped** frequency and period of the oscillations are given by:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}, \quad T = \frac{1}{f} = 2\pi\sqrt{LC} \tag{28.22}$$

The current as a function of time is therefore given by:

$$I = -\frac{dq}{dt} = Q\omega\sin(\omega t) = I_{\max}\sin(\omega t) \tag{28.23}$$

where the maximum current and charge are related by $I_{\max} = Q\omega$. The general solution of (28.19) is $q = Q\cos(\omega t + \phi)$, with ϕ is a phase angle.

Figure 28.11 displays the electric and magnetic fields as well as current of a complete cycle of an $L - C$ circuit.

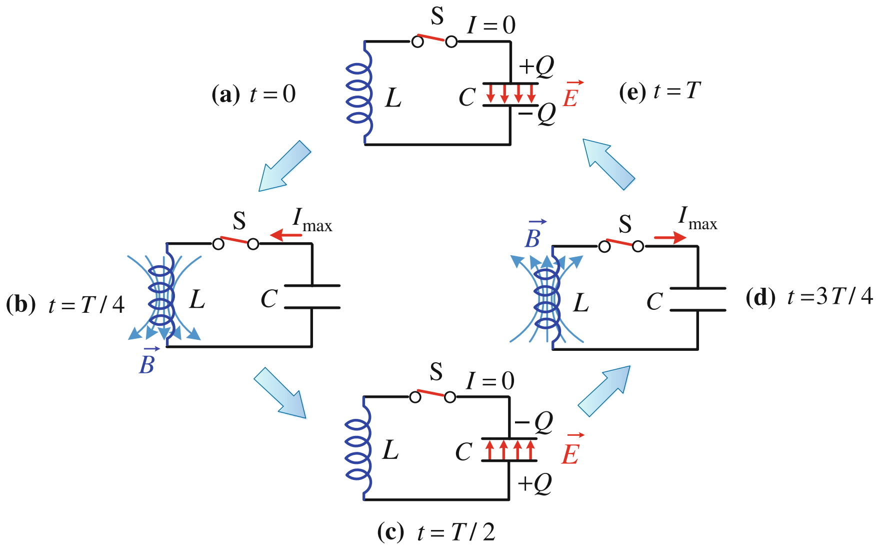
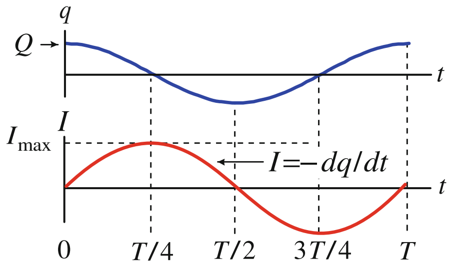


Fig. 28.11 (a) At $t = 0$, all of the energy is stored as an electric energy $Q^2/2C$ in the capacitor. (b) At $t = T/4$, all of the energy is stored as a magnetic energy $\frac{1}{2}LI^2_{max}$ in the inductor. (c) At $t = T/2$, all of the energy is stored again in the capacitor, but with opposite polarity. (d) At $t = 3T/4$, all of the energy is stored as a magnetic energy $\frac{1}{2}LI^2_{max}$ in the inductor. (e) At $t = T$, the circuit returns to its initial configuration at $t = 0$

For a complete cycle, Fig. 28.12 displays both the charge and current versus time for a resistanceless nonradiating LC circuit.

Fig. 28.12 Variation of q and I as a function of time t



Example 28.5

When S_1 is closed and S_2 is opened, as shown in the left part of Fig. 28.13, a capacitor of capacitance $C = 7.1$ pF is charged from a battery of emf $\mathcal{E} = 12$ V. Switch S_1 is then opened, and the capacitor remains charged. Switch S_2 is then closed, so the capacitor is connected directly to an inductor of inductance $L = 3.56\,\text{mH}$, as shown in the right part of Fig. 28.13. (a) Find the frequency of oscillation of the circuit. (b) Find both the maximum charge on the capacitor and current in the circuit. (c) Find the charge and current as a function of time.

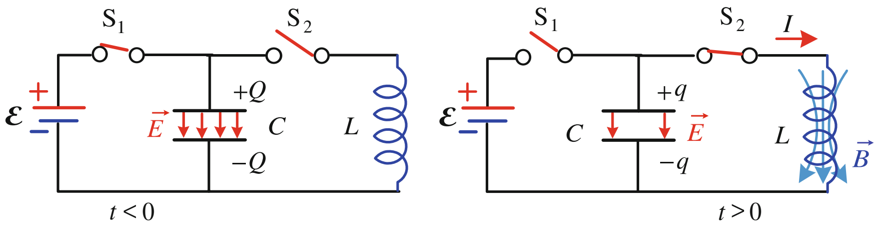


Fig. 28.13

Solution: (a) Eq. 28.22 gives for the frequency of the oscillating circuit as:

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(3.56\times10^{-3}\,\text{H})(7.1\times10^{-12}\,\text{F})}} = 1\times10^{6}\,\text{Hz} = 1\,\text{MHz}$$

(b) Using the relation $Q = C\Delta V = C\mathcal{E}$, we get the maximum charge as:

$$Q = C\mathcal{E} = (7.1\times10^{-12}\,\text{F})(12\,\text{V}) = 8.52\times10^{-11}\,\text{C} = 85.2\,\text{pC}$$

From Eq. 28.23 and the relation $\omega = 2\pi f$, the maximum current is given in terms of the maximum charge as:

$$I_{\text{max}} = Q\omega = (8.52\times10^{-11}\,\text{C})(2\pi\times10^{6}\,\text{s}^{-1}) = 5.35\times10^{-4}\,\text{A}$$

(c) Using Eqs. 28.21 and 28.23, the charge and current as a function of time are given as follows:

$$q = Q\cos(\omega t) = (8.52\times10^{-11}\,\text{C})\cos[(2\pi\times10^{6}\,\text{s}^{-1})\,t]$$

$$I = I_{\text{max}}\sin\omega t = (5.35\times10^{-4}\,\text{A})\sin[(2\pi\times10^{6}\,\text{s}^{-1})\,t]$$

28.6 The *L* – *R* – *C* Circuit

Now we consider a realistic $L-C$ circuit with some resistance R. In Fig. 28.14a, the switch S is open and the capacitor has an initial charge Q. This is the maximum charge that the capacitor can store. Consequently, the total energy stored in the capacitor is $U = Q^2/2C$.

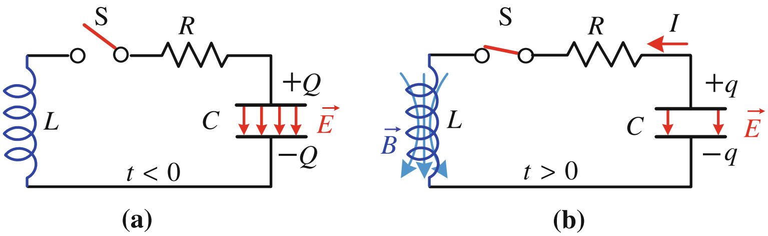


Fig. 28.14 (a) Before starting ($t < 0$), the switch S is open and the capacitor has an initial maximum charge Q. (b) After the switch is closed ($t > 0$), the charge has decreased to $q(t)$ and the current in the circuit $I = -dq(t)/dt$ establishes a magnetic field $\vec{B}$ in the inductor

The switch S is closed at $t = 0$. Figure 28.14b represents the case at $t > 0$. In this figure, the charge on the capacitor decreases to q (where $q < Q$) and the rate at which the charges leave (or enter) the capacitor is equal to the current I in the circuit. This current establishes a magnetic field $\vec{B}$ in the inductor. Applying Kirchhoff's loop rule and traversing the circuit counterclockwise (starting from the capacitor's negative plate), we get:

$$\frac{q}{C} - IR - L\frac{dI}{dt} = 0 \quad (\text{At time } t > 0) \tag{28.24}$$

Since $I = -dq/dt$, this equation becomes:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \tag{28.25}$$

This second-order differential equation in the variable q has the same form as the damped harmonic oscillator Eq. 14.25:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + k_{\text{H}}x = 0$$

Therefore, by comparison, Eq. 28.25 has the solution:

$$q(t) = Qe^{-(R/2L)t}\cos(\omega_d t + \phi) \tag{28.26}$$

where the angular frequency of the damped oscillation ω_d is given by:

$$\omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \quad \left(\omega_d \xrightarrow[\text{or } R\to 0]{R\ll 2\sqrt{1/LC}} \sqrt{\frac{1}{LC}} = \omega\right) \tag{28.27}$$

If the resistance R is relatively small, the circuit oscillates, but with damped oscillations. We refer to this as an **underdamped** circuit, see Fig. 28.15. If we increase R, the oscillations die out more rapidly. When R reaches a certain critical value $R_c = \sqrt{4L/C}$, the circuit does not oscillate and it is said to be **critically damped**, see Fig. 28.15. When R is greater than R_c, the circuit is said to be **overdamped**.

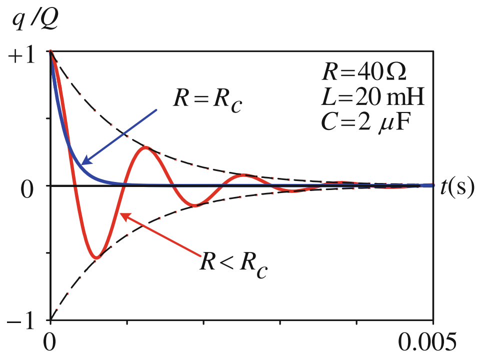


Fig. 28.15 When $\phi = 0$, the figure shows an underdamped circuit with $R < R_c$ (*red curve*) and critically damped circuit with $R = R_c$ (*blue curve*)

Example 28.6

In the circuit of Fig. 28.14, take $R = 40\,\Omega$, $L = 20\,\text{mH}$, and $C = 2\,\mu\text{F}$. (a) Show that this circuit oscillates. (b) Determine the frequency of the circuit. (c) When $\phi = 0$ and $t > 0$, find the first three times at which the cosine term of Eq. 28.26 becomes ∓ 1 and then find the ratio q/Q at these times. (d) What resistance R is required to make this circuit oscillate with one-half the undamped frequency?

Solution: (a) We first calculate the critical value R_c as follows:

$$R_c = \sqrt{\frac{4L}{C}} = \sqrt{\frac{4(20 \times 10^{-3}\,\text{H})}{2 \times 10^{-6}\,\text{F}}} = 200\,\Omega$$

Since $R < R_c$ is satisfied when $R = 40\,\Omega$, then this circuit oscillates.

(b) We use the relation $\omega_d = 2\pi f_d$ to find the frequency as follows:

$$f_d = \frac{\omega_d}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{(20 \times 10^{-3}\,\text{H})(2 \times 10^{-6}\,\text{F})} - \frac{(40\,\Omega)^2}{4(20 \times 10^{-3}\,\text{H})^2}} = 779.7\,\text{Hz}$$

(c) For $t>0$, the term $\cos(\omega_d t)$ equals ∓ 1 when $\omega_d t=\pi,\ 2\pi,\ldots,$ Thus, $t_n=n\pi/\omega_d=n/2f_d$, $(n=1,2,\ldots)$. Therefore, $\cos(\omega_d t)=\mp 1$ at:

$$t_1=0.641\,\text{ms},\quad t_2=1.28\,\text{ms},\quad t_3=1.924\,\text{ms},\ \ldots$$

For these calculated times, the ratio q_n/Q is:

$$\frac{q_n}{Q}=e^{-\left(\frac{R}{2L}\right)t_n}=e^{-\left(\frac{40\,\Omega}{2(20\times 10^{-3}\,\text{H})}\right)t_n}=e^{-(1000\,\text{s}^{-1})\,t_n}$$

Thus:

$$\frac{q_1}{Q}=e^{-(1000\,\text{s}^{-1})\,t_1}=e^{-(1000\,\text{s}^{-1})(0.641\times 10^{-3}\,\text{s})}=e^{-0.641}=0.53\quad(53\%)$$
$$\frac{q_2}{Q}=e^{-(1000\,\text{s}^{-1})\,t_2}=e^{-(1000\,\text{s}^{-1})(1.28\times 10^{-3}\,\text{s})}=e^{-1.28}=0.28\quad(28\%)$$
$$\frac{q_3}{Q}=e^{-(1000\,\text{s}^{-1})\,t_3}=e^{-(1000\,\text{s}^{-1})(1.924\times 10^{-3}\,\text{s})}=e^{-1.924}=0.15\quad(15\%)$$

(d) Using Eqs. 28.22 and 28.27, the required resistance R that makes this circuit oscillate with one-half the undamped frequency is obtained by setting $\omega_d=\omega/2$. Thus:

$$\omega_d=\tfrac{1}{2}\omega\quad\Rightarrow\quad\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}=\tfrac{1}{2}\sqrt{\frac{1}{LC}}$$

When we square both sides, we get:

$$R=\sqrt{\frac{3L}{C}}=\sqrt{\frac{3(20\times 10^{-3}\,\text{H})}{2\times 10^{-6}\,\text{F}}}=173.2\,\Omega$$

28.7 Circuits with an ac Source

In this section, we continue studying circuits containing elements such as resistors, inductors, and capacitors, but this time connecting them to a source of alternating voltage that produces an alternating current (ac). First, we examine these electronic components individually, considering a sinusoidal voltage (see Sect. 27.3) and current (see Sect. 27.4) that can be described by:

$$v=V\sin\omega t\tag{28.28}$$

$$i=I\sin\omega t\tag{28.29}$$

In these expressions, the lowercase v and i represent the instantaneous potential difference and current, respectively. The uppercase V and I represent the peak voltage and current, respectively. The angular frequency ω is equal to 2π times the frequency f of the oscillations.

Resistors in an ac Circuit

According to Eq. 28.28, we can write the alternating voltage across any resistor as:

$$v_R = V_R \sin \omega t \tag{28.30}$$

where V_R is the peak voltage across the resistor. From Ohm's law, the instantaneous current through such a resistor is:

$$i_R = \frac{v_R}{R} = \frac{V_R}{R} \sin \omega t = I_R \sin \omega t \tag{28.31}$$

Where the peak current I_R is given by $I_R = V_R/R$. According to this result, we have:

$$V_R = I_R\, R \tag{28.32}$$

In addition, the relations between the rms and peak values of the current and voltage; Ohm's law; and the average power delivered to a resistor as a heat, are all given Sect. 27.4 as follows:

$$I_{\rm rms} = \frac{I}{\sqrt{2}} = 0.707\, I \quad \text{and} \quad V_{\rm rms} = \frac{V}{\sqrt{2}} = 0.707\, V \tag{28.33}$$

$$V_{\rm rms} = I_{\rm rms}\, R \tag{28.34}$$

$$\overline{P} = I_{\rm rms}\, V_{\rm rms} = I_{\rm rms}^2\, R = V_{\rm rms}^2/R \tag{28.35}$$

Because the current i_R is zero when the voltage v_R is zero and the current reaches a peak when the voltage reaches a peak, they are both proportional to $\sin \omega t$ and we say that the current and voltage are **in phase**, see Fig. 28.16. That is:

> Spotlight
> The voltage across a resistor is in phase with current.

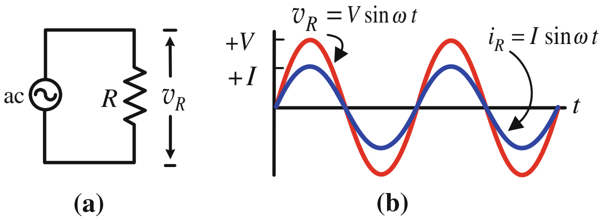


Fig. 28.16 (a) A resistor connected to an ac source. (b) Alternating voltage v_R (*red*) across R is in phase with alternating current i_R (*blue*)

Inductors in an ac Circuit

We replace the resistor in Fig. 28.16a with a pure inductor of inductance L and zero resistance as shown in Fig. 28.17a. The potential difference across the inductor can be written as:

$$v_L = V_L \sin \omega t \tag{28.36}$$

where V_L is the peak voltage. The voltage applied to the inductor will be equal to the back induced emf generated in the inductor by the changing alternating current. Thus:

$$v_L = L \frac{di_L}{dt} \tag{28.37}$$

If we combine the last two equations, we get:

$$\frac{di_L}{dt} = \frac{V_L}{L} \sin \omega t \tag{28.38}$$

To find the current, we integrate the last equation to get:

$$i_L = \frac{V_L}{L} \int \sin \omega t \, dt = -\frac{V_L}{\omega L} \cos \omega t \tag{28.39}$$

For reasons of symmetry of notation, we use trigonometric identities to replace $-\cos \omega t$ with a phase-shifted sine as follows:

$$-\cos \omega t = \sin(\omega t - \pi/2)$$

With this change, the current in the inductor becomes:

$$i_L = \frac{V_L}{\omega L} \sin(\omega t - \pi/2) = I_L \sin(\omega t - \pi/2) \tag{28.40}$$

where $I_L = V_L/\omega L$ is the peak current. Figure 28.17b shows the variation of v_L and i_L as a function of time. It is clear from the figure and Eqs. 28.36 and 28.40 that the voltage v_L and the current i_L are out of phase by a quarter cycle, which is equivalent to $\pi/2$ radians or 90°. That is:

> Soptlight
> The voltage across an inductor *leads* the current by 90°.

In other words, the current in an inductor reaches its peak quarter a cycle later than the voltage.

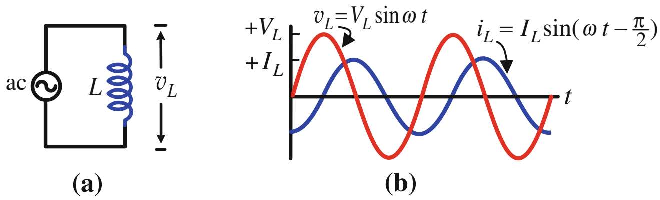


Fig. 28.17 (a) An inductor connected to an ac source. (b) Alternating voltage v_L (*red*) leads alternating current i_L (*blue*) by quarter a cycle or 90°

Because the current and voltage are out of phase by 90°, the average power dissipated is zero. Energy from the source is delivered to the inductor and stored as an increasing magnetic field between its turns. As the field decreases, the energy returns to the source. That is:

$$\overline{P}_L = 0 \tag{28.41}$$

Capacitors in an ac Circuit

Figure 28.18a shows a capacitor connected to a generator with an alternating emf. The applied potential difference of the ac source must equal the applied potential difference across the capacitor. Thus:

$$v_C = V_C \sin\omega t \tag{28.42}$$

where V_C is the peak voltage across the capacitor. According to the definition of capacitance, the instantaneous charge on the capacitor plates is:

$$q_C = C v_C = C V_C \sin\omega t \tag{28.43}$$

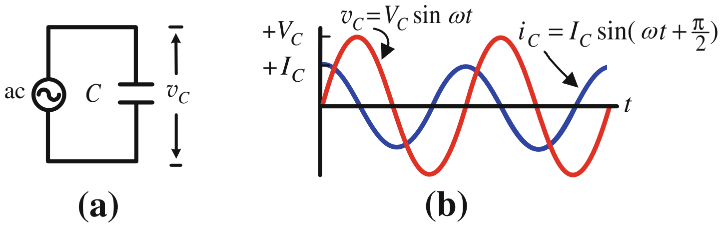


Fig. 28.18 (a) A capacitor connected to an ac source. (b) Alternating voltage v_C (*red*) lags alternating current i_C (*blue*) by quarter a cycle or 90°

The current in the circuit at any instant is thus:

$$i_C = \frac{dq_C}{dt} = \omega C V_C \cos \omega t \tag{28.44}$$

Again, for reasons of symmetry of notation, we use trigonometric identities to replace $\cos \omega t$ with a phase-shifted sine as follows:

$$\cos \omega t = \sin(\omega t + \pi/2)$$

With this change, the current in the capacitor becomes:

$$i_C = \omega\, C V_C \sin(\omega t + \pi/2) = I_C \sin(\omega t + \pi/2) \tag{28.45}$$

where $I_C = \omega\, C V_C$ is the peak current in the circuit. Figure 28.18b shows the variation of v_C and i_C as a function of time. It is clear from the figure and Eqs. 28.42 and 28.45 that the voltage v_C and the current i_C are out of phase by a quarter cycle, which is equivalent to $\pi/2$ radians or 90°. That is:

> Spotlight
> The voltage across a capacitor *lags* the current by 90°.

In other words, the current reaches its peak quarter a cycle ahead of the voltage.

Because the current and voltage are out of phase by 90°, the average power dissipated is zero. This is similar to an inductor. Energy from the source is delivered to the capacitor and stored as an increasing electric field between its plates. As the electric field decreases, the energy returns to the source. That is:

$$\overline{P}_C = 0 \tag{28.46}$$

Reactance and Phasors in an ac Circuit

We notice from Eqs. 28.40 and 28.45 that $V_L = I_L\,(\omega L)$ for inductors and $V_C = I_C(1/\omega C)$ for capacitors. As we search for additional symmetry in ac circuits, we introduce the two quantities X_L and X_C, called the **inductive reactance** of the inductor and the **capacitive reactance** of the capacitor, respectively, as follows:

$$X_L = \omega L \tag{28.47}$$

$$X_C = \frac{1}{\omega C} \tag{28.48}$$

where both quantities have the units of ohms. Just like the relation $V_R = I_R\,R$ for ohmic resistors, we can write similar relations for inductors and capacitors as follows:

$$V_L = I_L\,X_L \quad \text{(Peak or rms values)} \tag{28.49}$$

$$V_C = I_C\,X_C \quad \text{(Peak or rms values)} \tag{28.50}$$

Note that because the peak values of the current and voltage are not reached at the same time, these equations are valid only for peak or rms values and not for any other instant.

Note also that:

- The inductive reactance $X_L = \omega L$ is large for high frequencies f and/or larger inductances L. Consequently, the greater the value of X_L, the more it impedes the flow of charge and the smaller the current experienced in the inductor.
- The capacitive reactance $X_C = 1/\omega C$ is large for smaller frequencies f and/or smaller capacitances C. Consequently, the greater the value of X_C, the more it impedes the flow of charge and the smaller the current experienced in the capacitor. (For dc circuits $\omega = 0$ and $X_C = \infty$, and hence a capacitor does not pass dc current).

To simplify the analysis of complicated ac circuits, we use a graphical tool called the *phasor diagram*.

We define a phasor that represents a time-varying quantity to be a vector having the following properties:

- *Length*: Its length is proportional to the peak value of the variable
- *Angular frequency*: It rotates counterclockwise around the origin with the same angular frequency of the variable

- *Rotation angle*: Its rotation angle with respect to the horizontal axis is equal to the phase of the alternating quantity
- *Projection*: Its projection onto the vertical axis represents the instantaneous value of the variable

The time-varying quantities of v_R and i_R for a resistor, v_L and i_L for inductor, and v_C and i_C for capacitor are represented graphically in Fig. 28.19.

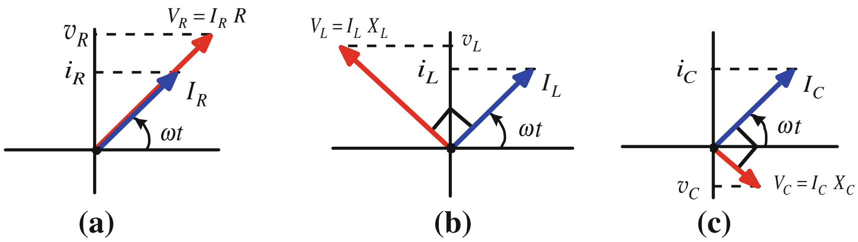


Fig. 28.19 Phasor diagrams. (a) For resistors, the voltage and current are in phase. (b) For inductors, the voltage leads the current by 90°. (c) For the capacitors, the voltage lags the current by 90°

Example 28.7

A coil has an inductance $L = 0.4$ H and a small resistance $R = 2\ \Omega$. Find the current in the coil when the applied voltage is: (a) 220-V dc, and (b) 220-V ac (rms) with a frequency $f = 50$ Hz.

Solution: (a) For a dc source, $\omega = 0$ and $X_L = 0$ and Ohm's law gives:

$$I_R = \frac{V_R}{R} = \frac{220\ \text{V}}{2\ \Omega} = 110\ \text{A}$$

(b) The value of the inductive reactance to be:

$$X_L = \omega L = 2\pi f L = 2\pi(50\ \text{cycle/s})(0.4\ \text{H}) = 126\ \Omega$$

Since X_L is much greater than R, we ignore its effect and use Eq. 28.49 to calculate the current as follows:

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_L} = \frac{220\ \text{V}}{126\ \Omega} = 1.75\ \text{A}$$

Example 28.8

A capacitor has a capacitance $C = 2\,\mu$F. Find the current in the capacitor if you apply a 50 Hz and 220-V ac (rms) voltage.

Solution: The value of the capacitive reactance is:

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2\pi\,(50\,\text{cycle/s})(2\times 10^{-6}\,\text{F})} = 1{,}592\,\Omega$$

We use Eq. 28.50 to calculate the current as follows:

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_C} = \frac{220\,\text{V}}{1{,}592\,\Omega} = 0.14\,\text{A}$$

28.8 *L* – *R* – *C* Series in an ac Circuit

Figure 28.20 shows an ac source connected to a circuit containing three elements in series: a resistor of resistance R, an inductor of inductance L, and a capacitor of capacitance C. Let us find the effect of R, X_L, and X_C on the peak current and the relation of the phase between the voltage and the current.

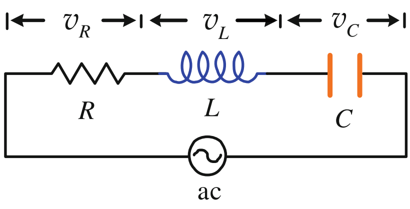


Fig. 28.20 An ac source connected in series with a resistor R, an inductor L, and a capacitor C

Since all elements in the circuit are in series, the current at any point in the circuit must be the same at any time. We choose the current i, at any time t to be:

$$i = I \sin \omega t$$

The peak currents in all elements are equal, i.e. $I_R = I_L = I_C = I$. Consequently, the peak voltages across the resistor, inductor, and capacitor are $V_R = IR$, $V_L = IX_L$, and $V_C = IX_C$, respectively. Based on the preceding section, the phases between the voltages across the elements and the current are summarized as follows:

1. The voltage across the resistor v_R is in phase with the current i
2. The voltage across the inductor v_L leads the current i by 90°
3. The voltage across the capacitors v_C lags the current i by 90°

We can express the relationships of these results as follows:

$$v_R = IR \sin \omega t = V_R \sin \omega t \tag{28.51}$$

$$v_L = IX_L \sin(\omega t + \pi/2) = V_L \sin(\omega t + \pi/2) \tag{28.52}$$

$$v_C = IX_C \sin(\omega t - \pi/2) = V_C \sin(\omega t - \pi/2) \tag{28.53}$$

The instantaneous voltage across the three elements equals the sum:

$$v = v_R + v_L + v_C \tag{28.54}$$

Although this analytical method is correct and leads to the final answer, it is actually simpler to use the phasor diagram. Figure 28.21a shows the phasor diagram for the three elements, based on the phasor diagram displayed in Fig. 28.19. To find the resultant phasor, we construct the difference phasor $V_L - V_C$ (assuming that the circuit is more inductive that capacitive), which is perpendicular to the phasor V_R, see Fig. 28.21b. From the Pythagorean theorem, the resultant voltage V is the hypotenuse of the right angle triangle shown in Fig. 28.21b. Thus:

$$V = \sqrt{V_R^2 + (V_L - V_C)^2} = I\sqrt{R^2 + (X_L - X_C)^2} \tag{28.55}$$

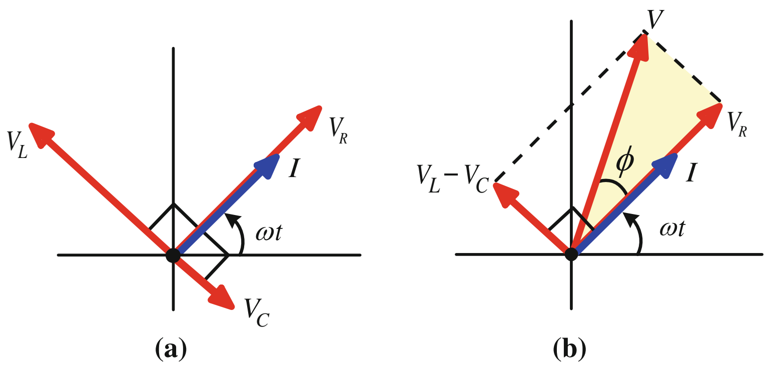


Fig. 28.21 (a) Phasor diagram for an ac source connected to a series *L-R-C* circuit. (b) The vector sum V of the three phasors V_R, V_L, and V_C

We define the impedance Z of an ac circuit as the ratio of the peak voltage across the circuit to the current peak in the circuit. Thus:

$$V = IZ \quad \text{or} \quad V_{\text{rms}} = I_{\text{rms}} Z \tag{28.56}$$

where

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{28.57}$$

Eq. 28.56 is known as the impedance version of Ohm's law.

According to Fig. 28.21, the phase angle ϕ between the peak voltage V and peak current I is giving by the two relations:

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR} = \frac{X_L - X_C}{R} \tag{28.58}$$

or

$$\cos\phi = \frac{V_R}{V} = \frac{IR}{IZ} = \frac{R}{Z} \tag{28.59}$$

In a series L-R-C circuit, only the resistor dissipates the power. Then, the average power dissipated is given by:

$$\overline{P} = I_{\text{rms}}^2 R \tag{28.60}$$

We can use $R = Z\cos\phi$, from Eq. 28.59, and $V_{\text{rms}} = I_{\text{rms}} Z$ from Eq. 28.56 to write the average power as follows:

$$\overline{P} = I_{\text{rms}} V_{\text{rms}} \cos\phi \tag{28.61}$$

where the quantity $\cos\phi$ is called the **power factor** of the circuit. When the circuit contains only a resistor, then $\phi = 0$, $\cos\phi = 1$, and consequently $\overline{P} = I_{\text{rms}} V_{\text{rms}}$. However, when the circuit does not contain a resistor but contains either an inductor or capacitor, $\phi = +90°$ and $\phi = -90°$, respectively, then no power is dissipated since $\cos\phi = 0$.

Example 28.9

An ac source of 220-V (rms) and angular frequency $\omega = 314$ rad/s is connected to a series L-R-C circuit, where $R = 35\ \Omega$, $L = 100$ mH, and $C = 650\ \mu$F. Find: (a) the inductive reactance, the capacitive reactance, and the impedance of the circuit, (b) the peak and rms current, (c) the peak voltage, the instantaneous voltage, and the rms voltage across each element, (d) the phase angle ϕ and the average power dissipated in the circuit.

Solution: (a) The reactance of the inductor and capacitor are:

$$X_L = \omega L = (314\,\text{rad/s})(100 \times 10^{-3}\,\text{H}) = 31.4\,\Omega$$
$$X_C = \frac{1}{\omega C} = \frac{1}{(314\,\text{rad.s})(650 \times 10^{-6}\,\mu\text{F})} = 4.9\,\Omega$$

The impedance of the circuit is thus:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(35\,\Omega)^2 + (31.4\,\Omega - 4.9\,\Omega)^2} = 43.9\,\Omega$$

(b) Using the impedance form of Ohm's law, Eq. 28.56, we get:

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{220\,\text{V}}{43.9\,\Omega} = 5.01\,A \simeq 5\,\text{A} \quad \text{and} \quad I = \sqrt{2}I_{\text{rms}} = 7.09\,\text{A}$$

(c) The peak and instantaneous voltages across each element are:

$$V_{\text{R}} = IR = (7.09\,\text{A})(35\,\Omega) = 248.2\,\text{V}$$
$$V_{\text{L}} = IX_L = (7.09\,\text{A})(31.4\,\Omega) = 222.6\,\text{V}$$
$$V_{\text{C}} = IX_C = (7.09\,\text{A})(4.9\,\Omega) = 34.7\,\text{V}$$
$$v_{\text{R}} = (248.2\,\text{V})\sin(314\,t)$$
$$v_{\text{L}} = (222.6\,\text{V})\sin(314\,t + \pi/2)$$
$$v_{\text{C}} = (34.7\,\text{V})\sin(314\,t - \pi/2)$$

The rms voltage across each element is:

$$(V_{\text{R}})_{\text{rms}} = I_{\text{rms}}\,R = (5.01\,\text{A})(35\,\Omega) = 175\,\text{V}$$
$$(V_{\text{L}})_{\text{rms}} = I_{\text{rms}}\,X_L = (5.01\,\text{A})(31.4\,\Omega) = 157\,\text{V}$$
$$(V_{\text{C}})_{\text{rms}} = I_{\text{rms}}\,X_C = (5.01\,\text{A})(4.9\,\Omega) = 24.5\,\text{V}$$

Notice that the peak and rms voltages across the elements do not add to equal the source voltage, 311 V (peak value) or 220 V (rms). This is because the different voltages are not in phase with each other. At a particular instant, one voltage across a particular element may be negative in order to compensate for the large positive voltage on the other, but the instantaneous voltages must add up to the source voltage. On the other hand, the rms voltages are always positive by definition.

(d) The phase angle ϕ is given by Eq. 28.59 as:

$$\cos\phi = \frac{R}{Z} = \frac{35\,\Omega}{43.9\,\Omega} = 0.797 \quad \Rightarrow \quad \phi = \cos^{-1}(0.797) = 37.1^\circ$$

The average power dissipated is given by Eq. 28.61 as:

$$\overline{P} = I_{\text{rms}}\,V_{\text{rms}}\cos\phi = (5\,\text{A})(220\,\text{V})(0.797) = 876.7\,\text{W}$$

28.9 Resonance in *L* – *R* – *C* Series Circuit

As we saw in Eq. 28.56, the rms current in an $L–R–C$ series circuit depends on the source's frequency f. This can be rewritten as:

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{28.62}$$

Such a circuit is said to be in **resonance** when the current is maximum at a certain frequency. The maximum current occurs when the impedance is minimum. This condition can happen when $X_L - X_C = 0$ at a certain frequency $\omega_\circ$, i.e. $X_L - X_C = \omega_\circ L - 1/\omega_\circ C = 0$. Therefore:

$$\omega_\circ = \frac{1}{\sqrt{LC}} \tag{28.63}$$

This frequency corresponds to the natural frequency of oscillation of an $L – C$ circuit as introduced in Sect. 28.5.

Figure 28.22 shows the variation of I_{rms} as a function of the angular frequency ω for a particular $L–R–C$ series circuit. The current I_{rms} is maximum at $\omega = \omega_\circ$ and decreases when $\omega < \omega_\circ$ and also when $\omega > \omega_\circ$. At resonance, the energy transferred from the source to the circuit is maximum and increases for small values of R.

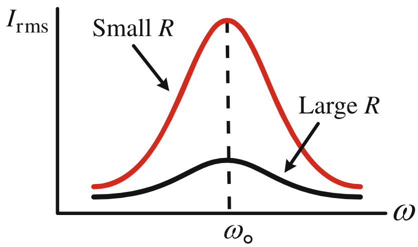


Fig. 28.22 Current in an $L–R–C$ circuit as a function of the angular frequency ω. At $\omega = \omega_\circ$ resonance occurs and the current is maximum

Resonance is used in Radio and TV sets for tuning to a station. By changing C of the $L–R–C$ circuit, the resonance frequency of the circuit matches a particular received EMW and the current flow is enhanced.

28.10 Exercises

Section 28.1 Self-Inductance

(1) The current in a coil of self-inductance $L = 75$ mH changes uniformly from zero to 1 A in 50 ms. What is the magnitude of the induced emf?

(2) The magnitude of the average induced emf in a coil is 5 V when its current changes from −30 to +120 mA during a period of 30 ms. What is the self-inductance (or inductance) of the coil?

(3) When a steady current $I = 10$ A passes through a solenoid of $N = 25$ turns, the magnetic flux through each turn is $\Phi_B = 10^{-2}$ Wb. What is the inductance of the coil?

(4) A solenoid has $N = 200$ turns, a length $\ell = 4$ cm, and a cross-sectional area $A = 10^{-2}$ m^2. What is the inductance of the solenoid?

(5) An air-filled cylindrical inductor of cross-sectional area $A = 5 \times 10^{-3}$ m^2 has a length $\ell = 4$ cm. How many turns of wire are required such that the inductor achieves an inductance $L = 125$ mH?

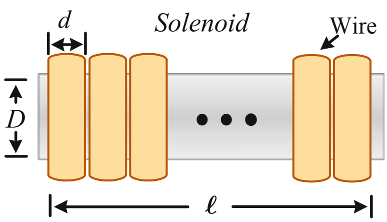
(6) If the core of the inductor of exercise 5 is filled with iron of relative permeability $K_m = \mu_m/\mu_\circ = 1{,}500$, how many turns are needed to obtain the same inductance?

(7) A solenoid of length $\ell = 20$ cm has $N = 500$ windings around an iron core of cross-sectional area $A = 2 \times 10^{-4}$ m^2 and relative permeability 500. (a) What is the inductance of the solenoid? (b) What is the average emf induced in the solenoid when its current decreases from 1.8 to 0.5 A in a period of 20 ms?

(8) A steady current $I = 5$ A passes through a coil of $N = 400$ turns and creates a magnetic flux $\Phi_B = 10^{-3}$ Wb through each turn of the coil. (a) Find the average emf induced in the coil when the current drops to zero in 40 ms. (b) What is the inductance of the coil? (c) What was the initial magnetic energy stored in the coil?

(9) A student wants to build an air-filled solenoid of inductance $L = 0.1$ H and diameter $D = 20$ cm by tightly winding one layer of insulated copper wire of diameter $d = 0.5$ mm around a plastic hollow tube, see Fig. 28.23. (a) What is the length ℓ of the solenoid? (b) What is the length of the required copper wire? (c) What will be the resistance of this wire if the resistivity of copper is $\rho = 1.68 \times 10^{-8}$ Ω.m?

Fig. 28.23 See Exercise (9)



(10) Two inductors having inductances $L_1 = 0.1$ H and $L_2 = 0.2$ H are assumed to be well separated. What is the equivalent self-inductance L_{eq} between terminals a and b when the two inductors are placed in: (a) series, and (b) parallel, see Fig. 28.24?

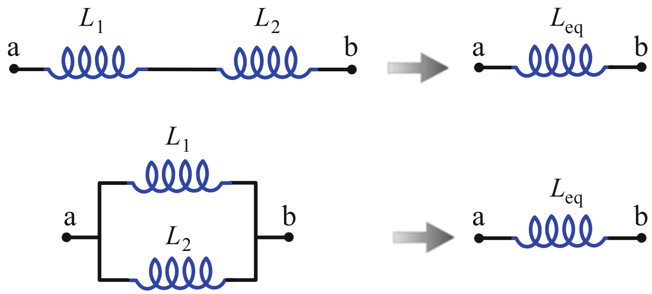


Fig. 28.24 See Exercise (10)

Section 28.2 Mutual Inductance

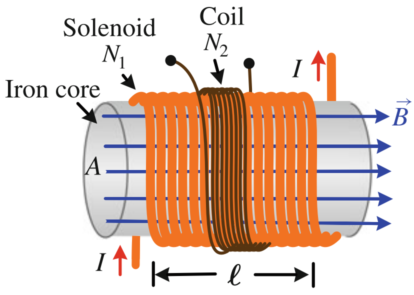
(11) When the current in a coil changes at a rate $dI/dt = 2$ A/s, an emf of 5 mV is induced in a nearby coil. What is the mutual inductance of the combination?

(12) The primary current in a transformer changes at a rate of 3 A/s. What is the induced emf in the secondary coil if the mutual inductance between the primary and secondary coils is 0.4 H?

(13) Primary and secondary coils have a common cylindrical iron core to allow for a common value of magnetic flux. A magnetic flux of 5×10^{-3} Wb is established in the primary coil when the current passing through it increases from zero to 5 A. What is the mutual inductance of the two coils if the secondary coil is an open circuit and has 20 loops?

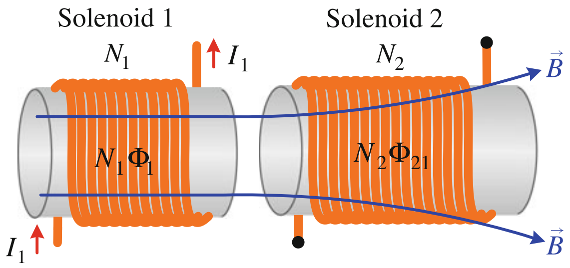
(14) A solenoid of length $\ell = 1.5$ m containing $N_1 = 500$ turns is wound around an iron core of cross-sectional area $A = 3 \times 10^{-3}$ m^2 and relative permeability $K_m = \mu_m/\mu_\circ = 2{,}100$. A second coil containing $N_2 = 40$ turns is wrapped around the solenoid such that the flux from the solenoid passes through the second coil, see Fig. 28.25. The current in the solenoid drops from 10 A to zero in 40 ms. (a) What is the mutual inductance of the combination? (b) What is the emf induced in the second coil?

Fig. 28.25 See Exercise (14)



(15) Two solenoids are close to each other and share the same cylindrical axle, see Fig. 28.26. The first solenoid has $N_1 = 250$ turns and the second solenoid has $N_2 = 500$ turns. A current $I_1 = 5$ A produces an internal magnetic flux per turn $\Phi_1 = 350\,\mu$Wb in the first solenoid and an external magnetic flux per turn $\Phi_{21} = 10\,\mu$Wb in the second solenoid. (a) What is the self-inductance of the first solenoid? (b) What is the mutual inductance of the two solenoids? (c) What is the emf induced in the second solenoid when the current in the first solenoid increases at a rate $dI_1/dt = 0.25$ A/s?

Fig. 28.26 See Exercise (15)



(16) Two inductors having self-inductances L_1 and L_2 and mutual inductance M_s when connected in series and M_p when connected in parallel, as shown in Fig. 28.27. Find the equivalent self-inductance L_{eq} of the system in both the series and parallel cases.

Section 28.3 Energy Stored in an Inductor

(17) An air-filled solenoid has length $\ell = 20$ cm and cross-sectional area $A = 10^{-4}$ m^2. The magnetic field inside the solenoid is uniform and has the value $B = 0.2$ T while the field outside the solenoid is very small (i.e. negligible).

(a) Find the magnetic energy density inside the solenoid. (b) How much magnetic energy is stored in this field?

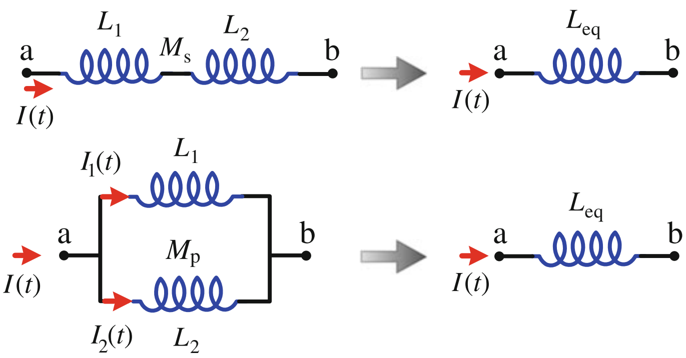


Fig. 28.27 See Exercise (16)

(18) An air-filled solenoid has $N = 500$ turns and carries a current $I = 1.5$ A in order to produce a magnetic flux per turn $\Phi_B = 3 \times 10^{-4}$ Wb. What is the energy stored in the magnetic field of the solenoid?

(19) An air-core solenoid has $N = 300$ turns, a length $\ell = 15$ cm, and a cross-sectional area $A = 10^{-4}$ m^2. How much magnetic energy is stored in its magnetic field when the current in the solenoid is $I = 0.5$ A?

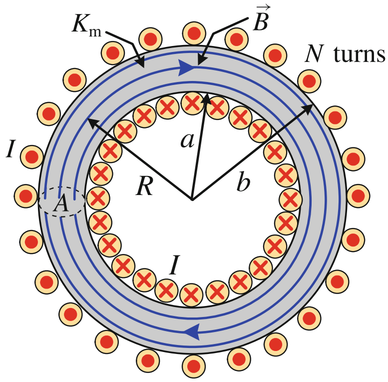
(20) Typical large experimental values of magnetic and electric fields that are used in laboratories are $B_{\text{large}} = 2$ T and $E_{\text{large}} = 10^4$ V/m. (a) Find and compare the energy density for each field. (b) Find the value of the electric field that produce the same energy as the magnetic field $B_{\text{large}} = 2$ T, and then compare this electric field with $E_{\text{breakdown}} = 3 \times 10^6$ V/m, the breakdown electric field in air.

(21) An electromagnet stores 800 J of magnetic energy when a current $I = 10$ A is used in its wires. What is the average emf induced if the current reduces to zero in 0.5 s?

(22) A loop of wire of radius $R = 30$ cm carries a current $I = 10$ A. What is the magnetic energy density at its center?

(23) A long narrow toroid has an average circumference $2\pi R$, cross-sectional area A, number of turns N, and permeability $K_m\, \mu_\circ$, see Fig. 28.28. (a) For circles of radii $a < r < b$ use the validity of $1/r \approx 1/R$ to show that the self-inductance of the toroid's coil is given by $L = K_m\, \mu_\circ N^2 A/(2\pi R)$. (b) Show that the energy stored per unit volume in the magnetic field of the toroid is $BH/2$.

Fig. 28.28 See Exercise (23)



(24) When the toroid of exercise 23 has $K_m = 250$, $A = 2.5 \times 10^{-4}$ m^2, $R = 0.05$ m, $N = 3000$, and current $I = 0.4$ A, find the values of: (a) the self-inductance L, (b) the energy stored in the magnetic field U_B, (c) the magnetic field B, (d) the magnetization field H, and (e) the magnetic energy density u_B.

Section 28.4 The *L* – *R* Circuit

(25) After how many multiplications of the time constant $\tau = L/R$ does the current in Fig. 28.6 reach: (a) 10%, (b) 50%, and (c) 90% of its final value?

(26) An inductor of inductance $L = 2$ H and resistance $R = 1.5\ \Omega$ is connected to a 6-V battery. (a) Find the time required for the current to rise to 80% of its final value. (b) Find the final current through the inductor.

(27) An inductor of inductance $L = 80$ mH is connected in series with a resistor of resistance $R = 4$ kΩ, a switch S, and a battery of emf $\mathcal{E} = 24$ V. (a) What is the time constant of the circuit? (b) How long after the switch S is closed will the current take to reach 99% of its final value? (c) Find the final current through the resistor.

(28) A circuit contains two elements, an inductor of inductance $L = 50$ mH and a resistor. When a battery is connected in series with the two elements, the current increases from zero to 80% of its maximum value in 4 ms. (a) Find the time constant of the circuit. (b) Find the resistance of the resistor.

(29) When an air-core solenoid is connected to a 12-V battery, the current passing through it rises from zero to 63% of its maximum value in 5 ms. However, if the core of the solenoid is filled with iron, the current rises from zero to 63%

of its maximum value in 1.5 s. (a) What is the relative permeability K_m of this iron core? (b) What is the resistance R of the solenoid and the inductance L_{air} of the coil if the maximum current is 0.75 A?

(30) An inductor of inductance L is connected in series with a resistor of resistance R, a switch S, and a battery of emf $\mathcal{E}$. After the switch S is closed at time $t = 0$, find the following: (a) the induced emf in the inductor $\mathcal{E}_L(t)$, (b) the power output of the battery $P_{output}(t)$, (c) the power dissipated in the resistor $P_{diss}(t)$, (d) the rate at which energy is stored in the inductor $dU_B(t)/dt$, and (e) evaluate parts (a–d) when $\tau = L/R$, where τ is the time constant of the circuit.

(31) In Fig. 28.29, $\mathcal{E} = 12$ V, $R_1 = 4\,\Omega$, $R_2 = 6\,\Omega$, and $R_3 = 3\,\Omega$. Determine the currents I_1, I_2, and I_3 at: (a) $t = 0$, when S is closed, (b) $t = \infty$, when S is closed for a very long time, (c) $t = 0$, when S is reopened (after being closed for a long time in part b, and (d) after a long time following part c.

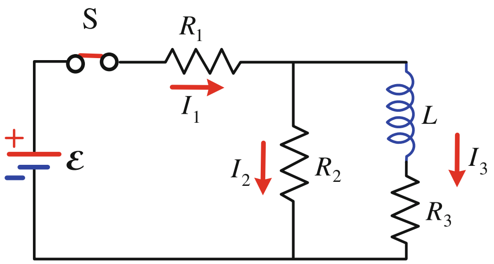


Fig. 28.29 See Exercise (31)

(32) In Fig. 28.8, take $\mathcal{E} = 9$ V, $R = 4\,\text{k}\Omega$, and $L = 40$ mH. The switch S in part a of the figure is connected to position a for a sufficient amount of time so that a steady current flows in the circuit. At $t = 0$, the switch S is disconnected from position a and connected instantaneously to position b to allow the current to decay exponentially through the resistor. (a) Find the induced emf $\mathcal{E}_L$ in the inductor as a function of time. (b) At what times does $\mathcal{E}_L(t)$ reach its maximum and minimum values?

Section 28.5 The Oscillating *L* – *C* Circuit

(33) Find the inductance of an L-C circuit that oscillates at 1 MHz when the capacitor's capacitance is 2 nF.

(34) An $L-C$ circuit has $L = 0.5$ H and $C = 8\,\mu$F. At $t = 0$, the initial charge on the capacitor is $Q = 400\,\mu$C. (a) What is the frequency of oscillation? (b) What is the maximum value of the current? (c) Represent the current as a function of time. (d) What is the maximum energy stored in the magnetic field of the inductor?

(35) When the capacitor of an $L-C$ circuit is originally charged to a potential difference of 10 V, the circuit oscillates at 1 kHz. A maximum current of 1 A is attained after quarter of a cycle and again after three quarters of a cycle. What are the values of the inductance L and capacitance C of the circuit?

(36) A radio tuner has an $L-C$ circuit of variable capacitance and a fixed inductance. The radio is tuned to a station of frequency 1.5 MHz when the tuner has a capacitance of 0.15 nF. (a) What must be the capacitance of the tuner in order to receive a station that broadcasts at a frequency of 0.8 MHz? (b) What is the inductance of the tuner?

(37) An $L-C$ circuit has an inductor of inductance $L = 20$ mH and a capacitor of capacitance $C = 2\,\mu$F. The capacitor is fully charged by a 50 V power supply and then discharged through the inductor. Use the concept of energy stored in the capacitor and inductor to find the maximum current in the oscillating circuit.

Section 28.6 The *L* – *R* – *C* Circuit

(38) In the circuit of Fig. 28.14, take $R = 8\,\Omega$, $L = 2.5$ mH, and $C = 2\,\mu$F. Does this circuit oscillate? If it does, then find the frequency of this oscillation.

(39) In the circuit of Fig. 28.14, take $R = 1.6\,\Omega$, $L = 1$ mH, and $C = 10^{-3}$ F. (a) Show that this circuit oscillates. (b) Determine the frequency of the circuit. (c) When $\phi = 0$ and $t > 0$, find the time when the cosine term of Eq. 28.26 first becomes -1 and then find the ratio q/Q at this time. (d) What resistance R is required to make this circuit oscillate with one-half the undamped frequency of the $L-C$ circuit?

(40) For the $L-R-C$ circuit of exercise 39, find the resistance R that will make the resistor dissipate only 5% of the circuit's energy in each cycle.

(41) An $L-R-C$ circuit executes a damped oscillation and its energy decreases by 2 % during each oscillation when it has a resistor of resistance $R = 10\,\Omega$. When the resistor is removed, the pure $L-C$ circuit oscillates at a frequency of 2 kHz. Find the inductance and capacitance of the circuit.

Sections 28.7 and 28.8 Circuits with ac Source—*L – R – C* Series in an ac Circuit

(42) A sinusoidal 50-cycle per second ac voltage is read to be 220 V by a voltmeter. (a) What is the peak (maximum) voltage of the source? (b) Find an equation that represents this voltage as a function of time.

(43) An ac voltage $v = (155.6\,\text{V})\sin(100\,\pi\,t)$ is applied across a resistor of resistance $R = 20\,\Omega$. (a) What will be the reading of an ac voltmeter placed in parallel with the resistor? (b) What will be the reading of an ammeter placed in series with the resistor? (c) What is the frequency of the ac voltage?

(44) A coil has an inductance $L = 0.5$ mH and a small resistance $R = 1\,\Omega$. Find the current in the coil when the applied voltage is: (a) 110-V dc, and (b) 110-V (rms) with a frequency $f = 60$ Hz.

(45) Repeat exercise 44 when the coil is replaced by a capacitor of capacitance $C = 2\,\mu\text{F}$.

(46) A coil has an inductance $L = 0.2$ mH and a resistance $R = 10\,\Omega$. When a voltage of 220-V (rms) with frequency $f = 50$ Hz is applied, find the impedance of the circuit and rms current in the coil.

(47) An ac source of frequency 50 Hz is connected in series with a resistor of resistance $R = 4\,\text{k}\Omega$ and an inductor of inductance $L = 0.5$ H. At what frequency does the circuit's impendence double?

(48) An $R-C$ circuit of $R = 3\,\text{k}\Omega$ and $C = 1\,\mu\text{F}$ is connected to a 50 Hz ac source of 220 V(rms). (a) What is the impedance of the circuit? (b) What is the rms current in the circuit? (c) What is the phase angle between the current and the voltage? (d) What is the power dissipated in the circuit? (e) What are the voltmeter readings across the resistor and capacitor?

(49) In Fig. 28.30, $R = 1\,\text{k}\Omega$ and $C = 1\,\mu\text{F}$. Digital voltmeters are used to measure the voltages across the ac source, the resistor, and the capacitor. Their measurements are $(V_{\text{rms}})_{\text{ac}} = 104.80\,\text{V}$, $(V_{\text{rms}})_R = 31.42\,\text{V}$, and $(V_{\text{rms}})_C = 100.00\,\text{V}$, respectively. (a) Find the frequency of the source. (b) Why is the voltage of the ac source not equal to the sum of the voltages across the resistor and the capacitor?

(50) An ac source of 110-V (rms) and frequency $f = 60$ Hz is connected to an $L-R-C$ series circuit which has a resistor of resistance $R = 8\,\Omega$, an inductor of inductive reactance $X_L = 9\,\Omega$, and capacitor of capacitive reactance $X_C = 3\,\Omega$.

(a) Find the impedance of the circuit. (b) Find the current in the circuit. (c) Find the voltage across the resistor, the inductor, and the capacitor.

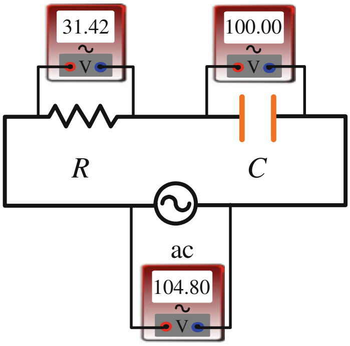


Fig. 28.30 See Exercise (49)

(51) For the circuit in exercise 50, find: (a) the inductance and capacitance of the circuit, (b) the power factor of the circuit, and (c) the power dissipated in the circuit.

(52) An ac source of 110-V (rms) and angular frequency $\omega = 377$ rad/s is connected to an $L-R-C$ series circuit, where $R = 35\,\Omega$, $L = 100$ mH, and $C = 650\,\mu$F. Find: (a) the inductive reactance, the capacitive reactance, and the impedance of the circuit, (b) the peak and rms current, (c) the peak voltage, the instantaneous voltage, and the rms voltage across each element, (d) the phase angle ϕ and the average power dissipated in the circuit.

(53) Show that the charge q on the capacitor of the $L-R-C$ series circuit of Fig. 28.20 has a peak value given by:

$$Q = \frac{V}{\sqrt{(\omega R)^2 + \left(\omega^2 L - \frac{1}{C}\right)^2}}$$

and show that $Q_{\max}$ occurs at an angular frequency ω' given by:

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$$

Section 28.9 Resonance in *L* – *R* – *C* Series Circuit

(54) An $L-R-C$ series circuit has $R = 4\,\text{k}\Omega$ and $L = 6$ mH. (a) What must the value of the capacitance C be in order to produce a resonance at frequency of 40 kHz?

(b) What is the maximum rms current in the circuit when the rms voltage of the source is 150 V?

(55) In the L–R–C series circuit of exercise 54, find: (a) the impedance of the inductor and capacitor, and (b) the power dissipated in the circuit.

(56) An L–R–C series circuit has $R = 20\,\Omega$, $L = 0.16\,\text{H}$, $C = 30\,\mu\text{F}$, and an ac source of peak voltage 250 V. For a certain angular frequency, the power factor of the circuit becomes unity and the circuit consumes the maximum power. (a) Find this angular frequency. (b) Find the inductive reactance, the capacitive reactance, the impedance of the circuit. (c) Find the phase angle ϕ and the maximum current in the circuit. (d) Find the peak voltage across the resistor, the peak voltage across the inductor, and the peak voltage across the capacitor.